COMP2610 / COMP6261 Information Theory
Lecture 21: Linear Codes

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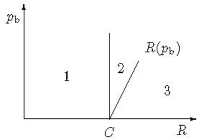
The Noisy-Channel Coding Theorem

Formal Statement

Recall: a rate is achievable if for any tolerance $\epsilon > 0$, an (N, K) code with rate $K/N \geq R$ exists with max. block error $p_{BM} < \epsilon$

The Noisy-Channel Coding Theorem (Formal)

1. Any rate $R < C$ is *achievable* for Q
2. If probability of bit error $p_b := p_B/K$ is acceptable, (N, K) codes exists with rates
$$\frac{K}{N} \leq R(p_b) = \frac{C}{1 - H_2(p_b)}$$
3. For any p, we **cannot** achieve a rate greater than $R(p)$ with probability of bit error p.



Theory and Practice

> *The difference between theory and practice is that, in theory, there is no difference between theory and practice but, in practice, there is.*
>
> — *Jan L. A. van de Snepscheut*

Theory vs. Practice

- The NCCT theorem tells us that good block codes exist for any noisy channel (in fact, most random codes are good)

- However, the theorem is non-constructive: it does not tell us **how** to create *practical* codes for a given noisy channel

- The construction of practical codes that achieve rates up to the capacity for general channels is ongoing research

NCCT Part 1: Comments

NCCT shows the existence of good codes; actually constructing practical codes is another matter

In principle, one could try the coding scheme outlined in the proof

- However, it would require a lookup in an exponential sized table (for the typical set decoding)!

Over the past few decades, some codes (e.g. Turbo codes) have been shown to achieve rate close to the Shannon capacity

- Beyond the scope of this course!

Types of Codes

When we talk about types of codes we will be referring to schemes for creating (N, K) codes for any size N. MacKay makes the following distinctions:

- **Bad**: Cannot achieve arbitrarily small error, or only achieve it if the rate goes to zero (i.e., either $p_{BM} \to a > 0$ as $N \to \infty$ or $p_{BM} \to 0 \implies K/N \to 0$)
- **Good**: Can achieve arbitrarily small error up to some maximum rate strictly less than the channel capacity (i.e, for any ϵ a good coding scheme can make a code with $K/N = R_{max} < C$ and $p_{BM} < \epsilon$)
- **Very Good**: Can achieve arbitrarily small error at any rate up to the channel capacity (i.e., for any $\epsilon > 0$ a very good coding scheme can make a code with $K/N = C$ and $p_{BM} < \epsilon$)
- **Practical**: Can be coded and decoded in time that is polynomial in the block length N.

Today's plan

Noisy Channel Coding Theorem proves there exists codes with rate $R < C$ with arbitrarily low probability of error.
But proof was non-constructive — we used a random code in order to be able to actually calculate error probability.
What about constructive codes?
We will focus on linear codes and look at two simple linear codes:

- repetition codes
- Hamming codes

We will sketch what can be said about the rate and reliability of the latter

1 Repetition Codes

2 The (7,4) Hamming code
- Coding
- Decoding
- Syndrome Decoding
- Multiple errors
- Error Probabilities

3 Coding: Review

Repetition Codes

Simplest channel code: add *redundancy* by repeating every bit of the message (say) 3 times:

Source sequence **s**	Transmitted sequence **t**
0	0 0 0
1	1 1 1

This *repetition code* is called R_3.

