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Exercises:

1.1 We know $\frac{1}{2}\sum_{v\in V} d(v) = m$, which means that $\sum_{v\in V} d(v)$ is even.
We have two types of nodes: let them belong to sets A and B
$v\in A$ iff $d(v)=3$, $v\in B$ iff $d(v)\leq 2$, $m=12$, $|A|=6$, $|B|=?$
We want the smallest sample

$$\frac{\sum_{v\in A} d(v) + \sum_{v\in B} d(v)}{2} = 12 \rightarrow \sum_{v\in B} d(v) = 24-18 = 6$$

Since $\min|G| = |A| + \min|B|$, then we should have $|B|=3$ where each node has degree of 2.

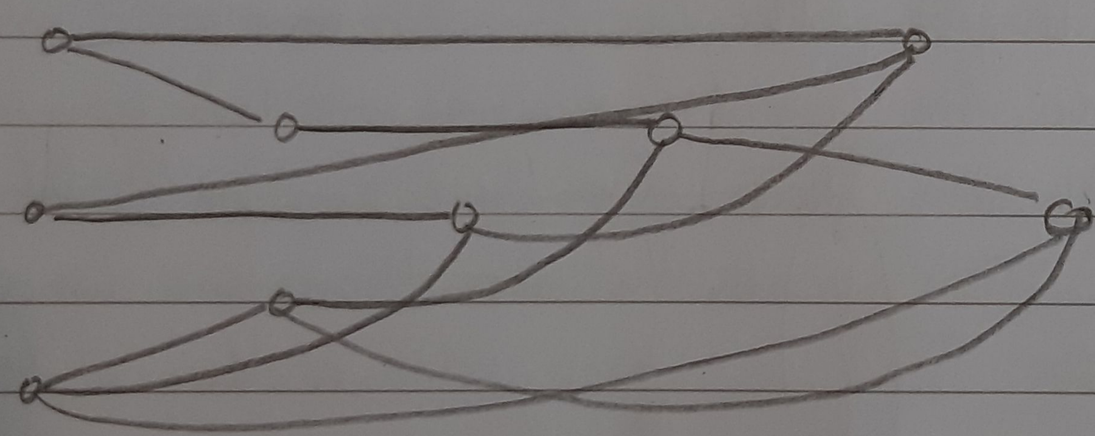
To get a sample graph we use score reduction:

$D = (2\ 2\ 2\ 3\ 3\ \underline{3\ 3\ 3}\ \boxed{3})$

$D' = (2\ 2\ 2\ 2\ \underline{2\ 2\ 3}\ \boxed{3})$

$D'' = (1\ 1\ 2\ 2\ \underline{2\ 2}\ \boxed{2}) \rightarrow D''' = (1\ 1\ 1\ \underline{1\ 2}\ \boxed{2})$

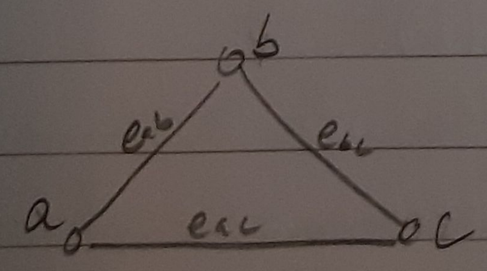
$D^{IV} = (0\ 1\ 1\ \underline{1}\ \boxed{1}) \rightarrow D^{V} = (0\ 0\ \underline{1}\ \boxed{1}) \rightarrow D^{VI} = (0\ 0\ 0)$



1.2 Given $G=(V,E)$ we want to prove or disprove that

$$\sum_{v\in V} d(v)^2 = \sum_{\{x,y\}\in E} (d(x)+d(y))$$

If we realize the summation over the edges, it means that a particular node x, its degree value, will be added a total of $d(x)$ times, which adds up to $d(x)^2$.

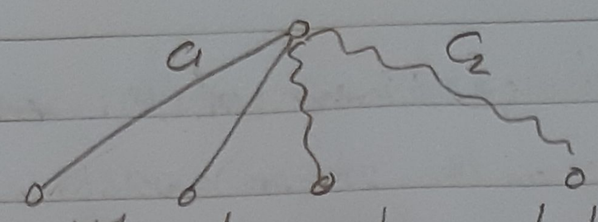


$$\sum_{\{x,y\}\in E} (d(x)+d(y)) = (d(a)+d(b))_{e_{ab}} + (d(a)+d(c))_{e_{ac}} + (d(b)+d(c))_{e_{bc}} = 12 = \sum_{v\in V} d(v)^2$$

<u>Formally</u>: $\sum_{\{x,y\}\in E}(d(x)+d(y)) = \sum_{\{x,y\}\in E} d(x) + \sum_{\{x,y\}\in E} d(y) = \sum_{x\in V}\sum_{\{x,y\}\in E} d(x) + \sum_{y\in V}\sum_{\{x,y\}\in E} d(y)$

$$= \sum_{v\in V}\left[\sum_{\{x,w\}\in E} d(w) + \sum_{\{y,v\}\in E} d(w)\right] = \sum_{v\in V}\sum_{i=1}^{d(v)} d(v) = \sum_{v\in V} d(v)\cdot d(v) = \sum_{v\in V} d(v)^2$$

1.3 b) Consider, for instance



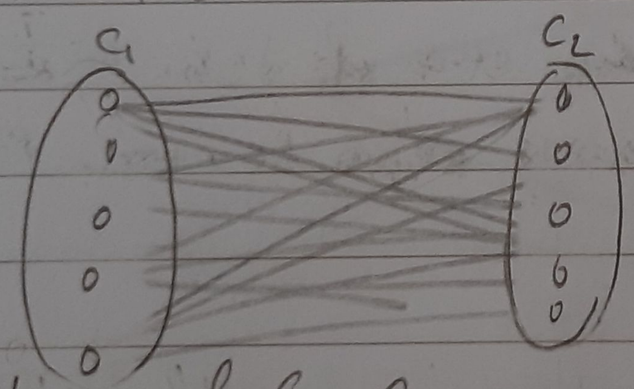
$|C_1| = |C_2|$, and they don't share a ~~vertex~~ edge. Therefore, it is not true that they will always share a common edge.

a) Prove or disprove: C_1 and C_2 share a common vertex

Assume they don't. C_1 is the longest path, C_2 the next longest. There must be a subpath $C_1[x]$ which is at least the same size as C_2. Because of this, $||C_1| - |C_2|| \leq 1$; ie, it cannot be more since it would be a contradiction because $C_2 = C_1 - v$ where $v \in C_1$, at an end. Now that we are considering cases where $C_2 \notin C_1$, the question is: given that the graph is connected, which means there is a path from each node to every node, can the two longest paths C_1 and C_2 exist without sharing a node and without creating the contradiction that the path between them is larger than one of them?

The answer is no.

Consider any two longest paths C_1, C_2 in a connected graph. Consider also that the nodes from C_1 and C_2 form a complete ~~[illegible]~~ In their connections graph.



Is there any path between C_1 and C_2 that doesn't create a longest path C_3 which is longer than, at least, C_2? No. Therefore we have a contradiction. Along the choice of C_1, C_2 since a longer path exists. ■

The minimum exists when both IC1I = IC2I and they are even.
Then at middle point we can create C3 which is IC1I = IC3I.
They share a vertex.

2.1 Prove or disprove that in a simple undirected graph G without loops and with $n > 2$ vertices, if $\delta(G) \geq \frac{n-1}{2}$ then G is connected.

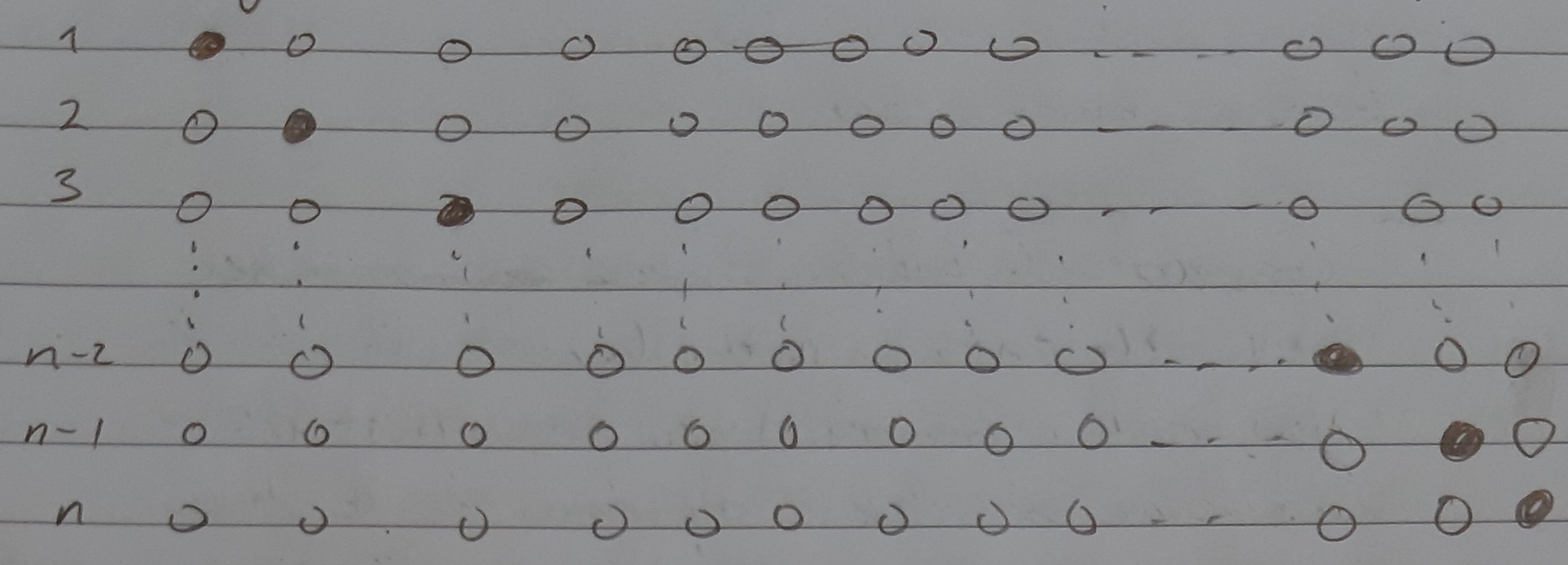
We will proceed by direct proof, considering the extreme $\delta(G) = \lceil \frac{n-1}{2} \rceil$

We have two cases: when n is odd, then $\delta(G) = \frac{n-1}{2}$

when n is even, then $\delta(G) = \frac{n}{2}$

Case #1: Since $\delta(G) = \frac{n-1}{2}$, this means that for each vertex $v \in V$ its degree $d(v) \geq \delta(G)$. In words, each vertex v is connected at least to half of the remaining vertices. By the Pigeonhole Principle it must be that the graph is connected. Consider a set of nodes with the given prescriptions, and at each iteration we create the edges of a particular node. Necessarily we will add edges that also belong to other nodes. As we continue, some of these nodes

will see their $d(v)$ exhausted, and we will have to add edges with other nodes, which also have their prescribed $d(w)$.



In other words, consider nodes i, j. Since node i connects to half of the rest, and also does j there must be an overlap between their connections since $V\backslash i \cap V\backslash j \equiv V\backslash i,j \rightarrow |V\backslash i,j| = n-2$ so $j \in V\backslash i$ and $i \in V\backslash j$.

If $\{i,j\} \notin E$ then $N(i) \cap N(j) \neq \emptyset$, where $N(\cdot)$ denotes the set of neighbors of the node. Here we can see that because $|V\backslash i,j| = n-2$ and $\delta(G) = \frac{n-1}{2}$, the Pigeonhole Principle can be used.

<u>Case #2</u>: Here we consider the case of $\delta(G) = \frac{n}{2}$ which is weaker than case #1. Same reasoning allows to infere that it also holds for case #2. ■

2.2 Prove or disprove: if G has a radius at most k and the greatest degree $\Delta(G)$ is at least 3, then G has less than $\frac{d}{d-2}(d-1)^k$ vertices, where $d = \Delta(G)$.

We will proceed by a direct proof.

<u>Proof</u>: The graph G for which the maximum amount of vertices is obtained under these conditions is a tree of branching factor $\Delta(G)-1$ with height $h = rad(G) - 1 = k-1$, repeated $\Delta(G)$ times and connected to one extra node.

