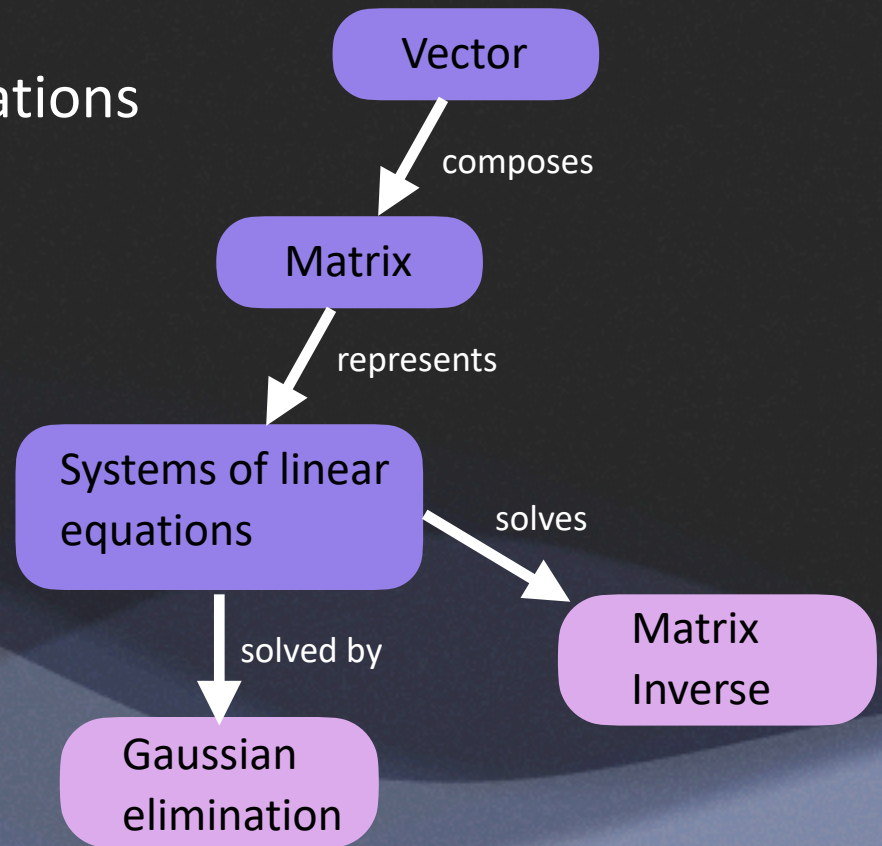# Linear Algebra II

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So far

- Base concepts: vectors and systems of linear equations
- Matrices
- Solving systems of linear equations

Vector

composes

Matrix

represents

Systems of linear equations

solves

Matrix Inverse

solved by

Gaussian elimination

Check your understanding

- Which of the following is correct?

(A) A vector, when multiplied by a scale, is still a vector.

(B) For a system of linear equations with n variables, it is possible that none of them are free variables.

(C) For a system of linear equations with n variables, the maximum number of pivots in the REF is $n - 1$.

(D) A matrix, when added by an identity matrix, stays as is.

(E) We can use matrix transpose in Gaussian Elimination.

(F) Two arbitrary matrices can be multiplied.

