# COMP2610 / COMP6261 Information Theory
# Lecture 14: Source Coding Theorem for Symbol Codes

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National
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# Last Time

- Variable-length codes
- Uniquely decodable and prefix codes
  - Prefix codes as trees
- Kraft's inequality:
  Lengths $\{\ell_i\}_{i=1}^{I}$ can form a prefix code $\iff \Sigma_{i=1}^{I} 2^{-\ell_i} \leq 1$.
- How to generate prefix codes?

# Prefix Codes (Recap)

A simple property of codes **guarantees** unique decodeability

## Prefix property

A codeword $\mathbf{c} \in \{0,1\}^+$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0,1\}^+$ if there exists a string $\mathbf{t} \in \{0,1\}^+$ such that $\mathbf{c}' = \mathbf{ct}$.

Can you create $\mathbf{c}'$ by gluing something to the end of $\mathbf{c}$?

- **Example**: 01101 has prefixes 0, 01, 011, 0110.

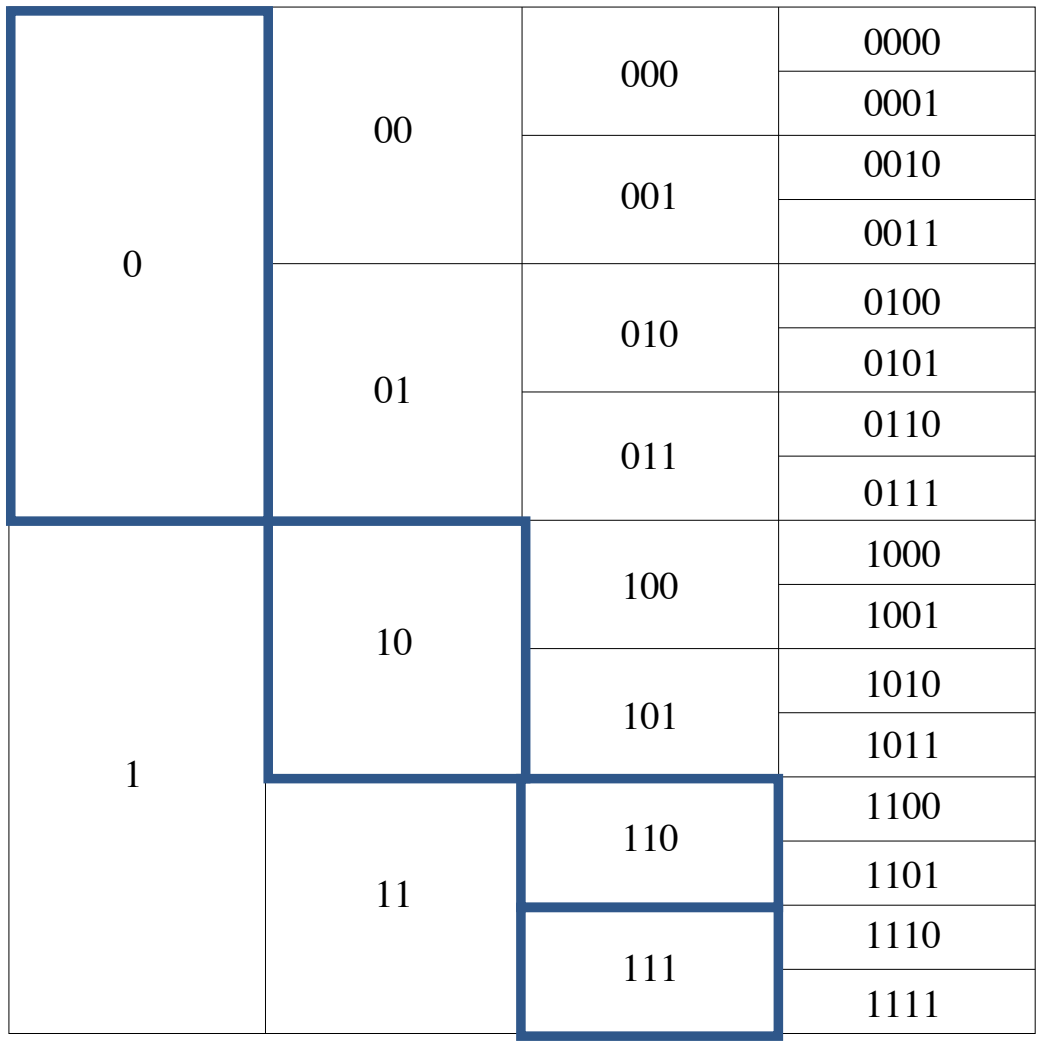
## Prefix Codes

A code $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$ is a **prefix code** if for every codeword $\mathbf{c}_i \in C$ there is no prefix of $\mathbf{c}_i$ in $C$.