Repetition Codes for the BSC

Example

On a binary symmetric channel with flip probability f, receiver sees

$$\boldsymbol{r} = \boldsymbol{t} + \boldsymbol{\eta}$$

where $\boldsymbol{\eta}$ is a *noise* vector

- $p(\boldsymbol{\eta}_i = 1) = f$

Repetition Codes for the BSC

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where $\boldsymbol{\eta}$ is a *noise* vector

- $p(\eta_i = 1) = f$

Example setting of $\boldsymbol{\eta}$, and resulting message $\boldsymbol{r}$:

$$\begin{array}{cccccccc}
\mathbf{s} & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\mathbf{t} & \overbrace{0\,0\,0} & \overbrace{0\,0\,0} & \overbrace{1\,1\,1} & \overbrace{0\,0\,0} & \overbrace{1\,1\,1} & \overbrace{1\,1\,1} & \overbrace{0\,0\,0} \\
\boldsymbol{\eta} & 0\,0\,0 & 0\,0\,1 & 0\,0\,0 & 0\,0\,0 & 1\,0\,1 & 0\,0\,0 & 0\,0\,0 \\
\hline
\mathbf{r} & 0\,0\,0 & 0\,0\,1 & 1\,1\,1 & 0\,0\,0 & 0\,1\,0 & 1\,1\,1 & 0\,0\,0
\end{array}$$

Note that elements of $\boldsymbol{\eta}$ are not replicated like those of $\boldsymbol{t}$

- noise acts independently on every bit

Beyond Repetition Codes

Goal: Communication with small probability of error and high rate:

- Repetition codes introduce redundancy on a per-bit basis
- Can we improve on this?

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Block Code

A block code is a rule for encoding a length-K sequence of source bits **s** into a length-N sequence of transmitted bits **t**.

- Introduce redundancy: $N > K$
- Focus on *Linear codes*

We will introduce a simple type of block code called
the (7,4) Hamming code

An Example

The (7, 4) Hamming Code

Consider $K = 4$, and a source message $\mathbf{s} = 1\ 0\ 0\ 0$

The repetition code R_2 produces

$$\mathbf{t} = 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0$$

The (7,4) Hamming code produces

$$\mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1$$

- Redundancy, but not repetition
- How are these magic bits computed?

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3 Coding: Review

The (7,4) Hamming code

Coding

Consider $K = 4$, $N = 7$ and $\mathbf{s} = 1\ 0\ 0\ 0$

The (7,4) Hamming code
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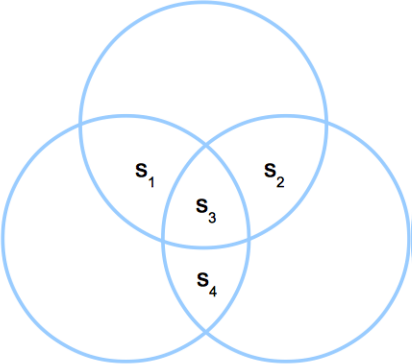
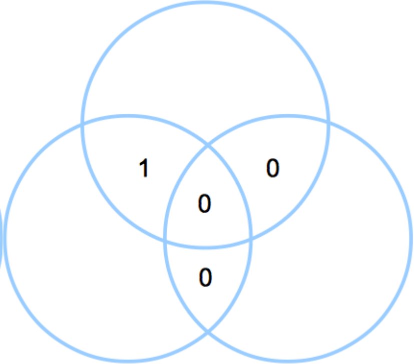
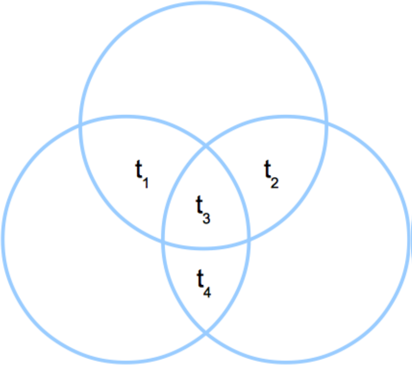


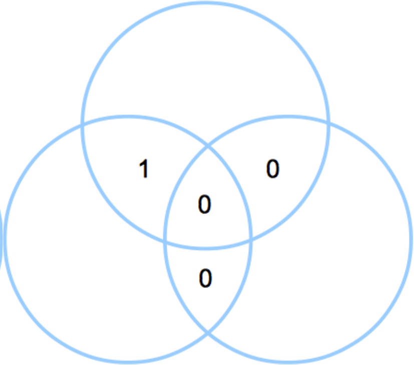


The (7,4) Hamming code

Coding

Copy the source bits into the the first 4 target bits:

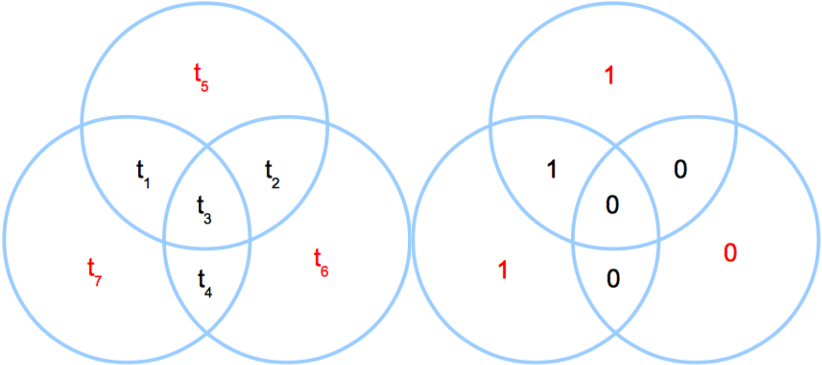




The (7,4) Hamming code

Coding

Set *parity-check* bits so that the number of ones within each circle is even:



So we have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1$

The (7,4) Hamming code

Coding

Algebraically, we have set:

$$
\begin{aligned}
t_i &= s_i \text{ for } i = 1, \ldots, 4 \\
t_5 &= s_1 \oplus s_2 \oplus s_3 \\
t_6 &= s_2 \oplus s_3 \oplus s_4 \\
t_7 &= s_1 \oplus s_3 \oplus s_4
\end{aligned}
$$

where we use modulo-2 arithmetic

The (7,4) Hamming code

Coding

In matrix form:

$$\mathbf{t} = \mathbf{G}^T \mathbf{s} \text{ with } \mathbf{G}^T = \begin{bmatrix} \mathbf{I}_4 \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

where $\mathbf{s} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix}^T$

G is called the *Generator matrix* of the code.

The Hamming code is linear!

The (7,4) Hamming code:

Codewords

Each (unique) sequence that can be transmitted is called a *codeword*.

Codeword examples:

s	Codeword (**t**)
0010	0010111
0110	0110001
1010	1010010
1110	?

For the (7,4) Hamming code we have a total of 16 codewords

The (7,4) Hamming code

Codewords

Write

$$\mathbf{G}^T = \begin{bmatrix}\mathbf{G}_{1\cdot} & \mathbf{G}_{2\cdot} & \mathbf{G}_{3\cdot} & \mathbf{G}_{4\cdot}\end{bmatrix}$$

where each $\mathbf{G}_{i\cdot}$ is a 7 dimensional bit vector

Then, the transmitted message is

$$\begin{aligned}\mathbf{t} &= \mathbf{G}^T\mathbf{s} \\ &= \begin{bmatrix}\mathbf{G}_{1\cdot} & \mathbf{G}_{2\cdot} & \mathbf{G}_{3\cdot} & \mathbf{G}_{4\cdot}\end{bmatrix}\mathbf{s} \\ &= s_1\mathbf{G}_{1\cdot} + \ldots + s_4\mathbf{G}_{4\cdot}\end{aligned}$$

The (7,4) Hamming code:

Codewords

There are 2^4 possible transmitted bit strings

- There are $2^7 - 2^4$ other bit strings that immediately imply corruption
- If we see a codeword, does that imply no corruption?

