COMP2610 / COMP6261 Information Theory
Lecture 13: Symbol Codes for Lossless Compression

Thushara Abhayapala

Audio & Acoustic Signal Processing Group
School of Engineering,
College of Engineering & Computer Science
The Australian National University,
Canberra, Australia.

Acknowledgement: These slides were originally developed by Professor Robert C. Williamson.

Australian National University

Last time

Proof of the source coding theorem

- Foundational theorem, but impractical
- Requires potentially very large block sizes

The theorem also only considers uniform coding schemes

- Could variable length coding help?
- Does entropy turn up for such codes as well?

This time

Variable-length codes

Prefix codes

Kraft's inequality

Codes: A Review

Notation:

- If $\mathcal{A}$ is a finite set then $\mathcal{A}^N$ is the set of all *strings of length N*.
- $\mathcal{A}^+ = \bigcup_N \mathcal{A}^N$ is the set of *all finite strings*

Examples:

- $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
- $\{0, 1\}^+ = \{0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$

Codes: A Review

Notation:

- If $\mathcal{A}$ is a finite set then $\mathcal{A}^N$ is the set of all *strings of length N*.
- $\mathcal{A}^+ = \bigcup_N \mathcal{A}^N$ is the set of *all finite strings*

Examples:

- $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
- $\{0,1\}^+ = \{0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$

Binary Symbol Code

Let X be an ensemble with $\mathcal{A}_X = \{a_1, \ldots, a_I\}$.
A function $c : \mathcal{A}_X \to \{0,1\}^+$ is a **code** for X.

- The binary string $c(x)$ is the **codeword** for $x \in \mathcal{A}_X$

Codes: A Review

Notation:

- If $\mathcal{A}$ is a finite set then $\mathcal{A}^N$ is the set of all *strings of length N*.
- $\mathcal{A}^+ = \bigcup_N \mathcal{A}^N$ is the set of *all finite strings*

Examples:

- $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
- $\{0,1\}^+ = \{0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$

Binary Symbol Code

Let X be an ensemble with $\mathcal{A}_X = \{a_1, \ldots, a_I\}$.
A function $c : \mathcal{A}_X \to \{0,1\}^+$ is a **code** for X.

- The binary string $c(x)$ is the **codeword** for $x \in \mathcal{A}_X$
- The **length** of the codeword for for x is denoted $\ell(x)$.
  Shorthand: $\ell_i = \ell(\mathbf{c}_i)$ for $i = 1 \ldots, I$.

Codes: A Review

Notation:

- If $\mathcal{A}$ is a finite set then $\mathcal{A}^N$ is the set of all *strings of length N*.
- $\mathcal{A}^+ = \bigcup_N \mathcal{A}^N$ is the set of *all finite strings*

Examples:

- $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
- $\{0,1\}^+ = \{0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$

Binary Symbol Code

Let X be an ensemble with $\mathcal{A}_X = \{a_1, \ldots, a_I\}$.
A function $c : \mathcal{A}_X \to \{0,1\}^+$ is a **code** for X.

- The binary string $c(x)$ is the **codeword** for $x \in \mathcal{A}_X$
- The **length** of the codeword for for x is denoted $\ell(x)$.
  Shorthand: $\ell_i = \ell(\boldsymbol{c}_i)$ for $i = 1 \ldots, I$.
- The **extension** of c assigns codewords to any sequence $x_1 x_2 \ldots x_N$ from $\mathcal{A}^+$ by $c(x_1 \ldots x_N) = c(x_1) \ldots c(x_N)$

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}\}$

Example 1 (Uniform Code):

- Let $c(\texttt{a}) = 0001$, $c(\texttt{b}) = 0010$, $c(\texttt{c}) = 0100$, $c(\texttt{d}) = 1000$

Codes: A Review

Examples

$$X \text{ is an ensemble with } \mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 000100100001

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 000100100001

Example 2 (Variable-Length Code):

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 000100100001

Example 2 (Variable-Length Code):

