COMP2610 / COMP6261 Information Theory
Lecture 5: Bernoulli, Binomial, Maximum Likelihood and MAP

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Announcements

Assignment 1

- Will be released this week
- Worth 10% of Course total
- Due Friday 25 August 2023, 5:00 pm

Last time

- Examples of application of Bayes' rule
  - Formalizing problems in language of probability
  - Eating hamburgers, detecting terrorists, ...
- Frequentist vs Bayesian probabilities

The Bayesian Inference Framework

Bayesian Inference

Bayesian inference provides us with a a mathematical framework explaining how to change our (prior) beliefs in the light of new evidence.

$$\underbrace{p(Z|X)}_{\text{posterior}} = \frac{\overbrace{p(X|Z)}^{\text{likelihood}}\overbrace{p(Z)}^{\text{prior}}}{\underbrace{p(X)}_{\text{evidence}}}$$

$$= \frac{p(X|Z)p(Z)}{\sum_{Z'} p(X|Z')p(Z')}$$

Prior: Belief that someone is sick

Likelihood: Probability of testing positive given someone is sick

Posterior: Probability of being sick given someone tests positive

This time

- The Bernoulli and binomial distribution (we will make much use of this henceforth in studying binary channels)

- Estimating probabilities from data

- Bayesian inference for parameter estimation

Outline

The Bernoulli Distribution

Introduction

Consider a binary variable $X \in \{0, 1\}$. It could represent many things:

- Whether a coin lands heads or tails
- The presence/absence of a word in a document
- A transmitted bit in a message
- The success of a medical trial

Often, these outcomes (0 or 1) are not equally likely

What is a general way to model such an X?

The Bernoulli Distribution

Definition

The variable X takes on the outcomes

$$X = \begin{cases} 1 & \text{probability } \theta \\ 0 & \text{probability } 1 - \theta \end{cases}$$

Here, $0 \leq \theta \leq 1$ is a parameter representing the probability of success

For higher values of θ, it is more likely to see 1 than 0

- e.g. a biased coin

The Bernoulli Distribution

Definition

By definition,

$$p(X = 1|\theta) = \theta$$
$$p(X = 0|\theta) = 1 - \theta$$

More succinctly,

$$p(X = x|\theta) = \theta^x (1 - \theta)^{1-x}$$

The Bernoulli Distribution

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$$p(X = x|\theta) = \theta^x(1-\theta)^{1-x}$$

This is known as a Bernoulli distribution over binary outcomes:

$$p(X = x|\theta) = \mathrm{Bern}(x|\theta) = \theta^x(1-\theta)^{1-x}$$

Note the use of the conditioning symbol for θ; will revisit later

The Bernoulli Distribution

Mean and Variance

The expected value (or mean) is given by:

$$\begin{aligned}\mathbb{E}[X|\theta] &= \sum_{x\in\{0,1\}} x \cdot p(x|\theta) \\ &= 1 \cdot p(X=1|\theta) + 0 \cdot p(X=0|\theta) \\ &= \theta.\end{aligned}$$

The variance (or squared standard deviation) is given by:

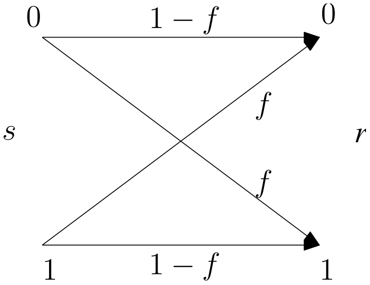
$$\begin{aligned}\mathbb{V}[X|\theta] &= \mathbb{E}[(X-\mathbb{E}[X])^2] \\ &= \mathbb{E}[(X-\theta)^2] \\ &= (0-\theta)^2 \cdot p(X=0|\theta) + (1-\theta)^2 \cdot p(X=1|\theta) \\ &= \theta(1-\theta).\end{aligned}$$

Example: Binary Symmetric Channel

Suppose a sender transmits messages *s* that are sequences of bits

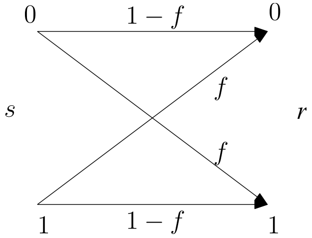
The receiver sees the bit sequence (message) *r*

Due to noise in the channel, the message is flipped with probability $0 \leq f \leq 1$