In a stream, no confusing one codeword with another

# Prefix Codes as Trees (Recap)

$$C_2 = \{0, 10, 110, 111\}$$



# This time

Bound on expected length for a prefix code

Shannon codes

Huffman coding

1. Expected Code Length
   - Minimising Expected Code Length
   - Shannon Coding
2. The Source Coding Theorem for Symbol Codes
3. Huffman Coding
   - Algorithm and Examples
   - Advantages and Disadvantages

# Expected Code Length

With uniform codes, the length of a message of $N$ outcomes is trivial to compute

With variable-length codes, the length of a message of $N$ outcomes will depend on the outcomes we observe

Outcomes we observe have some uncertainty

- On average, what length of message can we expect?

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## Expected Code Length

The **expected length** for a code $C$ for ensemble $X$ with $\mathcal{A}_X = \{a_1, \ldots, a_I\}$ and $\mathcal{P}_X = \{p_1, \ldots, p_I\}$ is

$$L(C, X) = \mathbb{E}\left[\ell(x)\right] = \sum_{x \in \mathcal{A}_X} p(x)\,\ell(x) = \sum_{i=1}^{I} p_i\,\ell_i$$

# Expected Code Length: Examples

**Example**: $X$ has $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\}$

1. The code $C_1 = \{0001, 0010, 0100, 1000\}$ has

$$L(C_1, X) = \sum_{i=1}^{4} p_i \, \ell_i = 4$$

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1. The code $C_1 = \{0001, 0010, 0100, 1000\}$ has

$$L(C_1, X) = \sum_{i=1}^{4} p_i \ell_i = 4$$

2. The code $C_2 = \{0, 10, 110, 111\}$ has

$$L(C_2, X) = \sum_{i=1}^{4} p_i \ell_i = \tfrac{1}{2} \times 1 + \tfrac{1}{4} \times 2 + \tfrac{1}{8} \times 3 + \tfrac{1}{8} \times 3 = 1.75$$

# Code Lengths and Probabilities

The *Kraft inequality* says that $\{\ell_1, \ldots, \ell_I\}$ are prefix code lengths **iff**

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

If it were true that

$$\sum_{i=1}^{I} 2^{-\ell_i} = 1$$

then we could interpret

$$\mathbf{q} = (2^{-\ell_1}, \ldots, 2^{-\ell_I})$$

as a probability vector over $I$ outcomes

General lengths $\ell$?

# Code Lengths and Probabilities

## Probabilities from Code Lengths

Given code lengths $\ell = \{\ell_1, \ldots, \ell_I\}$ such that $\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$, we define $\mathbf{q} = \{q_1, \ldots, q_I\}$, the **probabilities for** $\ell$, by

$$q_i = \frac{2^{-\ell_i}}{z}$$

where

$$z = \sum_i 2^{-\ell_i}$$

ensure that $q_i$ satisfy $\sum_i q_i = 1$.

Note: this implies $\ell_i = \log_2 \frac{1}{z q_i}$

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**Examples**:

1. Lengths $\{1, 2, 2\}$ give $z = 1$ so $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{4}$, and $q_3 = \frac{1}{4}$

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2. Lengths $\{2, 2, 3\}$ give $z = \frac{5}{8}$ so $q_1 = \frac{2}{5}$, $q_2 = \frac{2}{5}$, and $q_3 = \frac{1}{5}$

# Minimising Expected Code Length

The probability view of lengths will be useful in answering:

## Goal of compression

Given an ensemble $X$ with probabilities $\mathcal{P}_X = \mathbf{p} = \{p_1, \ldots, p_I\}$ how can we minimise the expected code length?

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## Limits of compression

Given an ensemble $X$ with probabilities $\mathbf{p}$, and prefix code $C$ with codeword length probabilities $\mathbf{q}$ and normalisation $z$,

$$\begin{aligned} L(C, X) &= H(X) + D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) + \log_2 \frac{1}{z} \\ &\geq H(X), \end{aligned}$$

with equality only when $\ell_i = \log_2 \frac{1}{p_i}$.