(G) Two arbitrary matrices can be added.
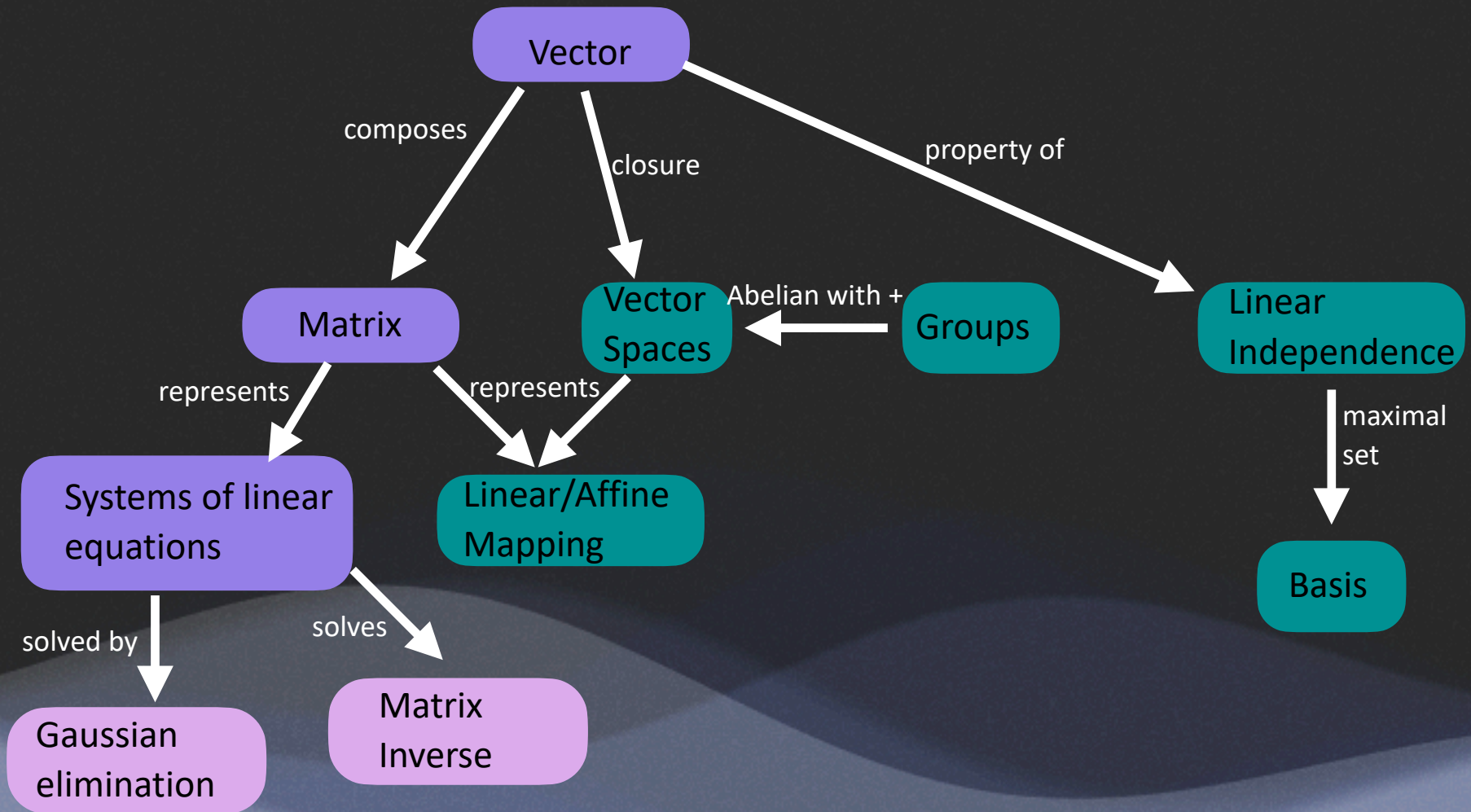
(H) An image with black borders is not a matrix.

Check your understanding

- Let A, B, C be 2x2 matrices.
- Which of the following is equivalent to A(B+C)?
  - AB+AC
  - BA+CA
  - A(C+B)
  - (B+C)A

- Which of the following expressions is equivalent to $I_2(AB)$?
  - AB
  - BA
  - $(AB)I_2$
  - $(BA)I_2$

Outline

- Vector spaces

- Linear Independence

- Spanning sets, bases and dimension

- Rank of a matrix

- Linear Mappings

Outline

Vector Spaces

2.4.1 Groups

- Consider a set $\mathcal{G}$ and an operation $\otimes: \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G :=$ $(\mathcal{G}, \otimes)$ is called a group if the following holds

  - Closure of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G}: x \otimes y \in \mathcal{G}$

  - Associativity: $\forall x, y, z \in \mathcal{G}: (x \otimes y) \otimes z = x \otimes (y \otimes z)$

  - Neutral element: $\exists e \in \mathcal{G} \ \forall x \in \mathcal{G}: x \otimes e = x$ and $e \otimes x = x$

  - Inverse element: $\forall x \in \mathcal{G} \ \exists y \in \mathcal{G}: x \otimes y = e$ and $y \otimes x = e$. We often write x^{-1} to denote the inverse element of x

- Additionally, If $\forall x, y \in \mathcal{G}, x \otimes y = y \otimes x$ (commutative), then $G := (\mathcal{G}, \otimes)$ is an Abelian group.

- Examples

- $(\mathbb{Z}, +)$ is a group and an Abelian group
  - …,-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, …

- $(\mathbb{Z}, -)$ is not a group: it does not satisfy associativity, has no neutral element or inverse element

> Closure: √
> Associativity: $(x + y) + z = x + (y + z)$ √
> Neutral element: 0 √
> Inverse element: $\forall x \in \mathbb{Z}, y = -x \in \mathbb{Z}$ √

> Associativity: $(x - y) - z \neq x - (y - z)$

Examples

$\left(\mathbb{R}^{m \times n}, \, +\right),$ the set of $m \times n$-matrices is Abelian (component-wise addition).