Any two codewords differ in at least three bits

- Each original bit belongs to at least two circles
- Useful in constructing reliable decoders

1 Repetition Codes

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3 Coding: Review

The (7,4) Hamming code:

Decoding

We can encode a length-4 sequence **s** into a length-7 sequence **t** using 3 parity check bits

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t can be corrupted by noise which can flip *any* of the 7 bits (including the parity check bits):

$$
\begin{array}{cc}
\mathbf{s} & 1\,0\,0\,0 \\
\mathbf{t} & \overbrace{1\,0\,0\,0\,1\,0\,1} \\
\eta & 0\,1\,0\,0\,0\,0\,0 \\
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\mathbf{r} & 1\,1\,0\,0\,1\,0\,1
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How should we decode **r**?

- We could do this exhaustively using the 16 codewords
- Assuming BSC, uniform $p(\mathbf{s})$: Get the most probable explanation
- Find **s** such that $\|\mathbf{t}(\mathbf{s}) \ominus \mathbf{r}\|_1$ is minimum

The (7,4) Hamming code:

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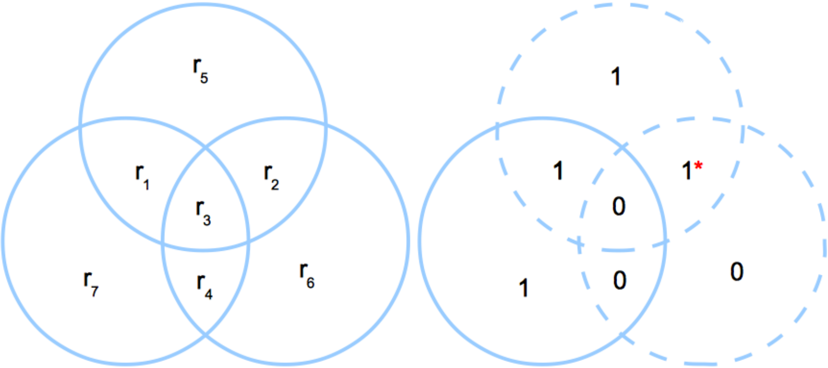
We can get the most probable source vector in an more *efficient* way.

The (7,4) Hamming code:

Decoding Example 1

We have $\mathbf{s} = 1\,0\,0\,0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\,0\,0\,0\,1\,0\,1 \xrightarrow{\text{noise}} \mathbf{r} = 1\,1\,0\,0\,1\,0\,1$:



(1) Detect circles with wrong (odd) parity

- What bit is responsible for this?

The (7,4) Hamming code:

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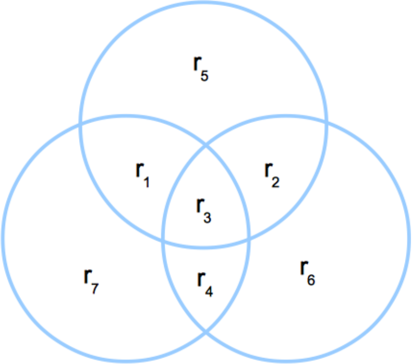
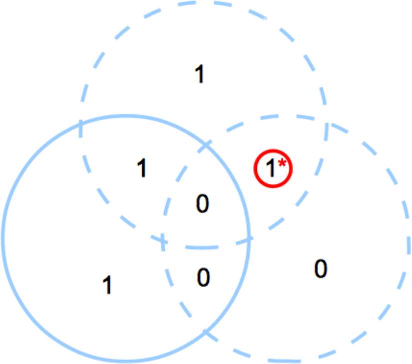




