The Australian National University Semester 2, 2023
School of Computing Tutorial 3
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COMP3670/6670: Introduction to Machine Learning

Exercises with a **!** *denote harder ones,* **!!** *denotes very difficult, and* **!!!** *denotes optional challenge exercises.*

This tutorial will be primarily about proofs in analytic geometry. There are far too many exercises to do in the 2 hours, so you should choose some particular ones to work on. Your tutor will present some in class, and feel free to post partial solutions on Ed if you get stuck.

Question 1 **Properties of the zero vector**

Show that for any vector space V with any inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, we have that $\mathbf{0}$ (the zero vector) is orthogonal to every vector $\mathbf{v} \in V$.

Also, show that for any vector $\mathbf{v} \in V$, that $\{\mathbf{v}, \mathbf{0}\}$ forms a linearly dependant set.

Question 2 **Inner products**

Prove that the standard Euclidean inner product on $\mathbb{R}^2$ given by

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2$$

is an inner product.

Question 3 **Pythagorus**

We have that any inner product induces a norm,

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

Show that for two orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ (that is $\langle \mathbf{x}, \mathbf{y} \rangle = 0$) that the following holds

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2$$

(This is a extension of Pythagorus' Theorem, that for a right angled triangle with hypotenuse of length c, and two other sides of length a and b, that $a^2 + b^2 = c^2$.)

Question 4 **! Parseval's Identity**

Let V be a vector space, together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. Given a set of orthogonal vectors $\{x_1, \ldots, x_n\}$, show that

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Question 5 **Norms**

1. Prove that the Manhatten norm (l_1 norm) on $\mathbb{R}^2$ defined by

$$\|\mathbf{x}\|_1 := |x_1| + |x_2|$$

   is a norm. (You will need the triangle inequality on $\mathbb{R}$, $|a + b| \leq |a| + |b|$, to help you.)

2. **!** Prove that the supremum norm (l_∞ norm) on $\mathbb{R}^2$ defined by

$$\|\mathbf{x}\|_\infty := \max\left(|x_1|, |x_2|\right)$$

   is a norm. (Hint: You will need triangle inequality on $\mathbb{R}$, and the property that if $A \subseteq B$, then $\max_{x \in A} f(x) \leq \max_{x \in B} f(x)$.)

Question 6 **! Basis of a vector space**

Let V be a finite dimensional vector space, and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. Suppose that for any two basis vectors $\mathbf{b}_i$ and $\mathbf{b}_j$, we can compute the inner product $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$. Then, show that for any two vectors $\mathbf{u}, \mathbf{v}$ in V, we can express the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ in terms of the inner product of basis vectors $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$

(Hint: Use the fact that B spans the space V.)

Question 7 **Orthogonal matrices preserve angles and norms**

Suppose we are in the vector space $\mathbb{R}^n$, together with the standard Euclidean dot product, that is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} := \mathbf{x}^T \mathbf{y}$$

Let

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix (that is, $\mathbf{A}^{-1} = \mathbf{A}^T$.)

Show that for any vector $\mathbf{x} \in \mathbb{R}^n$ that

$$\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$

Using the above result (or otherwise), show that if the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is θ then the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$ is either θ, or $-\theta$ (modulo 2π).

Given an example of an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, such that the angle between $\mathbf{x}$ and $\mathbf{y}$ is not the same as the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$.

Question 8 **Rotation matrices preserve norms**

Given a vector $\mathbf{x} \in \mathbb{R}^2$ and the rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Show that for any angle of rotation θ, we have

$$\|\mathbf{x}\|_2 = \|\mathbf{R}(\theta)\mathbf{x}\|_2$$

Question 9 **Gram-Schmidt**

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the standard basis vectors for $\mathbb{R}^2$. Let $\mathbf{v}$ be any vector in $\mathbb{R}^2$.

Define the projection operator

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) := \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

(if $\mathbf{u} = \mathbf{0}$, then we define $\mathrm{proj}_{\mathbf{0}}(\mathbf{v}) = \mathbf{0}$.

The Gram-Schmidt algorithm takes a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and proceeds as follows:

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{v}_1 \\
&\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) \\
&\mathbf{u}_3 = \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3) \\
&\ldots = \ldots \\
&\mathbf{u}_n = \mathbf{v}_n - \textstyle\sum_{j=1}^{n-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_n)
\end{aligned}$$

The output $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a set of orthonormal vectors that spans the same set as $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ (If the dimension of the space spanned by the $\mathbf{v}_i$'s is less than n, then some of the $\mathbf{u}_i$'s will be zero.)

Suppose we are considering vectors in the vector space of $\mathbb{R}^2$.
Show that if we input $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}\}$ to the Gram-Schmidt algorithm, the output is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{0}\}$

Question 10 **!!! Cauchy-Schwartz**

Prove the Cauchy-Schwartz inequality for a general inner product and corresponding induced norm:

$$\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

(Hint: Let $\mathbf{z} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$, and start with the fact that $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0.$)