Closure: addition of any two matrices in $\mathbb{R}^{m \times n}$ is a matrix in $\mathbb{R}^{m \times n}$

Associativity: $\forall A, \, B, C \in \mathbb{R}^{m \times n}, \, \left(A + B\right) + C = A + (B + C)$

Neutral element: $\mathbf{0}$

Inverse element: $\forall A \in \mathbb{R}^{m \times n}$, there exists its inverse element $-A$

Commutative: $\forall A, \, B \in \mathbb{R}^{m \times n}, \, A + B = B + A$

2.4.2 Vector spaces

- Definition
- A real-valued vector space $V = \left(\mathscr{V}, +, \bullet \right)$ is a set $\mathscr{V}$ with two operations

$$+ \ : \mathscr{V} \bigotimes \mathscr{V} \to \mathscr{V}$$

$$\cdot \ : \mathbb{R} \bigotimes \mathscr{V} \to \mathscr{V}$$

- where
  - $\left(\mathscr{V}, + \right)$ is an Abelian group
  - Distributivity:
    $$\forall \lambda \in \mathbb{R}, x, y \in \mathscr{V}: \qquad \lambda \cdot \left(x + y \right) = \lambda \cdot x + \lambda \cdot y$$
    $$\forall \lambda, \varphi \in \mathbb{R}, x \in \mathscr{V}: \qquad \left(\lambda + \varphi \right) \cdot x = \lambda \cdot x + \varphi \cdot x$$

  - Associativity (outer operation $\cdot$):
    $$\forall \lambda, \varphi \in \mathbb{R}, x \in \mathscr{V}: \qquad \lambda \cdot \left(\varphi \cdot x \right) = \left(\lambda \varphi \right) \cdot x$$

  - Neutral element (w.r.t to outer operation $\cdot$):
    $$\forall x \in \mathscr{V}: \qquad 1 \cdot x = x$$

2.4.2 Vector spaces

- Elements $x \in \mathscr{V}$ are called vectors

- The neutral element of $(\mathscr{V}, +)$ is the zero vector $\mathbf{0} = [0, \cdots, 0]^{\mathrm{T}}$

- $+$ is called vector addition

- Elements $\lambda \in \mathbb{R}$ are called scalars

- Outer operation $\cdot$ is a multiplication by scalars

2.4.2 Vector spaces

- Example

- $\mathcal{V} = \mathbb{R}^n,\ n \in \mathbb{N}$ is a vector space. Its operations are defined as

  - Addition: $\boldsymbol{x} + \boldsymbol{y} = \left[x_1, \cdots, x_n\right]^{\mathrm{T}} + \left[y_1, \cdots, y_n\right]^{\mathrm{T}} = \left[x_1 + y_1, \cdots, x_n + y_n\right]^{\mathrm{T}}$, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

  - Multiplication by scalars: $\lambda \boldsymbol{x} = \lambda\left[x_1, \cdots, x_n\right]^{\mathrm{T}} = \left[\lambda x_1, \cdots, \lambda x_n\right]^{\mathrm{T}}$, for $\boldsymbol{x} \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$

- We usually write $\boldsymbol{x} \in \mathbb{R}^n$ in a column vector

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

2.4.3 Vector Subspaces

- Sets contained in the original vector space

- "closed"
  - When we perform vector space operations on elements within this subspace, we will never leave it

- $U = \left(\mathcal{U}, + , \bullet \right)$ is called vector subspace of $V = \left(\mathcal{V}, + , \bullet \right)$, if

- $\mathcal{U} \subseteq \mathcal{V}$,

- $\mathcal{U} \neq \varnothing$, in particular $\mathbf{0} \in \mathcal{U}$

- Closure of U
  - $\forall x, y \in \mathcal{U}, \; x + y \in \mathcal{U}$
  - $\forall x \in \mathcal{U}, \; \lambda \in \mathbb{R}, \; \lambda x \in \mathcal{U}$