- Let $c(\mathtt{a}) = 0$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 110$, $c(\mathtt{d}) = 111$

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 000100100001

Example 2 (Variable-Length Code):

- Let $c(\mathtt{a}) = 0$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 110$, $c(\mathtt{d}) = 111$
- Shorthand: $C_2 = \{0, 10, 110, 111\}$

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 000100100001

Example 2 (Variable-Length Code):

- Let $c(\mathtt{a}) = 0$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 110$, $c(\mathtt{d}) = 111$
- Shorthand: $C_2 = \{0, 10, 110, 111\}$
- In this case $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = \ell_4 = 3$

Codes: A Review

Examples

X is an ensemble with $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$
- Shorthand: $C_1 = \{0001, 0010, 0100, 1000\}$
- All codewords have *length* 4. That is, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 4$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 000100100001

Example 2 (Variable-Length Code):

- Let $c(\mathtt{a}) = 0$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 110$, $c(\mathtt{d}) = 111$
- Shorthand: $C_2 = \{0, 10, 110, 111\}$
- In this case $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = \ell_4 = 3$
- The *extension* of c maps $\mathtt{aba} \in \mathcal{A}_X^3 \subset \mathcal{A}_X^+$ to 0100

Unique Decodeability

Recall that a code is lossless if for all $x, y \in \mathcal{A}_X$

$$x \neq y \implies c(x) \neq c(y)$$

This ensures that if we work with a single outcome, we can uniquely decode the outcome

When working with variable-length codes, it will be convenient to also require the following:

Unique Decodeability

Recall that a code is lossless if for all $x, y \in \mathcal{A}_X$

$$x \neq y \implies c(x) \neq c(y)$$

This ensures that if we work with a single outcome, we can uniquely decode the outcome

When working with variable-length codes, it will be convenient to also require the following:

Uniquely Decodable

A code c for X is **uniquely decodable** if no two strings from $\mathcal{A}_X^+$ have the same codeword. That is, for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}_X^+$

$$\mathbf{x} \neq \mathbf{y} \implies c(\mathbf{x}) \neq c(\mathbf{y})$$

This ensures that if we work with a sequence of outcomes, we can still uniquely decode the individual elements

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable
- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because

$$c(\texttt{aaa}) = c(\texttt{d}) = 111 \quad \text{and} \quad c(\texttt{ab}) = c(\texttt{c}) = 110$$

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable
- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because

$$c(\texttt{aaa}) = c(\texttt{d}) = 111 \quad \text{and} \quad c(\texttt{ab}) = c(\texttt{c}) = 110$$

  - The code is of course lossless

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable
- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because

$$c(\mathtt{aaa}) = c(\mathtt{d}) = 111 \quad \text{and} \quad c(\mathtt{ab}) = c(\mathtt{c}) = 110$$

  - The code is of course lossless
  - Lossless $\not\Rightarrow$ Uniquely decodable

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable
- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because

$$c(\texttt{aaa}) = c(\texttt{d}) = 111 \quad \text{and} \quad c(\texttt{ab}) = c(\texttt{c}) = 110$$

  - The code is of course lossless
  - Lossless $\not\Rightarrow$ Uniquely decodable
- $C_3 = \{0, 10, 110, 111\}$ is uniquely decodable

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable
- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because

$$c(\texttt{aaa}) = c(\texttt{d}) = 111 \quad \text{and} \quad c(\texttt{ab}) = c(\texttt{c}) = 110$$

  - The code is of course lossless
  - Lossless $\not\Rightarrow$ Uniquely decodable
- $C_3 = \{0, 10, 110, 111\}$ is uniquely decodable
  - We can easily segment a given code string scanning left to right

Examples of uniquely decodable codes

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is uniquely decodable
  - Uniform + Lossless $\Rightarrow$ Uniquely decodable
- $C_2 = \{1, 10, 110, 111\}$ is not uniquely decodable because

$$c(\texttt{aaa}) = c(\texttt{d}) = 111 \quad \text{and} \quad c(\texttt{ab}) = c(\texttt{c}) = 110$$

