The Ananke (Gravitational Closure) Theorem

A Classification of Closed Classical Gravitational Fields

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Abstract

We present the Ananke (Gravitational Closure) Theorem, a foundational classification result for classical gravity. Rather than proposing a model or modifying dynamics, the theorem asks which gravitational field structures are admissible once minimal structural requirements are imposed. Requiring classical covariance, quadratic closure with finite conserved energy, and orthogonal modes of response, we show that closure fixes the permissible response structure of gravity. In isolated vacuum regimes, closure enforces zero residual degrees of freedom, implying rigidity and unique exterior behaviour. In non-vacuum but symmetry-reduced regimes, closure admits exactly one residual redistributive degree of freedom, whose existence, form, and coupling are fixed by conservation and orthogonality. The admissible covariant action is thereby determined up to equivalence. We use *Ananke* in the classical philosophical sense of necessity: given the premises, *it can be no other way.*

1. Introduction

Classical gravity occupies a singular position in theoretical physics. Newtonian gravity and General Relativity (GR) describe isolated systems with extraordinary accuracy, from planetary motion to compact-object exteriors. In vacuum regimes, no confirmed deviations from GR exist, and any viable theory must recover vacuum gravity exactly.

At the same time, gravitational phenomena in extended systems and cosmology exhibit regularities not implied by a purely constraint-dominated description coupled only to visible matter. These are commonly interpreted as evidence for additional gravitational components inferred solely through their effects. Whether this reflects new ontology or incomplete field structure remains an open question.

This work adopts a prior, structural approach. Rather than asking how gravity behaves phenomenologically, or what additions reproduce observations, we ask:

What gravitational field structures are admissible once gravity is required to close as a classical field?

The Ananke Theorem is a classification result addressing necessity rather than phenomenology. It does not posit new forces or particles, modify GR by hand, or fit data. Instead, it classifies admissible gravitational structure under three minimal requirements, stated explicitly as axioms: classical covariance, quadratic closure with finite conserved energy, and orthogonal modes of response.

A separate upstream analysis establishes that, under minimal criteria for fundamentality, gravity is uniquely admissible as a closing classical interaction field.[1] The present theorem does not assume that result. Instead, it is conditional: given a gravitational field that closes, it determines the internal response structure that closure requires.

From these premises, vacuum gravity emerges as maximally closed: in isolated regimes it admits zero residual degrees of freedom and is therefore rigid and unique. Non-vacuum structure appears only as a controlled departure from this vacuum-closed limit and is uniquely restricted in number and coupling by symmetry and conservation. Newtonian gravity and vacuum GR thus emerge as structural consequences of closure rather than privileged assumptions.

The sections that follow develop the theorem deductively, eliminating structurally inadmissible possibilities step by step and culminating in the unique covariant action consistent with gravitational closure, together with a precise statement of domain of validity and falsification criteria.

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[1]See [1] for an upstream classification result. No part of that analysis is assumed in the present theorem.

2. Closure as the Central Principle

By *closure* we mean the requirement that the gravitational field regulate its own behaviour without appeal to external bookkeeping. A closed field admits a finite conserved energy, exhausts its degrees of freedom under symmetry reduction, and requires no regime-dependent prescriptions or hidden components to remain internally consistent.

Closure is not a speculative demand. It is already realised—decisively—in vacuum gravity. The exactness of Newtonian gravity and General Relativity in isolated regimes is not accidental; it is a direct expression of closure. In such regimes, the gravitational field admits no residual freedom beyond that fixed by sources and boundary conditions.

The difficulty arises when a single-mode description is extended beyond the regimes in which closure holds. In extended systems and homogeneous cosmology, gravity is required not only to enforce local constraint but also to organise influence across space or time. The conventional response has been to supplement the source sector. The Ananke Theorem instead asks whether closure itself already dictates the admissible structure of gravitational response.

3. Scope and Logical Status

It is necessary to be explicit about scope.

This work does:

- classify admissible classical gravitational field structures under minimal axioms,
- derive rigidity and uniqueness results in symmetry-reduced regimes,
- identify the unique admissible non-vacuum extension consistent with closure,
- recover Newtonian gravity and vacuum General Relativity as structural consequences.

This work does not:

- propose a new gravitational force or particle,
- modify General Relativity phenomenologically,
- assume the existence or non-existence of dark matter or dark energy as substances,
- rely on observational fitting.

Any phenomenological consequences arise only as corollaries of the classification and play no role in the theorem itself.

The axioms stated here are sufficient for the present classification and are not derived from any upstream result.