Example: Binary Symmetric Channel

We can think of r as the outcome of a random variable, with conditional distribution given by:



$$p(r=0|s=0)=1-f \quad p(r=0|s=1)=f$$
$$p(r=1|s=0)=f \quad p(r=1|s=1)=1-f$$

If E denotes whether an error occurred, clearly

$$p(E=e)=\text{Bern}(e|f), \qquad e\in\{0,1\}.$$

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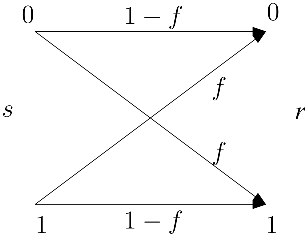

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$$p(E=e) = \text{Bern}(e|f), \qquad e \in \{0,1\}.$$

Why? $p(E=e) = p(r=1, s=0) + p(r=0, s=1)$ (mutually exclusive)
So $p(E=e) = p(r=1|s=0)p(s=0) + p(r=0|s=1)p(s=1)$
This equals f regardless of the value of $p(s=0)$.

Suppose we perform N independent Bernoulli trials

- e.g. we toss a coin N times
- e.g. we transmit a sequence of N bits across a noisy channel

Each trial has probability θ of success

What is the distribution of the number of times (m) that $X = 1$?

- e.g. the number of times we obtained m heads
- e.g. the number of errors in the transmitted sequence

The Binomial Distribution

Definition

Let

$$Y = \sum_{i=1}^{N} X_i$$

where $X_i \sim \text{Bern}(\theta)$.

Then Y has a binomial distribution with parameters N, θ:

$$p(Y = m) = \text{Bin}(m|N, \theta) = \binom{N}{m} \theta^m (1 - \theta)^{N-m}$$

for $m \in \{0, 1, \ldots, N\}$. Here

$$\binom{N}{m} = \frac{N!}{(N - m)!m!}$$

is the # of ways we can we obtain m heads out of N coin flips

The Binomial Distribution:

Mean and Variance

It is easy to show that:

$$\mathbb{E}[Y] = \sum_{m=0}^{N} m \cdot \mathrm{Bin}(m|N,\theta) = N\theta$$

$$\mathbb{V}[Y] = \sum_{m=0}^{N} (m - \mathbb{E}[m])^2 \cdot \mathrm{Bin}(m|N,\theta) = N\theta(1-\theta)$$

- Follows from linearity of mean and variance

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathbb{E}\left[X_i\right] = N\theta$$

$$\mathbb{V}[Y] = \mathbb{V}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathbb{V}\left[X_i\right] = N\theta(1-\theta)$$

The Binomial Distribution:

Example

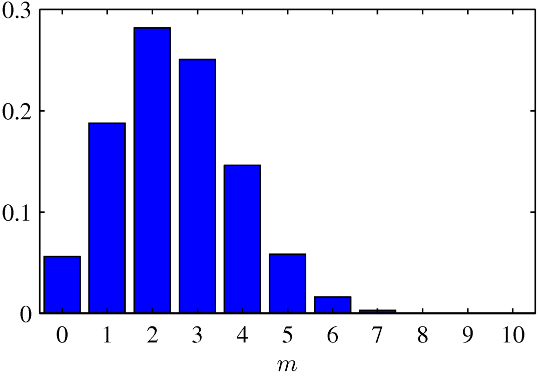
Ashton is an excellent off spinner. The probability of him getting a wicket during a cricket match is $\frac{1}{4}$. (That is, on each attempt, there is a 1/4 chance he will get a wicket.)

His coach commands him to make 10 attempts of wickets in a particular game.

1. What is the probability that he will get exactly three wickets?
   $\text{Bin}(3|10, 0.25)$
2. What is the expected number of wickets he will get?
   $\mathbb{E}[Y]$, where $Y \sim \text{Bin}(\cdot|10, 0.25)$.
3. What is the probability that he will get at least one wicket?
   $\sum_{m=1}^{10} \text{Bin}(m|N = 10, \theta = 0.25) = 1 - \text{Bin}(m = 0|N = 10, \theta = 0.25)$

The Binomial Distribution:

Example: Distribution of the Number of Wickets



Histogram of the binomial distribution with $N = 10$ and $\theta = 0.25$. From Bishop (PRML, 2006)

(A plot of the function $m \mapsto \text{Bin}(m|N = 10, \theta = 0.25)$, for $m \in \{0, \ldots, 10\}$)

The Bernoulli Distribution: Parameter Estimation

Consider the set of observations $\mathcal{D} = \{x_1, \ldots, x_N\}$ with $x_i \in \{0, 1\}$:

- The outcomes of a sequence of coin flips
- Whether or not there are errors in a transmitted bit string

Each observation is the outcome of a random variable X, with distribution

$$p(X = x) = \mathrm{Bern}(x|\theta) = \theta^x (1 - \theta)^{1-x}$$

for some parameter θ

The Bernoulli Distribution: Parameter Estimation

We know that

$$X \sim \text{Bern}(x|\theta) = \theta^x (1-\theta)^{1-x}$$

But often, we don't know what the value of θ is

- The probability of a coin toss resulting in heads
- The probability of the word *defence* appearing in a document about sports

What would be a reasonable estimate for θ from $\mathcal{D}$?

The Bernoulli Distribution: Parameter Estimation:

Maximum Likelihood

Say that we observe

$$\mathcal{D} = \{0, 0, 0, 1, 0, 0, 1, 0, 0, 0\}$$

Intuitively, which seems more plausible: $\theta = \frac{1}{2}$? $\theta = \frac{1}{5}$?

The Bernoulli Distribution: Parameter Estimation:

Maximum Likelihood

Say that we observe

$$\mathcal{D} = \{0, 0, 0, 1, 0, 0, 1, 0, 0, 0\}$$

If it were true that $\theta = \frac{1}{2}$, **then** the probability of this sequence would be

$$\begin{aligned} p(\mathcal{D}|\theta) &= \prod_{i=1}^{10} p(x_i|\theta) \\ &= \prod_{i=1}^{10} \frac{1}{2} \\ &= \frac{1}{2^{10}} \\ &\approx 0.001. \end{aligned}$$

The Bernoulli Distribution: Parameter Estimation:

Maximum Likelihood

Say that we observe

$$\mathcal{D} = \{0, 0, 0, 1, 0, 0, 1, 0, 0, 0\}$$

If it were true that $\theta = \frac{1}{5}$, **then** the probability of this sequence would be

$$\begin{aligned} p(\mathcal{D}|\theta) &= \prod_{i=1}^{10} p(x_i|\theta) \\ &= \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^8 \\ &\approx 0.007. \end{aligned}$$

The Bernoulli Distribution: Parameter Estimation:

Maximum Likelihood

We can write down how likely $\mathcal{D}$ is under the Bernoulli model. Assuming independent observations:

$$p(\mathcal{D}|\theta) = \prod_{i=1}^{N} p(x_i|\theta) = \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}$$

We call $L(\theta) = p(\mathcal{D}|\theta)$ the likelihood function

The Bernoulli Distribution: Parameter Estimation:

Maximum Likelihood

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Maximum likelihood principle: We want to maximize this function wrt θ

The parameter for which the observed sequence has the highest probability

The Bernoulli Distribution: Parameter Estimation:

Maximum Likelihood

Maximising $p(\mathcal{D}|\theta)$ is equivalent to maximising $\mathcal{L}(\theta) = \log p(\mathcal{D}|\theta)$

$$\mathcal{L}(\theta) = \log p(\mathcal{D}|\theta) = \sum_{i=1}^{N} \log p(x_i|\theta) = \sum_{i=1}^{N} [x_i \log \theta + (1 - x_i) \log(1 - \theta)]$$

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Setting $\frac{d\mathcal{L}}{d\theta} = 0$ we obtain:

$$\theta_{\mathrm{ML}} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

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The proportion of times $x = 1$ in the dataset $\mathcal{D}$!

The Bernoulli Distribution:

Parameter Estimation — Issues with Maximum Likelihood

Consider the following scenarios:

- After $N = 3$ coin flips we obtained 3 'tails'
  - What is the estimate of the probability of a coin flip resulting in 'heads'?
- In a small set of documents about sports, the words *defence* never appeared.
  - What are the consequences when predicting whether a document is about sports (using Bayes' rule)?

The Bernoulli Distribution:

Parameter Estimation — Issues with Maximum Likelihood

Consider the following scenarios:

- After $N = 3$ coin flips we obtained 3 'tails'
  - What is the estimate of the probability of a coin flip resulting in 'heads'?
- In a small set of documents about sports, the words *defence* never appeared.
  - What are the consequences when predicting whether a document is about sports (using Bayes' rule)?

