Exercise 1

(a) matrix is symmetric when $A = A^T$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Hence, matrix A is symmetric

(b) matrix A^2 is symmetric when $A^2 = (A^2)^T$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$(A^2)^T = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} = A^2$$

since $A^2 = (A^2)^T$, then A^2 is symmetric

(c) true, prove by $(A^2)^T = A^2$

$(A^2)^T = (A \times A)^T$

$= (A^T \times A^T)$ property $[(AB)^T = B^T \cdot A^T]$

$= A \times A$ by definition $[A^T = A]$

$= A^2$

since $(A^2)^T = A^2$. It is true.

(d) matrices $f(A)$ and $g(A)$ commute when

$f(A)g(A) = g(A)f(A)$

Define $f(A)$ and ~~f~~ $g(A)$ as:

$f(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n$

$g(A) = b_0 I + b_1 A + b_2 A^2 + \ldots + b_n A^n$

Suppose $f(A)g(A) = g(A)f(A)$ then

$f(A)g(A) = (a_0 I + a_1 A + \ldots + a_n A^n)(b_0 I + \ldots + b_n A^n)$

$= a_0 b_0 I + a_0 b_2 A + \ldots + a_n b_n A^{2n}$

$= (b_0 I + b_2 A + \ldots + b_n A^n)(a_0 I + a_1 A + \ldots + a_n A^n)$

$= g(A) f(A)$

since $f(A)g(A) = g(A)f(A)$, the matrices are commutative for arbitrary order of n.

(e) $X \otimes X$ is a magic square when its row sums, column sums, diagonal sum are all equal. $X = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$ has the sum of 15.

when $X \otimes X$ the matrix property still be the same but has a shape of $nr \times kc$ with the sum of $15^2 = 225$

Hence, $X \otimes X$ is a magic square.

(f) Show that given matrix X of $n \times n$, $X \otimes X$ is a ~~square~~ magic square.

let $X = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ the sum is 4

$$X \otimes X = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$ is 4 by 4 matrix with sum of 16.

(g) let $X = \begin{bmatrix} a \\ b \end{bmatrix}$, $y = \begin{bmatrix} c \\ d \end{bmatrix}$

$$X \otimes y = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

$$(X \otimes y) \otimes X = \begin{bmatrix} a \cdot c \cdot a \\ a \cdot c \cdot b \\ a \cdot d \cdot a \\ a \cdot d \cdot b \\ b \cdot c \cdot a \\ b \cdot c \cdot b \\ b \cdot d \cdot a \\ b \cdot d \cdot b \end{bmatrix}$$

the same also apply to $y \otimes X \otimes y$

Hence the condition are

$(a \cdot c \cdot a) = (c \cdot a \cdot c)$

$(a \cdot c \cdot b) = (c \cdot a \cdot d)$

$(a \cdot d \cdot a) = (d \cdot a \cdot c)$

$(a \cdot d \cdot b) = (d \cdot a \cdot d)$

$(b \cdot c \cdot a) = (c \cdot b \cdot c)$

$(b \cdot c \cdot b) = (c \cdot b \cdot d)$

$(b \cdot d \cdot a) = (d \cdot b \cdot c)$

$(b \cdot d \cdot b) = (d \cdot b \cdot d)$

Exercise 2 (a)

1) put it in augment matrix and elimination

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 6 & 2 & -5 & -2 \\ 2 & 1 & -2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

X_3 is a free variable denote $x_3 = t$

Hence, $x_1 + x_3 = 5$

$2x_2 - 3x_3 = 4$

then $x_3 = t$, $x_2 = 4 + 3t$

$x_1 = 5 - t$

the solution is

$$S = \left\{ \begin{bmatrix} 5-t \\ 4+3t \\ t \end{bmatrix} \middle| t \in R^3 \right\}$$

(a)

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 6 & 2 & -5 & -2 \\ 2 & 1 & -2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.4 \\ 0 & 2 & 0 & 3.8 \\ 0 & 0 & 2 & 3.6 \end{array}\right]$$

Since A is invertible matrix, there is a unique solution $S = \left\{ \begin{bmatrix} x_1 = 1.4 \\ x_2 = 3.8 \\ x_3 = 3.6 \end{bmatrix} \middle| x_i \in R \right\}$