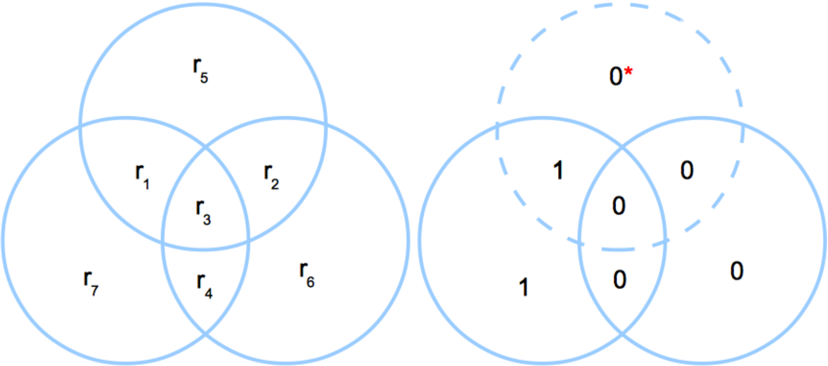
(2) Detect culprit bit and flip it

- The decoded sequence is $\hat{\mathbf{s}} = 1\ 0\ 0\ 0$

The (7,4) Hamming code:

Decoding Example 2

We have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1 \xrightarrow{\text{noise}} \mathbf{r} = 1\ 0\ 0\ 0\ 0\ 0\ 1$:



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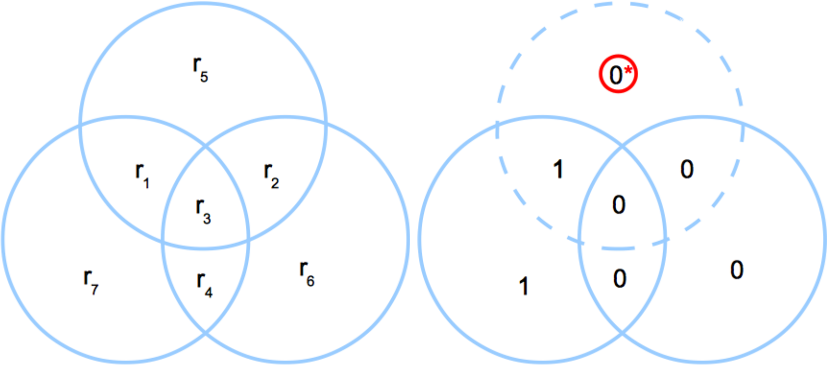

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Decoding Example 3

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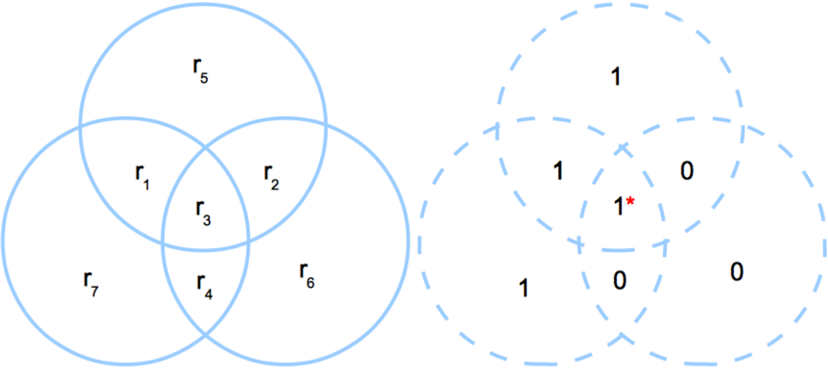

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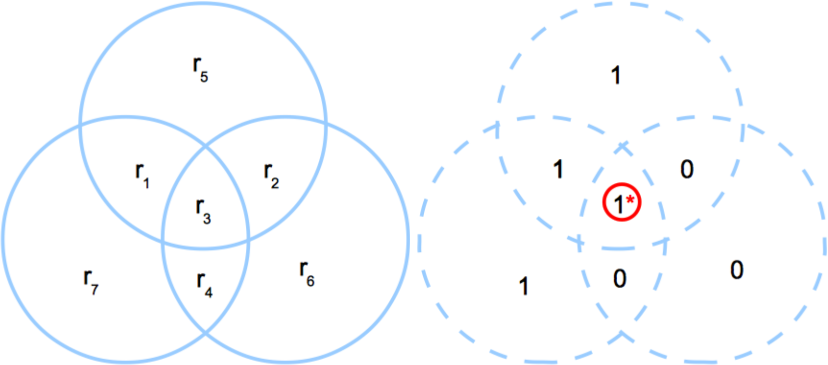