2.4.3 Vector Subspaces

- Examples
- For every vector space V, the trivial subspaces are V itself and $\{\mathbf{0}\}$
- Is it a subspace of $\mathbb{R}^2$?
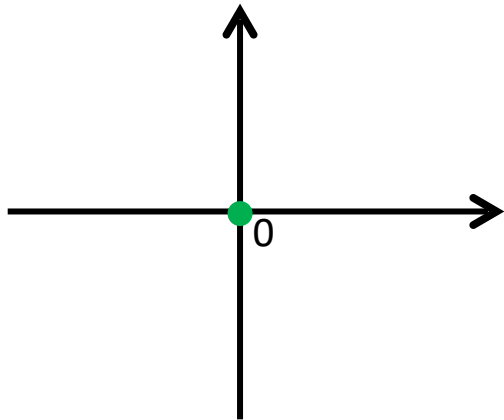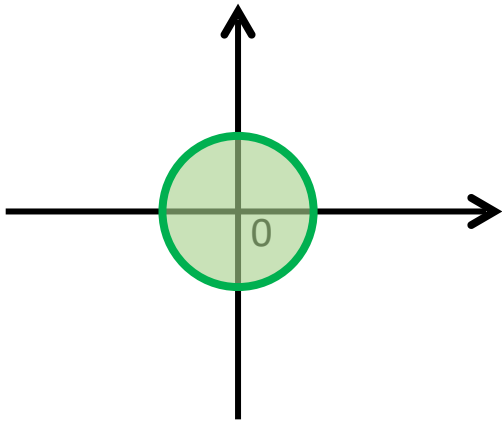

Is it a subset of $\mathbb{R}^2$?   Yes

Does it satisfy $\mathcal{U} \neq \varnothing$, in particular $\mathbf{0} \in \mathcal{U}$ Yes

Does it satisfy closure?   Yes

$$x + y \in \{0\}$$
$$\lambda x \in \{0\}$$

2.4.3 Vector Subspaces

- Examples
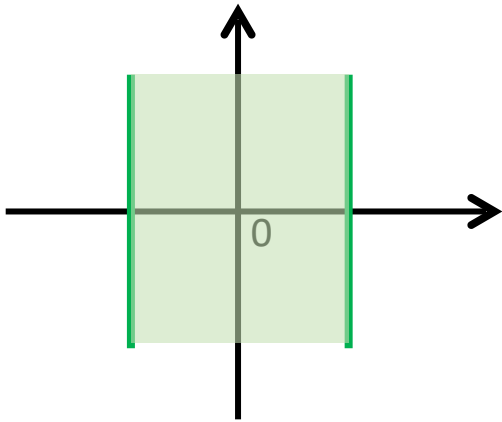- Is it a subspace of $\mathbb{R}^2$?

Is it a subset of $\mathbb{R}^2$? Yes

Does it satisfy $\mathscr{U} \neq \varnothing$, in particular $\mathbf{0} \in \mathscr{U}$ Yes

Does it satisfy closure? No

$$(0.8,\ 0) + (0.9,\ 0) = (1.7, 0) \notin \mathscr{U}$$

2.4.3 Vector Subspaces

- Examples
- Is it a subspace of $\mathbb{R}^2$?

Is it a subset of $\mathbb{R}^2$?   Yes

Does it satisfy $\mathscr{U} \neq \varnothing$, in particular $\mathbf{0} \in \mathscr{U}$ Yes

Does it satisfy closure?   No

2.4.3 Vector Subspaces

- Examples

- The solution set of a homogeneous system of linear equations $Ax = 0$ with n unknowns $x = \begin{bmatrix} x_1, \cdots, x_n \end{bmatrix}^{\mathrm{T}}$. Is it a subspace of $\mathbb{R}^n$?

Is it a subset of $\mathbb{R}^n$? Yes

Does it satisfy $\mathcal{U} \neq \varnothing$, in particular $0 \in \mathcal{U}$ Yes
Does it satisfy closure? Yes

$\forall x, \ y \in \mathcal{U}, \ $ we have $Ax = 0, \ Ay = 0$
1) We investigate whether $x + y \in \mathcal{U}$.
Because $A(x + y) = Ax + Ay = 0$,
We know $x + y$ is a solution, thus belonging to $\mathcal{U}$
2) We investigate whether $\lambda x \in \mathcal{U}$.
Because $A(\lambda x) = \lambda(Ax) = 0$,
We know λx is a solution, thus belonging to $\mathcal{U}$

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2.4.3 Vector Subspaces

- Examples

- The solution set of an inhomogeneous system of linear equations $Ax = b, \ b \neq 0.$ Is it a subspace of $\mathbb{R}^n$?

