COMP2610/COMP6261 – Information Theory

Tutorial 10

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Question 1.

Channel capacity. Consider the discrete memoryless channel $Y = X + Z \pmod{11}$, where

$$Z = \begin{pmatrix} 1, & 2, & 3 \\ 1/3, & 1/3, & 1/3 \end{pmatrix}$$

and $X \in \{0, 1, \ldots, 10\}$. Assume that Z is independent of X.

(a) Find the capacity.

(b) What is the maximizing $p^*(x)$?

Solution: *Channel capacity.*

$$Y = X + Z (\text{mod } 11)$$

where

$$Z = \begin{cases} 1 & \text{with probability} 1/3 \\ 2 & \text{with probability} 1/3 \\ 3 & \text{with probability} 1/3 \end{cases}$$

In this case,

$$H(Y|X) = H(Z|X) = H(Z) = \log 3,$$

independent of the distribution of X, and hence the capacity of the channel is

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) \\ &= \max_{p(x)} H(Y) - H(Y|X) \\ &= \max_{p(x)} H(Y) - \log 3 \\ &= \log 11 - \log 3, \end{aligned}$$

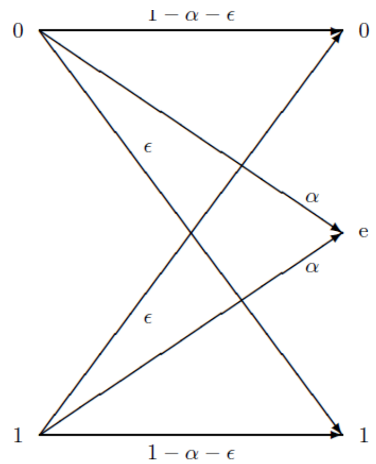
which is attained when Y has a uniform distribution, which occurs (by symmetry) when X has a uniform distribution.

(a) The capacity of the channel is $\log \frac{11}{3}$ bits/transmission.

(b) The capacity is achieved by an uniform distribution on the inputs. $p(X = i) = \frac{1}{11}$ for $i = 0, 1, \ldots, 10$.

Question 2.

Erasures and errors in a binary channel. Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be ϵ and the probability of erasure be α, so the the channel is as illustrated below:



(a) Find the capacity of this channel.

(b) Specialize to the case of the binary symmetric channel ($\alpha = 0$).

(c) Specialize to the case of the binary erasure channel ($\epsilon = 0$).

Solution:

(a) As with the examples in the text, we set the input distribution for the two inputs to be π and $1-\pi$. Then

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) \\ &= \max_{p(x)} (H(Y) - H(Y|X)) \\ &= \max_{p(x)} H(Y) - H(1-\epsilon-\alpha, \alpha, \epsilon). \end{aligned}$$

As in the case of the erasure channel, the maximum value for $H(Y)$ cannot be $\log 3$, since the probability of the erasure symbol is α independent of the input distribution. Thus,

$$\begin{aligned} H(Y) &= H(\pi(1-\epsilon-\alpha) + (1-\pi)\epsilon, \alpha, (1-\pi)(1-\epsilon-\alpha) + \pi\epsilon) \\ &\leq H(\alpha) + (1-\alpha) \end{aligned}$$

with equality iff $\frac{\pi+\epsilon-\pi\alpha-2\pi\epsilon}{1-\alpha} = \frac{1}{2}$, which can be achieved by setting $\pi = \frac{1}{2}$. (The fact that $\pi = 1-\pi = \frac{1}{2}$ is the optimal distribution should be obvious from the symmetry of the problem, even though the channel is not weakly symmetric.)

Therefore the capacity of this channel is

$$\begin{aligned} C &= H(\alpha) + 1 - \alpha - H(1-\alpha-\epsilon, \alpha, \epsilon) \\ &= H(\alpha) + 1 - \alpha - H(\alpha) - (1-\alpha) H\left(\frac{1-\alpha-\epsilon}{1-\alpha}, \frac{\epsilon}{1-\alpha}\right) \\ &= (1-\alpha)\left(1 - H\left(\frac{1-\alpha-\epsilon}{1-\alpha}, \frac{\epsilon}{1-\alpha}\right)\right) \end{aligned}$$

(b) Setting $\alpha = 0$, we get

$$C = 1 - H(\epsilon),$$

which is the capacity of the binary symmetric channel.

(c) Setting $\epsilon = 0$, we get

$$C = 1 - \alpha$$

which is the capacity of the binary erasure channel.

Question 3.

Binary multiplier channel

(a) Consider the channel $Y = XZ$ where X and Z are independent binary random variables that take on values 0 and 1. Z is Bernoulli(α), i.e. $P(Z=1) = \alpha$. Find the capacity of this channel and the maximizing distribution on X.

(b) Now suppose the receiver can observe Z as well as Y. What is the capacity?

Solution: *Binary Multiplier Channel*

(a) Let $P(X=1) = p$. Then $P(Y=1) = P(X=1)P(Z=1) = \alpha p$.

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - P(X=1) H(Z) \\ &= H(\alpha p) - p H(\alpha) \end{aligned}$$

We find that $p^* = \frac{1}{\alpha(2^{\frac{H(\alpha)}{\alpha}}+1)}$ maximizes $I(X;Y)$. The capacity is calculated to be $\log(2^{\frac{H(\alpha)}{\alpha}} + 1) - \frac{H(\alpha)}{\alpha}$.

(b) Let $P(X=1) = p$. Then

$$\begin{aligned} I(X;Y,Z) &= I(X;Z) + I(X;Y|Z) \\ &= H(Y|Z) - H(Y|X,Z) \\ &= H(Y|Z) \\ &= \alpha H(p) \end{aligned}$$

The expression is maximized for $p = 1/2$, resulting in $C = \alpha$. Intuitively, we can only get X through when Z is 1, which happens α of the time.

Question 4.

Can signal alternatives lower capacity? Show that adding a row to a channel transition matrix does not decrease capacity.

Solution: *Can signal alternatives lower capacity?*
Adding a row to the channel transition matrix is equivalent to adding a symbol to the input alphabet $\mathcal{X}$. Suppose there were m symbols and we add an $(m+1)$-st. We can always choose not to use this extra symbol.

Specifically, let C_m and C_{m+1} denote the capacity of the original channel and the new channel, respectively. Then

$$\begin{aligned} C_{m+1} &= \max_{p(x_1,\ldots,x_{m+1})} I(X;Y) \\ &\geq \max_{p(x_1,\ldots,x_m,0)} I(X;Y) \\ &= C_m. \end{aligned}$$