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The (7,4) Hamming code:

Optimal Decoding Algorithm: Syndrome Decoding

Given $\mathbf{r} = r_1, \ldots, r_7$, assume BSC with small noise level f:

1. Define the syndrome as the length-3 vector $\mathbf{z}$ that describes the pattern of violations of the parity bits r_5, r_6, r_7.
   - $\mathbf{z}_i = 1$ when the ith parity bit does not match the parity of $\mathbf{r}$
   - Flipping a single bit leads to a different syndrome

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3. Unflip the *single bit* responsible for this pattern of violation
   - This syndrome could have been caused by other noise patterns

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$\mathbf{z}$	0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
Flip bit	none	r_7	r_6	r_4	r_5	r_1	r_2	r_3

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The optimal decoding algorithm unflips at most one bit

The (7,4) Hamming code:

Syndrome Decoding: Matrix Form

Recall that we just need to compare the expected parity bits with the actual ones we received:

$$\begin{aligned} z_1 &= r_1 \oplus r_2 \oplus r_3 \ominus r_5 \\ z_2 &= r_2 \oplus r_3 \oplus r_4 \ominus r_6 \\ z_3 &= r_1 \oplus r_3 \oplus r_4 \ominus r_7, \end{aligned}$$

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but in modulo-2 arithmetic $-1 \equiv 1$ so we can replace $\ominus$ with $\oplus$ so we have:

$$
\mathbf{z} = \mathbf{H}\mathbf{r} \text{ with } \mathbf{H} = \begin{bmatrix} \mathbf{P} & \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
$$

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Homework: What is the syndrome for a codeword?

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3 Coding: Review

The (7,4) Hamming code:

Optimal Decoding Algorithm: Syndrome Decoding

When the noise level f on the BSC is small, it may be reasonable that we see only a single bit flip in a sequence of 4 bits

The syndrome decoding method exactly recovers the source message in this case

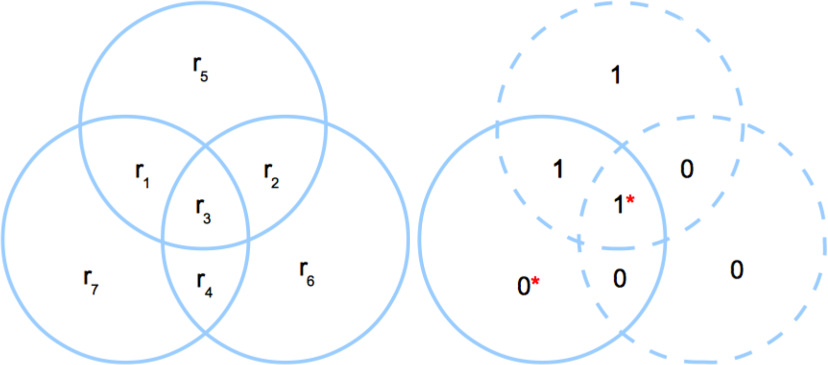
- c.f. Noise flipping one bit in the repetition code R_3

But what happens if the noise flips more than one bit?

The (7,4) Hamming code:

Decoding Example 4: Flipping 2 Bits

We have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1 \xrightarrow{\text{noise}} \mathbf{r} = 1\ 0\ {\color{red}1}\ 0\ 1\ 0\ {\color{red}0}$:



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The (7,4) Hamming code:

Decoding Example 4: Flipping 2 Bits

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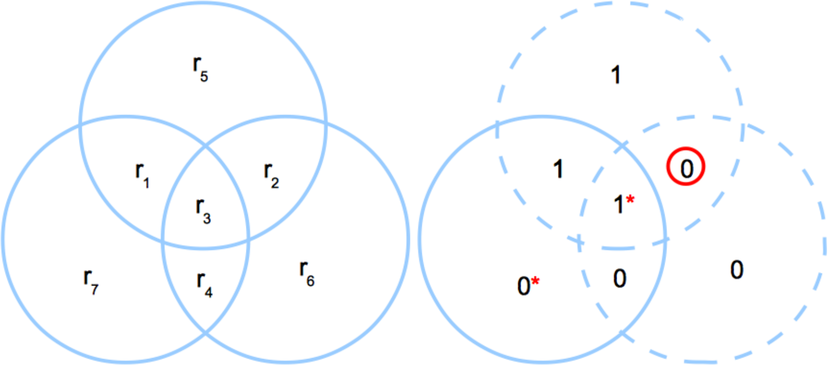

(2) Detect culprit bit and flip it

- The decoded sequence is $\hat{\mathbf{s}} = 1\ 1\ 1\ 0$
  - We have made 3 errors but only 2 involve the actual message

The (7,4) Hamming code:

Decoding Exercises

[Mackay, Ex 1.5]: Decode the following sequences using the syndrome decoding for the (7,4) Hamming code:

(a) $\mathbf{r} = 1101011 \rightarrow \hat{\mathbf{s}} =??$

(b) $\mathbf{r} = 0110110 \rightarrow \hat{\mathbf{s}} =??$

(c) $\mathbf{r} = 0100111 \rightarrow \hat{\mathbf{s}} =??$

(d) $\mathbf{r} = 1111111 \rightarrow \hat{\mathbf{s}} =??$

Work out the answers on your own.

The (7,4) Hamming code: Solution

For each exercise we simply compute the *syndrome* and use the optimal decoding algorithm (Table above) to determine which bit we should unflip.

(a) $\mathbf{r} = 1101011 \rightarrow$: $z_1 = r_1 \oplus r_2 \oplus r_3 \oplus r_5 = 0 \quad z_2 = r_2 \oplus r_3 \oplus r_4 \oplus r_6 = 1 \quad z_3 = r_1 \oplus r_3 \oplus r_4 \oplus r_7 = 1$ Therefore $\mathbf{z} = 011$, we unflip r_4, $\hat{\mathbf{s}} = 1100$

(b) $\mathbf{r} = 0110110 \rightarrow \mathbf{z} = 111$, we unflip r_3, $\hat{\mathbf{s}} = 0100$

(c) $\mathbf{r} = 0100111 \rightarrow \mathbf{z} = 001$, we unflip r_7, $\hat{\mathbf{s}} = 0100$

(d) $\mathbf{r} = 1111111 \rightarrow \mathbf{z} = 000$, we don't unflip any bit, $\hat{\mathbf{s}} = 1111$

The (7,4) Hamming code:

Zero-Syndrome Noise Vectors

[Mackay, Ex 1.7] Find some noise vectors that give the all-zero syndrome (so that the optimal decoding algorithm will not correct them). How many of these vectors are there?

Solution

By definition we have that the all-zero syndrome implies that the corresponding noise components should cancel out. For example for the first component we have:
$z_1 = r_1 \oplus r_2 \oplus r_3 \oplus r_5 = t_1 \oplus t_2 \oplus t_3 \oplus t_5 \oplus \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_5$. But $t_i = s_i$ for $i = 1, \ldots, 4$ and $t_5 = s_1 \oplus s_2 \oplus s_3$. Therefore
$z_1 = 2s_1 \oplus 2s_2 \oplus 2s_3 \oplus \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_5 = \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_5$. Thus, we have:

$$
\begin{aligned}
z_1 &= \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_5 = 0 \\
z_2 &= \eta_2 \oplus \eta_3 \oplus \eta_4 \oplus \eta_6 = 0 \\
z_3 &= \eta_1 \oplus \eta_3 \oplus \eta_4 \oplus \eta_7 = 0
\end{aligned}
$$

which is equivalent to:

$$
\begin{aligned}
\eta_5 &= \eta_1 \oplus \eta_2 \oplus \eta_3 \\
\eta_6 &= \eta_2 \oplus \eta_3 \oplus \eta_4 \\
\eta_7 &= \eta_1 \oplus \eta_3 \oplus \eta_4
\end{aligned}
$$

Solution (cont.)