A tree has a total of $\sum_{i=1}^{k} d^i$ vertices, where d is the branching factor. For our case $n = d\sum_{i=0}^{k-1}(d-1)^i + 1$

We know geometric series $\sum_{k=0}^{n-1} ar^k = \left(\frac{1-r^n}{1-r}\right)a$

So $\sum_{i=0}^{k-1} d(d-1)^i = d\left(\frac{1-(d-1)^k}{1-(d-1)}\right) = \left(\frac{1-(d-1)^k}{2-d}\right)d = \left(\frac{(d-1)^k-1}{d-2}\right)d$

Then $n = \sum_{i=0}^{k-1} d(d-1)^i + 1 = \frac{d(d-1)^k}{d-2} - \frac{d}{d-2} + \frac{d-2}{d-2} \boxed{= \frac{d(d-1)^k}{d-2} - \frac{2}{d-2}}$

So $n \leq \frac{d}{d-2}(d-1)^k$. ■

2.3 Let G be a simple undirected graph without loops with $\delta(G) > 1$.
Prove or disprove the following claims:

1) In G there exists a path of length $\delta(G)$.

<u>Proof</u>: We will proceed by direct proof
Consider the following
$v \in V$, $d(v) \geq \delta(G)$, $u \in N(v)$
Now, each u from $N(v)$ also has $d(u) \geq \delta(G)$
So, if we consider a sequence as follows:
$v \in V$, $u \in N(v)$, $w \in N(u) \setminus \{v\}$, $x \in N(w) \setminus \{v, u\}$, $y \in N(x) \setminus \{v, u, w\}$,
etc., we can see that we will find a path of length at least $\delta(G)$. We can aid/justify the intuition by invoking the Pigeonhole Principle.

If in N(v)\X for v in V we have that |X| < delta(G),
then we know that |N(v)\X| != 0

2) In G there exists a circuit of length at least $\delta(G)+1$.

<u>Proof</u>: We will proceed by direct proof.
We have already proved in 1 that in G there exists a path of length $\delta(G)$. We must now show that a circuit will be formed of length at least $\delta(G)+1$.
Consider the longest path P with $|P| \geq \delta(G)$
Since it is the longest path, any $v \in P$ has $N(v) \subseteq P$
(otherwise, if $N(v) \cap V \setminus P \neq \emptyset$, then $|P'| \geq |P|$, where $P' = P + u$ and $u \in N(v)$, $u \notin P$).
Then, consider a pair x, y in P such that $\{x, y\} \in E$, but in the path $P = AxByC$, where $A, B, C \subseteq P$
Choose x, y so that $|B|$ is maximal. Then, $|B|$ is at least $|N(x) \setminus y|$. by contradiction, then $z \in N(x)$ such that $z \notin B$, but we can have $zB'y$ where $B' = xB$, and $|B'| \geq |B|$, contradicting the choice of xBy.

N(x)\y is a subset of B,
and |B| >= delta(G) - 1
since |N(x)\y| = delta(G) - 1

Therefore, $|B| \geq \delta(G) - 1$, and xBy is a path involving a total of $\delta(G)+1$ vertices and $\delta(G)$ edges.
Since $\{x, y\} \in E$ we now have proved that a circuit of length $\delta(G)+1$ does exist under the given conditions.

$\blacksquare$

3.1 Prove or disprove: Every simple undirected graph G without loops with at least two vertices contains at least two vertices that are not cut vertices.

Recall that a cut vertex is a vertex v such that $G \setminus v$ is disconnected.
Proof: we will proceed by direct proof.
Consider a minimum spanning tree T of G
Given the conditions, the tree T will be at least a path, which means it will have at least two leaves.
Take any two leaves of the minimum spanning tree: this two nodes do not disconnect T, and since $T \subseteq G$, they neither disconnect G.
Therefore, it follows that the statement holds. ■

A cut vertex means that there is not path from some vertex to at least another. Removing the leaves doesn't change the paths traced by the minimum spanning tree, which reaches the rest of vertices.

3.2 Prove or disprove: A connected simple undirected graph G without loops is 2-connected iff every pair of distinct edges of G is contained in a common circuit.

Proof: we must proof necessity and sufficiency.
• Sufficiency: If every pair of distinct edges of G is contained in a common circuit, then a connected simple graph G without loops is 2-connected.
We will proceed by direct proof.
Consider any pair of edges e_1, e_2 which belong to common circuit C. Taking away any vertex $v \in C$ will not disconnect the graph. Since this holds for any pair of edges, it follows that the graph is 2-connected.

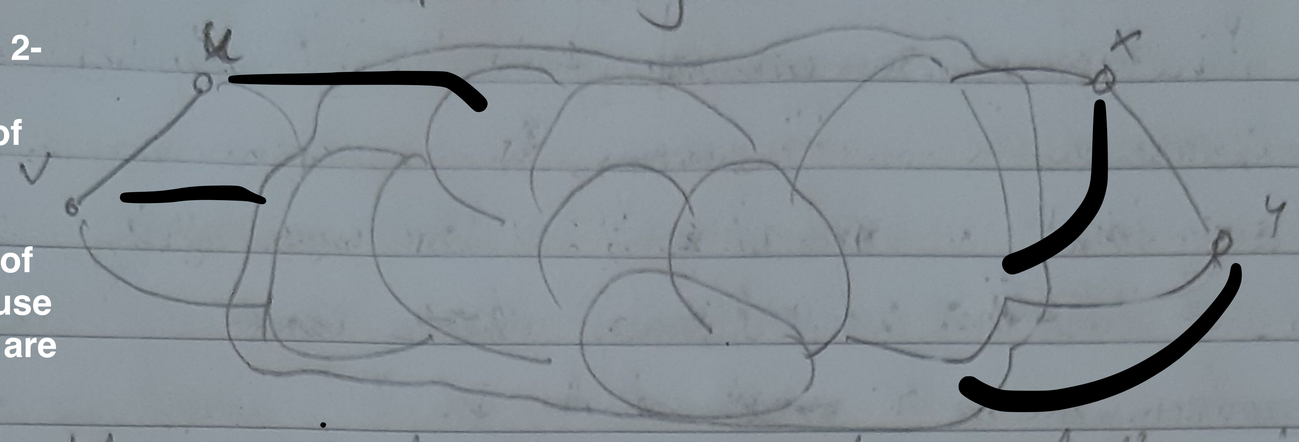
Removing v will not make e1 unreachable from e2.
Consider first that v is not an end vertex of e1, nor of e2. Then, since they are connected by a circuit, they are reachable through two distinct paths, and v doesn't belong to one of them.
Consider the case where v is an end vertex of one of the edges or of both.
Consider the remaining pair of edges in G after the removal of v. The first case applies for these pairs.
Since it holds for each pair, then the graph is 2-connected.

• Necessity: If a graph G which is connected, simple, without loops, is 2-connected, then every pair of distinct edges of G is contained in a common circuit.
Recall that a graph is 2-connected iff for every pair of distinct vertices there is a common circuit that contains them.
Now, we want to prove that this also holds analogously for edges.
Consider any pair of edges $e_1 = \{u, v\}$, $e_2 = \{x, y\}$.
We know there is a common circuit between all pairs we can think of: u, v; u, x; u, y; v, x; v, y; x, y

We have the following scenario:

We know only about the relationship of connectivity of the vertices (due to G being 2-connected).
Here we prove that it also holds for pairs of edges when it holds for vertices.

(*) It is irrelevant if the paths are subpaths of one another, the statement still holds, because we only want to show that indeed e1 and e2 are contained in a common circuit.



Let the circuits be named as follows:

$C_1 = uvP_{11}u$, $C_2 = uP_{21}xP_{22}u$, $C_3 = uP_{31}yP_{32}u$

$C_4 = vP_{41}xP_{42}v$, $C_5 = vP_{51}yP_{52}v$, $C_6 = xyP_{61}x$

We can now make a new circuit containing e_1 and e_2:

$C = uvP_{41}xyP_{32}u$

The originals were C1 (for u,v) and C6 (for x,y).

Therefore, it follows that, if such G is 2-connected, then every pair of distinct edges of G is contained in a common circuit. ∎

3.3 $G \in \mathcal{G}$, $|G| = n$ st $|E(G)|$ is maximal for the given conditions: which means, $\exists A \subseteq G : u, v \in V(A)$, $A + \{u,v\} \notin \mathcal{G}$, $u \neq v$

Prove or disprove: For the given conditions, the relation R defined on the set of vertices $V(G)$ by uRv iff $\{u,v\} \notin E(G)$ is an equivalence relation on $V(G)$.

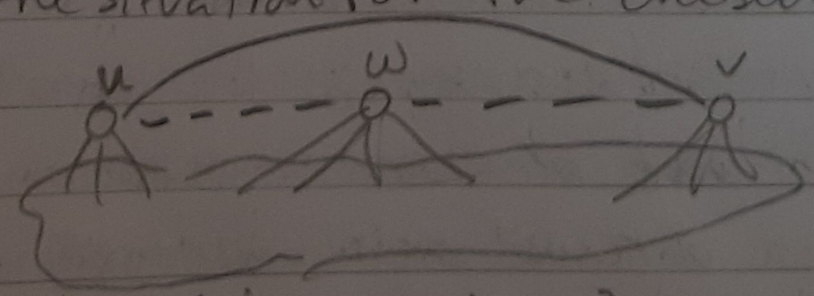
<u>Proof</u>: we must prove that R is reflexive, symmetric and transitive, in order to show that R is an equivalence relation

- <u>Reflexive</u>: R is reflexive since the graph does not have loops.
- <u>Symmetric</u>: R is symmetric because if $\{u,v\}$ are not joint by an edge, neither are $\{v,u\}$.
- <u>Transitive</u>: prove that if aRb, bRc then aRc.

In our case, if uRw, $\{u,w\} \notin E(G)$, wRv, $\{w,v\} \notin E(G)$ then uRv, $\{u,v\} \notin E(G)$.

We will proceed by contradiction, assuming that uRw and wRv hold as specified, but uRv has $\{u,v\} \in E(G)$.

Consider the situation for the chosen graph G



We can add edges $e_1 = \{u,w\}$ or $e_2 = \{w,v\}$ to G such that $G' = G + e_1$ (or e_2) and $G' \in \mathcal{G}$, which contradicts the choice of G (maximal for $|G| = n$). Therefore R is transitive.

We can add e_1 or e_2 because of the following: consider the induced subgraph $A = G[N(u) \cup N(v)] \in \mathcal{G}$. Note $u, v \in A$. Because G is maximal, there are pairs of vertices $x, y \in A$ such that $A + \{x, y\} \notin \mathcal{G}$. More concretely, if we consider the alternative $G' \in \mathcal{G}$ such that $\{u, v\} \notin G'$, there are edges $\{x, v\}$ (or $\{x, u\}$) which are in G' and not in G. We consider that the edges are $E'_v = \{\{x, v\} \in G', \{x, v\} \notin G\}$. Let $E'_{vw} = E'_v[w]$. Consider any 4 vertices such that $u, v, x : \{x, v\} \in E'_{vw}, y \in N(u) : \{x, y\} \in E$. Then, due to the existance in G of $\{u, v\} \in E$, and due to maximality of G, I cannot add any chosen $\{x, v\}$ to G.

However, considering now the same 4 vertices from above, and w: we can see that v, x, y, w don't form a maximal subgraph wrt. $\mathcal{G}$, since $\{v, x\} \notin G$ and also $\{v, w\} \notin G$.

Therefore, we can add $G + \{v, w\} \in \mathcal{G}$. We know that the induced subgraph $G[N(v) \cap N(w) + v + w]$ is not maximal because of the missing edges in G, denoted by E'_{vw}. Remark that the criteria of choice for u, v, x spans the whole of $N(v) \cap N(w)$, and therefore we can finish the proof with the certainty that adding $\{v, w\}$ to G will not create a graph G' that falls out of the class $\mathcal{G}$. ■

We only consider the case where we can add {w,v}. It holds analogously for the other case.

(from G' - G), but what about the neighborhood with edges also in G, can they form a K4 when adding {v,w}? Can there exist any vertex such that u, v, w, y1, y2 form K4 when adding {v,w}? It can be proven that such a graph would not be maximal. That we can make it bigger by removing {y1,y2} and adding {v,w} and {u,y1}.