  - The code is of course lossless
  - Lossless $\not\Rightarrow$ Uniquely decodable
- $C_3 = \{0, 10, 110, 111\}$ is uniquely decodable
  - We can easily segment a given code string scanning left to right
  - e.g. $0110010 \to 0, 110, 0, 10$

“Self-punctuating” property

The code $C_3 = \{0, 10, 110, 111\}$ has a “self-punctuating” property

“Self-punctuating” property

The code $C_3 = \{0, 10, 110, 111\}$ has a “self-punctuating” property

Trivial to segment a given code string into individual codewords

- Keep scanning until we match a codeword
- Once matched, add new segment boundary, and proceed to rest of string

“Self-punctuating” property

The code $C_3 = \{0, 10, 110, 111\}$ has a “self-punctuating” property

Trivial to segment a given code string into individual codewords

- Keep scanning until we match a codeword
- Once matched, add new segment boundary, and proceed to rest of string

Once our current segment matches a codeword, no ambiguity to resolve

- Why? No codeword is a prefix of any other

Not true for every uniquely decodable code, e.g. $C_4 = \{0, 01, 011\}$

- First bit 0 $\rightarrow$ no certainty what the symbol is

Prefix Codes

a.k.a *prefix-free* or *instantaneous* codes

A simple property of codes **guarantees** unique decodeability

Prefix property

A codeword $\mathbf{c} \in \{0,1\}^+$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0,1\}^+$ if there exists a string $\mathbf{t} \in \{0,1\}^+$ such that $\mathbf{c}' = \mathbf{ct}$.

Prefix Codes

a.k.a *prefix-free* or *instantaneous* codes

A simple property of codes **guarantees** unique decodeability

Prefix property

A codeword $\mathbf{c} \in \{0, 1\}^+$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0, 1\}^+$ if there exists a string $\mathbf{t} \in \{0, 1\}^+$ such that $\mathbf{c}' = \mathbf{ct}$.

Can you create $\mathbf{c}'$ by gluing something to the end of $\mathbf{c}$?

- **Example**: 01101 has prefixes 0, 01, 011, 0110.

Prefix Codes

a.k.a *prefix-free* or *instantaneous* codes

A simple property of codes **guarantees** unique decodeability

Prefix property

A codeword $\mathbf{c} \in \{0, 1\}^+$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0, 1\}^+$ if there exists a string $\mathbf{t} \in \{0, 1\}^+$ such that $\mathbf{c}' = \mathbf{ct}$.

Can you create $\mathbf{c}'$ by gluing something to the end of $\mathbf{c}$?

- **Example**: 01101 has prefixes 0, 01, 011, 0110.

Prefix Codes

A code $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_I\}$ is a **prefix code** if for every codeword $\mathbf{c}_i \in C$ there is no prefix of $\mathbf{c}_i$ in C.

In a stream, no confusing one codeword with another

Prefix Codes: Examples

Examples:

- $C_1 = \{0001, 0010, 0100, 1000\}$ is prefix-free
- $C_2 = \{0, 10, 110, 111\}$ is prefix-free
- $C_2' = \{1, 10, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 c_2$
- $C_2'' = \{1, 01, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 10$

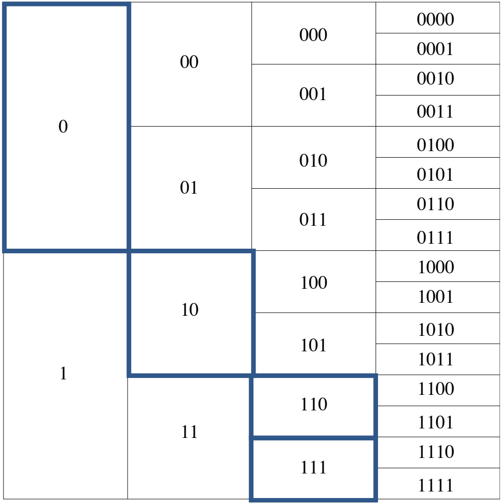
Prefix Codes as Trees

$$C_1 = \{0001, 0010, 0100, 1000\}$$

0	00	000	0000
			$\boxed{0001}$
		001	$\boxed{0010}$
			0011
	01	010	$\boxed{0100}$
			0101
		011	0110
			0111
1	10	100	$\boxed{1000}$
			1001
		101	1010
			1011
	11	110	1100
			1101
		111	1110
			1111

