**THE AUSTRALIAN NATIONAL UNIVERSITY**

*Assignment 3*

**COMP2610/COMP6261**

**Information Theory, Semester 2 2023**

**Release Date: Tuesday 26 September 2023**

**Due Date: Monday 23 October 2023, 9:00 am**

**Cut-off Date: Friday 28 October 2023, 5:00 pm**

**No submission allowed after Friday 28 October 2023, 5:00 pm.**

**Assignment 3 weighting is 20% of the course mark.**

## Instructions:

**Marks:**

- The mark for each question is indicated next to the question. For questions where you are asked to prove results, if you can not prove a precedent part, you can still attempt subsequent parts of the question assuming the truth of the earlier part.
- **For all students:** Answer all Questions. You will be marked out of 100.

**Submission:**

- Submit your assignment together **with a cover page** as a single PDF on Wattle.
- Submission deadlines will be strictly enforced. A late submission attracts a penalty of 5% per working day. If you submit after the cut-off date, you get zero marks (100% penalty). Extensions will be considered according to the ANU Policy.
- All assignments must be done individually. Plagiarism is a university offence and will be dealt with according to university procedures *http://academichonesty.anu.edu.au/UniPolicy.html*.

## Question 1: Entropy and Joint Entropy [10 marks total]

An ordinary deck of cards containing 13 clubs, 13 diamonds, 13 hearts, and 13 spades cards is shuffled and dealt out one card at time without replacement. Let $X_i$ be the suit of the $i$th card.

(a) Determine $H(X_1)$. [**4 marks**]

(b) Determine $H(X_1, X_2, \cdots, X_{52})$. [**6 marks**]

## Question 2: Source Coding [30 marks total]

### Question 2-I [12 marks total]

Construct a binary Huffman code and Shannon-Fano-Elias code for the following distribution on 5 symbols $p = (0.3, 0.3, 0.2, 0.1, 0.1)$. What is the average length of these codes?

### Question 2-II [12 marks total]

Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) Find a binary Huffman code for $X$. [**4 marks**]

(b) Find $H(X)$ and the expected codelength for this encoding. [**2 marks**]

(c) Find a ternary Huffman code for $X$. [**4 marks**]

### Question 2-III [6 marks total]

A random variable $X$ takes on three values, e.g., $a$, $b$, and $c$, with probabilities 0.55, 0.25, and 0.2.

(a) What are the lengths of the binary Huffman codewords for $X$? What are the lengths of the binary Shannon codewords for $X$? [**3 marks**]

(b) What is the smallest integer $D$ such that the expected Shannon codeword length with a $D$-ary alphabet equals the expected Huffman codeword length with a $D$-ary alphabet? [**3 marks**]

## Question 3: Channel Capacity [30 marks total]

### Question 3-I [15 marks total]

There is a discrete memoryless channel (DMC) with the channel input $X \in \mathcal{X} = \{1, 2, 3, 4\}$. The channel output $Y$ follows the following probabilistic rule.

$$Y = \begin{cases} X & \text{probability } \frac{1}{2} \\ 2X & \text{probability } \frac{1}{2} \end{cases}$$

Answer the following questions.

(a) Draw the schematic of the channel and clearly show possible channel outputs and the channel transition probabilities. **[5 marks]**

(b) Write the mutual information $I(X;Y)$ as a function of the most general input probability distribution. **[5 marks]**

(c) Find a way of using only a subset of the channel inputs such that the maximum mutual information (you need to quantify its value) can be achieved with zero error. **[5 marks]**

### Question 3-II [10 marks total]

The $Z$-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$p(y|x) = \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix} \quad x, y \in \{0, 1\}$$

Find the capacity of the $Z$-channel and the maximizing input probability distribution.

### Question 3-III [5 marks total]

Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$. Suppose that $Z_i$ has constant marginal probabilities $\Pr(Z_i = 0) = p$, but that $Z_1, Z_2, \cdots, Z_n$ are not necessarily independent. Assume that $Z_i$ is independent of the input $X_j$, for $\forall i, j$. Let $C = 1 - H(p, 1-p)$, Show that

$$\max_{p(x_1, x_2, \cdots, x_n)} I(X_1, X_2, \cdots, X_n; Y_1, Y_2, \cdots, Y_n) \geq nC.$$

## Question 4: Joint Typical Sequences [30 marks total]

### Question 4-I [15 marks total]

Let $(x^n, y^n, z^n)$ be drawn according to the joint distribution $p(x,y,z)$ in an independent and identically distributed (i.i.d.) manner. We say that $(x^n, y^n, z^n)$ is jointly $\varepsilon$-typical if all the following conditions are met

- $|\tilde{H}(x^n) - H(X)| \leq \varepsilon$
- $|\tilde{H}(y^n) - H(Y)| \leq \varepsilon$
- $|\tilde{H}(z^n) - H(Z)| \leq \varepsilon$
- $|\tilde{H}(x^n, y^n) - H(X,Y)| \leq \varepsilon$
- $|\tilde{H}(x^n, z^n) - H(X,Z)| \leq \varepsilon$
- $|\tilde{H}(y^n, z^n) - H(Y,Z)| \leq \varepsilon$
- $|\tilde{H}(x^n, y^n, z^n) - H(X,Y,Z)| \leq \varepsilon$

where $\tilde{H}(x^n) = -\frac{1}{n}\log_2(p(x^n))$. Now suppose that $(\tilde{x}^n, \tilde{y}^n, \tilde{z}^n)$ is drawn i.i.d. according to $p(x)$, $p(y)$, and $p(z)$. Therefore, $(\tilde{x}^n, \tilde{y}^n, \tilde{z}^n)$ have the same marginals as $p(x^n, y^n, z^n)$, but are independent. Find upper and lower bounds on the probability that $(\tilde{x}^n, \tilde{y}^n, \tilde{z}^n)$ is jointly typical in terms of $H(X,Y,Z)$, $H(X)$, $H(Y)$, $H(Z)$, $\varepsilon$, and $n$.

### Question 4-II [15 marks total]

Let $\mathbf{p} = [0.4, 0.33, 0.27]$ be the distribution of a random variable $X$ that takes symbols from $\{a,b,c\}$, respectively.

(a) Find the empirical entropy of the i.i.d. sequence

$$\mathbf{x} = aabaabbcabaccab$$

**[5 marks]**

(Hints: the empirical entropy $\tilde{H}(x^n) = -\frac{1}{n}\log_2(p(x^n))$.)

(b) Find whether it is a $\varepsilon$-typical sequence with $\varepsilon = 0.05$ **[5 marks]**

(c) Now assume the following joint probability distribution between $X$ and $Y$ that take symbols from $\{a,b,c\}$ and $\{d,e,f\}$ respectively.

$$p(x,y) = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.08 & 0.1 & 0.15 \\ 0.15 & 0.05 & 0.07 \end{bmatrix}$$

where in each row, $x$ is fixed. We observe two i.i.d. sequences

$$\mathbf{x} = aabaabbcabaccab$$

$$\mathbf{y} = deefdefddfddffd$$

Determine whether $(\mathbf{x}, \mathbf{y})$ are jointly $\varepsilon$-typical. **[5 marks]**