# Minimising Expected Code Length

Suppose we use code $C$ with lengths $\ell = \{\ell_1, \ldots, \ell_I\}$ and corresponding probabilities $\mathbf{q} = \{q_1, \ldots, q_I\}$ with $q_i = \frac{1}{z} 2^{-\ell_i}$. Then,

$$L(C, X) = \sum_i p_i \ell_i$$

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&= \sum_i p_i \left[ \log_2 \left( \frac{1}{p_i} \right) + \log_2 \left( \frac{p_i}{q_i} \right) + \log_2 \left( \frac{1}{z} \right) \right]
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&= \sum_i p_i \log_2 \frac{1}{p_i} + \sum_i p_i \log_2 \frac{p_i}{q_i} + \log_2 \left( \frac{1}{z} \right) \sum_i p_i \\
&= \quad H(X) \quad + \quad D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) \quad + \quad \log_2(1/z) \quad \cdot \quad 1
\end{aligned}
$$

# Minimising Expected Code Length

So if $\mathbf{q} = \{q_1, \ldots, q_l\}$ are the probabilities for the code lengths of $C$ then under ensemble $X$ with probabilities $\mathbf{p} = \{p_1, \ldots, p_l\}$

$$L(C, X) = H(X) + D_{\mathrm{KL}}(p \| q) + \log_2 \frac{1}{z}$$

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Thus, $L(C, X)$ is minimal (and equal to the entropy $H(X)$) if we can choose code lengths so that $D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) = 0$ and $\log_2 \frac{1}{z} = 0$

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But the *relative entropy* $D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) \geq 0$ with $D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) = 0$ iff $\mathbf{q} = \mathbf{p}$ (Gibb's inequality)

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For $\mathbf{q} = \mathbf{p}$, we have $z \stackrel{\text{def}}{=} \sum_i q_i = \sum_i p_i = 1$ and so $\log_2 \frac{1}{z} = 0$

# Entropy as a Lower Bound on Expected Length

We have shown that for a code $C$ with lengths corresponding to $\mathbf{q}$,

$$L(C, X) \geq H(X)$$

with equality only when $C$ has code lengths $\ell_i = \log_2 \frac{1}{p_i}$

Once again, the entropy determines a lower bound on how much compression is possible

- $L(C, X)$ refers to *average* compression
- Individual message length could be bigger than the entropy

# Shannon Codes

If we pick lengths $\ell_i = \log_2 \frac{1}{p_i}$, we get optimal expected code lengths

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## Shannon Code

Given an ensemble $X$ with $\mathcal{P}_X = \{p_1, \ldots, p_I\}$ define codelengths $\ell = \{\ell_1, \ldots, \ell_I\}$ by

$$\ell_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil \geq \log_2 \frac{1}{p_i}.$$

A code $C$ is called a **Shannon code** if it has codelengths $\ell$.

Here $\lceil x \rceil$ is "smallest integer not smaller than $x$". e.g., $\lceil 2.1 \rceil = 3$, $\lceil 5 \rceil = 5$.

This gives us code lengths that are "closest" to $\log_2 \frac{1}{p_i}$

# Shannon Codes: Examples

**Examples**:

1. If $\mathcal{P}_X = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$ then $\ell = \{1, 2, 2\}$ so $C = \{0, 10, 11\}$ is a Shannon code (in fact, this code has *optimal* length)

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2. If $\mathcal{P}_X = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ then $\ell = \{2, 2, 2\}$ with Shannon code $C = \{00, 10, 11\}$ (or $C = \{01, 10, 11\} \ldots$ )

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Shannon codes let us prove the following:

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For any ensemble $X$, there exists a prefix code $C$ such that

$$H(X) \leq L(C, X) < H(X) + 1.$$

In particular, **Shannon codes** $C$ — those with lengths $\ell_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil$ — have *expected code length within 1 bit of the entropy*.

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Entropy also gives a guideline upper bound of compression

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$$\begin{aligned} L(C, X) = \sum_i p_i \ell_i &< \sum_i p_i \log_2 \frac{1}{p_i} + 1 = \sum_i p_i \log_2 \frac{1}{p_i} + \sum_i p_i \\ &= H(X) + 1 \end{aligned}$$

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Furthermore, since $\ell_i \geq -\log_2 p_i$ we have $2^{-\ell_i} \leq 2^{\log_2 p_i} = p_i$, so $\sum_i 2^{-\ell_i} \leq \sum_i p_i = 1$. By Kraft there is a *prefix code* with lengths $\ell_i$