4. Informal Statement of the Result

Under the assumptions of classical covariance, quadratic closure with finite conserved energy, and orthogonal modes of response:

1. **Isolated vacuum regimes admit zero residual degrees of freedom:** the gravitational field closes entirely on its constraint sector, yielding rigidity and unique exterior behaviour. Newtonian vacuum gravity and vacuum General Relativity follow necessarily.
2. **Non-vacuum but symmetry-reduced regimes admit exactly one residual degree of freedom:** this degree of freedom is redistributive rather than sourcing, cannot propagate independently in vacuum, and is uniquely constrained in both form and coupling.
3. **The admissible covariant action is unique up to equivalence:** field content, interaction structure, and coupling to matter are fixed by necessity rather than choice.

The name *Ananke*—used here in its classical sense of inescapable constraint, *"it can be no other way"*—reflects the logic of the theorem. Once closure is imposed, alternative gravitational structures are not merely disfavoured; they are excluded.

5. Structure of the Theorem

The remainder of this work proceeds by strict deduction. Each section eliminates admissible freedom by ruling out incompatible structures, until a unique closed gravitational field remains. At no stage is phenomenological input required. Each step follows from the elimination of structural impossibilities.

Specifically:

- **Section 2** defines closure as the organising principle.
- **Section 3** states scope and logical status.
- **Section 4** gives an informal statement of the classification result.
- **Section 6** states the primitive axioms.
- **Section 7** derives immediate structural consequences, including exhaustion of functional freedom under symmetry (Proposition 1).
- **Section 8** derives orthogonality as a structural necessity (Lemma 1) and states the master statement.
- **Sections 9–11** classify admissible response structure and establish the forced two-mode decomposition (Proposition 2).
- **Sections 12–17** prove vacuum rigidity and its structural consequences, including exact recovery of Newtonian gravity and vacuum GR (Theorem 1).
- **Sections 18–20** transition to the non-vacuum case and prove existence and uniqueness of the residual degree of freedom (Theorem 2).
- **Sections 21–25** determine the admissible character of the residual structure, derive the forced interaction and conservation constraints (Proposition 3), and assemble the unique admissible covariant action.
- **Section 26** states the domain of validity and falsification criteria.
- **Section 27** summarises the result.
- **Appendices A–B** record logical dependencies and clarify the sense of action uniqueness.

6. Primitive Axioms

We state the primitive axioms underlying the Ananke (Gravitational Closure) Theorem. These axioms are minimal and purely structural. They do not presuppose a particular gravitational model, geometric ansatz, or particle ontology. Their sole function is to delimit the admissible class of closed classical gravitational fields.

6.1 Axiom I — Classical Covariance

Gravity is described by a classical physical field admitting a generally covariant formulation. Admissible structures are invariant under diffeomorphisms and require the existence of a covariant variational description,

$$S[\Psi] = \int \mathcal{L}(\Psi, \nabla\Psi)\, \sqrt{-g}\; d^4x,$$

whose physical content is independent of coordinate choice.

This axiom asserts physical equivalence under diffeomorphisms. It does not require the fundamental field variables to be identified with a metric, a connection, or any specific geometric object, nor does it presuppose a particular interpretation of spacetime.

6.2 Axiom II — Quadratic Closure

The gravitational field requires the existence of a quadratic action with a finite conserved bilinear field norm. That is, admissibility requires the existence of a quadratic functional

$$Q[\Psi] = \langle \Psi, \Psi \rangle,$$

which is finite on admissible configurations and conserved,

$$\delta Q = 0.$$

Quadratic closure ensures additive energetic bookkeeping between independent response components and exhaustion of functional freedom under symmetry reduction. Admissible solutions may retain only discrete integration constants fixed by conservation laws; no free functions of space or time are permitted.

Any non-quadratic contribution generically introduces additional dynamical scales, higher-order propagating structure, or implicit regime-dependent renormalisation. Such terms reintroduce independent degrees of freedom not fixed by symmetry or conservation and are therefore inadmissible.

6.3 Axiom III — Orthogonal Modes of Response

The gravitational field requires the existence of orthogonal modes of response with respect to the quadratic field norm. For any admissible decomposition,

$$\Psi = \sum_i \Psi_i,$$

orthogonality requires

$$\langle \Psi_i, \Psi_j \rangle = 0 \text{ for } i \neq j.$$

Under closure, the minimal admissible content consists of a constraint mode enforcing local balance and a redistributive mode governing internal organisation across extended domains. Orthogonality forbids redistribution from being absorbed into the constraint sector and fixes independent energetic bookkeeping between modes. This structure follows from necessity and does not depend on any particular dynamical realisation.

7. Immediate Structural Consequences

Although Axioms I–III are logically independent, their joint imposition places strong and immediate restrictions on admissible gravitational fields. The results derived in this section follow necessarily from the axioms alone and do not rely on additional assumptions.