We are considering the case where the edge between u and v is not added and the graph is maximal (G'). Then there are some vertices to which u and v are connected in such a graph, and they are not connected to, in our current graph (G). Why? Because since {u,v} in E, they cannot share as many neighbors, or they would form a K4 subgraph.

Let Ev' identify that difference in edges.

Evw' is the set of such edges whose endvertices not v have some other edge with an endvertex in w.

x is one of the endvertices of the edges in Evw' which is connected to v.

4.1 Prove or disprove: A directed graph is strongly connected iff it can be represented as a union of distinct cycles $C_1, C_2, \ldots, C_k$ such that every two cycles C_i, C_{i+1}, $i = 1, \ldots, k-1$ share a common vertex.

<u>Proof</u>: we must prove necessity and sufficiency of the statement.

Recall that a directed graph G is strongly connected if for any pair of vertices $u, v \in V(G)$ there is a directed path from u to v, and from v to u.

- <u>Necessity</u>: If a digraph is strongly connected, then it can be represented as a union of distinct cycles st every two C_i, C_{i+1} share a common vertex.

  We will proceed by direct proof.

  By the def. of strong connectivity we know that there has to be a directed path between each pair of vertices.

  Thus we can view the graph as made of such cycles, if we know it is SC: if a component couldn't be represented as a cycle it would be a contradiction on the assumption of strong connativity.

Let us assume that C_i and C_{i+1} are such that they don't share a node. Then, we have $C_i \xrightarrow{a} C_{i+1} \xrightarrow{b} C_i$, for instance (WLOG). Because there is a cycle C' such that $C' = b_i P_i a_i a_{i+1} P_{i+1} b_{i+1}$, where $P_i \subseteq C_i$, $P_{i+1} \subseteq C_{i+1}$ and $(a_i, a_{i+1}) = a$, $(b_{i+1}, b_i) = b$. Then, consider that $P_i' = b_i P_i a_i$, $P_{i+1}' = b_{i+1} P_{i+1} a_{i+1}$. We can rename $C_{i-1} = C_i$, $C' = C'$ and $C_{i+1}' = C_{i+1}$, where $a_i, b_i \in C_i$, $a_{i+1}, b_{i+1} \in C_{i+1}$.

This is actually a proof by contradiction on the maximality of the components.

<u>Sufficiency</u> If a digraph can be represented as a union of distinct cycles such that the neighbors C_i, C_{i+1} share at least a node in common, then the digraph G is strongly connected.

The proof follows trivially by the given assumptions, since such a directed graph is, by definition, strongly connected. ∎

4.2 Prove or disprove: Given a simple digraph G without loops and with n vertices and m edges, if G is connected but not strongly connected then $n-1 \le m \le (n-1)^2$.

<u>Proof</u>: we will proceed by direct proof

$n-1$ lower bound is given by the fact that G is connected

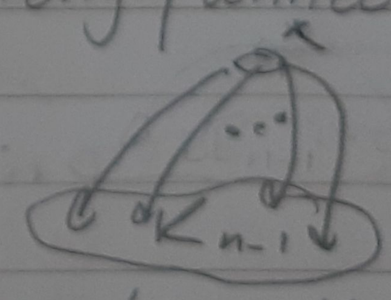
For K_n (complete digraph) with loops, connected $m = n^2$

For K_n (complete digraph) without loops, connected $m = n(n-1)$

A complete graph is strongly connected

$(n-1)^2$ upper bound means that there does not exist a digraph G with $m > (n-1)^2$ which is not strongly connected.

We can view it as follows:



1. K_{n-1} is a complete graph of $n-1$ vertices
2. $m' = |E(K_{n-1})| = (n-1)(n-2) = n^2 - 3n + 2$
3. Can add an edge from x to each of $n-1$ nodes in K_{n-1} and the digraph G will still not be strongly connected.
   $m = m' + (n-1) = n^2 - 2n + 1 = (n-1)^2$
4. Adding any edge from $v \in K_{n-1}$ to x will now make the graph strongly connected. ∎

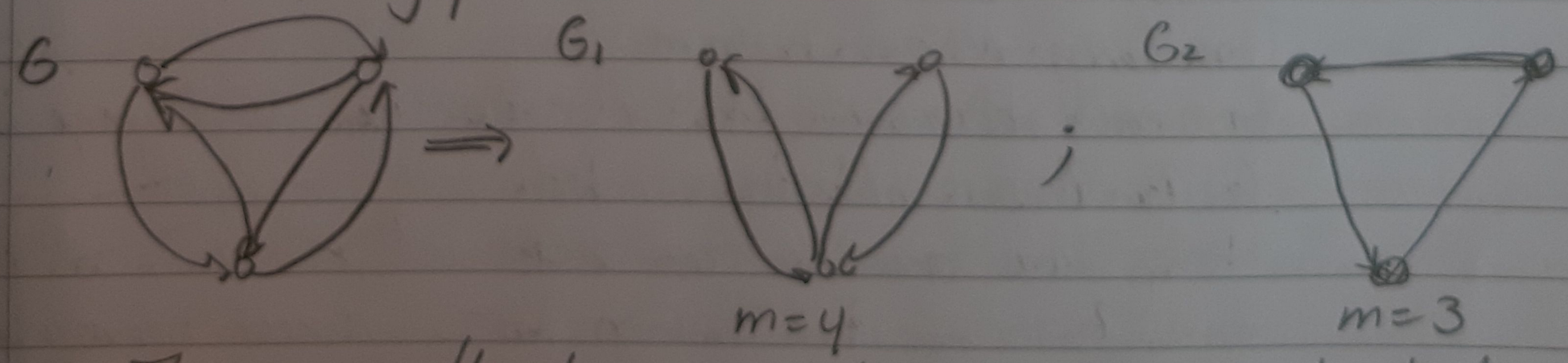
4.3 Prove: Every undirected graph G without loops has an orientation which is an acyclic graph.

<u>Proof</u>: we will proceed by direct proof.
Let us start by sorting the vertices $v_1, \dots, v_n \in V(G)$ such that $d(v_1) \leq d(v_2) : \dots \; d(v_{n-1}) \leq d(v_n)$.
Now, take sequentially the vertices from $v_1, \dots, v_n$ and give a direction to its undirected edges such that $(v_i, v_j) \in E(G')$, $i < j$ meaning that in the resulting directed graph G', edges will go from vertices with lower degree to those with higher degree.
We know this procedure produces an acyclic graph because it produces a topological sort for the digraph G'. ∎

4.4 We want a graph G such that $G_1 \subseteq G$ is a G^+ and $G_2 \subseteq G$ is a G^+; $G_1 \neq G_2$. We also want them to have the smallest number of vertices such that $|E(G_1)| \neq |E(G_2)|$.
Recall that the transitive reduction is the smallest graph (in terms of edges) G_1 which is a subgraph of the original G, and has the property of being a transitive closure of G.
Recall also that an inclusion-wise maximal set among a collection of sets is a set that is not a subset of some other set in the collection. An inclusion-wise minimal set among a collection of sets is a set in the collection that is not a superset of any other set in the collection.
Therefore, we want G_1, G_2 such that:

$$\nexists e \in E(G_1) \text{ such that } G_1 - e \in G^+, \quad |E(G_1)| \neq |E(G_2)|$$
$$\nexists e \in E(G_2) \text{ such that } G_2 - e \in G^+$$

The following pair meets those conditions:



The reason this happens is because we care only about each node being reachable from the nodes from which it has paths in the original graph G. We can make this happen using one cycle, or several, for a complete digraph G. ∎

5.1 Prove or disprove: Given $G \in \mathcal{G}$ with $n > 3$. Assume G does not contain $K_{1,3}$ as an induced subgraph. Then there are vertices x and y in G joined by an edge such that $G \setminus \{x,y\}$ is also connected, where $G \setminus \{x,y\} \equiv G - x - y$.

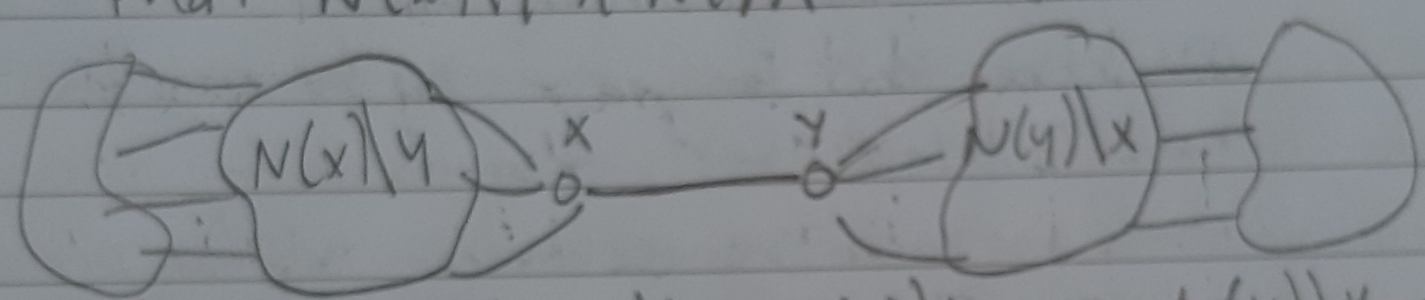
<u>Proof</u>: we want to show that $\exists x,y \in V(G) : \{x,y\} \in E(G)$, $G-x-y$ is connected.

This means that $\neg \forall x,y \in V(G) : \{x,y\} \in E(G)$, $G-x-y$ is disconnected.

We will proceed by contrapositive proof.

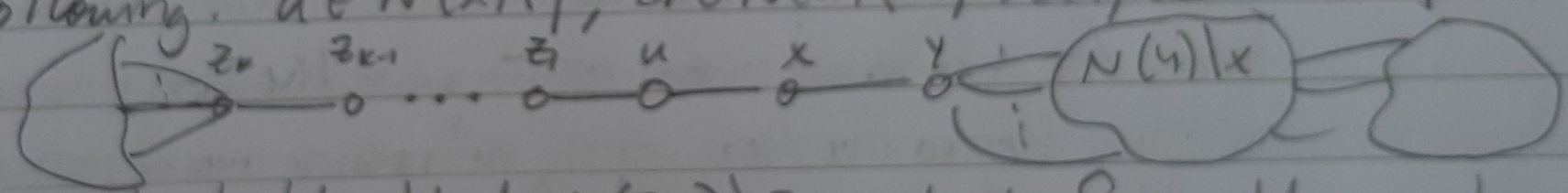
Let us assume that $\nexists x,y \in V(G) : \{x,y\} \in E(G)$, $G-x-y$ is connected then we want to prove that G contains a $K_{1,3}$ induced subgraph.

By our assumptions, removing any pair of adjacent vertices x,y united by an edge, disconnects the graph. This means that $N(x) \setminus y \cap N(y) \setminus x = \emptyset$.



We know that neither $N(x) \setminus y$ nor $N(y) \setminus x$ can be empty, otherwise the graph wouldn't disconnect by removing x,y.

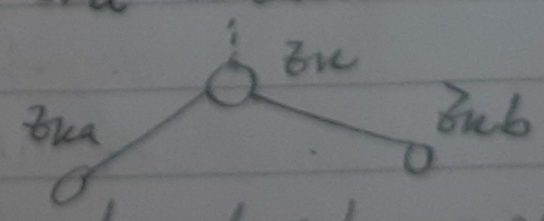
Since this holds for any pair x,y we can take the following: $u \in N(x) \setminus y$, $z_1 \in N(u) \setminus x$, ..., $z_k \in N(z_{k-1}) \setminus z_{k-2}$



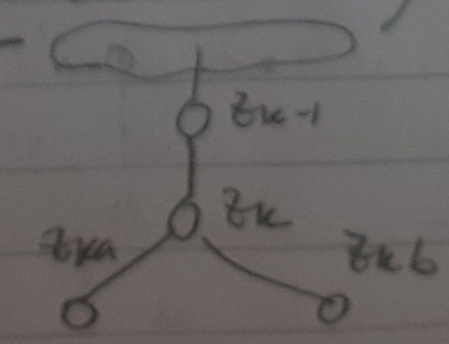
It also holds that $N(z_k) \setminus z_{k-1} \neq \emptyset$. Since the graph is finite there will exist such z_k that it doesn't have $|N(z_k) \setminus z_{k-1}| = 1$. We know that the neighborhoods of pairs x,y also separate G, otherwise it would remain connected.

