ASSIGNMENT COVER SHEET

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This coversheet must be attached to the front of your assessment

The assessment is due on 28/08/2023, 9:05 AM unless otherwise specified in the course outline.

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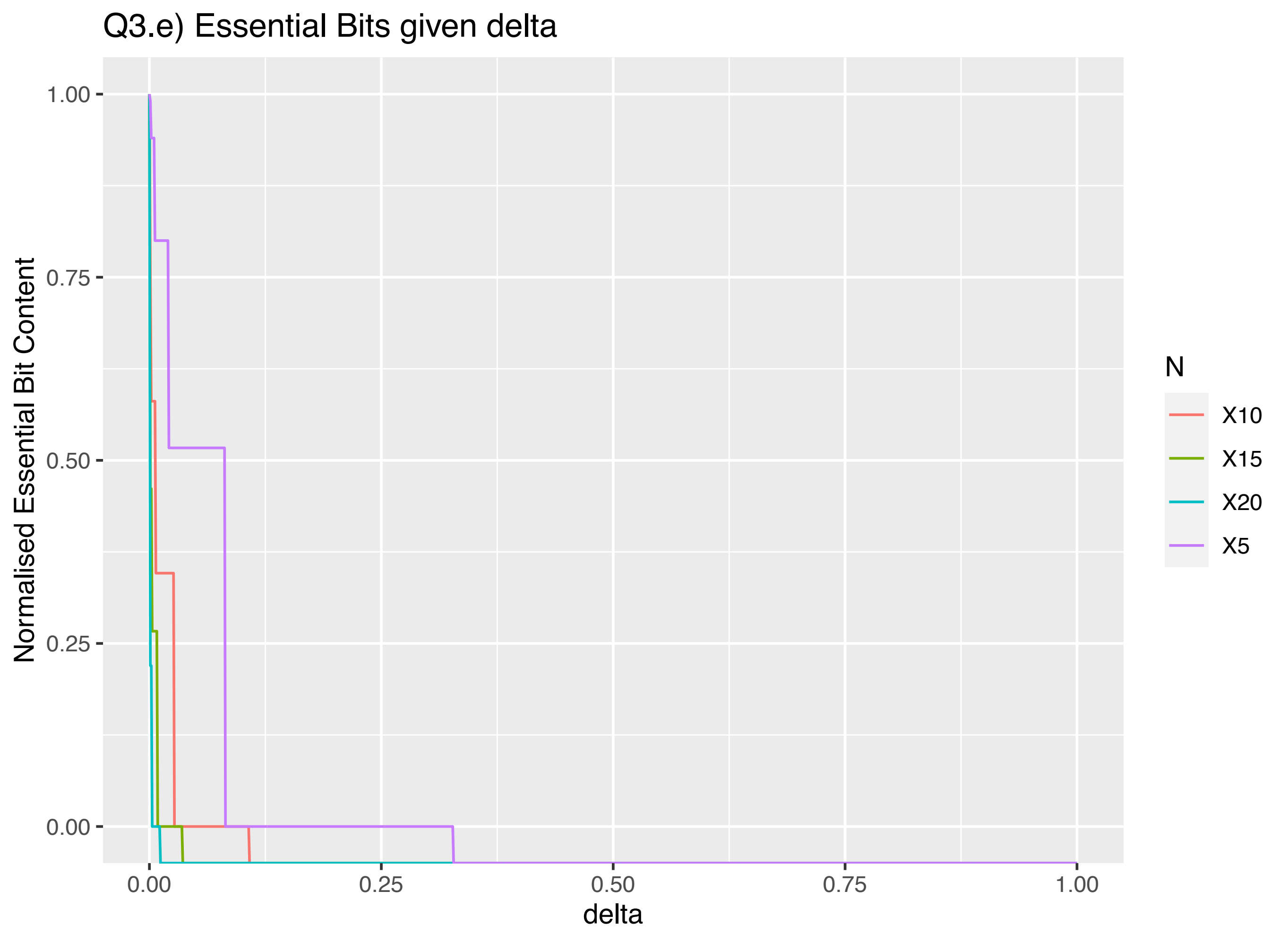
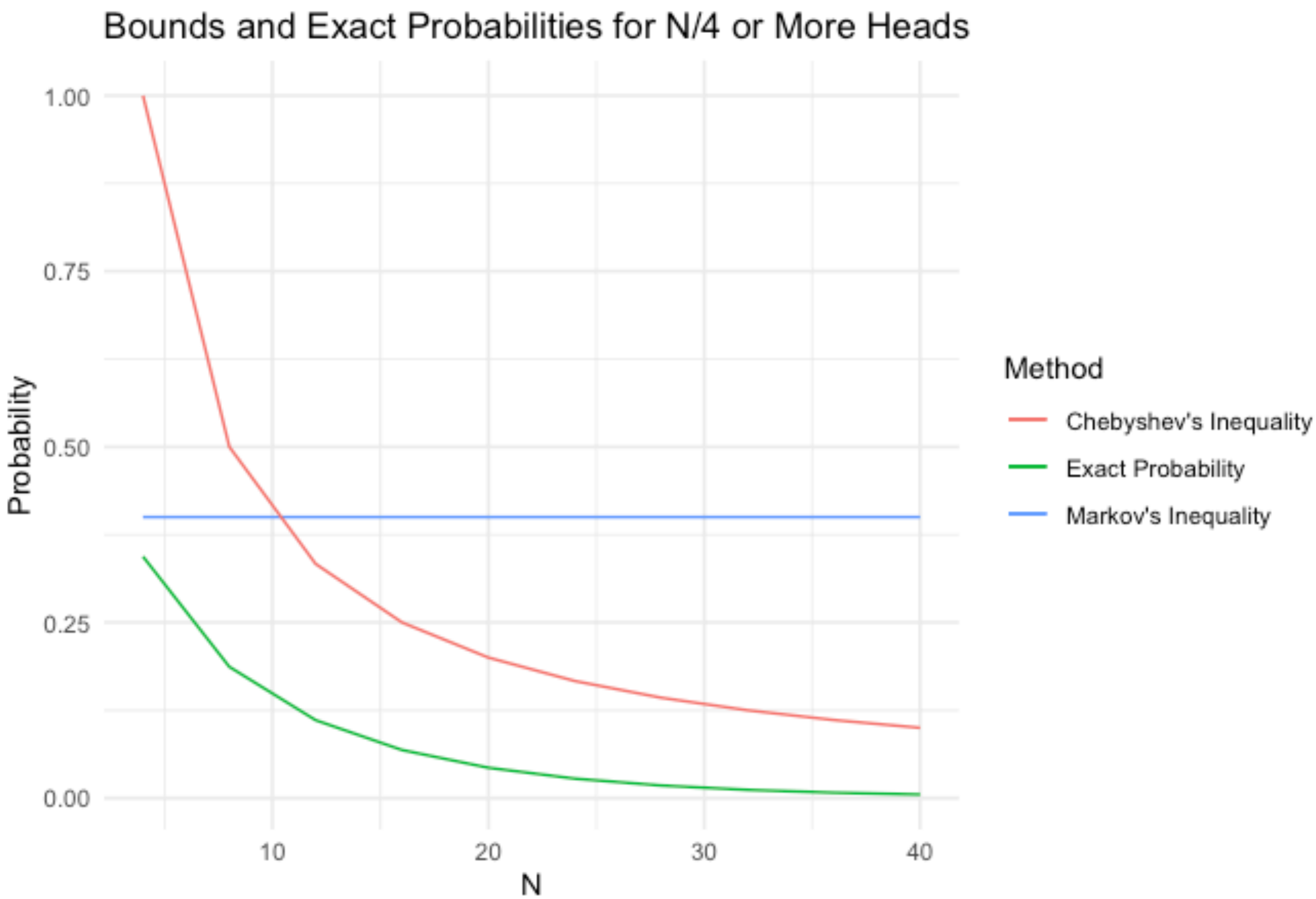
Course Code	COMP6261
Course Name	Information Theory
Assignment Item	Assignment 2
Due Date	25/09/2023,
Date Submitted	23/09/2023

