The Australian National University Semester 2, 2022
School of Computing Tutorial 2
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COMP3670/6670: Introduction to Machine Learning

Question 1 Matrix Properties

1. **Uniqueness of inverses**
   Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Assume $\mathbf{A}$ is invertible. Prove that the inverse of $\mathbf{A}$ is unique, (that is, there is only one matrix $\mathbf{B}$ that satisfies $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$)

2. **Inverse of an inverse**
   Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Assume $\mathbf{A}$ is invertable. Prove that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

3. **Distributing the transpose**
   For $\boldsymbol{A} \in R^{m \times n}$, $\boldsymbol{B} \in R^{m \times n}$, prove that $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$

4. **Matrix Cancellation**
   Let **A,B,C** all be square matrices of the same dimension. Assume $\mathbf{AB} = \mathbf{AC}$. Does it always follow that $\mathbf{B} = \mathbf{C}$?

Question 2 Moore-Penrose Inverse

Assuming $\mathbf{A}$ is invertable, prove that the Moore-Penrose inverse $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ equals $\mathbf{A}^{-1}$.

How does this show that the Moore-Penrose inverse is more general than the inverse?

Give an example of a matrix that does not have a Moore-Penrose inverse.

Question 3 Linear Equations

Prove that a system of linear equations $\mathbf{Ax} = \mathbf{b}$ either has no solutions, a unique solution or infinitely many solutions.

(This was done in lecture slides, but try to write the proof in great detail.)

(Hint: If there are at least two solutions $\mathbf{p}$ and $\mathbf{q}$, consider the vector $\mathbf{v}_\lambda = \lambda\mathbf{p} + (1-\lambda)\mathbf{q}$.)

Question 4 Vector Subspaces

Prove that the set of solutions to $\mathbf{Ax} = \mathbf{b}$ is a vector subspace [1] if and only if $\mathbf{b} = \mathbf{0}$.

Question 5 Linear Independence

Let $\mathbf{T} \in \mathbb{R}^{n \times m}$ be a matrix. Let $\{\mathbf{u}, \mathbf{v}\}$ be a set of linearly independent vectors in $\mathbb{R}^{m \times 1}$. Assume that $\{\mathbf{Tu}, \mathbf{Tv}\}$ are linearly dependant. Prove there exists non-zero $\mathbf{x} \in \mathbb{R}^{m \times 1}$ such that $\mathbf{Tx} = \mathbf{0}$.

Question 6 Combining vector subspaces

Let V be a vector space. Let $A \subseteq V$ and $B \subseteq V$ be vector subspaces of V.

1. Prove that $A \cap B$ is a vector subspace of V.

2. **(Tricky)** Prove that $A \cup B$ is a vector subspace of V if and only if A is contained in B, or B is contained in A.

   (This proof is easy in one direction, and tricky the other direction. As a hint, if the sets are not contained in each other, then there must lie a vector in $A \backslash B$ and in $B \backslash A$. Consider the sum of these vectors.)

[1] As a reminder, to check if a non-empty set $E \subseteq V$ is a vector subspace of V, we need to check two things:
Closure under addition: For every $\mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} + \mathbf{y} \in U$.
Closure under scalar multiplication: For every $\lambda \in \mathbb{R}, \mathbf{u} \in U$ we have $\lambda\mathbf{u} \in U$.