COMP3670/6670: Introduction to Machine Learning

Release Date. Aug 3th, 2023
Due Date. 11:59pm, Aug 28th, 2023
Maximum credit. 100

Exercise 1 **Properties of Matrices** (2+2+2+3+3+4+3 credits)

(a) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ be a square matrix. Show that $\mathbf{A}$ is symmetric.

(b) Compute the square of $\mathbf{A}$, that is $\mathbf{A}^2$ and show that $\mathbf{A}^2$ is also symmetric.

(c) Is it true for any symmetric matrix A, $\mathbf{A}^2$ is also symmetric? Show your working.

(d) Let $\mathbf{A}$ be a square matrix and $f(\mathbf{X})$ and $g(\mathbf{X})$ be n-th order polynomials, defined by $\sum_{i=0}^{n} a_i \mathbf{X}^i$ where a_i are arbitrary real numbers. Show that the matrices $f(\mathbf{A})$ and $g(\mathbf{A})$ commute, i.e, $f(\mathbf{A})g(\mathbf{A}) = g(\mathbf{A})f(\mathbf{A})$ for arbitrary order n.

(e) Let $\mathbf{A}$ and $\mathbf{B}$ be rectangular matrices of orders $n \times k$ and $r \times s$, respectively. The matrix of order $nr \times ks$ represented in a block form as

$$\begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1k}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2k}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \dots & a_{nk}\mathbf{B} \end{bmatrix}$$

is called the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of the matrices $\mathbf{A}$ and $\mathbf{B}$.

Let

$$\mathbf{X} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}.$$

$\mathbf{X}$ is a so-called magic square, since its row sums, column sums, principal diagonal sum, and principal counter diagonal sum are all equal. Is $\mathbf{X} \otimes \mathbf{X}$ a magic square?

(f) Determine if $\mathbf{X} \otimes \mathbf{X}$ is a magic square for any magic matrix $\mathbf{X}$ of order $n \times n$. Show an example of a magic square with n=2 for which $\mathbf{X} \otimes \mathbf{X}$ is also a magic square.

(g) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. What conditions should $\mathbf{x}, \mathbf{y}$ satisfy such that $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{x} = \mathbf{y} \otimes \mathbf{x} \otimes \mathbf{y}$?

Exercise 2 **Solving Linear Systems** (3+3 credits)

Find the set $\mathcal{S}$ of all solutions $\mathbf{x}$ of the following inhomogenous linear systems $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows. Write the solution space $\mathcal{S}$ in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 6 & 2 & -5 \\ 2 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 & 2 & -2 \\ 0 & 1 & 2 & 2 & 6 \\ 3 & 2 & 1 & 1 & -3 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 23 \\ -2 \\ 16 \end{bmatrix}$$

Exercise 3 **Inverses and rank** (3+4+6 credits)

(a) Let $\mathbf{A} \in \mathbb{M}_{n\times n}(\mathbb{R})$ be an invertible matrix. Show that the transpose of the inverse of $\mathbf{A}$, denoted $(\mathbf{A}^{-1})^T$, is equal to the inverse of the transpose of $\mathbf{A}$, denoted $(\mathbf{A}^T)^{-1}$.

(b) Find the values of $[a, b, c]^T \in \mathbb{R}^3$ for which the inverse of the following matrix exists.

$$\begin{bmatrix} 1 & 1 & b \\ 1 & a & c \\ 1 & 1 & 1 \end{bmatrix}$$

(c) Let $\mathbf{A}$ be an arbitrary matrix in $\mathbb{M}_{m\times n}(\mathbb{R})$, where m and n denote the number of rows and columns of $\mathbf{A}$, respectively. Prove that $\mathbf{rk}(\mathbf{A}) = \mathbf{rk}(\mathbf{A}^T)$, where $\mathbf{rk}(\mathbf{A})$ denotes the rank of matrix $\mathbf{A}$.

Exercise 4 **Subspaces** (2+2+2+3+4 credits)

(a) Which of the following sets are subspaces of $\mathbb{R}^n$? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(i) $A = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\}$

(ii) $B = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \text{at least one } x_i \text{ is irrational}\}$

(iii) $C = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} (-1)^{i+1} x_i \geq 0\right\}$

(iv) $D =$ The set of all solutions $\mathbf{x}$ to the matrix equation $\mathbf{Ax} = \mathbf{b}$, for some matrix $\mathbf{A}$ and some vector $\mathbf{b}$. (Hint: Your answer may depend on $\mathbf{A}$ and $\mathbf{b}$.)