7.1 Elimination of Functional Freedom Under Symmetry

Quadratic closure implies that, once sufficient spacetime symmetry is imposed, admissible field equations reduce to operators of fixed functional form. Orthogonality prevents deficiencies in one response mode from being absorbed by another, while covariance ensures that these restrictions are independent of coordinate choice.

Accordingly, in any sufficiently symmetric regime, admissible solutions cannot retain arbitrary functions of space or time. Any residual freedom must appear exclusively as discrete integration constants fixed by conservation laws.

7.2 Proposition 1 — Exhaustion of Residual Freedom by Symmetry

Proposition 1.
Under sufficient spacetime symmetry, the gravitational field equations admit no functional degrees of freedom. Only integration constants fixed by conservation may remain.

This proposition is not an additional axiom. It follows directly from Axioms I–III once finite-energy admissibility and symmetry reduction are imposed. Any surviving functional freedom would require supplementary regulating structure, thereby violating internal closure.

7.3 Clarification on Degrees of Freedom

Throughout this work, a degree of freedom refers exclusively to unfixed functional or parametric freedom that survives symmetry reduction and conservation, including residual integration constants. It does not refer to kinematical field components prior to the imposition of closure and admissibility.

With this convention, statements concerning zero, one, or multiple degrees of freedom are statements about what remains structurally admissible after all closure requirements are enforced.

7.4 Remark on Logical Status

The exhaustion of functional freedom under symmetry is a structural consequence rather than an assumption. It underlies vacuum rigidity, uniqueness of exterior solutions, and the classification of admissible non-vacuum extensions. No specific field decomposition or equation of motion has been assumed; the result applies to any classical gravitational field satisfying the axioms.

With functional freedom eliminated, the remaining task is to determine how many independent response modes are compatible with closure. Quadratic closure enforces additive energetic bookkeeping, while orthogonality ties independent physical roles to independent energetic sectors, constraining admissible modes prior to any dynamics.

8. Orthogonality as a Structural Necessity

This section establishes orthogonality of independent response sectors as a structural necessity of closure. The result completes the set of constraints required for the global classification statement presented at the end of this section. Orthogonality is shown to follow necessarily from quadratic closure and finite conserved energy, rather than by analogy with other field theories.

Having established that functional freedom is exhausted under symmetry (Section 7), we now determine the internal energetic structure admissible for a closed classical gravitational field. Quadratic closure enforces a decomposition of gravitational response into energetically independent sectors and strictly limits the number of admissible modes.

This result is deductive rather than heuristic. Orthogonality is not an assumption about dynamics or representation; it is required by closure itself.

8.1 Quadratic Closure and Energetic Structure

In any classical field theory governed by a quadratic action, the conserved energy defines a bilinear norm on the space of field configurations. Independent physical responses must therefore correspond to independent contributions to this norm.

Let F denote the total gravitational field response. Under quadratic closure, the conserved energy necessarily takes the schematic form

$$\mathcal{E} = \langle F, F \rangle.$$

If the field admits multiple response sectors, the most general quadratic decomposition is

$$\mathcal{E} = \langle F_L, F_L \rangle + \langle F_T, F_T \rangle + 2\langle F_L, F_T \rangle,$$

where F_L and F_T denote candidate independent responses. The cross term represents energetic exchange between sectors.

8.2 Lemma 1 — Orthogonality Is Forced

Lemma 1.
In a closed classical gravitational field satisfying Axioms I–III, independent physical responses must be orthogonal with respect to the quadratic field norm.

Proof (structural).
If $\langle F_L, F_T \rangle \neq 0$, energy may be transferred arbitrarily between sectors without altering the total conserved norm. This permits one sector to compensate deficiencies in the other, reintroducing hidden functional freedom not fixed by sources or symmetry. Such freedom violates closure. Therefore,

$$\langle F_L, F_T \rangle = 0$$

is required identically.

This result holds independently of field representation or dynamics. Orthogonality is a consequence of closure, not a modelling choice.

With the admissible response structure fully constrained by closure, covariance, and forced orthogonality, the Ananke Theorem may now be stated in consolidated form.

The Ananke (Gravitational Closure) Theorem — Master Statement

Under the axioms of classical covariance, quadratic closure with finite conserved energy, and orthogonal modes of response:

1. **Vacuum rigidity:** in isolated vacuum regimes, the gravitational field admits zero residual degrees of freedom. Closure is exact, exterior solutions are unique, and Newtonian gravity and General Relativity are recovered necessarily.

2. **Residual uniqueness:** in non-vacuum but symmetry-reduced regimes, closure admits exactly one residual degree of freedom. This degree of freedom is redistributive rather than sourcing, introduces no independent vacuum modes or universal constants, and is uniquely constrained by conservation and orthogonality.

3. **Action uniqueness:** the admissible covariant gravitational action is unique up to equivalence. Field content, interaction structure, and matter coupling are fixed by structural necessity rather than modelling choice.