Is it a subset of $\mathbb{R}^n$? Yes

Does it satisfy $\mathcal{U} \neq \varnothing,$ in particular $0 \in \mathcal{U}$ No

Does it satisfy closure? No

Linear Independence

Linear combination

- Consider a vector space V and k vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in V$. For $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$, $\boldsymbol{v} \in V$ is called a linear combination of vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$, if

$$\boldsymbol{v} = \lambda_1 \, \boldsymbol{x}_1 + \cdots + \lambda_k \, \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \, \boldsymbol{x}_i \in V$$

2.5 Linear Independence

- Consider a system of linear functions $\lambda_1 \, \boldsymbol{x}_1 \; + \; \cdots \; + \lambda_k \, \boldsymbol{x}_k = \boldsymbol{0}$

- If there is a non-trivial solution, $\lambda_1, \ldots, \lambda_k$, with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ are linearly dependent

- If only the trivial solution exists, i.e., $\lambda_1 = \cdots = \lambda_k = 0$, then vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ are linearly independent

- Intuitively, a set of linearly independent vectors consists of vectors that have no redundancy, *i.e.,* if we remove any of those vectors from the set, we will lose something.

How to determine linear (in)dependence

- Write all vectors $x_1, \cdots, x_k$ as columns of a matrix A
- Perform Gaussian elimination until the matrix is in row echelon form
- The pivot columns correspond to independent vectors

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$x_1 \quad x_2 \quad x_3$

$\boxed{x_2 = 3x_1}$

- All column vectors are linearly independent if and only if all columns are pivot columns.
- If there is at least one non-pivot column, the columns are linearly dependent.

Basis & dimension

Determine linear (in)dependence

- Consider three vectors in $\mathbb{R}^3$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{R1+R2->R2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Swap R2 and R3}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{R3-2R2->R3}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\quad x_1 \quad x_2 \quad x_3$$
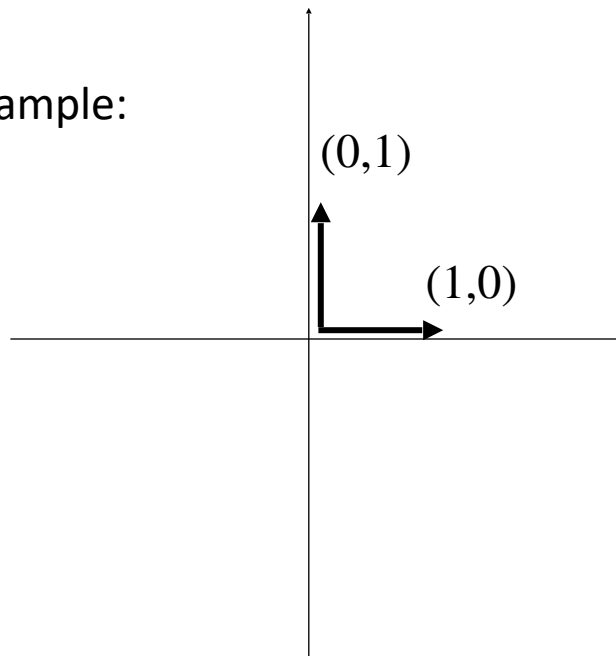
$$x_3 = x_1 + 2x_2$$

The Basis of a vector space

- A set of vectors $\{x_1, \cdots, x_k\}$ is said to form a basis for a vector space if

(1) The vectors $\{x_1, \cdots, x_k\}$ span the vector space: every vector in this space can be represented by a linear combination of $\{x_1, \cdots, x_k\}$

(2) The vectors $\{x_1, \cdots, x_k\}$ are linearly independent.

Example:

(0,1)

(1,0)

- Example
- In $\mathbb{R}^3$, the canonical/standard basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

First, this REF has three pivots, so the three bases are linearly independent.

- Different bases in $\mathbb{R}^3$ are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Second, do the three bases span $\mathbb{R}^3$?

Specifically, $\forall \begin{bmatrix} a, b, c \end{bmatrix}^T \in \mathbb{R}^3$, we examine whether it can be obtained by a linear combination by the three bases.

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We can obtain the solution

$$\begin{cases} \lambda_3 = c \\ \lambda_2 = b - c \\ \lambda_1 = a - b \end{cases}$$

- Another different basis in $\mathbb{R}^3$ is

$$\mathscr{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

- Another example

$$\mathscr{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

is linearly independent, but not a basis of $\mathbb{R}^4$: For instance, the $[1,0,0,0]^\top$ cannot be obtained by a linear combination of elements in $\mathscr{A}$.

So, a couple of things about basis

- Let $V = (\mathcal{V}, +, \bullet)$ be a vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \varnothing$ be a basis of V.

- $\mathcal{B}$ is a maximal linearly independent set of vectors in V, i.e., adding any other vector to this set will make it linearly dependent.