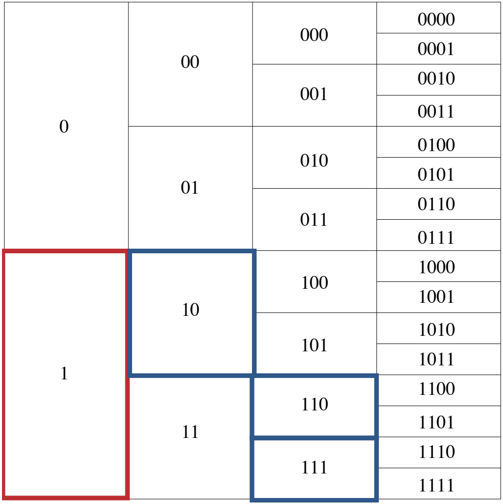
Prefix Codes as Trees

$$C_2 = \{0, 10, 110, 111\}$$



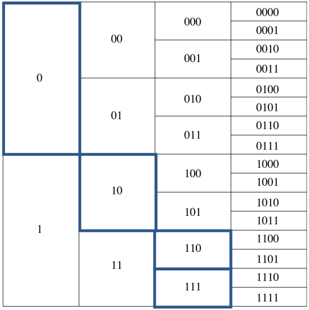
Prefix Codes as Trees

$$C_2' = \{1, 10, 110, 111\}$$



Each codeword choice eliminates its descendants

Prefix Codes are Uniquely Decodeable



- If $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$ then symbol is decodeable after seeing at most ℓ^* bits
- Consider $C_2 = \{0, 10, 110, 111\}$
  - If $c(\mathbf{x}) = 0 \ldots$ then $x_1 = \mathtt{a}$
  - If $c(\mathbf{x}) = 1 \ldots$ then $x_1 \in \{\mathtt{b}, \mathtt{c}, \mathtt{d}\}$
  - If $c(\mathbf{x}) = 10 \ldots$ then $x_1 = \mathtt{b}$
  - If $c(\mathbf{x}) = 11 \ldots$ then $x_1 \in \{\mathtt{c}, \mathtt{d}\}$

Uniquely Decodeable Codes are Not Always Prefix Codes

A uniquely decodeable code is not necessarily a prefix code

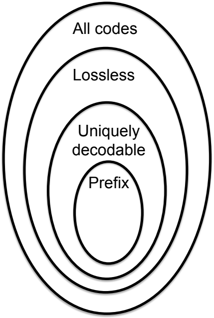
Example: $C_1 = \{0, 01, 011\}$

- $00\ldots \to$ first codeword
- $010\ldots \to$ second codeword
- $011\ldots \to$ third codeword

Example: $C_2 = \{0, 01, 011, 111\}$

- This is the reverse of the prefix code $C_2' = \{0, 10, 110, 111\}$

Relating various types of codes



Note that e.g.

$$\text{Prefix} \implies \text{Uniquely Decodable}$$

but

$$\text{Not Prefix} \not\Longrightarrow \text{Not Uniquely Decodable}$$

Why prefix codes?

