COMP2610/COMP6261 - Information Theory

Tutorial 2: Probability and Bayesian Inference

Manish Kumar

Week 2, Semester 2, 2023

1. (From Bishop, 2006) Suppose that we have three coloured boxes r (red), b (blue), and g (green). Box r contains 3 apples, 4 oranges, and 3 limes. Box b contains 1 apple, 1 orange, and 0 limes. Box g contains 3 apples, 3 oranges, and 4 limes. A box is chosen at random with probabilities $p(r) = 0.2$, $p(b) = 0.2$, $p(g) = 0.6$ and a piece of fruit is removed from the box (with probability of selecting any of the items in the box):

   (a) What is the probability of selecting an apple?

   (b) If we observe that the selected fruit is in fact an orange, what is the probability that it came from the green box?

2. A scientist conducts 1000 trials of an experiment involves variables X, Y, each with possible values $\{0,1\}$. He records the outcomes of the trials in the following table.

Counts		X = 0	X = 1
Y	0	100	250
	1	150	500

   Compute each of the following, showing all your working.

   (a) $p(X = 1, Y = 1)$.

   (b) $p(X = 1)$.

   (c) $E[X]$.

   (d) $p(Y = 1|X = 1)$.

   (e) $p(Y = 1|X = 0)$.

   (f) Let Z be a noisy version of the XOR of X and Y, with

$$p(Z = 1|X = x, Y = y) = \begin{cases} 0.9 & \text{if } (x, y) = (0, 1) \text{ or } (x, y) = (1, 0) \\ 0.1 & \text{if } (x, y) = (0, 0) \text{ or } (x, y) = (1, 1) \end{cases}.$$

   Compute $p(X = 1, Y = 1|Z = 1)$.

3. (From Barber, 2011) Two balls are placed in a box as follows:

   - a fair coin is tossed
   - a white ball is placed in the box if a head occurs; otherwise, a red ball is placed in the box
   - the coin is tossed again
   - a red ball is placed in the box if a tail occurs; otherwise, a white ball is placed in the box.

   Balls are drawn from the box three times in succession (always with replacing the drawn ball back in the box). It is found that on all three occasions a red ball is drawn. What is the probability that both balls in the box are red?

4. Several of the results we have seen in lectures generalise in the expected way when we condition on additional random variables.

   (a) For random variables X,Y , the conditional probability $p(X|Y)$ may be defined as

   $$p(X|Y) = \frac{p(X,Y)}{p(Y)}.$$

   Using this definition, show that for random variables X,Y,Z, the following conditional version of Bayes rule holds:

   $$p(X|Y,Z) = \frac{p(Y|X,Z)p(X|Z)}{p(Y|Z)}.$$

   (b) The sum rule (or marginalisation) says that for random variables $X_1,\ldots,X_n$,

   $$p(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n) = \sum_x p(X_1,\ldots,X_{i-1},X_i = x,X_{i+1},\ldots,X_n).$$

   Using this fact, and the definition of conditional probability, show that

   $$p(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n|Y) = \sum_x p(X_1,\ldots,X_{i-1},X_i = x,X_{i+1},\ldots,X_n|Y)$$

   where Y is another random variable.