Thus, $N(z_k) \setminus z_{k-1}$ cannot contain nodes $z_1, \dots, z_{k-2}, u, x, y, \dots$

Therefore, $N(z_k) \setminus z_{k-1}$ must contain two vertices disconnected from each other such that $G - z_k - z_{ka}$ is disconnected, and $G - z_k - z_{kb}$ is also disconnected.



Then, the four vertices now $z_{k-1}, z_k, z_{ka}, z_{kb}$ form a $K_{1,3}$ induced subgraph:



Therefore we have proved that if G does not contain $K_{1,3}$ induced subgraph, then there exists a pair $x,y \in V(G)$ such that $\{x,y\} \in E(G)$ and $G-x-y$ is connected. ∎

5.2 Let $G \in \mathcal{G}$, $n \geq 3$ vertices. Let x, y be two vertices of G for which $\{x,y\} \notin E$ and $d(x)+d(y) \geq n$

Prove or disprove: G contains a Hamiltonian circuit iff also does $G+\{x,y\}$ contain a Hamiltonian circuit.

<u>Proof</u>: we must prove sufficiency and necessity.

- <u>Necessity</u>: if G contains a Hamiltonian circuit, then $G+\{x,y\}$ also does.

  Indeed, if G already contains a Hamiltonian circuit, adding an edge to G preserves the Hamiltonian circuit so also $G+\{x,y\}$ has one.

- <u>Sufficiency</u>: if $G+\{x,y\}$ contains a Hamiltonian circuit, then also does G.

  We will proceed by direct proof.

  We know that for pair $x,y \in V$ such that $\{x,y\} \notin E$ we have $d(x)+d(y) \geq n$. Also note that max degree is $n-1$.

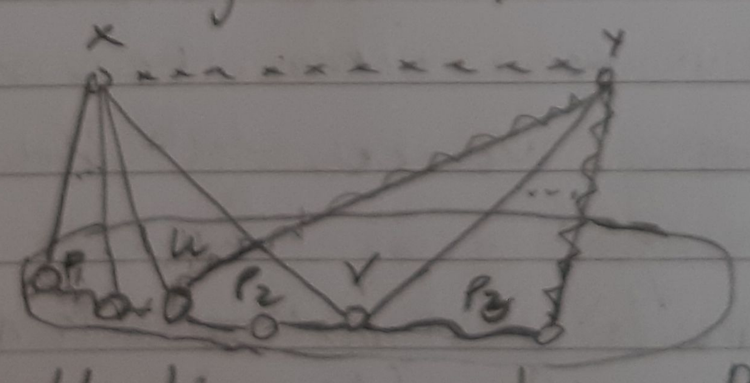
  Then, for such x,y we have $|N(x) \cap N(y)| \geq 2$, because $\{x,y\} \notin E$, $d(x)+d(y) \geq n$, and $|V-x-y| = n-2$ (apply Pigeonhole Principle)

  $d(x)+(n-2) \geq n \rightarrow d(x) \geq 2$, $d(y) \geq 2$

  Now, if $G+\{x,y\}$ contains a Hamiltonian circuit, it follows that G must contain a Hamiltonian path from x to y.

  We must prove that, given the conditions, we can use that Hamiltonian path to form a Hamiltonian circuit in G.

  Consider x,y and $u,v \in N(x) \cap N(y)$:

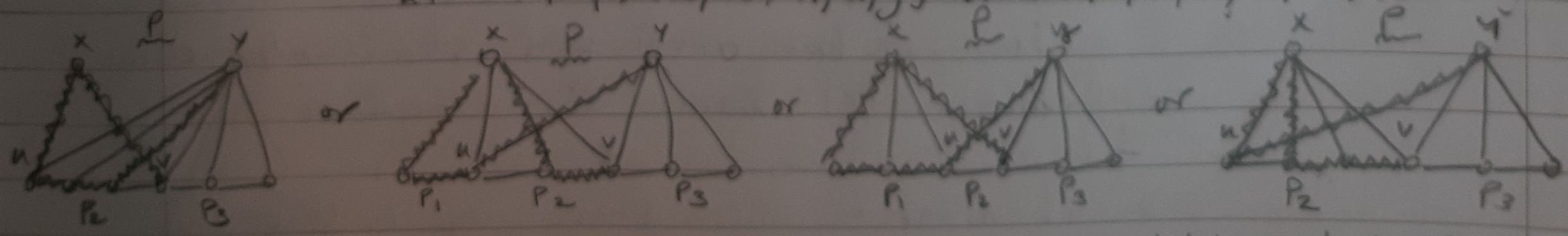


Let $HP(G)$ denote the Hamiltonian path that we know exists in G between x,y:

$HP(G) = \{x, P_1, u, P_2, v, P_3, y\}$

Note that we can have $P_1 = \emptyset$, or $P_2 = \emptyset$, or $P_3 = \emptyset$ and also P_2 being an edge $\{u,v\}$, for instance.

Without loss of generality, consider the alternate path starting at v: $P = \{v, P_2, x, P_1, u, y\}$ or $P = \{v, x, P_1, u, P_2, y\}$.



Note that we always need only two edges connected to x, which we know exist.

Then, we form a Hamiltonian circuit $HC(G) = \{v, P_3, y, P', v\}$, where $P' = \text{reversed}(P) - v = \{y, u, P_1, x, P_2\}$ or $\{y, P_2, u, P_1, x\}$.

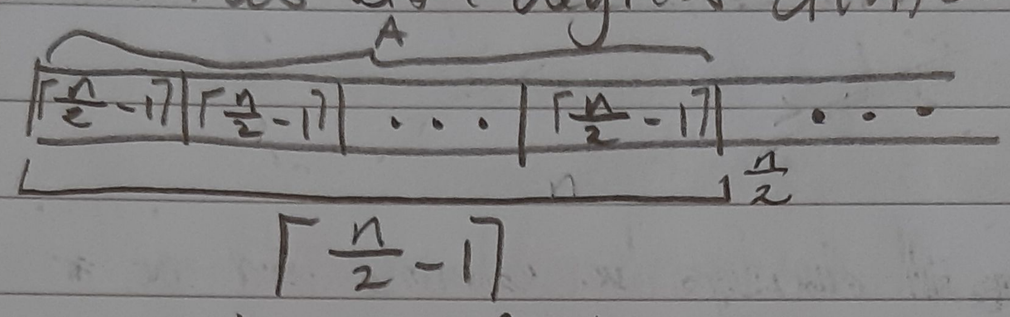
5.3. Prove or disprove: Given $G \in \mathcal{G}$, $n > 2$ vertices. Assume for G every $\{u, v\} \notin E$ has $d(u) + d(v) \geq n$, then the sequence of degrees satisfies: for every $1 \leq k < \frac{n}{2}$ it is $d_k > k$.

<u>Proof</u>: we will proceed by contrapositive proof

Let us assume that there exists $1 \leq k \leq \frac{n}{2} - 1$ with $d_k \leq k$

Consider the case where the induced subgraph formed by the vertices with degrees $d(V_1) \leq \ldots \leq d(V_{\frac{n}{2}-1})$ is maximal.

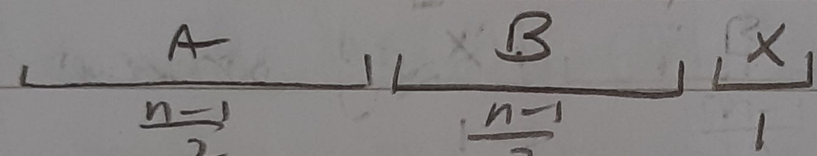
Then,



$d(u) = \lceil \frac{n}{2} - 1 \rceil$, for all $u \in A$

Now, we have two cases: for odd and for even.

<u>Case #1</u>: n is odd. Then the range is $1 \leq k \leq \frac{n-1}{2}$ (A)



; Consider $d(u) = \frac{n-1}{2}$ for $u \in A$.

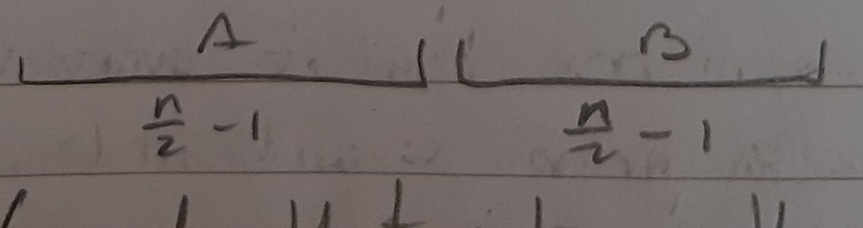
Consider that vertices within the range are not connected to the other $\frac{n-1}{2}$ group (B).

Consider that $d(x) = n-1$, for maximal B.

Now, the maximum degree for any node $v \in B$ is $d(v) = \frac{n-1}{2} - 1 + 1 = \frac{n-1}{2}$ (all of $B \setminus v + x$)

Then, we have $u \in A$, $v \in B$ st. $\{u, v\} \notin E$ and $d(u) + d(v) = 2\left(\frac{n-1}{2}\right) = n - 1 < n$.

<u>Case #2</u>: n is even. Then the range is $1 \leq k \leq \frac{n}{2} - 1$ (A)



; Consider $d(u) = \frac{n}{2} - 1$ for $u \in A$

Consider that vertices within the range are not connected to the other $\frac{n}{2} - 1$ group (B).

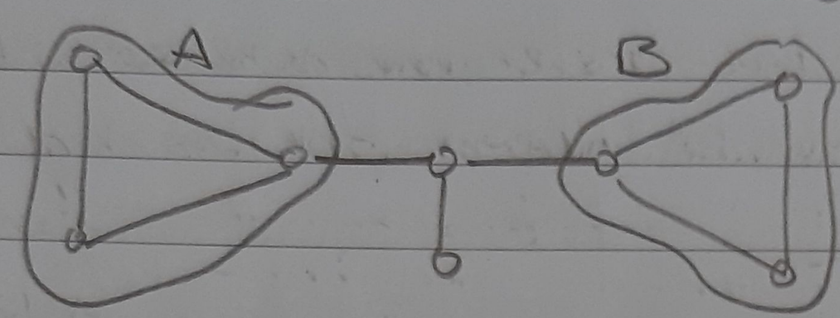
Consider that $d(x) = d(y) = n - 1$, for maximal B and A

Now, the maximum degree for any node $v \in B$ is $d(v) = \frac{n}{2} - 1 - 1 + 2 = \frac{n}{2}$ (all of $B \setminus v + x + y$)

Then, we have $u \in A$, $v \in B$ st. $\{u, v\} \notin E$ and $d(u) + d(v) = \frac{n}{2} - 1 + \frac{n}{2} = n - 1 < n$. ∎

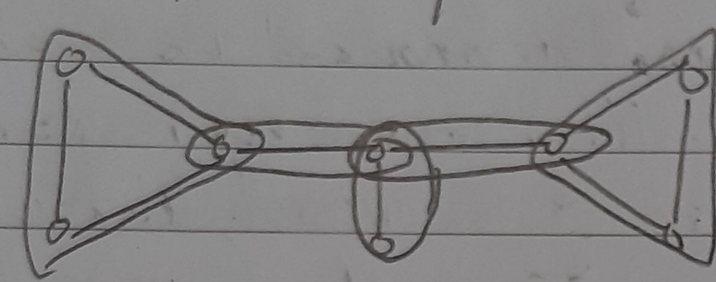
6.1 A block is a maximally connected multigraph with no cut vertices. Fully characterize how they can look like.
Assume that we have two distinct blocks in G. What is the smallest and the biggest number of vertices they have in common?

a) Two blocks don't need to be connected.



$|V(A) \cap V(B)| = \emptyset$

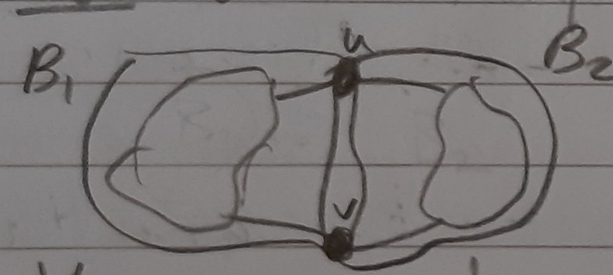
b) A block will always be connected to at least another block.



Therefore, a block will always share at least one node with another block.

c) Two blocks cannot have more than one node in common.