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$$L(C, X) = \sum_i p_i \ell_i < \sum_i p_i \log_2 \frac{1}{p_i} + 1 = \sum_i p_i \log_2 \frac{1}{p_i} + \sum_i p_i$$
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2. If $\mathcal{P}_X = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ then $\ell = \{2, 2, 2\}$ and $H(X) = \log_2 3 \approx 1.58 \leq L(C, X) = 2 \leq 2.58 \approx H(X) + 1$

# The Source Coding Theorem for Symbol Codes

The previous arguments have established:

## Source Coding Theorem for Symbol Codes

For any ensemble $X$ there exists a *prefix code* $C$ such that

$$H(X) \leq L(C, X) < H(X) + 1.$$

In particular, **Shannon codes** $C$ — those with lengths $\ell_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil$ — have *expected code length within 1 bit of the entropy*.

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This is good, but is it **optimal**?

# Shannon codes are suboptimal

**Example**: Consider $p_1 = 0.0001$ and $p_2 = 0.9999$. (Note $H(X) \approx 0.0013$)

- The Shannon code $C$ has lengths $\ell_1 = \lceil \log_2 10000 \rceil = 14$ and $\ell_2 = \left\lceil \log_2 \frac{10000}{9999} \right\rceil = 1$
- The expected length is $L(C, X) = 14 \times 0.0001 + 1 \times 0.9999 = 1.0013$
- But clearly $C' = \{0, 1\}$ is a prefix code and $L(C', X) = 1$

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Shannon codes do not necessarily have **smallest** expected length

This is perhaps disappointing, as these codes were constructed very naturally from the theorem

- Fortunately, there is another simple code that **is** provably optimal

1. Expected Code Length
   - Minimising Expected Code Length
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# Constructing a Huffman Code

**Huffman Coding** is a procedure for making provably optimal prefix codes

It assigns the longest codewords to least probable symbols

Basic algorithm:

- Take the two least probable symbols in the alphabet
- Prepend bits 0 and 1 to current codewords of symbols
- Combine these two symbols into a single “meta-symbol”
- Repeat

# Huffman Coding: Example 1

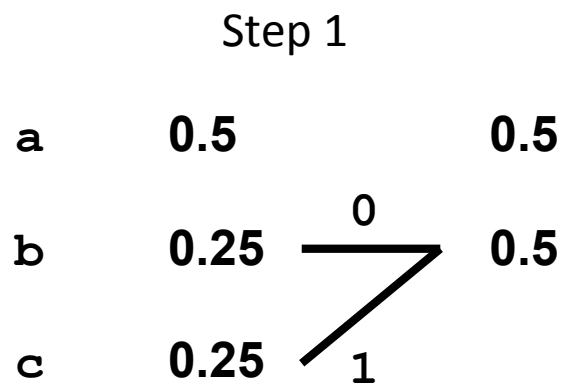
Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

Step 1

| | |
|---|---|
| **a** | **0.5** |
| **b** | **0.25** |
| **c** | **0.25** |

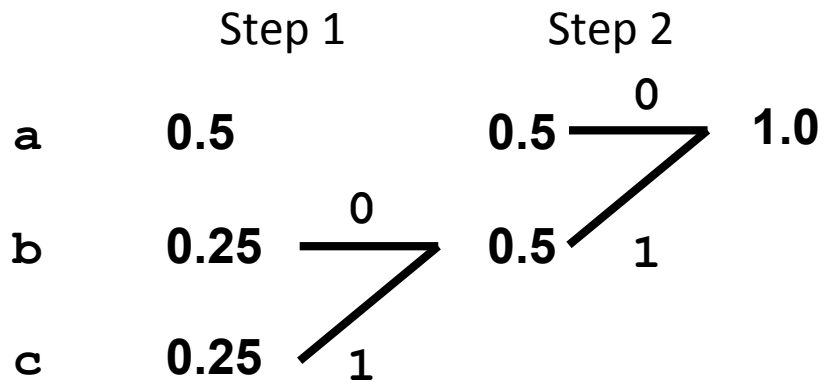
# Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$



