COMP2610/COMP6261 – Information Theory

Tutorial 7

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Question 1. Inequality

Suppose a coin is tossed n times. The coin is known to land "heads" with probability p. The number of observed "heads" is recorded as a random variable X.

(a) What is the exact probability of X being $n-1$ or more?

(b) Using Markov's inequality, compute a bound on the same probability as the previous part.

(c) Suppose $n = 2$. For what values of p will the bound from Markov's inequality be within 1% of the exact probability?

Solution:
(a). We know that X is a binomial with parameters n, p. Thus,

$$p(X \geq N-1) = p(X = N-1) + p(X = N) = np^{n-1}(1-p) + p^n$$

(b). By Markov's inequality,

$$p(X \geq N-1) = \frac{\mathbb{E}[X]}{n-1} = \frac{n}{n-1}p$$

(c). When $n = 2$,

$$p(X \geq 1) = 2p(1-p) + p^2 = p(2-p).$$

The bound from Markov's inequality is

$$p(X \geq 1) = 2p$$

The difference between the two is

$$2p - p(2-p) = p^2.$$

Thus, for the Markov bound to be within 0.01 of the exact probability, we will need $p \leq 0.1$.

Question 2. AEP and Source Coding

A sequence of bits its generated by i.i.d. draws from an ensemble with probabilities $p_0 = 0.995$ and $p_1 = 0.005$. Sequences are coded in 100-bit blocks. Every 100-bit block with at most three 1s is assigned a codeword. Those blocks with more than three 1s are not assigned codewords.

(a) What is the minimum required length of the assigned codewords if they are all to be of the same length?

(b) Calculate the probability of observing a 100-bit block that has no associated codeword

(c) (Harder) Use Chebyshev's inequality to bound the probability of observing a 100-bit block for which no codeword has been assigned. Compare the bound to the probability just calculated.

Solution:
(a). We need to code all sequences of length 100 with three or less 1's. There is 1 sequence with zero 1's. There are 100 sequences with one 1 (one sequence for each possible position in which the 1 appears.) Similarly, there are $\binom{100}{2}$ and $\binom{100}{3}$ sequences with two and three 1's respectively. In total we have

$$1 + 100 + \binom{100}{2} + \binom{100}{3} = 166751$$

If we want a uniform code over this number of elements, we need $\lceil \log_2 166751 \rceil = 18$.
(b). We need to find the probability of observing more than 3 1's. Let K be the number of 1's observed. Then

$$\begin{aligned} P(K > 3) = 1 - P(K \leq 3) =& 1 - 0.995^{100} - 100 \times 0.995^{99} \times 0.005 \\ & - \binom{100}{2} \times 0.995^{98} \times 0.005^2 - \binom{100}{3} \times 0.995^{97} \times 0.005^3 = 0.00167 \end{aligned}$$

(c). By using Chebyshev's inequality, we have

$$P(K \geq 4) = P(|K - 0.5| \geq 3.5) \leq \frac{V[K]}{3.5^2} = \frac{0.4957}{12.25} = 0.0406$$

Question 3. Typical Set

Let X_N be an extended ensemble for X with $\mathcal{A}_X = \{0,1\}$ and $P_X = \{0.4, 0.6\}$.

(a) Calculate the entropy $H(X)$.

(b) Let $N = 25$ and $\beta = 0.1$.

i. Which sequences in X_N fall in the typical set $T_{N\beta}$?

ii. Compute $P(x \in T_{N\beta})$, the probability of a sequence from X_N falling in the typical set.

iii. How many elements are there in $T_{N\beta}$?

iv. How many elements are in the smallest δ-sufficient subset S_δ for $\delta = 0.9$?

v. What is the essential bit content $H_\delta(X_N)$ for $\delta = 0.9$?

Solution:
(a).

$$H(X) = -0.4\log 0.4 - 0.6\log 0.6 = 0.971$$

(b).

i. Note that

$$T_{N\beta} = \{x : |-\frac{1}{N}\log P(x) - H(X)| < \beta\}$$

Hence, we find

$$0.871 = H(X) - 0.1 < -\frac{1}{N}\log P(x) < H(X) + 0.1 = 1.071$$

So we find that the length of 1 is from 11 to 19.
ii. The probability is

$$P = \sum_{k=11}^{19} \binom{25}{k} 0.6^k \times 0.4^{25-k} = 0.936247$$

iii. The number is

$$\sum_{k=11}^{19} \binom{25}{k} = 26366510$$

iv. Here we want the smallest subset of sequences. Note that the highest probability sequences are the ones with the most 1s. We find that

$$P(K \geq 19) = 0.073564$$

So we need to add $\lceil \frac{0.1-0.073564}{0.4^7 \times 0.6^{18}} \rceil = 158876$ sequences in S_δ. So we have

$$\sum_{k=0}^{6} \binom{25}{k} + 158876 = 404382.$$

v. It is $\log_2 |S_\delta| = 18.625$.