While prefix codes do not represent **all** uniquely decodable codes, they are convenient to work with

It will be easy to generate prefix codes (Huffman coding, next lecture)

Further, we can quickly establish if a given code is **not** prefix

- Testing for unique decodability is non-trivial in general

Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

- $L_1 = \{4, 4, 4, 4\}$
- $L_2 = \{1, 2, 3, 3\}$
- $L_3 = \{2, 2, 3, 4, 4\}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$

<table>
<tr><td rowspan="8">0</td><td rowspan="4">00</td><td rowspan="2">000</td><td>0000</td></tr>
<tr><td>0001</td></tr>
<tr><td rowspan="2">001</td><td>0010</td></tr>
<tr><td>0011</td></tr>
<tr><td rowspan="4">01</td><td rowspan="2">010</td><td>0100</td></tr>
<tr><td>0101</td></tr>
<tr><td rowspan="2">011</td><td>0110</td></tr>
<tr><td>0111</td></tr>
<tr><td rowspan="8">1</td><td rowspan="4">10</td><td rowspan="2">100</td><td>1000</td></tr>
<tr><td>1001</td></tr>
<tr><td rowspan="2">101</td><td>1010</td></tr>
<tr><td>1011</td></tr>
<tr><td rowspan="4">11</td><td rowspan="2">110</td><td>1100</td></tr>
<tr><td>1101</td></tr>
<tr><td rowspan="2">111</td><td>1110</td></tr>
<tr><td>1111</td></tr>
</table>

Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

- $L_1 = \{4,4,4,4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1,2,3,3\}$
- $L_3 = \{2,2,3,4,4\}$
- $L_4 = \{1,3,3,3,3,4\}$

<table>
<tr><td rowspan="8">0</td><td rowspan="4">00</td><td rowspan="2">000</td><td>0000</td></tr>
<tr><td>0001</td></tr>
<tr><td rowspan="2">001</td><td>0010</td></tr>
<tr><td>0011</td></tr>
<tr><td rowspan="4">01</td><td rowspan="2">010</td><td>0100</td></tr>
<tr><td>0101</td></tr>
<tr><td rowspan="2">011</td><td>0110</td></tr>
<tr><td>0111</td></tr>
<tr><td rowspan="8">1</td><td rowspan="4">10</td><td rowspan="2">100</td><td>1000</td></tr>
<tr><td>1001</td></tr>
<tr><td rowspan="2">101</td><td>1010</td></tr>
<tr><td>1011</td></tr>
<tr><td rowspan="4">11</td><td rowspan="2">110</td><td>1100</td></tr>
<tr><td>1101</td></tr>
<tr><td rowspan="2">111</td><td>1110</td></tr>
<tr><td>1111</td></tr>
</table>

Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$

<table>
<tr><td rowspan="8">0</td><td rowspan="4">00</td><td rowspan="2">000</td><td>0000</td></tr>
<tr><td>0001</td></tr>
<tr><td rowspan="2">001</td><td>0010</td></tr>
<tr><td>0011</td></tr>
<tr><td rowspan="4">01</td><td rowspan="2">010</td><td>0100</td></tr>
<tr><td>0101</td></tr>
<tr><td rowspan="2">011</td><td>0110</td></tr>
<tr><td>0111</td></tr>
<tr><td rowspan="8">1</td><td rowspan="4">10</td><td rowspan="2">100</td><td>1000</td></tr>
<tr><td>1001</td></tr>
<tr><td rowspan="2">101</td><td>1010</td></tr>
<tr><td>1011</td></tr>
<tr><td rowspan="4">11</td><td rowspan="2">110</td><td>1100</td></tr>
<tr><td>1101</td></tr>
<tr><td rowspan="2">111</td><td>1110</td></tr>
<tr><td>1111</td></tr>
</table>

Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

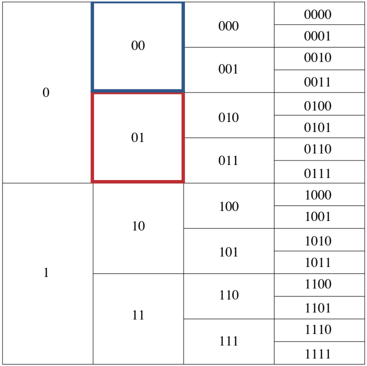
- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$ — $C_3 = \{00, \quad , \quad , \quad , \quad \}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$