(b) Let V be an inner product space, and let W be a subspace of V. The orthogonal complement of W, denoted $W^\perp$, is defined as the set of all vectors in V that are orthogonal to every vector in W. Show that $W^\perp$ is also a subspace of V.

Exercise 5 **Linear Independence** (4+4+4+4+4 credits)

Let V and W be vector spaces. Let $T : V \to W$ be a **linear** transformation.

The *image* of T is defined as:

$$\mathbf{Im}(T) = \{w \in W \,|\, \exists v \in V \text{ such that } w = T(v)\}.$$

The *kernel* of T is defined as:

$$\mathbf{Ker}(T) = \{v \in V \,|\, T(v) = 0\}.$$

We say that T is *injective* if for all $u, v \in V$, $T(u) = T(v)$ implies $u = v$.

(a) Show that $T(\mathbf{0}) = \mathbf{0}$.

(b) For any integer $n \geq 1$, show that given a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ in V and a set of coefficients $\{c_1, \ldots, c_n\}$ in $\mathbb{R}$, that

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n)$$

(c) Prove that $\mathbf{Im}(T)$ is a vector subspace of W and $\mathbf{Ker}(T)$ is a vector subspace of V.

(d) The Rank-Nullity Theorem states that for a linear map $T : V \to W$, the dimension of the finite-dimensional domain V is equal to the sum of the dimensions of the kernel and the image of T, i.e.,

$$\mathbf{dim}(V) = \mathbf{dim}(\mathbf{Ker}(T)) + \mathbf{dim}(\mathbf{Im}(T)).$$

Give an example of a linear map T such that $\mathbf{dim}(\mathbf{Im}(T)) = 3$ and $\mathbf{dim}(\mathbf{Ker}(T)) = 2$.

(e) Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & a & b \\ 1 & 1 & c \\ 1 & 1 & 1 \end{bmatrix}$$

with $[a, b, c]^T \in \mathbb{R}^3$. Find the conditions on a, b, and c for which this transformation is injective.

Exercise 6 **Inner Products** (3+3+4+3+5 credits)

(a) Show that if an inner product $\langle \cdot, \cdot \rangle$ is symmetric and linear in the second argument, then it is bilinear.

(b) Given a 2×2 rotation matrix $\mathbf{R}$ represented as

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

show that it preserves the standard inner product, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we have $\mathbf{x}^T\mathbf{y} = (\mathbf{R}\mathbf{x})^T(\mathbf{R}\mathbf{y})$.

(c) Now, let us consider an inner product in $\mathbb{R}^2$ defined by the 2×2 matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Find the matrix $\mathbf{D}'$ (in terms of $\mathbf{R}$ and $\mathbf{D}$) such that the inner product defined by $\mathbf{D}$ is preserved under the rotation by $\mathbf{R}$, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we have $\mathbf{x}^T\mathbf{D}\mathbf{y} = (\mathbf{R}\mathbf{x})^T\mathbf{D}'(\mathbf{R}\mathbf{y})$.

(d) For $\theta = \pi/4$, compute $\mathbf{D}'$ explicitly.

(e) Consider $\mathbf{u} = [1, 1]^T \in \mathbb{R}^2$ and $\mathbf{v} = [2, -1]^T \in \mathbb{R}^2$. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$ under the inner product defined by $\mathbf{D}$, and the angle between $\mathbf{R}\mathbf{u}$ and $\mathbf{R}\mathbf{v}$ under the inner product defined by $\mathbf{D}'$.

Exercise 7 **Orthogonality** (7+4 credits)

(a) Let V denote a vector space together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

Let $\mathbf{x}, \mathbf{y}$ be **non-zero** vectors in V.

Prove or disprove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then they are linearly independent.

(b) Determine if the 'vectors' defined by the functions $p(x) = 3x^2 - 1$ and $q(x) = 2x + 1$ in the inner product space of continuous functions on the interval $[0, 1]$ with the inner product defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$ are orthogonal.

You may find the formulae helpful:

$$\int_a^b \alpha x^n \,dx = \left[\frac{\alpha x^{n+1}}{n+1}\right]_a^b = \frac{\alpha b^{n+1}}{n+1} - \frac{\alpha a^{n+1}}{n+1}$$

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$