# Huffman Coding: Example 1

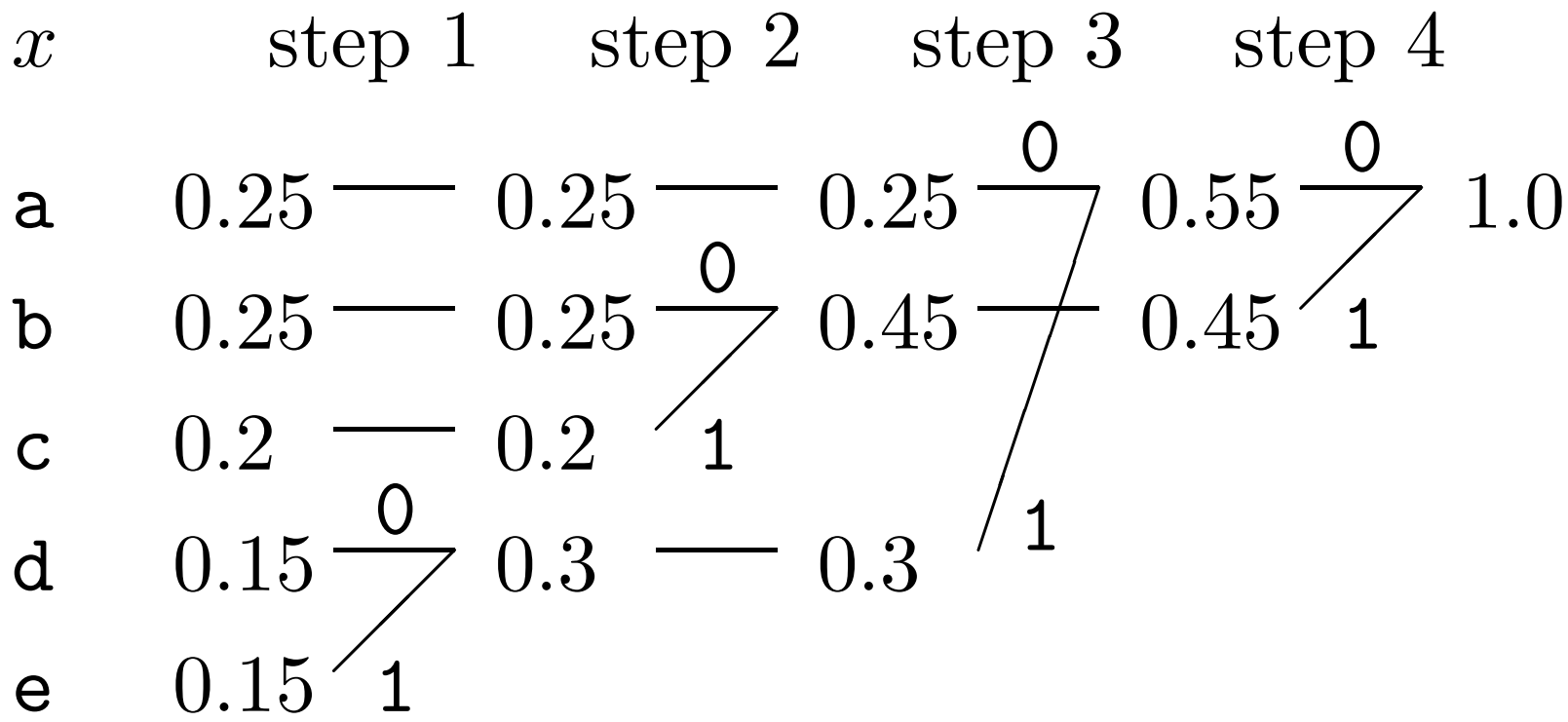
Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$




Now we read off the labelling implied by path from the last meta-symbol to each of the original symbols: $C = \{0, 10, 11\}$

# Huffman Coding: Example 2

$\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P}_X = \{0.25, 0.25, 0.2, 0.15, 0.15\}$



