Linear Algebra

Textbook Sec. 2.1 - 2.3

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Also, please call me Jo :-).

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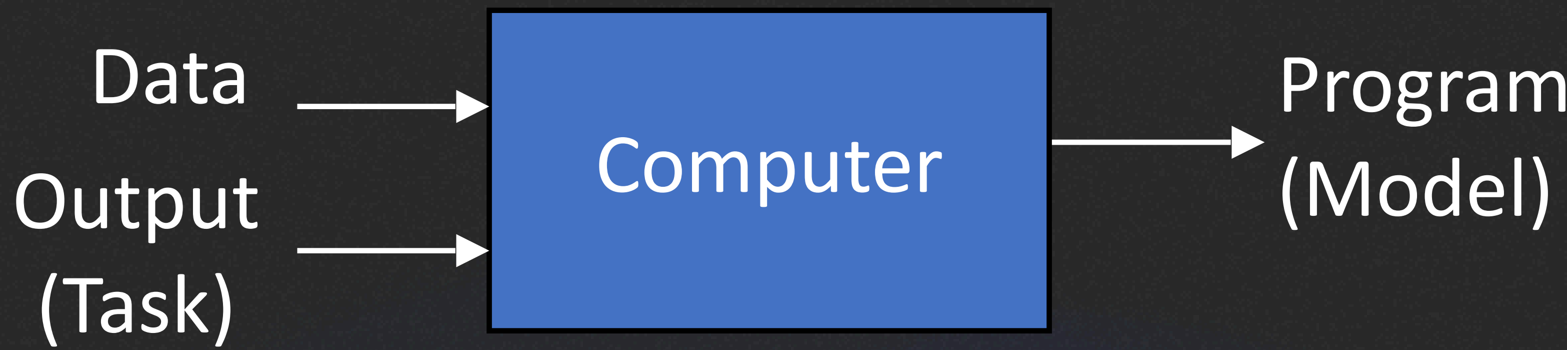
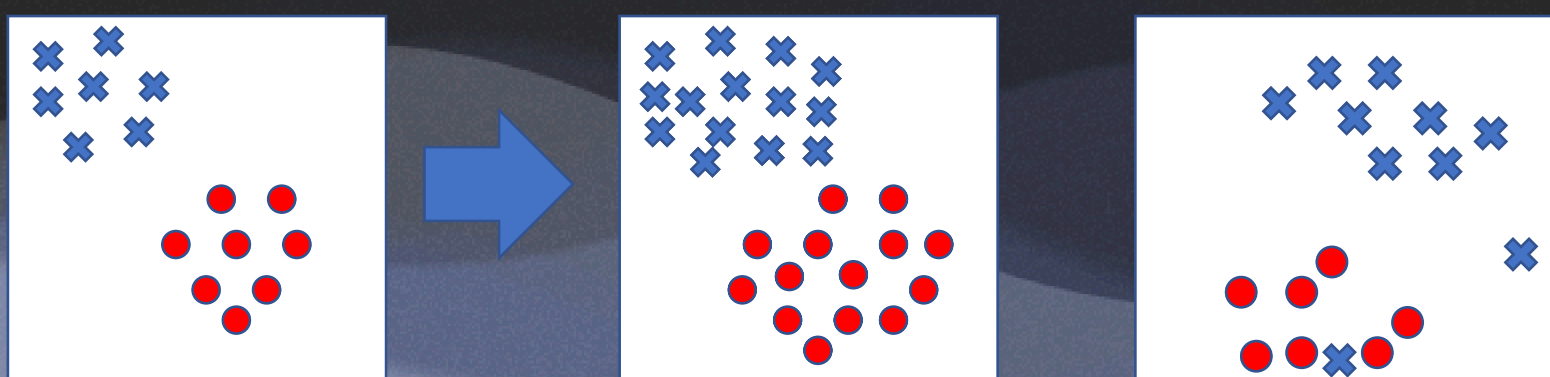
Quick review



Quick review

- **Training** is the process of making the system able to learn.