These issues are usually referred to as overfitting

- Need to "smooth" out our parameter estimates

- Alternatively, we can do Bayesian inference by considering priors over the parameters

The Bernoulli Distribution:

Parameter Estimation: Bayesian Inference

Recall:

$$\underbrace{p(\theta|X)}_{\text{posterior}} = \frac{\overbrace{p(X|\theta)}^{\text{likelihood}}\overbrace{p(\theta)}^{\text{prior}}}{\underbrace{p(X)}_{\text{evidence}}}$$

If we treat θ as a random variable, we may have some prior belief $p(\theta)$ about its value

- e.g. we believe θ is probably close to 0.5

Our prior on θ quantifies what we believe θ is likely to be, before looking at the data

Our posterior on θ quantifies what we believe θ is likely to be, after looking at the data

The Bernoulli Distribution:

Parameter Estimation: Bayesian Inference

The likelihood of X given θ is

$$\mathrm{Bern}(x|\theta) = \theta^x (1-\theta)^{1-x}$$

For the prior, it is mathematically convenient to express it as a Beta distribution:

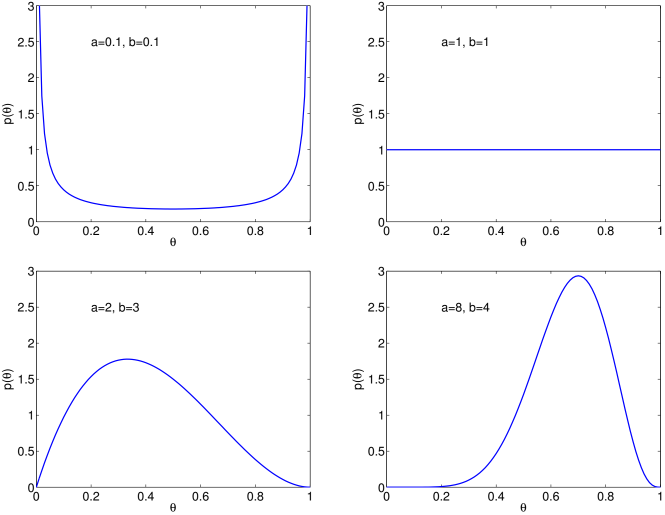
$$\mathrm{Beta}(\theta|a,b) = \frac{1}{Z(a,b)} \theta^{a-1}(1-\theta)^{b-1},$$

where $Z(a,b)$ is a suitable normaliser

We can tune a, b to reflect our belief in the range of likely values of θ

Beta Prior

Examples



Beta Prior and Binomial Likelihood:

Beta Posterior Distribution

Recall that for $\mathcal{D} = \{x_1, \ldots, x_N\}$, the likelihood under a Bernoulli model is:

$$p(\mathcal{D}|\theta) = \theta^m (1-\theta)^\ell,$$

where $m = \sharp(x=1)$ and $\ell \stackrel{\text{def}}{=} N - m = \sharp(x=0)$.

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For the prior $p(\theta|a, b) = \text{Beta}(\theta|a, b)$ we can obtain the posterior:

$$\begin{aligned} p(\theta|\mathcal{D}, a, b) &= \frac{p(\mathcal{D}|\theta)p(\theta|a, b)}{p(\mathcal{D}|a, b)} \\ &= \frac{p(\mathcal{D}|\theta)p(\theta|a, b)}{\int_0^1 p(\mathcal{D}|\theta)p(\theta|a, b)d\theta} \\ &= \text{Beta}(\theta|m + a, \ell + b). \end{aligned}$$

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Can use this as our new prior if we see more data!

Beta Prior and Binomial Likelihood:

Beta Posterior Distribution

Now suppose we choose θ_{MAP} to maximise $p(\theta|\mathcal{D})$
(MAP= Maximum *A Posteriori*)

One can show that

$$\theta_{\text{MAP}} = \frac{m + a - 1}{N + a + b - 2}$$

cf. the estimate that did not use any prior,

$$\theta_{\text{ML}} = \frac{m}{N}$$

The prior parameters *a* and *b* can be seen as adding some “fake” trials!

What values of *a* and *b* ensure $\theta_{\text{MAP}} = \theta_{\text{ML}}$? $a = b = 1$. Make sense?
(Note that the choice of the beta distribution was not accidental here — it is the “conjugate prior” for the binomial distribution.)

Summary

- Distributions involving binary random variables
  - Bernoulli distribution
  - Binomial distribution
- Bayesian inference: Full posterior on the parameters
  - Beta prior and binomial likelihood $\rightarrow$ Beta posterior
- Reading: Mackay §23.1 and §23.5; Bishop §2.1 and §2.2

Next time

- Entropy

Acknowledgement

These slides were originally developed by Professor Robert C. Williamson.