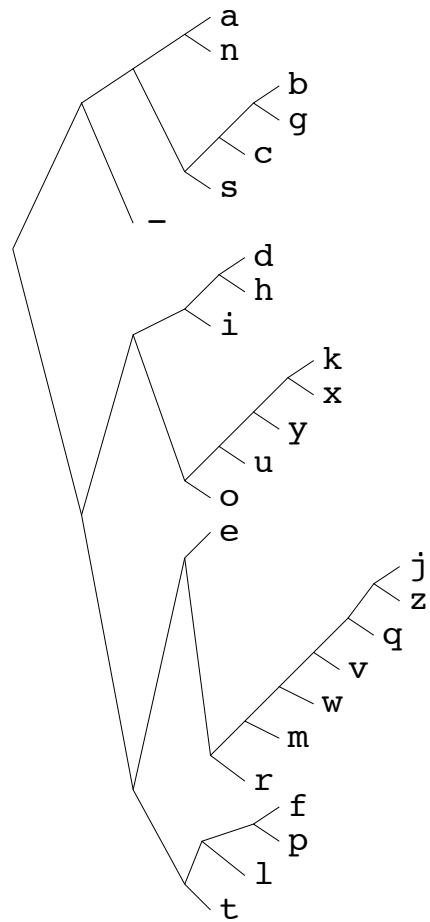

From Example 5.15 of MacKay

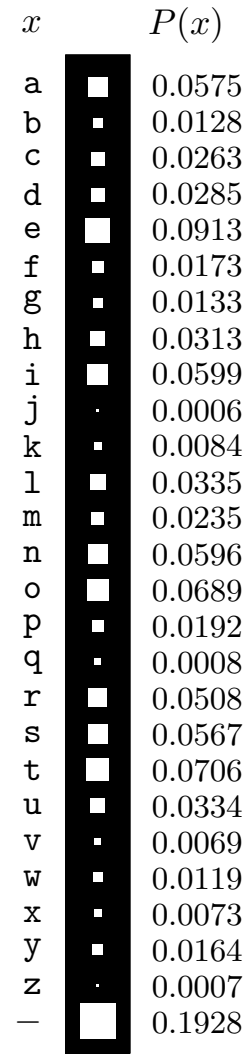
$C = \{00, 10, 11, 010, 011\}$

# Huffman Coding: Example 3

## English letters – Monogram statistics

| $a_i$ | $p_i$ | $\log_2 \frac{1}{p_i}$ | $l_i$ | $c(a_i)$ |
|---|---|---|---|---|
| a | 0.0575 | 4.1 | 4 | 0000 |
| b | 0.0128 | 6.3 | 6 | 001000 |
| c | 0.0263 | 5.2 | 5 | 00101 |
| d | 0.0285 | 5.1 | 5 | 10000 |
| e | 0.0913 | 3.5 | 4 | 1100 |
| f | 0.0173 | 5.9 | 6 | 111000 |
| g | 0.0133 | 6.2 | 6 | 001001 |
| h | 0.0313 | 5.0 | 5 | 10001 |
| i | 0.0599 | 4.1 | 4 | 1001 |
| j | 0.0006 | 10.7 | 10 | 1101000000 |
| k | 0.0084 | 6.9 | 7 | 1010000 |
| l | 0.0335 | 4.9 | 5 | 11101 |
| m | 0.0235 | 5.4 | 6 | 110101 |
| n | 0.0596 | 4.1 | 4 | 0001 |
| o | 0.0689 | 3.9 | 4 | 1011 |
| p | 0.0192 | 5.7 | 6 | 111001 |
| q | 0.0008 | 10.3 | 9 | 110100001 |
| r | 0.0508 | 4.3 | 5 | 11011 |
| s | 0.0567 | 4.1 | 4 | 0011 |
| t | 0.0706 | 3.8 | 4 | 1111 |
| u | 0.0334 | 4.9 | 5 | 10101 |
| v | 0.0069 | 7.2 | 8 | 11010001 |
| w | 0.0119 | 6.4 | 7 | 1101001 |
| x | 0.0073 | 7.1 | 7 | 1010001 |
| y | 0.0164 | 5.9 | 6 | 101001 |
| z | 0.0007 | 10.4 | 10 | 1101000001 |
| – | 0.1928 | 2.4 | 2 | 01 |



# Huffman Coding: Formally

**HUFFMAN($\mathcal{A}, \mathcal{P}$):**

1. If $|\mathcal{A}| = 2$ return $C = \{0, 1\}$; else
2. Let $a, a' \in \mathcal{A}$ be *least probable* symbols.
3. Let $\mathcal{A}' = \mathcal{A} - \{a, a'\} \cup \{aa'\}$
4. Let $\mathcal{P}' = \mathcal{P} - \{p_a, p_{a'}\} \cup \{p_{aa'}\}$ where $p_{aa'} = p_a + p_{a'}$
5. Compute $C' = \text{HUFFMAN}(\mathcal{A}', \mathcal{P}')$
6. Define $C$ by
   - $c(a) = c'(aa')0$
   - $c(a') = c'(aa')1$
   - $c(x) = c'(x)$ for $x \in \mathcal{A}'$
7. Return $C$

# Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_{\mathtt{a}} = p_{\mathtt{b}} = \frac{1}{4}$

# Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_{\mathtt{a}} = p_{\mathtt{b}} = \frac{1}{4}$
  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$

# Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_{\mathtt{a}} = p_{\mathtt{b}} = \frac{1}{4}$
  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  - Call HUFFMAN($\mathcal{A}', \mathcal{P}'$):
    - $|\mathcal{A}| = |\{\mathtt{a}, \mathtt{bc}\}| = 2$
    - Return code with $c'(\mathtt{a}) = 0$, $c'(\mathtt{bc}) = 1$

# Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_{\mathtt{a}} = p_{\mathtt{b}} = \frac{1}{4}$
  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
    - $|\mathcal{A}| = |\{\mathtt{a}, \mathtt{bc}\}| = 2$
    - Return code with $c'(\mathtt{a}) = 0$, $c'(\mathtt{bc}) = 1$
  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = \mathbf{1}0$
    - $c(\mathtt{c}) = c'(\mathtt{bc})1 = \mathbf{1}1$
    - $c(\mathtt{a}) = c'(\mathtt{a}) = 0$

# Huffman Coding: Example 1

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  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
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  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = \mathbf{1}0$
    - $c(\mathtt{c}) = c'(\mathtt{bc})1 = \mathbf{1}1$
    - $c(\mathtt{a}) = c'(\mathtt{a}) = 0$
  - Return $C = \{0, 10, 11\}$

# Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_{\mathtt{a}} = p_{\mathtt{b}} = \frac{1}{4}$
  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
    - $|\mathcal{A}| = |\{\mathtt{a}, \mathtt{bc}\}| = 2$
    - Return code with $c'(\mathtt{a}) = 0$, $c'(\mathtt{bc}) = 1$
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    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = \mathbf{1}0$
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  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
    - $|\mathcal{A}| = |\{\mathtt{a}, \mathtt{bc}\}| = 2$
    - Return code with $c'(\mathtt{a}) = 0$, $c'(\mathtt{bc}) = 1$
  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = \mathbf{1}0$
    - $c(\mathtt{c}) = c'(\mathtt{bc})1 = \mathbf{1}1$
    - $c(\mathtt{a}) = c'(\mathtt{a}) = 0$
  - Return $C = \{0, 10, 11\}$

The constructed code has $L(C, X) = \frac{1}{2} \times 1 + \frac{1}{4} \times (2 + 2) = 1.5$.
The entropy is $H(X) = 1.5$.

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($\mathcal{A}', \mathcal{P}'$):

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($\mathcal{A}', \mathcal{P}'$):
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

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  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($\mathcal{A}', \mathcal{P}'$):
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$
    - Call HUFFMAN($\mathcal{A}'', \mathcal{P}''$):
      - $\mathcal{A}''' = \{\mathbf{ade}, \mathtt{bc}\}$ and $\mathcal{P}''' = \{\mathbf{0.55}, 0.45\}$
      - Return $c'''(\mathtt{ade}) = 0, c'''(\mathtt{bc}) = 1$

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
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  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$
    - Call HUFFMAN$(\mathcal{A}'', \mathcal{P}'')$:
      - $\mathcal{A}''' = \{\mathbf{ade}, \mathtt{bc}\}$ and $\mathcal{P}''' = \{\mathbf{0.55}, 0.45\}$
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, $c''(\mathtt{de}) = \mathbf{0}1$

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$
    - Call HUFFMAN$(\mathcal{A}'', \mathcal{P}'')$:
      - $\mathcal{A}''' = \{\mathbf{ade}, \mathtt{bc}\}$ and $\mathcal{P}''' = \{\mathbf{0.55}, 0.45\}$
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, $c''(\mathtt{de}) = \mathbf{0}1$
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{1}0$, $c'(\mathtt{c}) = \mathbf{1}1$, $c'(\mathtt{de}) = 01$

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($\mathcal{A}', \mathcal{P}'$):
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$
    - Call HUFFMAN($\mathcal{A}'', \mathcal{P}''$):
      - $\mathcal{A}''' = \{\mathbf{ade}, \mathtt{bc}\}$ and $\mathcal{P}''' = \{\mathbf{0.55}, 0.45\}$
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, $c''(\mathtt{de}) = \mathbf{0}1$
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{1}0$, $c'(\mathtt{c}) = \mathbf{1}1$, $c'(\mathtt{de}) = 01$
- Return $c(\mathtt{a}) = 00$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 11$, $c(\mathtt{d}) = \mathbf{01}0$, $c(\mathtt{e}) = \mathbf{01}1$

# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($\mathcal{A}', \mathcal{P}'$):
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$
    - Call HUFFMAN($\mathcal{A}'', \mathcal{P}''$):
      - $\mathcal{A}''' = \{\mathbf{ade}, \mathtt{bc}\}$ and $\mathcal{P}''' = \{\mathbf{0.55}, 0.45\}$
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, $c''(\mathtt{de}) = \mathbf{0}1$
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{1}0$, $c'(\mathtt{c}) = \mathbf{1}1$, $c'(\mathtt{de}) = 01$
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# Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN$(\mathcal{A}', \mathcal{P}')$:
    - $\mathcal{A}'' = \{\mathtt{a}, \mathbf{bc}, \mathtt{de}\}$ and $\mathcal{P}'' = \{0.25, \mathbf{0.45}, 0.3\}$
    - Call HUFFMAN$(\mathcal{A}'', \mathcal{P}'')$:
      - $\mathcal{A}''' = \{\mathbf{ade}, \mathtt{bc}\}$ and $\mathcal{P}''' = \{\mathbf{0.55}, 0.45\}$
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, $c''(\mathtt{de}) = \mathbf{0}1$
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{1}0$, $c'(\mathtt{c}) = \mathbf{1}1$, $c'(\mathtt{de}) = 01$
- Return $c(\mathtt{a}) = 00$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 11$, $c(\mathtt{d}) = \mathbf{01}0$, $c(\mathtt{e}) = \mathbf{01}1$

The constructed code is $C = \{00, 10, 11, 010, 011\}$.
It has $L(C, X) = 2 \times (0.25 + 0.25 + 0.2) + 3 \times (0.15 + 0.15) = 2.3$.
Note that $H(X) \approx 2.29$.

# Huffman Coding in Python

## See full example code with examples at:

`https://gist.github.com/mreid/fdf6353ec39d050e972b`

```
def huffman(p):
    '''Return a Huffman code for an ensemble with distribution p.'''
    assert(sum(p.values()) == 1.0) # Ensure probabilities sum to 1

    # Base case of only two symbols, assign 0 or 1 arbitrarily
    if(len(p) == 2):
        return dict(zip(p.keys(), ['0', '1']))

    # Create a new distribution by merging lowest prob. pair
    p_prime = p.copy()
    a1, a2 = lowest_prob_pair(p)
    p1, p2 = p_prime.pop(a1), p_prime.pop(a2)
    p_prime[a1 + a2] = p1 + p2

    # Recurse and construct code on new distribution
    c = huffman(p_prime)
    ca1a2 = c.pop(a1 + a2)
    c[a1], c[a2] = ca1a2 + '0', ca1a2 + '1'

    return c
```

# Advantages of Huffman coding

- Produces prefix codes automatically (by design)
- Algorithm is simple and efficient
- Huffman Codes are provably optimal [Exercise 5.16 (MacKay)]

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- Algorithm is simple and efficient
- Huffman Codes are provably optimal [Exercise 5.16 (MacKay)]

  If $C_{\text{Huff}}$ is a Huffman code, then for any other uniquely decodable code $C'$,

  $$L(C_{\text{Huff}}, X) \leq L(C', X)$$

  It follows that

  $$H(X) \leq L(C_{\text{Huff}}, X) < H(X) + 1$$

# Disadvantages of Huffman coding

- Assumes a fixed distribution of symbols

- The extra bit in the SCT
  - If $H(X)$ is large – not a problem
  - If $H(X)$ is small (e.g., $\sim 1$ bit for English) codes are $2\times$ optimal

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- The extra bit in the SCT
  - If $H(X)$ is large – not a problem
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Huffman codes are the best possible symbol code
but symbol coding is not always the best type of code

**Next Time**: Stream Codes!

# Summary

**Key Concepts**:

1. The expected code length $L(C, X) = \sum_i p_i \ell_i$
2. Probabilities and codelengths are interchangeable $q_i = 2^{-\ell_i} \iff \ell_i = \log_2 \frac{1}{q_i}$
3. Relative entropy $D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q})$ measures excess bits over the entropy $H(X)$ for using the wrong code $\mathbf{q}$ for probabilities $\mathbf{p}$
4. The Source Coding Theorem for symbol codes: There exists prefix (Shannon) code $C$ for ensemble $X$ with $\ell_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil$ so that

$$H(X) \leq L(C, X) \leq H(X) + 1$$

5. Huffman codes are optimal symbol codes

**Reading**:

- §5.3-5.7 of MacKay
- §5.3-5.4, §5.6 & §5.8 of Cover & Thomas