No phenomenological input or observational constraint is required at any stage.

9. Classification of Admissible Response Structure

We now determine how many orthogonal response sectors a closed classical gravitational field may admit under Axioms I–III.

9.1 Exclusion of a Single-Sector Theory

A theory with a single energetic sector can enforce local constraint exactly and therefore closes in isolated vacuum regimes, where gravitational response is fixed by sources and boundary conditions. In extended or homogeneous configurations, however, a single-sector description provides no mechanism for organising gravitational influence across space or time, so closure fails unless additional structure is introduced.

A purely single-sector theory is therefore admissible only regime-by-regime and cannot constitute a globally closed gravitational field. This accounts for the exact success of Newtonian gravity and vacuum General Relativity without assuming their global completeness.

9.2 Exclusion of Three or More Independent Sectors

Suppose the gravitational field admitted three or more orthogonal energetic sectors. Each would contribute additively to the conserved quadratic norm, while Proposition 1 requires exhaustion of functional freedom under symmetry.

With more than two sectors, closure would require additional constants to regulate their relative amplitudes. Such constants are not fixed by conservation and would reintroduce hidden degrees of freedom, violating quadratic closure and finite-energy admissibility. Accordingly, any theory admitting three or more independent gravitational response sectors is excluded from the admissible class.

9.3 Forced Two-Sector Classification

The only admissible possibility is therefore exactly two orthogonal modes of response.

These modes are not postulated but forced by elimination:

- one mode enforces local constraint and source response;
- one mode governs redistribution across extended spatial or temporal domains.

Any alternative description is equivalent up to representation.

10. Structural Roles of the Two Admissible Modes

We now characterise the roles of the two response modes fixed by the preceding classification.

10.1 Constraint (Longitudinal) Mode

The constraint mode enforces local balance between sources and gravitational response. In symmetry-reduced regimes it closes exactly, admitting no functional freedom beyond integration constants fixed by boundary conditions. In isolated vacuum regimes it fully determines the gravitational field and yields Newtonian gravity and vacuum General Relativity as necessary consequences of closure rather than assumed dynamics.

10.2 Redistributive (Transverse) Mode

The redistributive mode governs the internal organisation of the gravitational field across extended spatial or temporal domains. It introduces no new sources and cannot act independently in isolated vacuum regimes.

Its role is purely internal: redistribution of gravitational influence subject to global conservation. The redistributive mode vanishes identically, or reduces to pure gauge, in vacuum and becomes admissible only as a controlled departure from the vacuum-closed limit.

11. Proposition 2 — Uniqueness of the Two-Mode Structure

Proposition 2.
A closed classical gravitational field satisfying Axioms I–III admits exactly two orthogonal modes of response: one constraint mode and one redistributive mode.

Proof.
A single response mode enforces local constraint but fails to close in extended or homogeneous regimes, where organisation across space or time is required. Conversely, admitting more than two independent response modes violates quadratic closure under symmetry by introducing uncontrolled residual freedom not fixed by conservation laws. Exactly two orthogonal modes satisfy all axioms simultaneously.

12. Transition to Vacuum Rigidity

At this stage, no equations of motion or geometric interpretation have been assumed. The preceding classification establishes only that a closed classical gravitational field must consist of two orthogonal response sectors with distinct structural roles.

We now examine the behaviour of this two-mode structure in isolated vacuum regimes, where closure enforces complete rigidity and eliminates all residual degrees of freedom.

13. Isolated Vacuum Regimes

We consider the behaviour of the classified two-mode gravitational field in isolated vacuum regimes. Such regimes are characterised by:

- absence of matter sources,
- absence of external fluxes,
- sufficient spacetime symmetry to exhaust functional freedom, and
- admissible boundary conditions ensuring finite conserved energy.

These include exterior regions surrounding compact sources and asymptotically isolated systems. They constitute the empirical domain in which Newtonian gravity and vacuum General Relativity achieve exact success.

The question addressed here is not whether gravity behaves rigidly in vacuum, but why such rigidity is structurally unavoidable.

14. Behaviour of the Redistributive Mode in Vacuum

By construction, the redistributive mode governs internal organisation across extended spatial or temporal domains. In an isolated vacuum regime:

- no internal domains exist to be organised,
- no gradients of source density are present, and
- no admissible channels for redistribution exist.

Any non-trivial excitation of the redistributive mode would therefore constitute a propagating vacuum degree of freedom carrying energy independently of sources or boundary data. Such behaviour is incompatible with closure.

14.1 Lemma 2 — Vanishing of Redistribution in Vacuum

Lemma 2.
In an isolated vacuum regime, the redistributive mode must vanish identically (or reduce to pure gauge).