I declare that this work:

- ☐ upholds the principles of academic integrity, as defined in the ANU Policy: Code of Practice for Student Academic Integrity;
- ☐ is original, except where collaboration (for example group work) has been authorized in writing by the course convener in the course outline and/or Wattle site;
- ☐ is produced for the purposes of this assessment task and has not been submitted for assessment in any other context, except where authorized in writing by the course convener.
- ☐ gives appropriate acknowledgment of the ideas, scholarship, and intellectual property of others insofar as these have been used.
- ☐ in no part involves copying, cheating, collusion, fabrication, plagiarism, or recycling.

Signature Nanthawat Anancharoenpakorn


Q3.e) Essential Bits given delta
Normalised Essential Bit Content
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
delta
N
X10
X15
X20
X5


Bounds and Exact Probabilities for N/4 or More Heads
Probability
1.00
0.75
0.50
0.25
0.00
10
20
30
40
N
Method
Chebyshev's Inequality
Exact Probability
Markov's Inequality

Question 1

Question 1(a)

1) $p(X \geq a) \leq \frac{E[X]}{a}$ given $E[X] = 2000$, $a = 2400$

$p(X \geq 2400) \leq \frac{2000}{2400}$ or $\frac{5}{6}$ #

2) $P(|X-\mu| \geq k \times \sigma) \leq \frac{1}{k^2}$ given $\mu = 2000$, $\sigma = 200$ Hence, $k = \frac{2400-2000}{200} = 2$

$p(1600 \leq X \leq 2400) \geq 1 - \frac{1}{2^2}$ or $\frac{3}{4}$ #

Question 1(b)

1) $p(X = \text{head}) < \frac{1}{8}$, coin flipped N times $N \in \mathbb{N}$

$p(X^N \geq \frac{N}{4}) \leq \frac{E[X^N]}{N/4} = \frac{1}{2}$ #

2) $p < \frac{1}{8}$, coin flipped N times

$\mu = \frac{N}{8}$, $\sigma^2 = \frac{7N}{64}$ $k = \frac{\frac{N}{4} - \frac{N}{8}}{\frac{\sqrt{7N}}{8}} = \frac{N}{\sqrt{7N}}$

$p(X^N \geq \frac{N}{4}) \leq \frac{7}{N}$

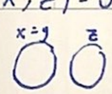
3) plot

Question 2 : Markov Chain

Question 2(a)

1) yes it's possible, for example let X be a random variable where $x \in \{0,1\}$ with $P(X=0) = P(X=1) = \frac{1}{2}$ then let $Y = X$ and $Z = X$. Hence $I(X;Y) = I(X;Z) = H(X)$ since Y and Z are complety determined by X

2) yes. let X be random v. that $P(X=0) = P(X=1) = \frac{1}{2}$ and $Y = X$. Z is a variable that independent to Y. Hence, $I(X;Y) = H(X) = H(Y)$ and $I(X;Z) = 0$, then $I(X;Y) \geq I(X;Z)$



3) according to chain rule for mutal information :

$I(X;Y,Z) = I(X;Y) + I(X;Z|Y) = I(X;Z) + I(X;Y|Z)$

Hence $I(X;Y) + I(X;Z|Y) = I(X;Z) + I(X;Y|Z)$

according to markov chain assumption where $I(X;Z|Y) = 0$

Proof : $I(X;Z|Y) = H(Z|Y) - H(Z)$

$= E\left[\log_2 \frac{p(X,Z|Y)}{p(X,Y)\,p(Z|Y)}\right] = E[\log_2 1] = 0$

Then $I(X;Y) = I(X;Z) + I(X;Y|Z)$ Hence $I(X;Y) \geq I(X;Z)$

4) according to the chain rule on previous question, we get

$I(X;Y) + I(X;Z|Y) = I(X;Z) + I(X;Y|Z)$

$I(X;Y|Z) = I(X;Y) - I(X;Z)$ since $I(X;Z)$ must be equal or more than 0 Hence $I(X;Y) \geq I(X;Y|Z)$

this because when we know Z, the uncertainty can only decreze or stay the same. In other words, the dependence between X and Y can't be increased by the observation of a downstream variable.