<u>Proof</u>: assume they could



By definition of a block it needs to be maximal. But creating a new block $B' = B_1 \cup B_2$ is possible if they share more than one vertex, since we can remove either u or v, and $B' \setminus u$ ($B' \setminus v$) will remain connected. Therefore, B' is a block. ∎

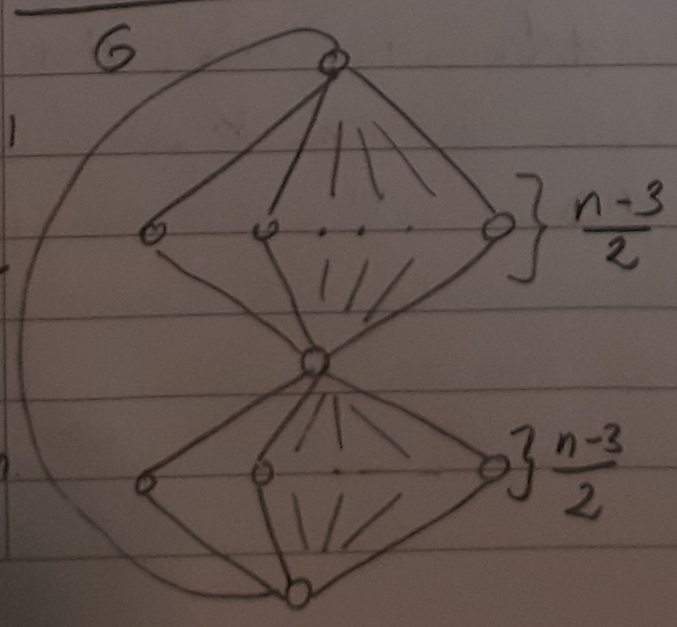
6.2 Given $n \geq 5$. Is it true that for every such n there exists a 2-conn. simple, undirected graph G without loops which satisfies:

- G has diameter $diam(G) = 2$.
- G has $2n-5$ edges

Yes. The following are examples of it for cases n odd and even

<u>n is odd:</u>

G



$m = 4\left(\frac{n-3}{2}\right) + 1$

$m = 2n - 6 + 1$

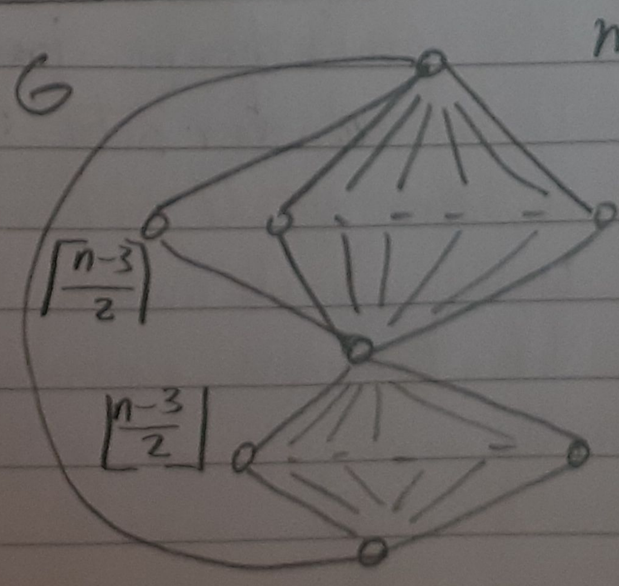
$m = 2n - 5$

$diam(G) = 2$

G is 2-conn.

<u>n is even:</u>

G



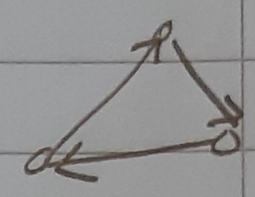
$m = 2\left(\left\lceil \frac{n-3}{2} \right\rceil\right) + 2\left(\left\lfloor \frac{n-3}{2} \right\rfloor\right) + 1 = 2(n-3) + 1 = 2n - 5$

$diam(G) = 2$

G is 2-connected

∎

6.3 Prove or disprove: Let G be a tournament with odd number of vertices n, where $n \geq 3$. If G contains a Hamiltonian cycle, then there exists a vertex v such that $G \setminus v$ also contains a Hamiltonian cycle.



<u>Disproof</u>: for $n=3$ it doesn't hold. Take the instance

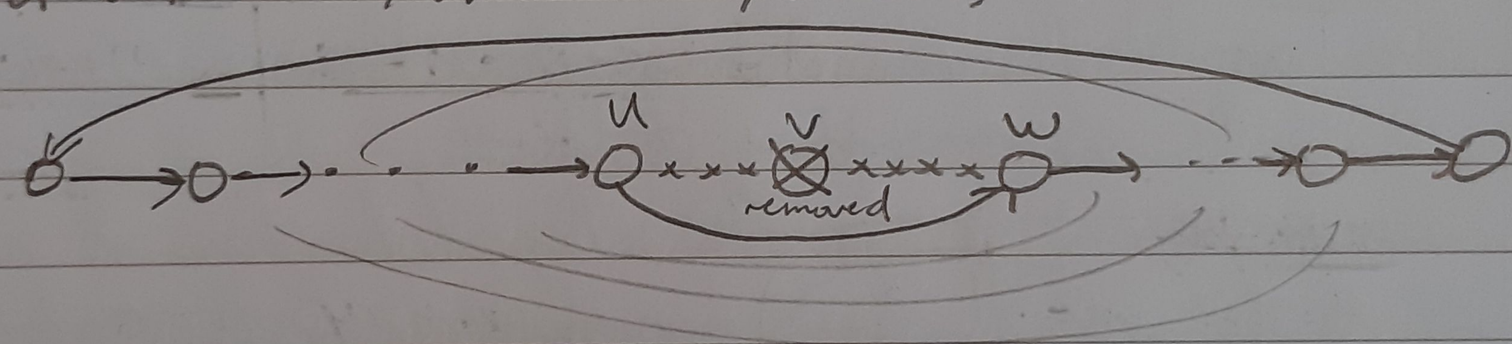
<u>Proof</u>: for $n \geq 4$. Let us first enumerate some facts.

In tournament graphs there must be an edge between all pairs of nodes.

If G contains a Ham. cycle, then $d^+(v) \geq 1$, $\forall v \in V(G)$

Also, if G contains a Ham. cycle, then $G \setminus v$ contains at minimum a Ham path, for any $v \in V(G)$

Finally, as we have $d^+(v) \geq 1$, $\forall v \in V(G)$ given a Ham. cycle in G, then there exists a vertex v such that $(u,v) \in E$, $d^+(u) \geq 2$ and $(v,w) \in E$, $d^-(w) \geq 2$, with $(u,w) \in E(G)$.



Then, by the definition of a tournament and a Ham. cycle the proof will follow.

Assume that all nodes have only edges going in the direction of the Ham. cycle. This is only virtual, and says nothing about the degrees: any outgoing edge from a vertex x is an incoming edge of some vertex y.

Again, assuming that G contains a Ham. cycle:

if $(u,w) \in E$, then $d^+(u) \geq 1$, $d^-(v) \geq 1$

Since $(u,v) \in E$ and $(v,w) \in E$ then some Ham. cycle exists such that it contains (u,v) and (v,w).

Therefore, removing v, we still have a Ham. cycle by keeping the same Ham. path, and adding the edge (u,w).

Therefore, for any G which is a tournament and contains a Ham. cycle, there exists a vertex v with neighbours u,w which meet the conditions described above, such that $G \setminus v$ contains also a Ham. cycle. ∎

7.1 Given a tree G with $n \geq 4$ vertices such that every vertex of degree greater than 1 has the same degree d. For which pairs of numbers n and d such tree exists? In the case such tree exists, determine the number of vertices of degree 1 of G, wrt. n and d.

Draw at least two such trees for $d > 2$, $n > 7$.

<u>Proof</u>: we will proceed by direct proof.

Let the tree have k vertices of degree d, and $l = n-k$ vertices of degree 1.

Then, $\sum_{v \in V} d(v) = kd + l = 2m$

Recall that $m = n-1$ in trees. Then $\sum_{v \in V} d(v) = kd + l = 2(n-1)$

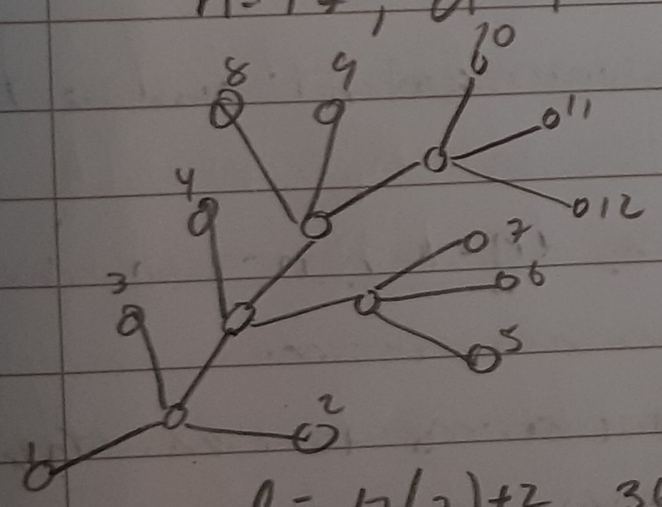
$\sum_{v \in V} d(v) = kd + (n-k) = 2n-2 \rightarrow k(d-1) = n-2 \rightarrow k = \frac{n-2}{d-1}$

It follows: $l = n-k = \frac{n(d-1)-(n-2)}{d-1}$

$= \frac{nd-n-n+2}{d-1} \boxed{= \frac{n(d-2)+2}{d-1}}$ (# leaves)

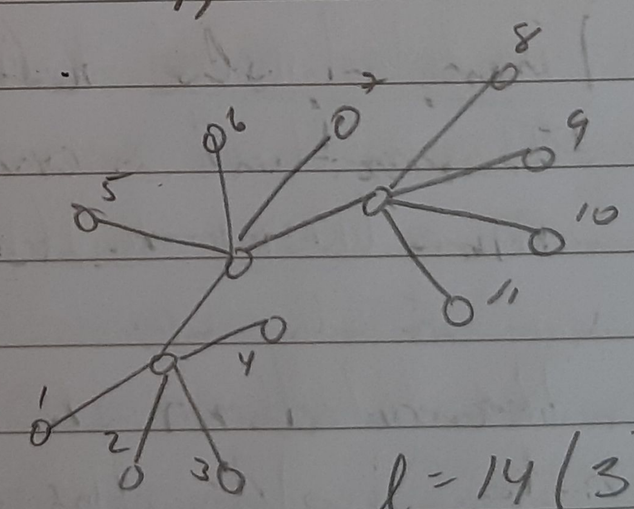
<u>Examples</u>

$n = 17$, $d = 4$



$l = \frac{17(2)+2}{3} = \frac{36}{3} = 12$

$n = 14$, $d = 5$



$l = \frac{14(3)+2}{4} = \frac{42+2}{4} = 11$

7.2 Given $G \in \mathcal{G}$, with n vertices, m edges and k components of connectivity. Prove or disprove the following:

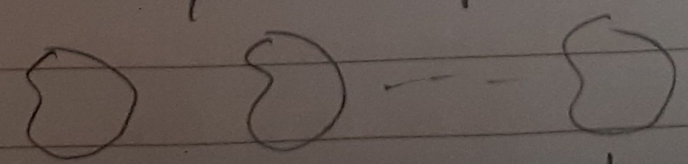
a) $n-k \leq m \leq \frac{(n-k)(n-k+1)}{2}$

b) If G does not have circuits of odd length, then

$$n-k \leq m \leq \begin{cases} \frac{(n-k)(n-k+2)}{4} & \text{for } n-k \text{ even} \\ \frac{(n-k+1)^2}{4} & \text{for } n-k \text{ odd} \end{cases}$$

<u>Proof</u>: we will proceed by direct proofs.

a) <u>$m \geq n-k$</u>:



k components of connectivity.

Since we want to find the lower bound, consider a minimally connected graph (a tree) in each component. We know that adding an edge between consecutive pairs of components $C_1 - C_2 - C_3 - \dots - C_k$ will connect the graph, which will still be a tree and have $m = n-1$. Now, we have $k-1$ edges extra joining the components, so our real $m = n-1-(k-1) = n-k$.