<table>
<tr><td rowspan="8">0</td><td rowspan="4">00</td><td rowspan="2">000</td><td>0000</td></tr>
<tr><td>0001</td></tr>
<tr><td rowspan="2">001</td><td>0010</td></tr>
<tr><td>0011</td></tr>
<tr><td rowspan="4">01</td><td rowspan="2">010</td><td>0100</td></tr>
<tr><td>0101</td></tr>
<tr><td rowspan="2">011</td><td>0110</td></tr>
<tr><td>0111</td></tr>
<tr><td rowspan="8">1</td><td rowspan="4">10</td><td rowspan="2">100</td><td>1000</td></tr>
<tr><td>1001</td></tr>
<tr><td rowspan="2">101</td><td>1010</td></tr>
<tr><td>1011</td></tr>
<tr><td rowspan="4">11</td><td rowspan="2">110</td><td>1100</td></tr>
<tr><td>1101</td></tr>
<tr><td rowspan="2">111</td><td>1110</td></tr>
<tr><td>1111</td></tr>
</table>

Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

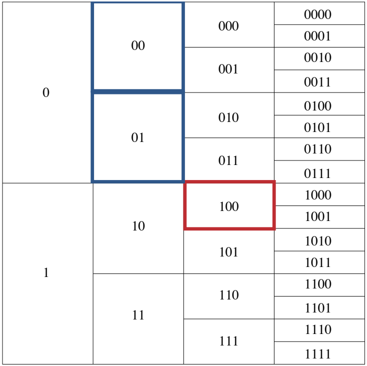
- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$ — $C_3 = \{00, 01, \quad , \quad , \quad \}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$



Lengths and Trees

Suppose someone said "I want prefix codes with codewords lengths":

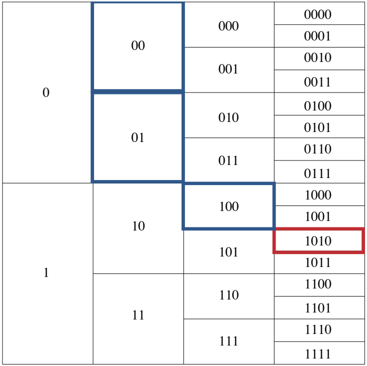
- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$ — $C_3 = \{00, 01, 100, \qquad , \qquad \}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$



Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

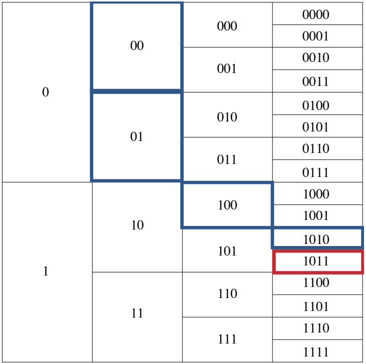
- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$ — $C_3 = \{00, 01, 100, 1010, \qquad \}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$



Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

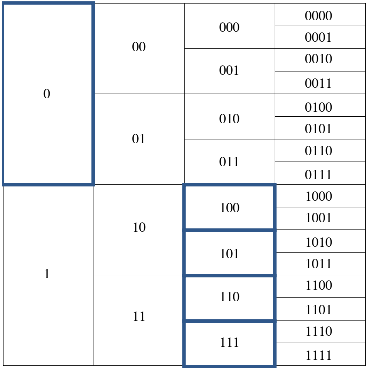
- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$ — $C_3 = \{00, 01, 100, 1010, 1011\}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$



Lengths and Trees

Suppose someone said “I want prefix codes with codewords lengths”:

- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
- $L_2 = \{1, 2, 3, 3\}$ — $C_2 = \{0, 10, 110, 111\}$
- $L_3 = \{2, 2, 3, 4, 4\}$ — $C_3 = \{00, 01, 100, 1010, 1011\}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$ — Impossible!



The Kraft Inequality

a.k.a. The Kraft-McMillan Inequality

Kraft Inequality

For any prefix (binary) code C, its codeword lengths $\{\ell_1, \dots, \ell_I\}$ satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1 \tag{1}$$

Conversely, if the set $\{\ell_1, \dots, \ell_I\}$ satisfy (1) then there exists a prefix code C with those codeword lengths.