Question 2(b)

1) we know that $I(X;Y,Z) = H(X) - H(X|Y,Z)$ and $I(X;T) = H(X) - H(X|T)$ since $p(t|y,z,x) = p(t|y,z)$, according Markov, this means $H(X|Y,Z) \leq H(X|T)$, then we substitute and get $I(X;Y,Z) = H(X) - H(X|Y,Z) \geq H(X) - H(X|T) = I(X;T)$ #

2) In order for $I(X;Y,Z) = I(X;T)$ knowing T must give same amout of information with Y,Z or $H(X|Y,Z) = H(X|T)$. Hence, T must be a function of Y, Z & that $p(t|y,z) = p(t|y,z,x)$

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Question 2 (c)

1) By definition of mutual information, we get

$$I(X_1; X_2, \dots, X_n) = H(X_1) - H(X_1 | X_2, \dots, X_n)$$

where $H(X_1 | X_2, \dots, X_n) = H(X_1 | X_2, \dots, X_{n-1}) + H(X_1 | X_2, \dots, X_{n-2}, X_n) + \dots + H(X_1 | X_2, X_n) + H(X_1 | X_n)$

and we plug H back, we get: $I(X_1; X_2, \dots, X_n) = H(X_1) - H(X_1 | X_2, \dots, X_{n-1}) - H(X_1 | X_2, \dots, X_{n-2}, X_n) - \dots - H(X_1 | X_2, X_n) - H(X_1 | X_n)$

2) we can simplify expression above and get:

$$I(X_1; X_2, \dots, X_n) = H(X_1) - H(X_1 | X_n)$$

Question 3: AEP

a) $H(X) = 0.8 \times \log_2\left(\frac{1}{0.8}\right) + 0.2 \times \log_2\left(\frac{1}{0.2}\right) = 0.7219$ bits

b) since there are two outcome and $N \in \{0, 1, 2, \dots, N\}$ the size of $A_X^N = 2^N$

c) since $|A_{X^4}| = 16$, then $H_0(X^4) = \log_2 |A_{X^4}| = 4$

d) $H(X^N) = N \sum_x p(x) \log p(x) = N H(X)$

e) As we increase N from small to large, the slope of the line become more flat meaning that it become ~~be~~ less sensitive to change in error. When N get large, such sequence occupy most of the probability mass, and are equally likely.

Question 5: AEP

collection
- Alina (7)
- Beyonce (12)
- Cecilia (15)
- Derek (14)

(4)

1) a. $\log_2(4) = 2$ bits, $\{00, 01, 10, 11\}$

b. $\lceil \log_2(7) \rceil = 3$ bits, $\{000, 001, 010, 011, 100, 101, 110\}$

c. $\lceil \log_2(48) \rceil = 6$ bits

2) $H_0(X) = \lceil \log_2(48) \rceil = 6$ bits

3) a. Ensemble $A = (x, A_X, P_X)$ where $A_X = \{$Alina, Beyonce, Cecilia, Derek$\}$ and $P_X = (7/48, 12/48, 15/48, 14/48)$ such that the first element of P_X is Alina and so on.

b. $\log_2(256) = 8$ bit

c. smallest is 0

d. largest is 0.3958

4) a. $H(A) = -\left(\frac{7}{48}\log_2\frac{7}{48} + \frac{12}{48}\log_2\frac{12}{48} + \frac{15}{48}\log_2\frac{15}{48} + \frac{14}{48}\log\frac{14}{48}\right)$