$m \leq \frac{(n-k)(n-k+1)}{2}$: we have as well k components of connectivity. Now we want to prove the upper bound; so let us assume maximally connected components (complete/cliques). We also want the distribution of nodes across the components which maximizes the number of edges. We have two candidates:

i) Uniformly distributed: each component contains $\frac{n}{k}$ nodes.

$$m_i = \frac{\frac{n}{k}\left(\frac{n}{k}-1\right)}{2} = \frac{\frac{n}{k}\left(\frac{n}{k}-\frac{k}{k}\right)}{2} = \frac{n(n-k)}{2k^2}$$

$$m = k\,m_i = \frac{n(n-k)}{2k} \leq \frac{(n-k)(n-k+1)}{2}$$

ii) Isolated nodes: $k-1$ components are $k-1$ isolated vertices. Then, there is one component with $n_k = n-(k-1)$

$$m_k = \frac{(n+1-k)(n+1-k-1)}{2} = \frac{(n-k)(n-k+1)}{2} \quad \text{(max)}$$

The total $m = m_k$ since the others have no edges.

b) $m \geq n-k$: holds by the same reasoning as before.

$m \leq \frac{(n-k)(n-k+2)}{4}$ (for $n-k$ even) : we proceed similarly than in a).

We want the maximal number of edges in order to obtain the upper bound. In b) we require to not have circuits of odd length.

- Lemma: Complete bipartite graphs do not contain circuits of odd length.
  - Proof: Assume they did. This would mean that two vertices which belong to the same partition share an edge, which contradicts the definition of partite graphs.

Thus, we will use complete bipartite graphs in order to find the upper bounds.

Same reasoning than in a). have $k-1$ components contain each an isolated node, and component k contain $n-k+1$. Then, have the vertices in component k form a bipartite graph with parts with sizes $\frac{n-k}{2}$, and $\frac{n-k}{2}+1$

Then, the total number of edges $m_k = \frac{n-k}{2}\left(\frac{n-k}{2}+1\right)$

$m = m_k = \frac{(n-k)(n-k+2)}{4}$ OK

$m \geq \frac{(n-k+1)^2}{4}$ (for $n-k$ odd)

we follow the same reasoning.

We have $k-1$ components with one isolated vertex each, and a component, the kth component, with $n-k+1$ vertices. Then, have the vertices in component k form a bipartite graph with parts with sizes $\frac{n-k+1}{2}$, $\frac{n-k+1}{2}$.

Then, $m_k = \frac{n-k+1}{2}\left(\frac{n-k+1}{2}\right)$

$m = m_k = \left(\frac{n-k+1}{2}\right)^2 = \frac{(n-k+1)^2}{4}$ OK

7.3 Prove or disprove: Given an k edge connected simple graph without loops which has n vertices and m edges. Then it holds that $m \geq \frac{k \cdot n}{2}$

Proof: we will proceed by direct proof.

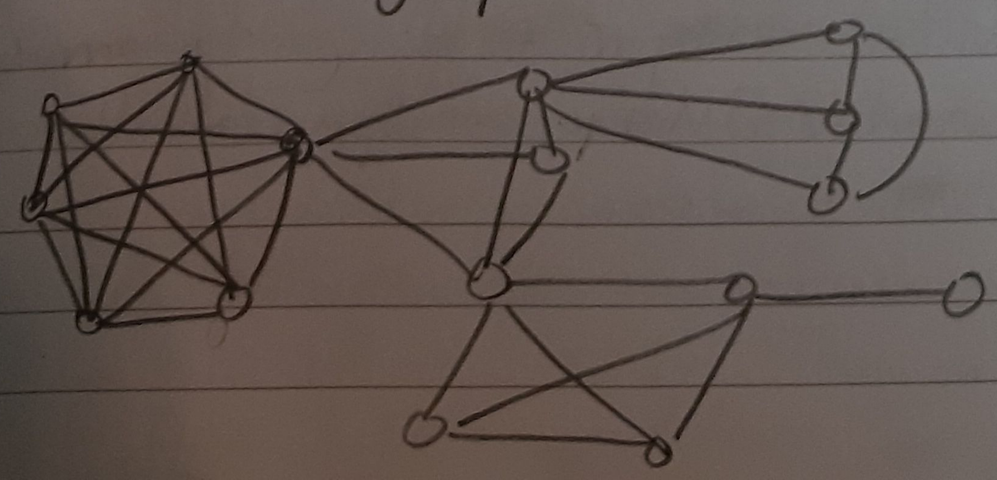
Since we have k-edge connectivity, we will have each node with at least k edges.

Then, $\sum_{v \in V} d(v) = 2m \geq k \cdot n \Rightarrow \boxed{m \geq \frac{k \cdot n}{2}}$

8.1 Prove or disprove: Let G be a connected undirected graph of even number of vertices and such that it does not contain $K_{1,3}$ as an induced subgraph. Then G has a perfect matching.

Proof: Let us proceed first by a study of the situation

Consider a graph G as described.



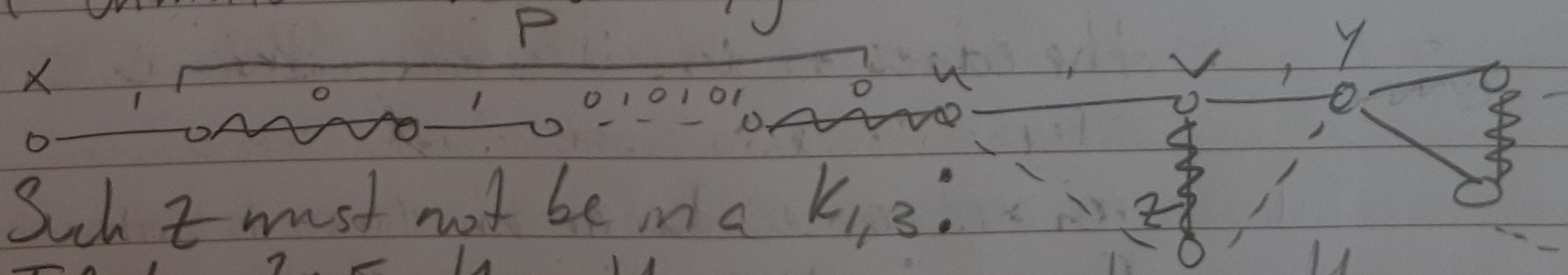
In order for the graph to meet the condition of not having $K_{1,3}$ induced subgraph, its neighborhoods for every vertex will be cliques.

... In other words, each node v which has neighbors $u, w \in N(v)$ such that $\{u, w\} \notin E$ will be a shared vertex between two cliques formed by the induced subgraphs $G[N(u)+u+v]$ and $G[N(w)+w+v]$
Otherwise, we would have $K_{1,3}$ given by v, u, w, and some $x \in N(v)$ such that $x \notin N(u) \cup N(w)$
Now, we know that any complete graph of even size on its vertices has a perfect matching.
Also we know that graphs with odd number of vertices don't have a perfect matching. In the case of a complete graph with odd number of vertices we have a matching $|M_{max}| = \frac{n-1}{2}$, which cover $n-1$ vertices.

<u>Proof</u>: we will proceed by contradiction. Assume maximal matching.
Let us assume that G does not contain $K_{1,3}$ and that it doesn't contain a perfect matching.
Then, there exist an even number of vertices which are unmatched, since n is even.

- <u>Lemma</u>: there exists an alternating path between each pair of unmatched nodes, given our conditions.
  - <u>Proof</u>: since graph is union of cliques we can always go in and out of the clique through an alternating path, since it is maximally connected.

Now, consider the alternating path between a pair of unmatched vertices x, y:



Such z must not be in a $K_{1,3}$:
If $\{u, z\} \in E$, then there is an augmenting path $xPuzvy$, contradicting the maximality of the matching.
If $\{z, y\} \in E$, then there is an augmenting path $xPuvzy$, contradicting the maximality of the matching.

$\blacksquare$

8.2 Prove or disprove: Every tree has at most one perfect matching.

<u>Proof</u>: Assume for the sake of contradiction that the tree T contains two perfect matchings, M_1 and M_2.
Since $M_1 \neq M_2$, $\exists e \in E$ such that $e \in M_1$, $e \notin M_2$
Consider $v \in V$ from $\{v, w\} = e$
Build a path starting from v and an edge $e' \in M_2$.
Choose an edge of v which is not e (in M1) but through which it is matched in M2.
Then, consecutively add an edge either from M_1 or M_2, incident to the last vertex of the path.
Since the graph is finite, at some point we will match w through an edge from M_2, which means that there is a circuit in the graph. This contradicts the fact that the graph is a tree, since trees don't have circuits. ∎

8.3 Let $G \in \mathcal{G}$, 3-regular. Prove or disprove: If G does not contain a bridge then in G there is a perfect matching.

<u>Proof</u>: we will proceed by direct proof and use Tutte's theorem.
Recall that if we have a subset $S \subseteq V(G)$, then we look at the odd components of $O(V(G)-S)$. For the graph to have a perfect matching it must be that $|O(V(G)-S)| \leq |S|$.
This is known by Tutte's theorem.
Now, consider a subset $S \subseteq V(G)$, and an odd component $O_i \in O(V(G)-S)$. Since the graph is 3-regular, each vertex has exactly 3 neighbors. Then, let $E(O_i)$ denote the edges of O_i. We know that O_i will have a number of edges m_i, going out of the component, which is bigger than 1 ($m_i > 1$), since the graph doesn't contain bridges.
Then, $\sum_{v \in V(O_i)} d(v) = 2|E(O_i)| + m_i$, since each edge within the component is counted twice.
Also, by 3-regularity: $\sum_{v \in V(O_i)} d(v) = 3|V(O_i)|$.
Since we know that $V(O_i)$ is odd, then $\sum_{v \in V(O_i)} d(v)$ is also odd.
Since $2|E(O_i)|$ is even, then m_i must be odd. So, $m_i \geq 3$.
Let m be the total edges with one vertex in S, and the other in $V(G)-S$. Every component $O(V(G)-S)$ contributes with 3 edges to m. Therefore, we have $|O(V(G)-S)| \leq \frac{1}{3}m$. Also, in worst case every edge from S is counted in m.
Then $m \leq 3|S| \rightarrow \frac{1}{3}m \leq |S|$. Thus: $\boxed{|O(V(G)-S)| \leq \frac{1}{3}m \leq |S|}$, which satisfies Tutte's condition. ∎

9.1 Given a system $S = \{S_1, S_2, \dots, S_k\}$ of non-empty subsets of a finite set $X = \{1, \dots, n\}$ (ie. $S_i \subseteq X$).

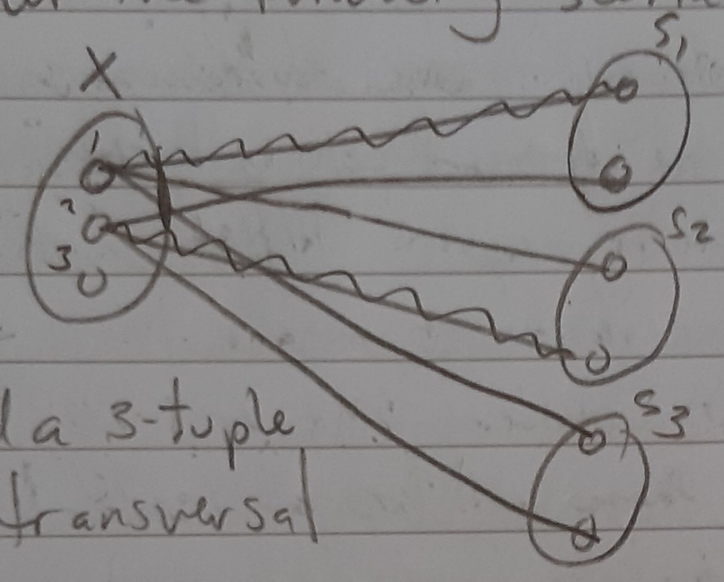
We call transversal a k-tuple $(x_1, \dots, x_k)$ of elements of X if:

1. For every $i \neq j$ we have $x_i \neq x_j$.
2. For every i we have $x_i \in S_i$.

Prove or disprove: The system $S = \{S_1, S_2, \dots, S_k\}$ has a transversal iff for every $I \subseteq \{1, \dots, k\}$ we have $\left|\bigcup_{i \in I} S_i\right| \geq |I|$

Let us first understand the problem.