- A model that explains a certain situation well may fail in another situation.
  - The training set and test set come from the same distribution (in-distribution vs. out-of-distribution)
  - Before applying a model, check the assumptions!



Training data

Test data

Quick review

Supervised

Input: Data **X** and label **y**

Goal: Learn how to map **X** to **y**

Examples: Regression, classification

Unsupervised

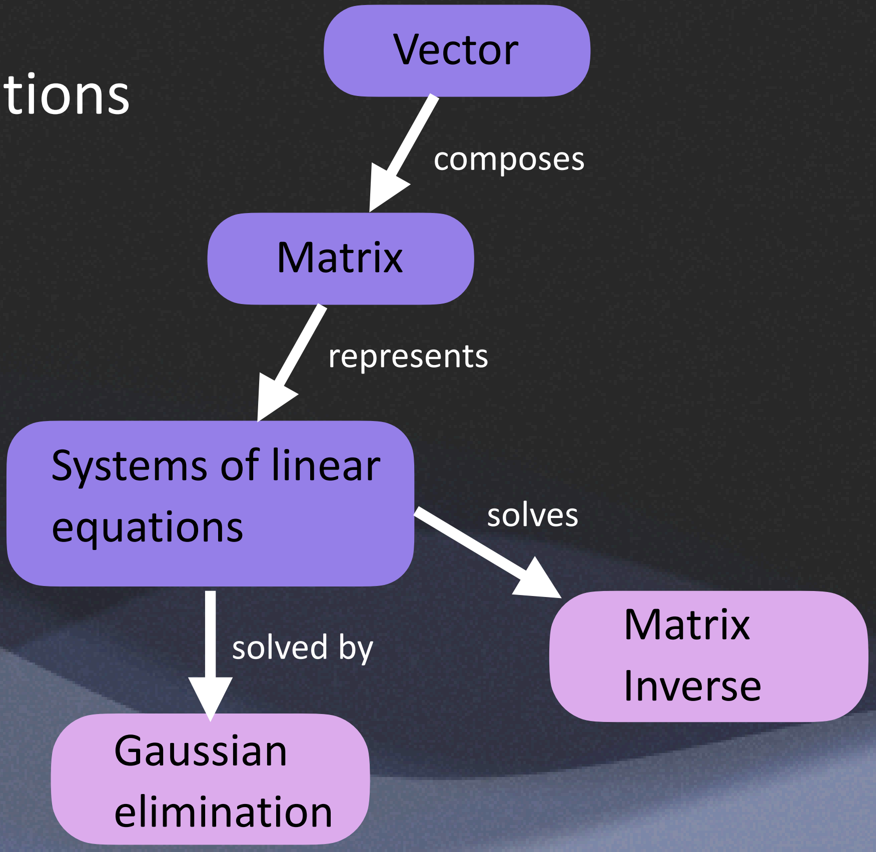
Input: Data **X**, no label

Goal: Learn underlying structure of data

Examples: Clustering, dimensionality reduction

Outline

- Base concepts: vectors and systems of linear equations
- Matrices
- Solving systems of linear equations



2.1 Base concepts

Linear Algebra

- Algebra comes from the Arabic word ‘al-jabr’ (restoration/ completion) introduced by the Persian astronomer and mathematician Muhammad ibn Musa al-Khwarizmi in the 9th century
- Linear Algebra: “the study of vectors and certain rules to manipulate vectors”
- Add vectors -> vector ; scale vector -> vector
- The “mathematics of data” since data is represented by vectors and matrices

Vectors as elements of $\mathbb{R}^n$

- A simple example of vector, an element of $\mathbb{R}^n$:

$$x = \begin{bmatrix} 1 \\ 3 \\ 2 \\ \vdots \\ -1 \end{bmatrix} \in \mathbb{R}^n \qquad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$$

- Adding two vectors (component-wise) $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} \in \mathbb{R}^n$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}$$

- Multiplying $\boldsymbol{a} \in \mathbb{R}^n$ by $\lambda \in \mathbb{R}$ results in a scaled vector:

$$\lambda \boldsymbol{a} \in \mathbb{R}^n$$

2.1 Systems of Linear Equations

You only have 5 AUD to buy pens, notebooks or snacks.

