COMP2610/COMP6261 – Information Theory

Tutorial 5: Fundamental and Probabilistic

inequalities (Week 5, Semester 2, 2023)

1. Let X, Y and Z be joint random variables. Prove the following inequalities and find conditions for equality.

   a) $H(X, Y | Z) \geq H(X|Z)$.

   b) $I(X, Y; Z) \geq I(X; Z)$.

   c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

   d) $I(X; Z|Y) = I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

2. Which of the following inequalities are generally ≥, =, ≤? Label each with ≥, =, or ≤.

   a) $H(5X)$ vs. $H(X)$

   b) $I(g(X); Y)$ vs. $I(X; Y)$

   c) $H(X_0|X_{-1})$ vs. $H(X_0|X_{-1}, X_1)$

   d) $H(X, Y) / (H(X) + H(Y))$ vs. 1

3. (Markov's inequality for probabilities) Let p(x) be a probability mass function.

   Prove, for all $\mathbf{d \geq 0}$, $\mathbf{Pr\{p(X) \leq d\} \log\left(\frac{1}{d}\right) \leq H(X).}$

4. (Asymptotic Equipartition Property) A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary code word is provided for every sequence of 100 digits containing three or fewer ones.

   (a) Assuming that all code words are the same length, find the minimum length require to provide code words for all sequences with three or fewer ones.

   (b) Calculate the probability of observing a source sequence for which no code word has been assigned.

   (c) Use Chebyshev's inequality to bound the probability of observing a source sequence for which no codeword has been assigned. Compare this bound with the actual probability computed in part (b).