Consider the following scenario where there is <u>no transversal</u>.



$X = \{1, 2, 3\}$, $k = \{1, 2, 3\}$

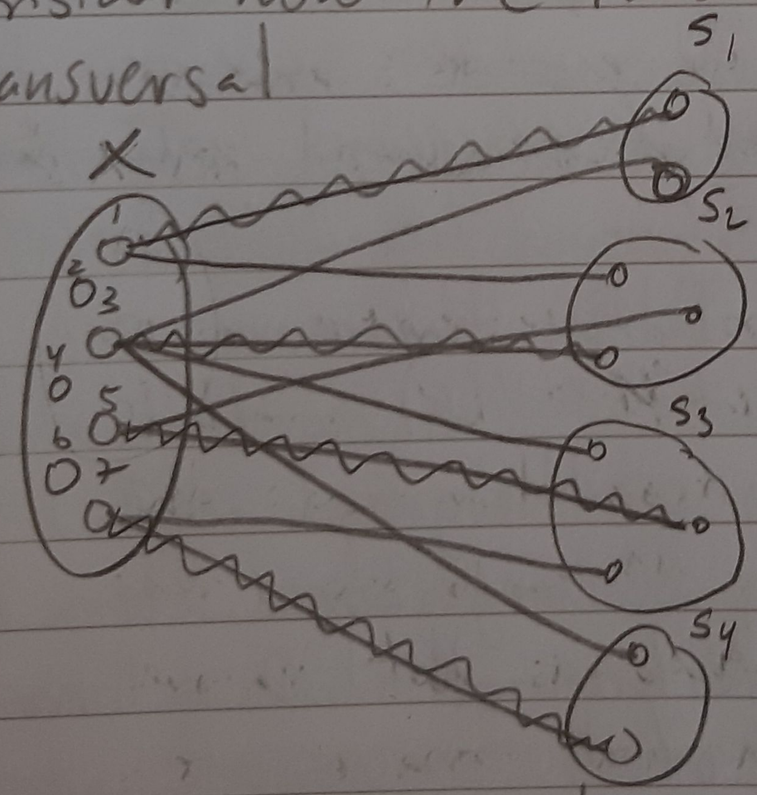
The theorem holds for all $I \in 2^k \setminus \{1, 2, 3\}$

But for $I = \{1, 2, 3\}$ we get $\left|\bigcup_{i \in I} S_i\right| = |\{1, 2\}| = 2$, $|I| = 3$

And, indeed, it is impossible to get a transversal in this case.

We need a 3-tuple for a transversal

Consider now the following scenario where <u>there</u> is a transversal



$X = \{1, \dots, 7\}$, $K = \{1, \dots, 4\}$

$I \in 2^K$

The 4-tuple $(1, 3, 5, 7)$ is a transversal.

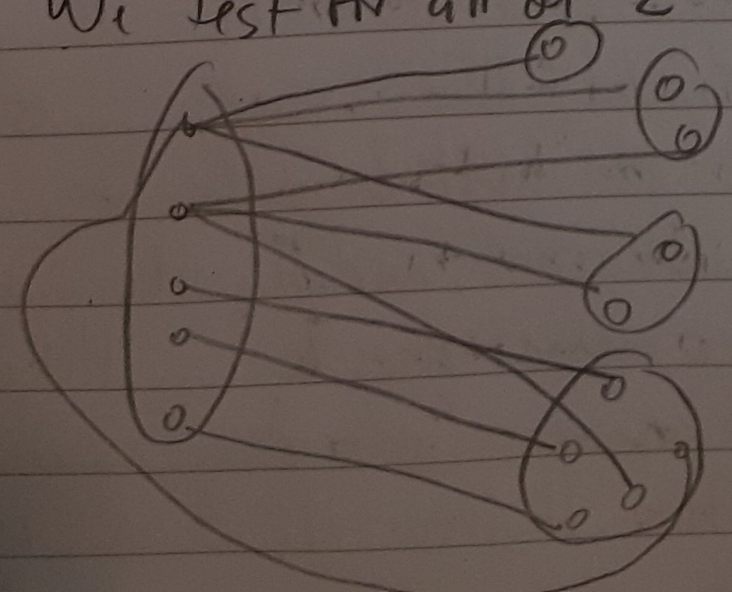
Let us examine the theorem for a couple of cases:

$I = \{2, 4\} \to \left|\bigcup_{i \in I} S_i\right| = |\{1, 3, 5, 7\}| = 4 > |I|$

$I = \{3, 4\} \to \left|\bigcup_{i \in I} S_i\right| = |\{3, 5, 7\}| = 3 > |I|$

The main idea is that we need to have all S_i connected to a variety big enough of X (tested by $\left|\bigcup_{i \in I} S_i\right|$, $I \in 2^K$) such that one is selected for each $i \in K$ (second condition for transversal).

We test for all of 2^K because we could ignore specifics like:



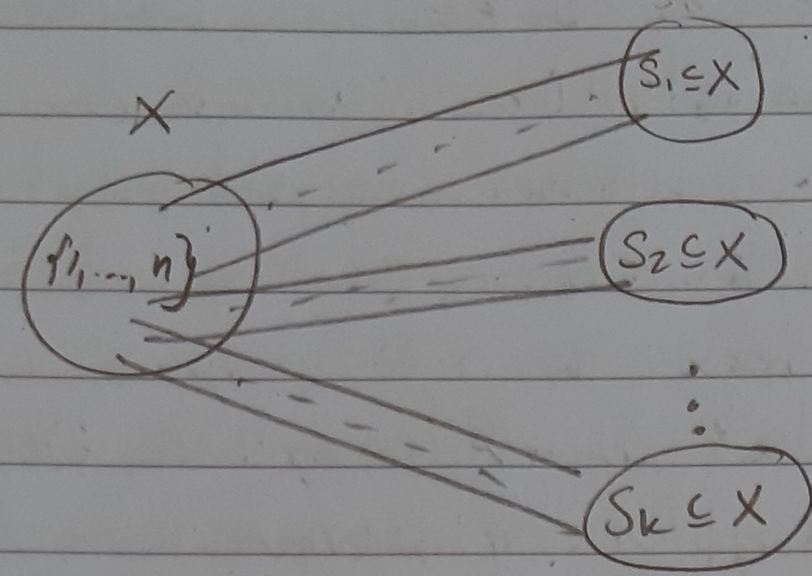
Here $I = K$ gives $\left|\bigcup_{i \in I} S_i\right| = 5$, $|I| = 4$

But $I = \{1, 2, 3\}$ gives $\left|\bigcup_{i \in I} S_i\right| = 2$, $|I| = 3$

Therefore, we must see that Hall's theorem holds (marriage) for all $I \in 2^K$, to ensure that a transversal exists.

<u>Formalizing</u>: The suitable graph for the scenario is:

G



Where G is bipartite.
In X each of $\{1, \dots, n\}$ is a vertex
In S_i we will have a vertex corresponding to the element from X it contains (each) and there will be edges between $x_j \in X$ and $x_j' \in S_i$ where the labels of x_j and x_j' are the same (from X).

<u>Proof</u>: we must prove necessity and sufficiency.

- <u>Necessity</u>: Suppose for the sake of contradiction that S has a transversal but the marriage condition is not satisfied. $\exists I' \in 2^k : |\bigcup_{i \in I'} S_i| < |I'|$
However, since S has a transversal, it means that k sets S_i were selected, each with some $x_j \in S_i$ where $\{x_j, x_j'\} \in E$, $x_j' \in X$. This means that for any $I' \in 2^k$ there will be at least $|\bigcup_{i \in I'} S_i| = |I'|$, since the transversal requires that each $(x_1, \dots, x_k)$ from the k-tuple to be different from each other, and $I' \in 2^k$ is basically selecting the sets S_i to be united. At least this unification will amount to the different elements of the transversal tuple that belong to the selected S_i, $i \in I'$.

- <u>Sufficiency</u>: If the marriage condition is satisfied for each $I' \in 2^k$ such that $|\bigcup_{i \in I'} S_i| \geq |I'|$, then S has a transversal.
Suppose for the sake of contradiction that the marriage condition holds for all $I' \in 2^k$ but there is no transversal. Assume we have a maximal tuple which is not transversal. This means there are some S_i such that none of its elements $x_j' \in S_i$ is in the maximal tuple. This is equivalent to having matched $x_j \in X$ and x_j' through $\{x_j, x_j'\} \in E$, as discussed previously.
Since marriage theorem holds for all $I' \in 2^k$ we know that there is some I' such that $i \in I'$ selects one such S_i with no matched element, so that $|\bigcup_{j \in I'} S_j| \geq |I'|$. We can select one element per cluster (S_j) and if conflicts appear with the first condition of transversal, we consider another I' which includes the S_j involved in the conflict, and swap selections to fix it. We can also do it thanks to $|\bigcup_{j \in I'} S_j| \geq |I'|$ holds. Since we can increase the size of the tuple, then it contradicts its maximality. A contradiction. ∎

Basically, if there is one Si with nothing matched, there is one element of X to which it is connected which is not in the k-tuple. Otherwise, can just add it. We proceed to match it. If there is a conflict with some other Sj which had matched it, we know that this Sj will have more connections to which it can propagate this conflict. Since the graph is finite, we can see that the conflict will resolve, due to |USj| >= |I| for all I subset of K.

* 9.2 Given a bipartite graph G with sides X and Y, its maximum matching P_{max}. For every edge $e \in P_{max}$, we choose one endvertex of e (and put it into a set A) as follows:

Let $e = \{x, y\} \in P_{max}$ we choose y and put it in A if there exists a path $a_1, e_1, a_2, e_2, \dots, e_{2k+1}, y$ where:

1) $a_1 \in X$ is free in P_{max}
2) for $i > 0$ it holds that $e_{2i-1} \notin P_{max}$, $i > 0$, $e_{2i} \in P_{max}$

If such a path does not exist, we put x to A.

Prove or disprove: The set A constructed above is a vertex cover of G.

<u>Proof</u>: In the case the algorithm fails, we must suppose that computing A as above indicates, it is not a vertex cover. Then, there is some $\{x, y\} \in E$ such that neither x nor y belongs to A ($x \notin A$, $y \notin A$).

Let us analyze the following cases.

<u>Case 1</u>: P_{max} is a perfect matching.

Then, the path described never exists (no free vertex), and $\forall \{x, y\} \in P_{max}$ we choose x. Because matching is perfect, this yields a vertex cover.

<u>Case 2</u>: P_{max} is not a perfect matching.

Let us assume that the algorithm fails, as described above.

Then, $\exists \{x, y\} \in E : x \notin A, y \notin A$.

We know that $\{x, y\} \notin P_{max}$, otherwise $x \in A$ or $y \in A$.

Remark that $x \in X$, $y \in Y$. So $N(x) \subseteq Y$, $N(y) \subseteq X$.

Since $x \notin A$, a path $a_1, e_1, a_2, \dots, e_{2k+1}, v$ existed for his $v \in N(x) \setminus y$, $\{x, v\} \in P_{max}$, or x is free in X

Since $y \notin A$, the path $a_1, e_1, a_2, \dots, e_{2k+1}, u$ didn't exist for his $u \in N(y) \setminus x$, $\{u, y\} \in P_{max}$, or Y is free in Y

This is because if path exists we select vertex from Y, otherwise we select the vertex from X.

Now, we have again cases: either both x and y are free, or one of them is, or both are not free in P_{max}.

i) If both are free, add $\{x, y\}$ to P_{max}. Contradiction.

ii) If x is free and not y, then y should be selected in A instead of u (by first step of algorithm).

iii) If y is free and not x, then consider the case where there are no free vertices in X. Then y must be covered by one of its neighbors, which are matched.

Consider now the case where there are free vertices in X, then since none of y neighbors $u \in N(y)$ which are not free, belongs to A, then there was always a path such that an element from Y was selected. Consider x: its neighbor $v \in N(x)$, $\{x, v\} \in P_{max}$. It means that there is an alternating path as described, up to v. If we extend this path with $\{v, x\}, x, \{x, y\}, y$, we will have an augmenting path, and we can augment P_{max}, which contradicts its maximality.

iv) If both are not free, again, if we don't have free vertices in X, then x should have been selected by the algorithm to be in A. If there are free vertices in X then there is an alternating path reaching to $v \in N(x)$, $\{x, v\} \in P_{max}$. Then by taking this path and adding $\{v, x\}, x, \{x, y\}, y$, we have that y is reachable and should be selected by the algorithm instead of $u \in N(y)$. Therefore, a contradiction is found in the execution of the procedure.