- Every vector $x \in V$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., with

$$x = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i = \sum_{i=1}^{k} \psi_i \boldsymbol{b}_i$$

Think about:

(0,1)

(1,0)
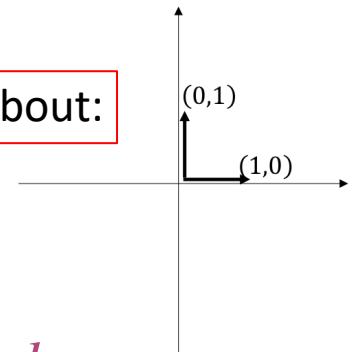
and $\lambda_i, \psi_i \in \mathbb{R}, \boldsymbol{b}_i \in B$ it follows that $\lambda_i = \psi_i, i = 1, \cdots, k.$

- Every vector space V possesses a basis $\mathscr{B}$.

- There can be many bases of a vector space.

- All bases possess the same number of elements, called the basis vectors

$$\mathscr{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \text{ then } \dim(\mathscr{B}) = 3$$

- Dimension of (V): number of basis vectors of V. We write $\dim(V)$

- If $U \subseteq V$ is a subspace of V, then $\dim(U) \leq \dim(V)$

- $\dim(U) = \dim(V)$ if and only if $U = V$

Determining a Basis

- Write the spanning vectors as columns of a matrix A

- Determine the row-echelon form of A.

- The spanning vectors associated with the pivot columns are a basis of U.

- Example

- For a vector subspace $U \subseteq \mathbb{R}^5$, spanned by the vectors

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad x_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \in \mathbb{R}^5$$

Determining a Basis - Example

- Which vectors of $x_1, \ldots, x_4$ are a basis for U?
- Check whether $x_1, \ldots, x_4$ are linearly independent.

$$\sum_{i=1}^{4} \lambda_i x_i = 0$$

- A homogeneous system of equations with matrix

$$[x_1, x_2, x_3, x_4] = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

- Through Gaussian Elimination, we obtain the row-echelon form

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix}$$

x_1, x_2, x_4 are linearly independent. Therefore, $\{x_1, x_2, x_4\}$ is a basis of U

Rank

2.6.2 Rank

- The number of linearly independent columns of a matrix $A \in \mathbb{R}^{m \times n}$ is called the rank of A, denoted by $\mathrm{rk}(A)$

- $\mathrm{rk}(A)$ also equals the number of linearly independent rows

- Rank gives us an idea of how much information a matrix contains

Important properties

- $\mathrm{rk}\left(\boldsymbol{A}\right) = \mathrm{rk}\left(\mathbf{A}^{\mathrm{T}}\right)$

- Columns and rows of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ can both span subspaces of the same dimension $\mathrm{rk}\left(\boldsymbol{A}\right)$

- The basis of the subspace spanned by columns (rows) can be found by Gaussian elimination to $\boldsymbol{A}\,(\mathbf{A}^{\mathrm{T}})$ to identify the pivot column

- For all $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ it holds that $\boldsymbol{A}$ is regular (invertible) if and only if $\mathrm{rk}\left(\boldsymbol{A}\right) = n$.

$$\begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{bmatrix}_{n \times n}$$

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- Example

- We use Gaussian elimination to determine the rank

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$
$$ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x_1 \quad x_2 \quad x_3$$

- 2 pivot columns. So $\mathrm{rk}(A) = 2$

More properties

- For all $A \in \mathbb{R}^{m \times n}$ and all $b \in \mathbb{R}^m$ it holds that the linear equation system $A\,x = b$ can be solved if and only if $\mathrm{rk}(A) = \mathrm{rk}(A\,|\,b)$, where $A\,|\,b$ denotes the augmented matrix

- For $A \in \mathbb{R}^{m \times n}$ the subspace of solutions for $A\,x = 0$ possesses dimension $n - \mathrm{rk}(A)$.

  Let's look at a simpler case where $A \in \mathbb{R}^{n \times n}$ and $\mathrm{rk}(A) = n$.

  In this scenario, the dimension of the solution space is $n - \mathrm{rk}(A) = 0$.

  The only solution is $x = 0$.

More properties

- A matrix $A \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions.

- The rank of a full-rank matrix is the lesser of the number of rows and columns, i.e., $\mathrm{rk}(A) = \min(m, n)$.

  For example, for $A \in \mathbb{R}^{5 \times 3}$, $\mathrm{rk}(A)$ does not exceed 3.

- A matrix is said to be rank deficient if it does not have full rank.