The Kraft Inequality

a.k.a. The Kraft-McMillan Inequality

Kraft Inequality

For any prefix (binary) code C, its codeword lengths $\{\ell_1, \ldots, \ell_I\}$ satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1 \tag{1}$$

Conversely, if the set $\{\ell_1, \ldots, \ell_I\}$ satisfy (1) then there exists a prefix code C with those codeword lengths.

Examples:

1. $C_1 = \{0001, 0010, 0100, 1000\}$ is prefix and $\sum_{i=1}^{4} 2^{-4} = \frac{1}{4} \leq 1$

The Kraft Inequality

a.k.a. The Kraft-McMillan Inequality

Kraft Inequality

For any prefix (binary) code C, its codeword lengths $\{\ell_1, \ldots, \ell_I\}$ satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1 \tag{1}$$

Conversely, if the set $\{\ell_1, \ldots, \ell_I\}$ satisfy (1) then there exists a prefix code C with those codeword lengths.

Examples:

1. $C_1 = \{0001, 0010, 0100, 1000\}$ is prefix and $\sum_{i=1}^{4} 2^{-4} = \frac{1}{4} \leq 1$
2. $C_2 = \{0, 10, 110, 111\}$ is prefix and $\sum_{i=1}^{4} 2^{-\ell_i} = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} = 1$

The Kraft Inequality

a.k.a. The Kraft-McMillan Inequality

Kraft Inequality

For any prefix (binary) code C, its codeword lengths $\{\ell_1, \ldots, \ell_I\}$ satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1 \tag{1}$$

Conversely, if the set $\{\ell_1, \ldots, \ell_I\}$ satisfy (1) then there exists a prefix code C with those codeword lengths.

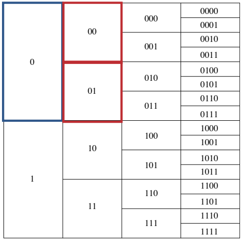
Examples:

1. $C_1 = \{0001, 0010, 0100, 1000\}$ is prefix and $\sum_{i=1}^{4} 2^{-4} = \frac{1}{4} \leq 1$
2. $C_2 = \{0, 10, 110, 111\}$ is prefix and $\sum_{i=1}^{4} 2^{-\ell_i} = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} = 1$
3. Lengths $\{1, 2, 2, 3\}$ give $\sum_{i=1}^{4} 2^{-\ell_i} = \frac{1}{2} + \frac{2}{4} + \frac{1}{8} > 1$ so no prefix code

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:

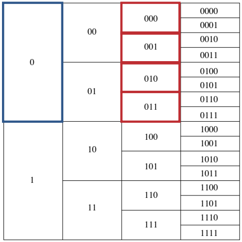


- 2 x 2-bit codewords: $\{00, 01\}$

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:

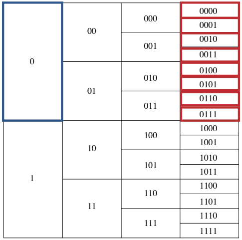


- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:

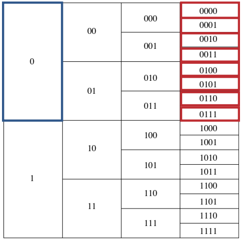


- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:

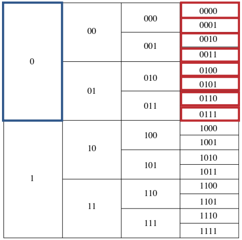


- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$
- In general, an ℓ-bit codeword excludes $2^{k-\ell}$ x k-bit codewords

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:

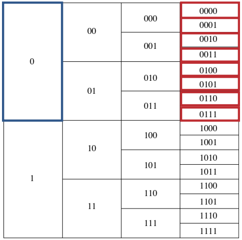


- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$
- In general, an ℓ-bit codeword excludes $2^{k-\ell}$ x k-bit codewords

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:



- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$
- In general, an ℓ-bit codeword excludes $2^{k-\ell}$ x k-bit codewords

For lengths $L = \{\ell_1, \ldots, \ell_I\}$ and $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$, there will be

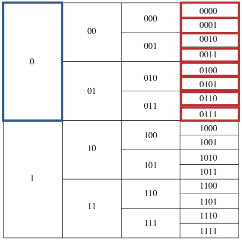
$$\sum_{i=1}^{I} 2^{\ell^* - \ell_i}$$

excluded ℓ^*-bit codewords.