Proof.
A non-zero redistributive mode in vacuum would carry finite energy independent of sources or boundary conditions. Under symmetry reduction, this energy could not be fixed by conservation, contradicting Proposition 1. Closure therefore requires the redistributive mode to vanish in vacuum.

This result is structural. It is independent of dynamics, field representation, or geometrical interpretation.

15. Zero-Degree-of-Freedom Closure and Vacuum Primacy

In isolated vacuum regimes, the redistributive mode is inadmissible. With no extended domains to organise and no gradients requiring internal coordination, any non-trivial redistributive excitation would constitute an independent vacuum degree of freedom. Such freedom is excluded by closure.

The gravitational field in vacuum therefore reduces entirely to the constraint sector. Under symmetry reduction, this sector admits no freedom beyond boundary conditions. Quadratic closure, orthogonality, and the absence of redistribution together exhaust all functional freedom, leaving no residual degrees of freedom.

Vacuum primacy follows as a structural consequence. The vacuum configuration represents the fully closed state of the gravitational field: a regime in which all admissible degrees of freedom are exhausted. Any non-vacuum behaviour must therefore be continuous with, and subordinate to, this zero-degree-of-freedom limit.

15.1 Theorem 1 — Vacuum Zero-Degree-of-Freedom Rigidity

Theorem 1 (Vacuum Rigidity).
In any isolated vacuum regime admitting sufficient spacetime symmetry, a closed classical gravitational field satisfying Axioms I–III possesses zero residual degrees of freedom. The gravitational field configuration is uniquely fixed by boundary conditions.

Proof.
By Lemma 2, the redistributive mode vanishes identically (or reduces to pure gauge) in vacuum. The remaining constraint mode closes exactly under symmetry and, by Proposition 1, admits no functional freedom. With no admissible degrees of freedom remaining, the gravitational field is fully determined by boundary data.

This result establishes vacuum primacy as a necessity of closure rather than a contingent property of particular field equations. The redistributive sector is not an independent component of gravity, but a closure-preserving extension whose admissibility, magnitude, and form are fixed entirely by continuity with the zero-degree-of-freedom vacuum limit.

16. Structural Consequences of Vacuum Closure

Several foundational consequences follow immediately from vacuum zero-degree-of-freedom closure.

16.1 Uniqueness of Exterior Solutions

Because no residual freedom exists, exterior gravitational fields surrounding isolated sources are unique. No additional parameters, profiles, screening mechanisms, or environmental dependence are admissible.

This result is structural and does not depend on any particular choice of field equations.

16.2 Inverse-Square Scaling as Structural Necessity

In three spatial dimensions, uniqueness, scale freedom, and covariance together admit only a single admissible radial dependence for a vacuum gravitational field. Under zero-degree-of-freedom closure, the exterior solution must be isotropic, source-determined, and free of adjustable parameters or intrinsic length scales.

These requirements uniquely fix inverse-square radial scaling as the only scale-free, isotropic solution compatible with closure in three dimensions. This behaviour is therefore not a dynamical coincidence or empirical fit, but a structural consequence of vacuum rigidity once functional freedom is exhausted.

Inverse-square scaling follows here as a classification result: any alternative radial dependence would either introduce an intrinsic scale, violate isotropy, or reintroduce residual degrees of freedom, all of which are excluded by closure.

16.3 Birkhoff-Type Rigidity

Time-independent exterior behaviour follows immediately. With no redistributive mode and no residual degrees of freedom, spherically symmetric vacuum configurations cannot evolve dynamically.

Birkhoff-type rigidity is therefore a structural consequence of closure rather than a special property of particular gravitational field equations.

17. Structural Recovery of Newtonian Gravity and Vacuum General Relativity

At no stage have Newtonian gravity or General Relativity been assumed. Nevertheless, the preceding results enforce unique exterior behaviour, inverse-square scaling, absence of vacuum dynamics, and universal coupling to sources.

Any admissible gravitational theory satisfying Axioms I–III must therefore recover Newtonian gravity and vacuum General Relativity exactly in isolated regimes. Their empirical success is a consequence of structural closure rather than contingent modelling choice.

18. Transition to the Non-Vacuum Case

Vacuum rigidity represents the maximally closed state of the gravitational field. We now relax the vacuum condition while preserving sufficient symmetry for admissibility.

In extended non-vacuum regimes, closure cannot eliminate all degrees of freedom. The relevant question is therefore not whether residual freedom appears, but how much may remain without violating the axioms.

19. Controlled Departure from Vacuum Closure

Vacuum closure represents the fully exhausted state of the gravitational field: in isolated vacuum regimes, all admissible degrees of freedom are eliminated and the field closes exactly on its constraint sector. This zero-degree-of-freedom configuration is structurally primary.