Linear Mappings

2.7 Linear Mappings

- For vector spaces $V,\ W,$ a mapping $\Phi\colon V \to W$ is called a linear mapping if

$$\forall x, y \in V,\ \forall \lambda, \psi \in \mathbb{R} : \Phi\big(\lambda x + \psi y\big) = \lambda \Phi(x) + \psi \Phi(y)$$

- It implies the following

$$\Phi\big(x + y\big) = \Phi(x) + \Phi(y) \qquad\qquad \Phi(\lambda x) = \lambda\, \Phi(x)$$

Example

- The mapping $\Phi: \mathbb{R}^2 \to \mathbb{C}, \Phi(\boldsymbol{x}) = x_1 + i x_2$, is a linear mapping:

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = (x_1 + y_1) + i(x_2 + y_2) = x_1 + ix_2 + y_1 + iy_2$$

$$= \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \Phi\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$\Phi\left(\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \lambda x_1 + \lambda i x_2 = \lambda(x_1 + ix_2) = \lambda \Phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

2.7 Linear Mappings

- For linear mappings $\Phi: V \to W$ and $\Psi: W \to X$, the mapping $\Phi \circ \Psi: V \to X$ is also linear

- If $\Phi: V \to W$ and $\Psi: V \to W$ are both linear mappings, then $\Phi + \Psi$ and $\lambda\Phi, \lambda \in \mathbb{R}$ are also linear.

Coordinates of a vector

- Consider a vector space V and an ordered basis $B = \left(b_1, \cdots, b_n \right)$ of V. For any $x \in V$ we obtain a unique representation
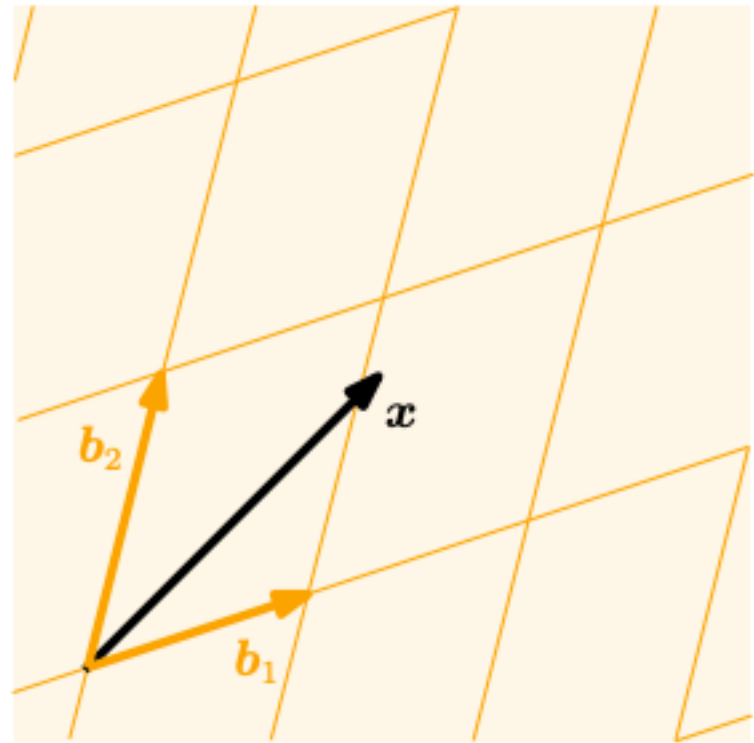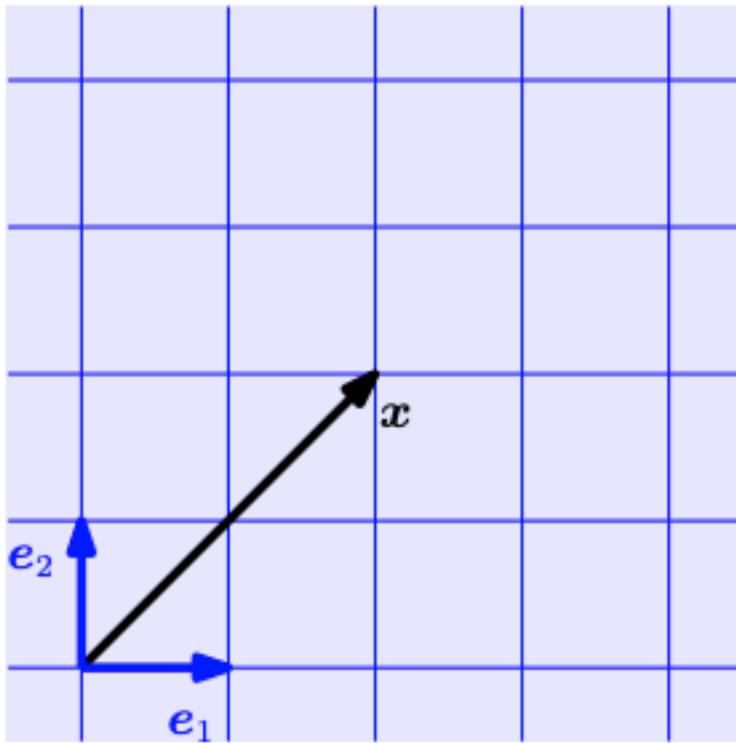
$$x = a_1 b_1 + \ldots + a_n b_n$$

