COMP 2610/6261: Information Theory

Week 12 Tutorial Questions

Question 1: Suppose X and Y are random variables with the following joint distribution *p (x, y):*

X \ Y	0	1	2
0	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{6}$
1	$\frac{1}{4}$	$\frac{1}{3}$	0

Answer the following questions.

(i) Are X and Y independent? Explain your answer.

(ii) Compute the expected value of X and Y, i.e., $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(iii) Compute the expected value of XY, i.e., $\mathbb{E}[XY]$.

Solution:

(i). No, they are not independent, as can be seen from the fact that $p(X=1,Y=1) = \frac{1}{3} \neq \frac{7}{12} \times \frac{5}{12} = p(X=1)p(Y=1)$.

(ii). $\mathbb{E}[X] = \sum_{x,y} p(x,y)x = \frac{7}{12}$ and $\mathbb{E}[Y] = \sum_{x,y} p(x,y)y = \frac{3}{4}$.

(iii). $\mathbb{E}[XY] = \sum_{x,y} p(x,y)xy = \frac{1}{3}$.

Question 2: Three random variables X, Y, Z are given,

(i) Comparing the term H (Y) + H (X, Y, Z) with the term H (X, Y) + H (Y, Z) in general and determine which terms is greater than or equal to the other.

(ii) What is the condition for random variables X, Y, Z that these two terms are equal to each other, i.e., H(Y) + H(X,Y,Z) = H(X,Y) + H(Y,Z)

Solution:

(i). Since conditioning cannot increase entropy, we have

$$\begin{aligned} H(Y)+H(X,Y,Z)-H(X,Y)-H(Y,Z) &= H(X,Y,Z)-H(X,Y)-(H(Y,Z)-H(Y)) \\ &= H(Z|X,Y)-H(Z|Y) \leq 0. \end{aligned}$$

Thus, $H(Y)+H(X,Y,Z) \leq H(X,Y)+H(Y,Z)$.

(ii). If $H(Y)+H(X,Y,Z) = H(X,Y)+H(Y,Z)$, we have $H(Z|X,Y) = H(Z|Y)$, which means Z is independent with X given Y. Thus, we obtain $X \to Y \to Z$ forms a Markov chain.

Question 3: Consider the Random Variable X and answer the following questions.

X	x_1	x_2	x_3	x_4	x_5
$p(x_i)$	0.4	0.25	0.15	0.1	0.1

(a) What is the entropy H(X)?

(b) Find the binary Huffman code for X.

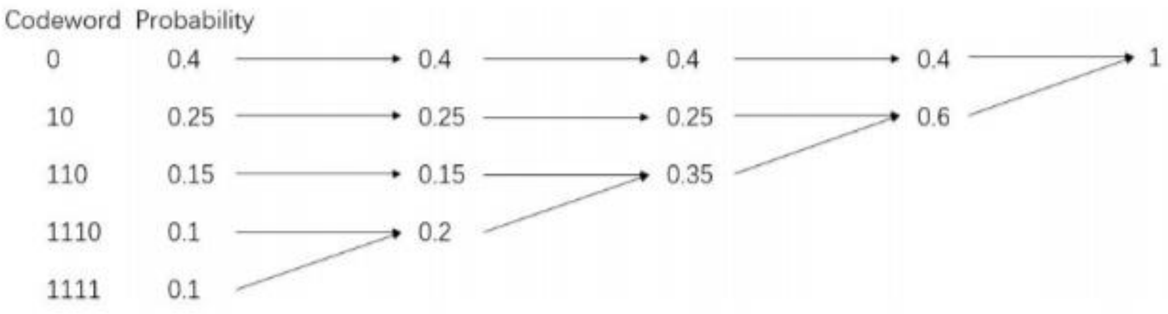
(c) Find the Shannon-Fano-Elias code for X.

Solution:

(a). The entropy is calculated as

$$H(X) = -\sum_{x_i} p(x_i) \log p(x_i)$$
$$= -0.4\log 0.4 - 0.25\log 0.25 - 0.15\log 0.15 - 0.1\log 0.1 - 0.1\log 0.1$$
$$= 2.1037\text{bits}.$$

(b). For the Huffman code, the code constructed by the standard Huffman procedure



(c). For the Shannon-Fano-Elias code, the code is constructed as follows

X_i	p_i	l_i	$\sum p_j + \frac{p_i}{2}$	$\sum p_j + \frac{p_i}{2}$ in binary	Code word
x_1	0.4	3	0.2	0.001	001
x_2	0.25	3	0.525	0.100	100
x_3	0.15	4	0.725	0.1011	1011
x_4	0.1	5	0.85	0.11011	11011
x_5	0.1	5	0.95	0.11110	11110