Non-vacuum regimes are admissible only as controlled departures from this vacuum-closed limit. Any additional gravitational structure must remain continuous with vacuum rigidity, introduce no vacuum degrees of freedom, and preserve closure under symmetry and conservation.

The role of the non-vacuum extension is therefore not to modify vacuum behaviour, but to accommodate redistribution in regimes where closure cannot be maintained by constraint alone. The question addressed in the following sections is not whether residual structure appears, but how much is permitted and in what form. The answer is fixed by the same principles that enforce vacuum rigidity.

20. Counting Residual Degrees of Freedom

Quadratic closure and orthogonality restrict the gravitational field to two sectors: a constraint sector and a redistributive sector. In non-vacuum regimes, the redistributive sector need not vanish identically. However, Proposition 1 continues to apply: functional freedom must be exhausted under symmetry.

This requirement leaves only one admissible possibility.

20.1 Lemma 3 — At Most One Residual Degree of Freedom

Lemma 3.
In any symmetry-reduced non-vacuum regime, a closed classical gravitational field admits at most one residual degree of freedom.

Proof.
Any residual freedom must reside in the redistributive sector, since the constraint sector closes exactly under symmetry. If more than one independent degree of freedom remained, their relative amplitudes would require additional regulating constants or functions. Such freedom would violate quadratic closure and the requirement of finite conserved energy. Therefore, at most one residual degree of freedom is admissible.

20.2 Lemma 4 — At Least One Residual Degree of Freedom

Lemma 4.
In extended non-vacuum regimes, closure requires at least one residual degree of freedom.

Proof.
A purely constraint-dominated field enforces local balance but provides no mechanism for organising gravitational influence across extended spatial or temporal domains. In non-vacuum systems with spatial or temporal extension, such organisation is required for global consistency. In the absence of a residual redistributive degree of freedom, closure fails.

20.3 Theorem 2 — Existence and Uniqueness of the Residual Degree of Freedom

Theorem 2.
In any admissible non-vacuum but symmetry-reduced regime, a closed classical gravitational field possesses exactly one residual degree of freedom.

Proof.
Lemma 3 excludes more than one residual degree of freedom. Lemma 4 excludes none. Exactly one remains.

This result is purely structural. It establishes existence and uniqueness but does not yet specify the nature of the residual degree of freedom.

21. Character of the Residual Degree of Freedom

We now determine the admissible character of the unique residual degree of freedom consistent with Axioms I–III and the preceding classification results.

21.1 Exclusion of an Independent Scalar

If the residual degree of freedom were realised as an independent scalar field, it would modify the constraint sector directly. Such a modification would persist in vacuum unless tuned to vanish identically, thereby reintroducing functional arbitrariness.

An independent scalar degree of freedom therefore violates vacuum rigidity and is excluded from the admissible class.

21.2 Exclusion of a Free Vector Field

If the residual degree of freedom were realised as a free vector field with an independent kinetic term, it would admit propagating vacuum modes carrying energy independently of sources or boundary data.

This possibility contradicts Lemma 2 and Theorem 1. A free vector field is therefore inadmissible.

21.3 Structural Requirements on the Residual Degree of Freedom

The residual degree of freedom must satisfy all of the following:

- it resides entirely within the redistributive sector;
- it admits no independent degrees of freedom in vacuum;
- it introduces no additional universal constants;
- it vanishes identically, or reduces to pure gauge, in the vacuum limit;
- its magnitude and behaviour are fixed by the constraint sector and conservation alone.

No admissible realisation permits this residual to exist as an independent fundamental field.

22. Forced Slaving of the Redistributive Mode

The only admissible possibility is that the redistributive mode is slaved to the constraint mode.

Accordingly, the redistributive field must be constructed entirely from the constraint field and a direction field whose freedom is fixed by symmetry and admissibility rather than by independent dynamics. Formally, the redistributive potential must take the form

$$A_\mu = \Phi \, u_\mu,$$

where Φ denotes the scalar associated with the constraint sector and u_μ is a unit or tangent field encoding the direction of redistribution. No alternative construction satisfies all admissibility conditions simultaneously.

23. Conservation and the Continuity Constraint

Redistribution must occur without creation or destruction of the scalar content it transports. Global conservation therefore imposes a continuity-type constraint.

23.1 Proposition 3 — Continuity of Redistribution

Proposition 3.
The slaved redistributive mode must satisfy a continuity constraint of the form

$$\nabla_\mu(\Phi\, u^\mu) = 0.$$

Justification.
This condition is the unique covariant expression of redistribution without creation compatible with quadratic closure and finite conserved energy. Any alternative would introduce sources, sinks, or additional degrees of freedom, thereby violating closure.