Therefore, indeed the algorithm returns a vertex cover of G ■

9.3 Let G be a simple graph without loops with a vertex v. We say that v is critical if $\beta_0(G \setminus v) < \beta_0(G)$

Prove or disprove: A vertex v is critical in G iff it belongs to some vertex cover with the smallest possible number of vertices.

<u>Proof</u>: we must proof necessity and sufficiency.

Recall that $\beta_0(G)$ is a minimum vertex cover in the number of vertices. $\beta_0(G) = |B_{min}|$

· <u>Sufficiency</u>: If v is in some vertex cover with the smallest number of vertices, then v is critical.

Let us proceed by contradiction.

Assume v is in min. vertex cover, but v is not critical

Then $\beta_0(G \setminus v) = \beta_0(G)$. However, since v belongs to some min. vertex cover, this means that there is at least one edge e such that $v \in e$, and no other node u in the cover is an end vertex of e. If we remove v, now $e \notin E(G \setminus v)$. But we still have $\beta_0(G \setminus v) = \beta_0(G)$. This contradicts the fact that the rest of vertices are covering all edges such that $E(G) \setminus \{e : v \in e, e \in E(G)\}$.

Necessity: If v is critical then it belongs to some vertex cover with the smallest number of vertices.
Suppose v is critical and we have some minimal cover.
Proceed by cases.

Case 1: v belongs to the vertex cover. Then we are done.

Case 2: Some neighbor of v, $u \in N(v)$ belongs to the vertex cover in G, but is not necessary in $G \setminus v$. In this case we can swap the vertices u and v in the cover in G, and the statement is proven.

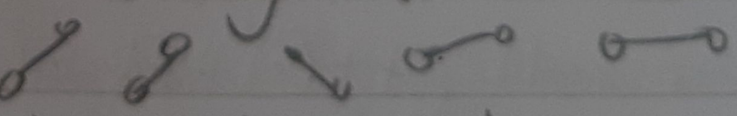
Case 3: If neither case holds, then v is not critical and we have a contradiction.

Therefore, sufficiency and necessity hold for our statement. ∎

10.1 Prove that in every undirected graph G it holds that

$$\alpha_1(G) \leq \beta_0(G)$$

where $\alpha_1(G)$ is the number of edges in a maximum matching in G, and $\beta_0(G)$ is the number of vertices of the minimum vertex cover of G.

Proof: we will proceed by direct proof.
Consider a minimum vertex cover VC with $|VC| = \beta_0(G)$
Since VC is minimum, there is one edge per each node v in VC that is only covered by v.
Consider the case where all such vertices have an independent set of end vertices in all their edges.
Then, a maximum matching M_{max} has $|M_{max}| = |VC|$ since the graph is 
In any other case $|M_{max}| \leq |VC|$ since it might happen that the end vertex of some edge $\{v, u\}$ is already matched by another edge $\{w, u\}$ (considering that $v, w \in VC$).
Therefore, $\alpha_1(G) \leq \beta_0(G)$, since $|M_{max}| \leq |VC|$. ∎

10.2 Prove that in every bipartite graph G, it holds that
$$\alpha_1(G) = \beta_0(G)$$

<u>Proof</u>: we proved by direct proof.
We know that the maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover (König's theorem).
Then, for us: $|M_{max}| = \alpha_1(G)$, $|VC_{min}| = \beta_0(G)$
Therefore, $\alpha_1(G) = \beta_0(G)$. ∎

10.3 Let G be a simple undirected graph without loops. Prove that in some optimal coloring (ie, $\chi(G)$ colors) every neighborhood $N(v)$ of any vertex v has the same color iff G is a bipartite graph.

<u>Proof</u>: we must show necessity and sufficiency.
Let us assume for simplicity that G is connected

- <u>Necessity</u>: If every $N(v)$ for each $v \in V$ has the same color, then G is bipartite.
Assume that $\forall v \in V$, it holds that $\forall u \in N(v)$ have same color.
Then, no vertex in G has neighbors with more than one different color. Apply now the following coloring alg.:
1) Select $v \in V$. Color it white
2) Color all its neighbors $N(v)$ red.
3) Now go through each node $u \in N(v)$ and repeat the steps 1 and 2. Since they are colored already, instead let them keep their color and color their respective neighbors with the opposite color (white or red)

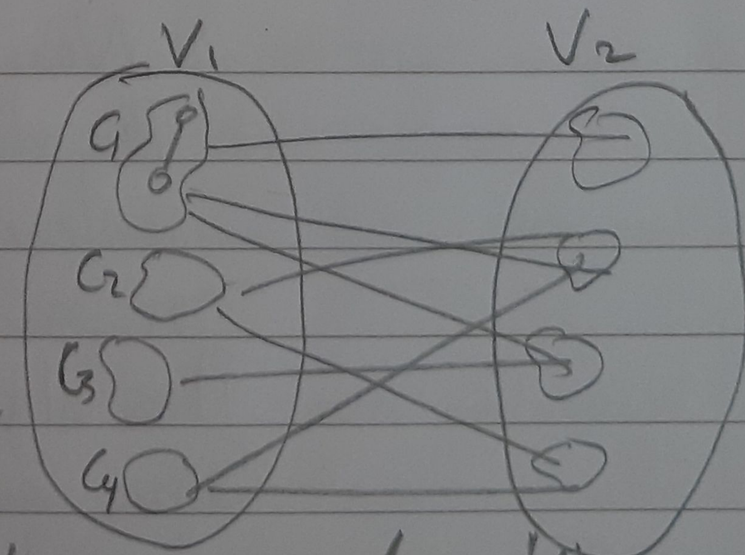
This algorithm yields a 2 coloring.
Since any 2-coloring entails a 2-partition of vertices, we conclude that G must be bipartite.

- <u>Sufficiency</u>: If G is bipartite, then every $N(v)$ for each $v \in V$ has the same color.
Assume G is bipartite.
Then G is 2-colorable: start at any vertex and go through all vertices by choosing an edge going to an unvisited vertex. This will make a path that alternates between partitions. This yields a sequence $V_1 V_2 V_3 \ldots V_k$

Color all nodes V_{2i+1} white, and all V_{2i} red.
We have a 2-colored bipartite graph G.
Let parts be A, B.
For any node $v \in A$ we have all neighbors $w \in N(w) \subseteq B$ and the converse also $u \in B$, then $w \in N(u) \subseteq A$.
Therefore, all neighbors will belong to the same partition and thus they will share the same color.
Therefore, it holds that if G is bipartite, every $N(v)$ for each $v \in V$ has the same color. ∎

11.1 Let $G = (V, E)$ be a connected undirected graph with edges E. A subset $S \subseteq E$ is called a mincut if it is a minimal set of edges such that $G \setminus S$ is disconnected.
Let V_1, V_2 be two non-empty disjoint sets of vertices such that $V_1 \cup V_2 = V$. Then the cut $\langle V_1, V_2 \rangle$ is the set of edges $e = \{u, v\}$ for which $u \in V_1$, $v \in V_2$.
Prove or disprove: Every cut $\langle V_1, V_2 \rangle$ of a connected, undir. graph is a union of mincuts that are pairwise disjoint (no two mincuts share a common edge).

<u>Proof</u>: we will proceed by direct proof.
A cut is minimal if removing any edge reconnects the graph. Therefore, we only need to look at the components of connectivity.
Consider the following:

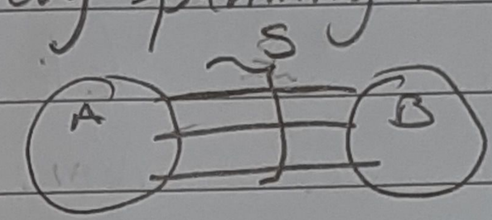


Take any component of V_1 C_i. Consider the components in V_2 that its edges reach (the span of C_i in V_2).
Let this be $N(C_i)$. Now, consider the number of edges going out of the component of each $C_j \in N(C_i)$ and C_i. This is the number of the size of the cut.
Select $C \in N(C_i) \cup C_i$ such that the cut is the smallest.
Remove those edges from the graph. Repeat this process now for another component which is still connected.
Do this until $\langle V_1, V_2 \rangle$ is created. All the removed edges are $\langle V_1, V_2 \rangle$. ∎

11.2 Let G be a connected undirected graph with the set of edges E. Prove or disprove: A subset $S \subseteq E$ is a minwt iff it is minimal set of edges satisfying: for every spanning tree T of G we have $S \cap T \neq \emptyset$.

<u>Proof</u>: we must show necessity and sufficiency.

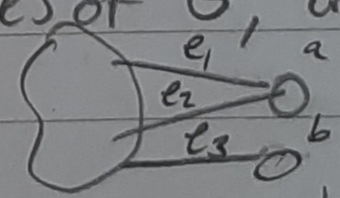
- <u>Necessity</u>: If $S \subseteq E$ is a minwt, then it is a minimal set of edges satisfying that for every spanning tree T of G we have $S \cap T \neq \emptyset$.



Consider the situation:
Consider also any spanning tree on component A, T_A; any spanning tree T_B on component B.
In order to have <u>any</u> spanning tree T of G, we must include one edge connecting A to B; ie, an edge of minwt S is necessary in <u>any</u> $T = T_A \cup T_B \cup \{e\}, e \in S$.
Therefore, for every T, $S \cap T \neq \emptyset$. By definition of a minwt we know that such S is the minimal possible.

- <u>Sufficiency</u>: If a subset $S \subseteq E$ is a minimal set of edges satisfying that for every spanning tree T of G we have $S \cap T \neq \emptyset$, then S is a minwt.

We proceed by direct proof.
Remark that S is minimal, which means that $S' = S - e$, $e \in S$ has some T_x with $S' \cap T_x = \emptyset$.
Now, any set S'' for which the condition holds $S'' \cap T \neq \emptyset$ for all trees of G, creates a cut.
Consider any graph



It is clear that the minwt is e_3, but if we do $S'' = \{e_1, e_2, e_3\}$ it will also hold for all $T \subseteq G$.

Proving the necessity of it being minimal

However, $S'' - e_2$ and $S'' - e_3$ also meet the condition but $S - e_3$ doesn't. In general we can do:
Set $S = G$. It meets $T \cap S \neq \emptyset$, $\forall T \subseteq G$. Select $e \in S$ such that $S' = S - e$ also fulfills $T \cap S' \neq \emptyset$, $\forall T \subseteq G$. Repeat until no such e can be found. This is a minimal S fulfilling the condition, and also an algorithm to find a minwt, since always some edge from a minwt is used in each $T \subseteq G$ (proven in necessity part).

11.3 Let G be a connected undirected graph with the set of edges E. Denote by W_K the circuits space and W_R the cut space of G. Prove or disprove: A subset $S \subseteq E$ belongs to W_R iff for every K in W_K we have $S \cap K$ contains an even number of edges.

Proof: we proceed by direct proof. Assume that we have $S \subseteq E$ with $|S \cap K| = 2a$, for every $K \in W_K$

Let T be a fixed spanning tree of G

Now, obtain $S \cap T = \{b_1, \dots, b_\ell\}$ (all branches from T in S)

Take $Z = S_{b_1} \oplus S_{b_2} \oplus \dots \oplus S_{b_\ell}$, $Z \in W_R$

Now, $|Z \cap K| = 2a$, for every circuit of $K \in W_K$

Then, for $S \oplus Z$, $|(S \oplus Z) \cap K|$ is even (since $S \oplus Z \in W_R$)

Assume that $S \oplus Z \neq \emptyset$. Note that $S \oplus Z$ contains no branches, since any branch (wrt T) which is in S will also be in Z, and $x \oplus x = \emptyset$.

Then, since $|S \oplus Z| \neq \emptyset$, $S \oplus Z$ must contain a chord $c \in S \oplus Z$ such that $(S \oplus Z) \cap K_c$ is even, for $K_c \in W_K$

$K_c = \{c\} \cup K'$, $K' \subseteq T$. Then $(S \oplus Z) \cap K_c = \{c\}$, (because $S \oplus Z$ does not contain branches, and K' only contains branches). We derive a contradiction, since $|(S \oplus Z) \cap K_c| = 1$ is even. ∎