(b)

$$A = \left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 2 & 2 & 2 & 6 & 23 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ -2 & 0 & 1 & 1 & 1 & 16 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -5 & -16 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \end{array}\right]$$ let X_3 be free variable

since A is non-invertible, there are many solution

Hence,

$$S = \left\{ \begin{bmatrix} -16 \\ 23 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} : \alpha \in R \right\}$$

Exercise 3

(a) Show that $(A^{-1})^T = (A^T)^{-1}$

when A is invertible matrix, there is

$A \cdot A^{-1} = I$

$(A \cdot A^{-1})^T = I^T$

$(A^{-1})^T \cdot A^T = I$ [from $(AB)^T = B^T \cdot A^T, I^T = I$]

Since $(A^{-1})^T \cdot A^T = I$, $(A^{-1})^T$ is the inverse of A^T [$A^{-1} \cdot A = I$]

then $(A^{-1})^T = (A^T)^{-1}$

(b) inverse of matrix will exist only when determinant is not equal to 0, $\det(A) \neq 0$

$$A = \begin{bmatrix} 1 & 1 & b \\ 1 & a & c \\ 1 & 1 & 1 \end{bmatrix}$$

$\det(A) = 1(a-c) - 1(1-c) + b(1-a)$

$= a - c - 1 + c + b(1-a)$

$= a - 1 + b - ba$

Hence, $\det(A) \neq 0$ or $a - 1 + b - ba \neq 0$

(c) Show that $\text{rank}(A) = \text{rank}(A^T)$

since $\text{rank}(A^T)$ is the dimension of column space of A^T which is a number of basis for $C(A^T)$, pivot rows in rref(A) are the basis for $C(A^T)$. Since rank(A) is the dimension of column space A which is the basis for $C(A)$. Hence, the number of pivot entry of rref(A) is Rank(A) and Rank(A^T)

Exercise 4 (a)

i) 1) since $x_1, \ldots, x_n \geq 0$, vector $0 \in A$.

2. If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ then $u + v = (u_1+v_1, \ldots, u_n+v_n)$. The sum is also in A because $u_i, v_i \geq 0$

3. A is not closed under scalar multiplication let $u \in A$ and $c = -1$ then cu is not element in A.

Hence, set A is not subspace of R^n

ii) $0 \notin B$ because 0 is rational and 0 vector contain all rational number. It is not closure under addition and multiplication since rational plus irrational can be irrational number.

Hence, B is not subspace of R^n

iii) vector $0 \in C$, it also closed under addition, since

$$\sum_{i=1}^{n} (-1)^{i+1} u_i \geq 0, \quad \sum_{i=1}^{n} (-1)^{i+1} v_i \geq 0$$

then

$$\left(\sum_{i=1}^{n} (-1)^{i+1} (u_i + v_i)\right) = \left(\sum_{i=1}^{n} (-1)^{i+1} u_i + \sum_{i=1}^{n} (-1)^{i+1} v_i\right)$$

3. Closed under multiplication let $c \in R$

$$\sum_{i=1}^{n} (-1)^{i+1} (c \cdot u_i) = c \cdot \sum_{i=1}^{n} (-1)^{i+1} (u_i)$$

since negative scalar is not satisfied. It's not closed under multiplication

Hence, C is not the subspace of R^n

iv) let $b = 0$, then zero vector is the solution of $Ax = b$ and 0 vector $\in D$. If $b \neq 0$, 0 vector may or may not be a solution depend on A

D is subspace when $b = 0$. If $b \neq 0$ then D is not subspace as it not satisfy the closure properties. If $b = 0$, then D is null space of A.

Hence, H depend on whether $b = 0$ or not.

(b) $0 \in W^{\perp}$ since zero vector is orthogonal to every vector in V and in W

let u, v be vector in $W^{\perp}$ for all vector w in W, $\langle u, w \rangle = 0$ and $\langle v, w \rangle = 0$

then for $u+v$ vector, $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = 0$ Hence it is true that $\forall w \in W$, $u + v \in W^{\perp}$ which is closed under vector addition. It's also closed under multiplication since $\forall w \in W$, $\langle cu, w \rangle = c\langle u, w \rangle = 0$ then cu is in $W^{\perp}$.