24. Summary of the Non-Vacuum Classification

In non-vacuum symmetry-reduced regimes, exactly one residual degree of freedom is admissible. It cannot appear as an independent scalar or vector field, must be slaved to the constraint sector, and is fixed entirely by conservation.

With this classification complete, the admissible form of gravitational interaction is fully determined. The remaining task is to assemble the unique covariant action consistent with these constraints.

25. Assembly of the Unique Admissible Action

At this stage, no freedom remains in the admissible gravitational structure. The action is neither postulated nor selected by modelling preference. It is recovered as the unique representative of the equivalence class permitted by closure, orthogonality, and vacuum rigidity.

Each sector is fixed by necessity. Any omission would violate closure. Any addition would introduce independent degrees of freedom, universal constants, or residual vacuum structure excluded by the classification.

25.1 Metric Sector

Vacuum rigidity requires that, in isolated vacuum regimes, the gravitational field admit zero residual degrees of freedom and exhibit unique exterior behaviour. In four spacetime dimensions, the only generally covariant, second-order metric action satisfying this requirement is the Einstein–Hilbert action, up to boundary terms.

Accordingly, the metric sector is fixed as

$$S_g = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R.$$

No primitive cosmological term is permitted, as such a term would survive in vacuum and introduce an independent universal constant, contradicting vacuum rigidity. Curvature-like contributions may arise only as integration constants in non-vacuum or homogeneous regimes.

25.2 Constraint Sector

The constraint mode governs local gravitational balance and must close exactly under symmetry reduction. Quadratic closure requires that its contribution enter solely through a quadratic norm.

The admissible scalar contribution is therefore

$$S_c = \int d^4x\,\sqrt{-g}\,\frac{1}{2}(\nabla\phi)^2.$$

No potential term is allowed, as this would introduce an additional universal constant and modify vacuum behaviour.

25.3 Redistributive Sector

Sections 21–22 established that the redistributive mode cannot be an independent field. It must be slaved to the constraint sector and constructed as a composite object fixed by symmetry and admissibility.

The redistributive potential therefore takes the form

$$A_\mu = \phi\, u_\mu,$$

where u_μ is a direction field whose freedom is fixed by symmetry rather than by independent dynamics.

Quadratic closure and orthogonality permit only a Maxwell-type norm built from this composite field. The redistributive contribution is therefore

$$S_r = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu},$$

with

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu.$$

No cross-terms with the constraint sector are permitted, as these would violate orthogonality and reintroduce hidden degrees of freedom.

25.4 Conservation Constraint

Redistribution must occur without creation or destruction of the scalar content it transports. Global conservation therefore imposes a continuity constraint.

This constraint is enforced by a Lagrange multiplier field λ through the term

$$S_\lambda = \int d^4x\,\sqrt{-g}\,\lambda\,\nabla_\mu(\phi\, u^\mu).$$

The form of this constraint is fixed uniquely by covariance and closure.

25.5 Matter Coupling

Universality and closure forbid direct coupling of matter to either the constraint or redistributive sectors. Matter therefore couples minimally to the metric alone,

$$S_{\text{matter}} = S_{\text{matter}}[g,\psi].$$

No additional couplings are admissible.

25.6 The Unique Closed Action

Collecting the metric, constraint, redistributive, conservation, and matter sectors, the admissible closed classical gravitational action is fixed uniquely once closure, orthogonality, and vacuum rigidity are imposed. At the structural level, the action necessarily decomposes as

$$S \;=\; S_{\text{EH}}[g] \;+\; S_{\text{C}}[g,\phi] \;+\; S_{\text{R}}[g,\phi,u] \;+\; S_{\text{cons}}[g,\phi,u,\lambda] \;+\; S_{\text{matter}}[g,\psi].$$

Here the redistributive sector contains no independent degrees of freedom and is fully slaved to the constraint sector through the conservation constraint. Matter couples minimally to the metric alone. No additional sectors are admissible.

Up to boundary terms, invertible local field redefinitions, and normalisation conventions, a representative of this unique equivalence class is given by

$$\boxed{\begin{aligned} S \;&= \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R + \int d^4x\,\sqrt{-g}\,\frac{1}{2}(\nabla\phi)^2 - \frac{1}{4}\int d^4x\,\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu} \\ &\quad + \int d^4x\,\sqrt{-g}\,\lambda\,\nabla_\mu(\phi\, u^\mu) + S_{\text{matter}}[g,\psi]. \end{aligned}}$$

Under the requirements of closure, orthogonality, and vacuum rigidity, this action is fixed uniquely up to equivalence. Any modification introduces independent degrees of freedom or violates closure; any omission renders the structure incomplete.

26. Domain of Validity and Falsification

The Ananke Theorem is explicit about both its domain of applicability and the conditions under which it would be falsified.