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathrm{a}) = 0$ — excludes:



- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$
- In general, an ℓ-bit codeword excludes $2^{k-\ell}$ x k-bit codewords

For lengths $L = \{\ell_1, \ldots, \ell_I\}$ and $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$, there will be

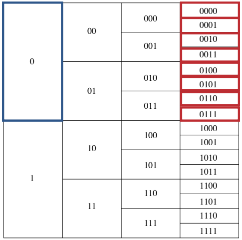
$$\sum_{i=1}^{I} 2^{\ell^* - \ell_i} \leq 2^{\ell^*}$$

excluded ℓ^*-bit codewords. But there are only 2^{ℓ^*} possible ℓ^*-bit codewords

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathtt{a}) = 0$ — excludes:



- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$
- In general, an ℓ-bit codeword excludes $2^{k-\ell}$ x k-bit codewords

For lengths $L = \{\ell_1, \ldots, \ell_I\}$ and $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$, there will be

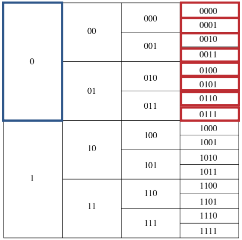
$$\frac{1}{2^{\ell^*}} \sum_{i=1}^{I} 2^{\ell^* - \ell_i} \leq 1$$

excluded ℓ^*-bit codewords. But there are only 2^{ℓ^*} possible ℓ^*-bit codewords

Prefixes Exclude Codes

We are constrained when constructing prefix codes, as selecting a codeword eliminates a whole subtree

Choosing a prefix codeword of length 1 — e.g., $c(\mathrm{a}) = 0$ — excludes:



- 2 x 2-bit codewords: $\{00, 01\}$
- 4 x 3-bit codewords: $\{000, 001, 010, 011\}$
- 8 x 4-bit codewords: $\{0000, 0001, \ldots, 0111\}$
- In general, an ℓ-bit codeword excludes $2^{k-\ell}$ x k-bit codewords

For lengths $L = \{\ell_1, \ldots, \ell_I\}$ and $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$, there will be

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

excluded ℓ^*-bit codewords. But there are only 2^{ℓ^*} possible ℓ^*-bit codewords

Kraft inequality: other direction

Suppose we are given lengths satisfying

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

Then, we can construct a code by:

- Picking the first (remaining) node at depth ℓ_1, and using it as the first codeword
- Removing all descendants of the node (to ensure the prefix condition)
- Picking the next (remaining) node at depth ℓ_2, and using it as the second codeword
- Removing all descendants of the node (to ensure the prefix condition)
- $\vdots$

Kraft inequality: comments

Kraft's inequality actually holds more generally for uniquely decodable codes

- Harder to prove

Note that if a **given** code has lengths that satisfy

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

it does not mean the **given** code necessarily is prefix

- Just that we can **construct** a prefix code with these lengths

Summary

Key ideas from this lecture:

- **Prefix** and **Uniquely Decodeable** variable-length codes
- Prefix codes are tree-like
- Every Prefix code is Uniquely Decodeable but not *vice versa*
- The **Kraft Inequality**:
  - Code lengths satisfying $\sum_i 2^{-\ell_i} \leq 1$ implies Prefix/U.D. code exists
  - Prefix/U.D. code implies $\sum_i 2^{-\ell_i} \leq 1$

Relevant Reading Material:

- MacKay: §5.1 and §5.2
- Cover & Thomas: §5.1, §5.2, and §5.5