As η_5 is determined by η_1, η_2, η_3 we have $2^3 = 8$ possibilities here.
Now, for fixed η_1, η_2 (and η_3) in the previous step we only have two possibilities for η_4, which determines η_6.
We have now that all the variables are set and η_7 is fully determined by their values.
Thus, we have $8 \times 2 \times 1$ possible noise vectors that yield the all-zero syndrome.
The trivial noise vectors that yield this syndrome are: $\boldsymbol{\eta} = 0000000$ and $\boldsymbol{\eta} = 1111111$.
However, we can follow the above procedure and set the corresponding variables.
This is equivalent to having arbitrary settings for η_1, η_2, η_3 and η_4 which gives us 16 possible noise vectors which exactly correspond to the 16 codewords of the (7,4) Hamming code.

1 Repetition Codes

2 The (7,4) Hamming code
- Coding
- Decoding
- Syndrome Decoding
- Multiple errors
- Error Probabilities

3 Coding: Review

The (7,4) Hamming code:

Error Probabilities

Decoding Error : Occurs if at least one of the decoded bits $\hat{s}_i$ does not match the corresponding source bit s_i for $i = 1, \ldots 4$

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What is the probability of block error for the (7,4) Hamming code with $f = 0.1$?

The (7,4) Hamming code:

Leading-Term Error Probabilities

Block Error: This occurs when 2 or more bits in the block of 7 are flipped

We can approximate p_B to the leading term:

$$p_B = \sum_{m=2}^{7} \binom{7}{m} f^m (1-f)^{7-m}$$
$$\approx \binom{7}{2} f^2 = 21 f^2.$$

The (7,4) Hamming code:

Leading-Term Error Probabilities

Bit Error: Given that a block error occurs, the noise must corrupt 2 or more bits

The most probable case is when the noise corrupts 2 bits, which induces 3 errors in the decoded vector:

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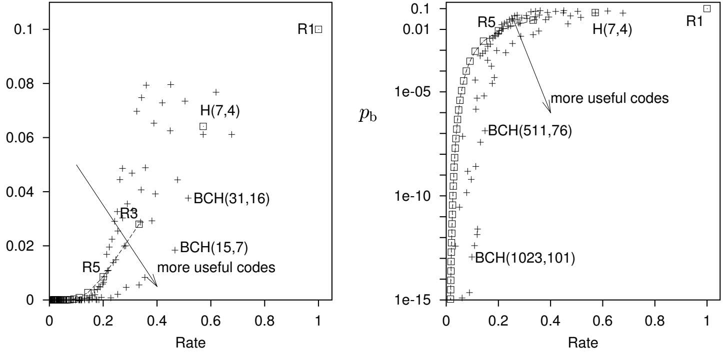
- All bits are equally likely to be corrupted (due to symmetry)
- $p_b \approx \frac{3}{7} p_B \approx 9f^2$

What Can Be Achieved with Hamming Codes?



- H(7,4) improves p_b at a moderate rate $R = 4/7$
- BCH are a generalization of Hamming codes.

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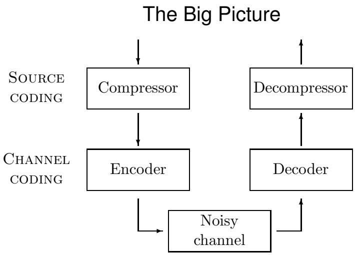

Source Coding for Compression

- Shrink sequences
- Identify and remove redundancy
- Size limited by entropy
- Source Coding Theorems (Block & Variable Length)

Channel Coding for Reliability

- Protect sequences
- Add known form of redundancy
- Rate limited by capacity
- Noisy-Channel Coding Theorem