$= 1.95$

b. $|T_N^\beta| \approx 2^{100(1.98+0.1)} \approx 2^{100 \times 2.08}$

c. no, it's not possible because the minimum of bit of uniform code with zero loss is 1.95 which is the minimum for uniform code and 1.5 bit is less than 1.95

```
# Function for toss coin
library(ggplot2)
coinTossOutcomes <- function(N, p) {
  # Define the sample space and probabilities
  sample_space <- c("H", "T")
  probabilities <- c(p, 1 - p)

  # Generate all possible outcomes for N coin tosses
  all_outcomes <- expand.grid(replicate(N, sample_space, simplify = FALSE))

  # Initialize a vector to store the probabilities of each outcome
  outcome_probabilities <- numeric(nrow(all_outcomes))

  # Calculate the probability of each outcome
  for (i in 1:nrow(all_outcomes)) {
    outcome <- as.character(unlist(all_outcomes[i, ]))
    prob <- prod(ifelse(outcome == "H", probabilities[1], probabilities[2]))
    outcome_probabilities[i] <- prob
  }

  # Add the probabilities to the data frame
  all_outcomes$Probability <- outcome_probabilities

  # Combine the toss outcomes into a single string for easier viewing
  all_outcomes$X <- do.call(paste0, all_outcomes[, 1:N])

  # Create the final table
  final_table <- all_outcomes[, c("X", "Probability")]

  # Normalize table
  return(final_table)

}

# 1. N = 5
N <- 5
p <- 0.8
result <- coinTossOutcomes(N, p)
x_axis = seq(0,1,0.001) ;print(x_axis)
y = c()
for (x in x_axis) {
  events = sum(result$Probability > x) # >= (1-x)
  y_axis = log2(events)
```

```
  y = c(y,y_axis)

}
final_result = data.frame(y,x_axis)
final_result$normal = final_result$y/N
a = final_result$normal

# 2. N = 10
rm(final_table)
rm(result)
N <- 10
p <- 0.8
result <- coinTossOutcomes(N, p)
x_axis = seq(0,1,0.001) ;print(x_axis)
y = c()
for (x in x_axis) {
  events = sum(result$Probability > x) # >= (1-x)
  y_axis = log2(events)
  y = c(y,y_axis)

}
final_result = data.frame(y,x_axis)
final_result$normal = final_result$y/N
b = final_result$normal



# 3. N = 15
rm(final_table)
rm(result)
N <- 15
p <- 0.8
result <- coinTossOutcomes(N, p)
x_axis = seq(0,1,0.001) ;print(x_axis)
y = c()
for (x in x_axis) {
  events = sum(result$Probability > x) # >= (1-x)
  y_axis = log2(events)
  y = c(y,y_axis)

}
final_result = data.frame(y,x_axis)
final_result$normal = final_result$y/N
c = final_result$normal
```

```
# 4. N = 20
rm(final_table)
rm(result)
N <- 20
p <- 0.8
result <- coinTossOutcomes(N, p)
x_axis = seq(0,1,0.001) ;print(x_axis)
y = c()
for (x in x_axis) {
  events = sum(result$Probability > x) # >= (1-x)
  y_axis = log2(events)
  y = c(y,y_axis)

}
final_result = data.frame(y,x_axis)
final_result$normal = final_result$y/N
d = final_result$normal




# 3. Combine all
df <- data.frame(x_axis = final_result$x_axis, "5" = a, "10" = b, "15" = c, "20" = d)

# Reshape the data to long format
df_long <- tidyr::gather(df, variable, value, -x_axis)

# Generate the plot
ggplot(df_long, aes(x = x_axis, y = value, color = variable)) +
  geom_line() +
  labs(title = "Q3.e) Essential Bits given delta",
     x = "delta",
     y = "Normalised Essential Bit Content") +
  scale_color_discrete(name = "N")
```