26.1 Domain of Validity

The exact classification results apply only in regimes admitting sufficient spacetime symmetry, including:

- isolated vacuum exteriors,
- stationary extended systems, and
- homogeneous or homogeneous–isotropic configurations.

In less symmetric situations, closure does not uniquely fix detailed behaviour. Variability, scatter, and system-dependent structure are therefore expected and should be interpreted as diagnostics of symmetry breaking rather than as violations of the theorem.

26.2 Falsification Criteria

The theorem would be falsified by empirical evidence for any of the following:

- independent propagating degrees of freedom in vacuum;
- more than one universal residual gravitational constant;
- screening or shielding of gravitational interaction;
- environment-dependent gravitational coupling;
- additional gravitational charges.

It would also be falsified by evidence that closure requires intrinsically non-quadratic regulating structure, or that functional freedom persists in regimes admitting sufficient symmetry.

In the absence of such evidence, the class of closed classical gravitational fields identified by the theorem remains the only admissible class under Axioms I–III.

27. Conclusion

We have shown that the Ananke (Gravitational Closure) Theorem classifies the admissible structure of classical gravity once minimal structural requirements are imposed. Classical covariance, quadratic closure with finite conserved energy, and orthogonal modes of response together eliminate all but one permissible gravitational field structure.

In isolated vacuum regimes, structural closure is complete. The redistributive mode vanishes, all functional freedom is exhausted under symmetry, and the gravitational field admits zero residual degrees of freedom. Vacuum gravity is therefore rigid and exact: exterior solutions are unique, inverse-square scaling is forced in three spatial dimensions, and no propagating or adjustable vacuum degrees of freedom are permitted.

Non-vacuum regimes arise only as controlled departures from this fully constrained vacuum limit. When symmetry is preserved but vacuum isolation is relaxed, exactly one residual redistributive degree of freedom is admitted. Its existence, uniqueness, and coupling are fixed by conservation and orthogonality. It cannot propagate independently in vacuum and introduces no additional universal constants. The resulting covariant action is therefore not selected by modelling preference, but obtained by elimination of all structurally inadmissible alternatives.

Newtonian gravity and vacuum General Relativity thus follow as necessary consequences of structural exhaustion rather than as independent postulates. Their empirical success reflects the fact that they describe the maximally constrained regime of the gravitational field. Apparent limitations outside vacuum arise only when a purely constraint-dominated description is extended beyond the domain in which full closure holds.

The geometric interpretation of gravity established by General Relativity is not discarded. It is completed. A fully closed classical geometry necessarily comprises both a constraint structure and a redistributive structure, each fixed by the same principles of covariance, conservation, and admissibility.

Gravity therefore conforms to the general behaviour of closed classical fields: rigidity when no internal degrees of freedom are available, and controlled redistribution when additional structure is admitted. Once the axioms are imposed, no alternative admissible classical gravitational structure remains. In this sense, gravity exhibits *Ananke*, *it can be no other way*.

Appendix A. Logical Dependencies and Scope

This appendix records the logical structure and scope of the Ananke (Gravitational Closure) Theorem.

The theorem proceeds deductively from a minimal set of primitive axioms: classical covariance, quadratic closure with finite conserved energy, and orthogonal modes of response. From these axioms, propositions and lemmas establish exhaustion of functional freedom under symmetry, vacuum rigidity, and the admissibility of residual structure only in non-vacuum, symmetry-reduced regimes. The main theorems then classify the number, character, and coupling of admissible degrees of freedom and fix the unique covariant action consistent with closure.

No phenomenological input, observational fitting, or regime-specific assumptions enter at any stage. All results follow solely from structural elimination: admissible structures are identified by excluding those incompatible with closure, conservation, or orthogonality. The theorem is therefore independent of any particular gravitational model, geometric ansatz, or matter content, and applies to the full class of closed classical gravitational fields satisfying the axioms.

Appendix B. Uniqueness of the Admissible Action

This appendix clarifies the sense in which the covariant gravitational action assembled in Section 25 is unique.

Uniqueness is understood up to equivalence under boundary terms, invertible local field redefinitions, and overall normalisation conventions. Such transformations leave the field equations, conserved quantities, and physical content invariant and are therefore immaterial to closure.

Beyond these equivalences, no additional terms are admissible. In particular, closure and orthogonality exclude:

- independent dynamical fields beyond the identified constraint and redistributive sectors;
- potential terms or additional universal constants that would survive in vacuum;
- cross-couplings between sectors that would violate orthogonality;
- higher-order or non-quadratic contributions that would reintroduce functional freedom or regime-dependent behaviour.

Accordingly, any action differing from that assembled in Section 25 by more than the equivalences noted above necessarily violates at least one of the structural requirements of the theorem. The admissible action is therefore fixed uniquely by necessity (*Ananke*) rather than by modelling choice.