The Australian National University Semester 2, 2023
School of Computing Self-assessment
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COMP3670: Introduction to Machine Learning

Problem 1: Matrix addition and Multiplication

(1pt) We have three matrices: $\boldsymbol{A} \in \mathbb{R}^{3\times 2}$, *i.e.*, real-valued 3 by 2 matrix; $\boldsymbol{B} \in \mathbb{R}^{2\times 1}$; $\boldsymbol{C} \in \mathbb{R}^{3\times 1}$.

$\boldsymbol{A} = \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ 1 & -3 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$. Calculate $\boldsymbol{AB}+\boldsymbol{C}$.

Solution.

$$\boldsymbol{AB}+\boldsymbol{C} = \begin{bmatrix} 9 \\ 9 \\ 6 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$$

Problem 2: Gaussian Elimination for System of Linear Equations

(2 pts) Solve the following system of linear equations. You can use any method you know of, such as intuitively solving it, or using the constructive Gaussian Elimination method.

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ x_2 + 2x_3 = 2 \end{cases}$$

Solution.

From the second equation

$$x_2 = 2 - 2x_3$$

From the first equation, inserting the second:

$$x_1 + (2 - 2x_3) + x_3 = 8$$

$$x_1 = 6 + x_3$$

Note that x_3 is a free variable. So the solution set is

$$\left\{ \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R} \right\}$$

Can also be solved via gaussian elimination.

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 2 & 2 \end{array}\right] \\ \sim &\left[\begin{array}{ccc|c} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & 2 \end{array}\right] (R_1 := R_1 - R_2) \end{aligned}$$

At which point you can read off the solutions

$$\begin{aligned} x_1 - x_3 &= 6 \\ x_2 + 2x_3 &= 2 \end{aligned}$$

and hence

$$\begin{aligned} x_1 &= 6 + x_3 \\ x_2 &= 2 - 2x_3 \end{aligned}$$

Note x_3 is free, and we get the same solution obtained above.

Problem 3: Group

(1pt) Consider the set $\{1, -1\}$ together with the operation multiplication (*i.e.,* $\times$). Is this set a Group? Please explain.

Solution. Yes, it's a group.

Closure: Multiplying any of $\{-1, 1\}$ results in either -1 or 1.

Associativity: Trivial, as multiplication is an associative operator.

Identity: 1 is the identity, as anything multiplied by 1 is 1.

Inverse: Every element is it's own inverse.

Problem 4: Abelian Group

(2pt) Determine if the set of matrices of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where θ is a real number, forms an Abelian Group under matrix multiplication.

Solution. The matrix in the question is also called a **2D rotation matrix**. Abelian Group requires an additional property: commutativity. We prove by:

Closure: Consider arbitrary matrices $\mathbf{A} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$ where $\theta_1, \theta_2 \in \mathbb{R}$.

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \end{aligned}$$

To prove this more rigorously, since $\mathbb{R}$ is a group under addition, and $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 + \theta_2$ is also in $\mathbb{R}$ due to the property of closure in $\mathbb{R}$. Thus, **AB** is also in the group, the closure property is satisfied.

Associativity: Consider three arbitrary matrices $\mathbf{A} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix}$ where $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$.

According to what has been proved above,

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2+\theta_3) & -\sin(\theta_2+\theta_3) \\ \sin(\theta_2+\theta_3) & \cos(\theta_2+\theta_3) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) \end{bmatrix} = (\mathbf{AB})\mathbf{C} \end{aligned}$$

So associativity is also satisfied.

Identity: The identity is the 2×2 Identity matrix, as any 2×2 real matrices multiply the identity matrix is itself.

Inverse: For arbitrary matrix $\mathbf{A} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \theta_1 \in \mathbb{R}$, its inverse is always valid, and can be calculated as

$$\mathbf{A}^{-1} = \frac{1}{\cos^2\theta_1 + \sin^2\theta_1} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

by applying the rule $\forall\theta \in \mathbb{R}, \cos^2\theta + \sin^2\theta = 1$, and the inverse formula for 2×2 matrices $\mathbf{X}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for arbitrary real invertible matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

To tell if the inverse is an element in the group, we need to write $\mathbf{A}^{-1}$ as the form where the position of the negative sign is appended to the first sin entry. If we perform a changed of variable, and let $-\theta_2 \in \mathbb{R} = \theta_1$,

$$\mathbf{A}^{-1} = \begin{bmatrix} \cos(-\theta_2) & \sin(-\theta_2) \\ -\sin(-\theta_2) & \cos(-\theta_2) \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Since $\mathbb{R}$ is a group under addition, the inverse element of $-\theta_2$, θ_2 is also in $\mathbb{R}$. Hence, we proved the inverse property.

Commutativity: Finally, we prove commutativity. From above, consider arbitrary matrices $\mathbf{A} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$ where $\theta_1, \theta_2 \in \mathbb{R}$. Replicate what has been done above we have

$$\mathbf{AB} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} \cos(\theta_2 + \theta_1) & -\sin(\theta_2 + \theta_1) \\ \sin(\theta_2 + \theta_1) & \cos(\theta_2 + \theta_1) \end{bmatrix} = \mathbf{AB}$$

Thus, the group is commutative. In conclusion, 2D rotation matrices form an Abelian group under matrix multiplication.

Problem 5: properties of matrix transpose

(1pt) For $\boldsymbol{A} \in R^{m\times n}$, $\boldsymbol{B} \in R^{m\times n}$, prove that $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$

Solution. We check the $i-j$th element, and verify they both match.

$$
\begin{aligned}
&(\mathbf{A}+\mathbf{B})^T_{i,j} \\
&= (\mathbf{A}+\mathbf{B})_{j,i} \\
&= \mathbf{A}_{j,i} + \mathbf{B}_{j,i} \\
&= \mathbf{A}^T_{i,j} + \mathbf{B}^T_{i,j} \\
&= (\mathbf{A}^T + \mathbf{B}^T)_{i,j}
\end{aligned}
$$

The above proof works as addition is performed elementwise.

Problem 6: Matrix Inverse

(1pt) Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix}.$$

Solution. We can either use the rule mentioned in the previous question, or we can reduce an augmented matrix.

Using the rule:

$$\mathbf{A}^{-1} = \frac{1}{19} \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix}$$

Reduce the matrix $[\mathbf{A}|\mathbf{I}]$ to $\left[\mathbf{I}|\mathbf{A}^{-1}\right]$.

$$\begin{bmatrix} 4 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 -= 1/4\ R_2} \begin{bmatrix} 4 & 1 & 1 & 0 \\ 0 & 19/4 & -1/4 & 1 \end{bmatrix} \xrightarrow{R_1 /= 4} \begin{bmatrix} 1 & 1/4 & 1/4 & 0 \\ 0 & 19/4 & -1/4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 -= 1/19\ R_2} \begin{bmatrix} 1 & 0 & 5/19 & -1/19 \\ 0 & 19/4 & -1/4 & 1 \end{bmatrix} \xrightarrow{R_2 \times= 4/19} \begin{bmatrix} 1 & 0 & 5/19 & -1/19 \\ 0 & 1 & -1/19 & 4/19 \end{bmatrix}$$

Clearly,

$$\mathbf{A}^{-1} = \frac{1}{19} \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix}$$