Hence, $W^{\perp}$ is a subspace of V.

Exercise 5 (a)

Since H is linear transformation, it must satisfy $T(u+v) = T(u) + T(v)$. Let $u, v = 0$ then $T(0+0) = T(0) + T(0)$

$T(0) = T(0) + T(0)$

$T(0) = 0$

(b) Let $n = 1$

$T(c_1 v_1) = c_1 T(v_1)$ by definition. Induction step: let $n = k$, prove $n = k+1$

$T(c_1 v_1 + \ldots + c_k v_k + c_{k+1} v_{k+1}) = T(c_1 v_1 + \ldots + c_k v_k) + T(c_{k+1} v_{k+1})$

$= c_1 T(v_1) + \ldots + c_k T(v_k) + c_{k+1} T(v_{k+1})$

(c) since $T(0)=0$, 0 vector is in $Im(T)$. Let $w_1, w_2 \in Im(T)$. Then $\exists\, u_1, u_2 \in V \mid T(u_1)=w_1$ and $T(u_2)=w_2$. Then $T(u_1+u_2) = w_1+w_2$. So $Im(T)$ is closed under addition. let $w \in Im(T) \wedge c \in R$. Then $\exists\, u \in V \mid T(u) = w$. Then $T(cu)=cw$. So it's closed under multiplication. For kernel of T ($ker(T)$). Since $T(0)=0$, then zero vector is in $ker(T)$. Let $u_1, u_2 \in ker(T)$. Then $T(u_1)=T(u_2)=0$, $T(u_1+u_2)=0$ and $u_1+u_2 \in ker(T)$. Let $u \in ker(T)$ and $c \in R$. Then $T(u)=0$, $T(cu)=0$ and $cu \in ker(T)$

(d) let $T: R^5 \to R^3$, $T(x)=Ax$

where
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$dim(Im(T))=3$ and $dim(ker(T))=2$

(e) matrix A need to be invertible to satisfy $T(x)=Ax$ (injective)

Then $det(A) \neq 0$ prove:

$1(1-c) - a(1-c) + b(1-1) \neq 0$

$1-c-a+ac \neq 0$ #

Exercise 6

(a) Inner product is symmetric and linear in second argument mean $\langle u, cv+w \rangle = c\langle u,v\rangle + \langle u,w\rangle$ Then

$$\begin{aligned} \langle cu+w, v\rangle &= \langle v, cu+w\rangle \\ &= c\langle v,u\rangle + \langle v,w\rangle \\ &= c\langle u,v\rangle + \langle w,v\rangle \\ &= c\langle u,v\rangle + \langle w,v\rangle \end{aligned}$$

(b) let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ show that

$x^T y = (Rx)^T(Ry)$

LHS: $x^T y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (x_1y_1 + x_2y_2)$

RHS: $(Rx)^T(Ry) = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta & x_1\sin\theta + x_2\cos\theta \end{bmatrix}\begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix}$

$= x_1y_1(\cos^2\theta + \sin^2\theta) + x_2y_2(\sin^2\theta + \cos^2\theta)$

$= x_1y_1 + x_2y_2$.

Hence, LHS = RHS, rotation matrix R preserve standard inner product.

(c) find $D' \mid x^T D y = (Rx)^T D' (Ry)$

$D' = R^T D R$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \cos^2\theta \\ -\sin\theta\cos\theta + \sin^2\theta & 2\sin^2\theta + \cos^2\theta \end{bmatrix}$$

(d) let $\theta = \frac{\pi}{4}$ Then

$$D' = \begin{bmatrix} 2\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4} & -\sin\frac{\pi}{4}\cos\frac{\pi}{4} + \cos^2\frac{\pi}{4} \\ -\sin\frac{\pi}{4}\cos\frac{\pi}{4} + \sin^2\frac{\pi}{4} & 2\sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

(e)

Exercise 7

(a) Assume x, y are linearly dependent. Then $x = cy$ where $c \in R \mid c \neq 0$

$cy \cdot y = 0$ $[x \cdot y = 0]$

$= c(y \cdot y) = c\|y\|^2 = 0$

since c is not equal to 0 then $\|y\|^2$ must be zero.

Hence it's a contradiction since $\|y\|^2$ must equal to 1.

(b)