of x with respect to B. Then $\alpha_1, \cdots, \alpha_n$ are the coordinates of x with respect to B, and the vector

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$

is the coordinate vector/coordinate representation of x with respect to the ordered basis B.
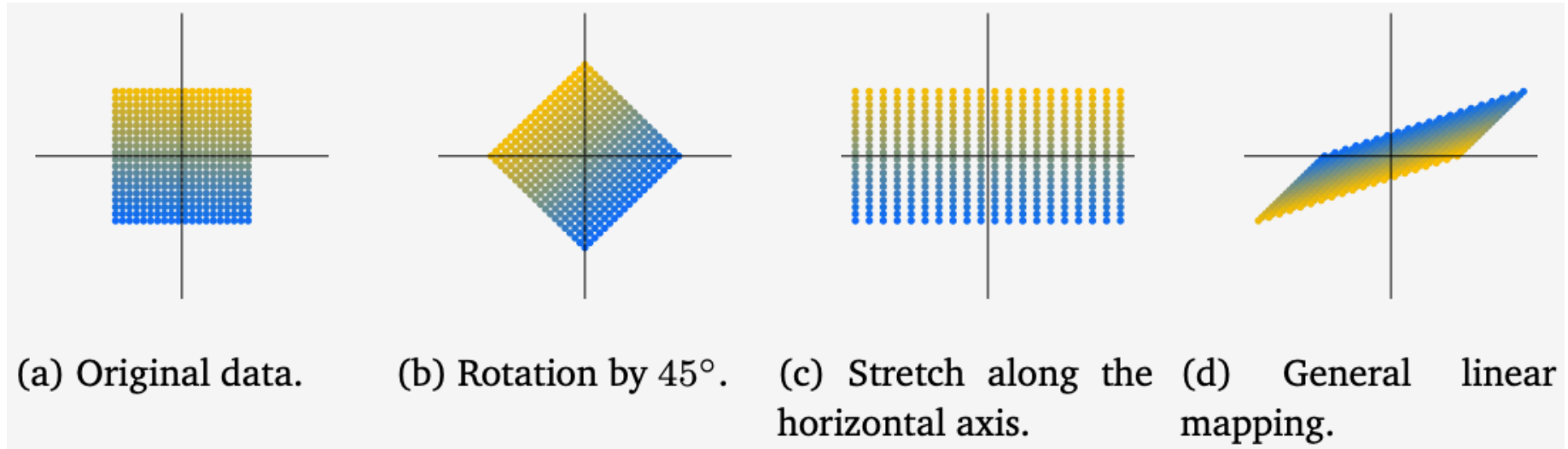
Coordinates of a vector

- [Left] A Cartesian coordinate system in two dimensions, which is spanned by the canonical basis vectors e_1, e_2.

- The same vector x may have different coordinates under different basis.

2.7.1 Matrix Representation of Linear Mappings

- Example - Linear Transformations of Vectors

(a) Original data.   (b) Rotation by $45°$.   (c) Stretch along the horizontal axis.   (d) General linear mapping.

- The following three linear transformations are used

$$A_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_3 = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

- Consider vector spaces V, W with corresponding bases $B = (b_1, \cdots, b_n)$ and $C = (c_1, \cdots, c_m)$. We consider a linear mapping $\Phi: V \to W$. For $j \in \{1, \cdots, n\}$,

$$\Phi(b_j) = \alpha_{1j} c_1 + \cdots + \alpha_{mj} c_m = \sum_{i=1}^{m} \alpha_{ij} c_i$$

. is the unique representation of $\Phi(b_j)$ with respect to C. Then, we call the $m \times n$-matrix A_Φ the transformation matrix of Φ, whose elements are given by

$$A_\Phi(i, j) = \alpha_{ij}$$

- If $\hat{x}$ is the coordinate vector of $x \in V$ with respect to B, and $\hat{y}$ the coordinate vector of $y = \Phi(x) \in W$ with respect to C, then

$$\hat{y} = A_\Phi \hat{x}$$