Because it's Canberra, we have Coles, Supabarn, Aldi…

Coles: one pen + 2 notebooks + no snack = 10 AUD
Supabarn: 2 pens + 1 notebook + 1 snack = 15 AUD
Aldi: no pen + 2 notebooks + 1 snack looking sadder than usual = 7 AUD

What can you buy?

2.1 Systems of Linear Equations

$$
\begin{aligned}
a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Unknowns

2.1 Systems of Linear Equations

- Example 1: Does it have a solution?

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \qquad (1)\\
x_1 + x_2 + 2x_3 &= 2 \qquad (2)\\
+\,3x_3 &= 6 \qquad (3)
\end{aligned}
$$

2.1 Systems of Linear Equations

- Example 1: Does it have a solution?

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \qquad (1)\\
x_1 + x_2 + 2x_3 &= 2 \qquad (2)\\
+\, 3x_3 &= 6 \qquad (3)
\end{aligned}
$$

Yes, it has infinitely many solutions.

$$
\boxed{\begin{aligned} &x_3 = 2\\ &x_1 + x_2 = -2 \end{aligned}}
$$

2.1 Systems of Linear Equations

- Example 2: Does it have a solution?

$$-x_1 + x_2 + 3x_3 = 3 \qquad (1)$$

$$x_1 + x_2 + 2x_3 = 2 \qquad (2)$$

$$2x_2 + 5x_3 = 1 \qquad (3)$$

3 unknowns

$x_1 \quad x_2 \quad x_3$

2.1 Systems of Linear Equations

- Example 2: Does it have a solution?

$$
\begin{aligned}
-x_1 + x_2 + 3x_3 &= 3 \qquad (1)\\
x_1 + x_2 + 2x_3 &= 2 \qquad (2)\\
2x_2 + 5x_3 &= 1 \qquad (3)
\end{aligned}
$$

3 unknowns

$x_1 \quad x_2 \quad x_3$

No.

Adding the first two equations yields $2x_2 + 5x_3 = 5$.
It contradicts Equation (3).

2.2 Matrices

2.2 Matrices

- A rectangular scheme consisting of m rows and n columns:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R}\,.$$

ith row, jth column

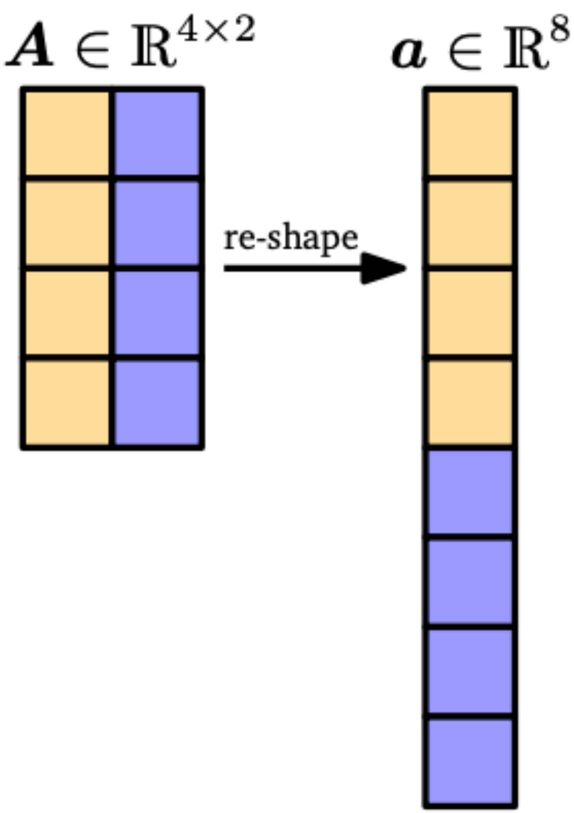
- By convention $(1,n)$-matrices are called rows and $(m,1)$-matrices are called columns. These special matrices are also called row/ column vectors.

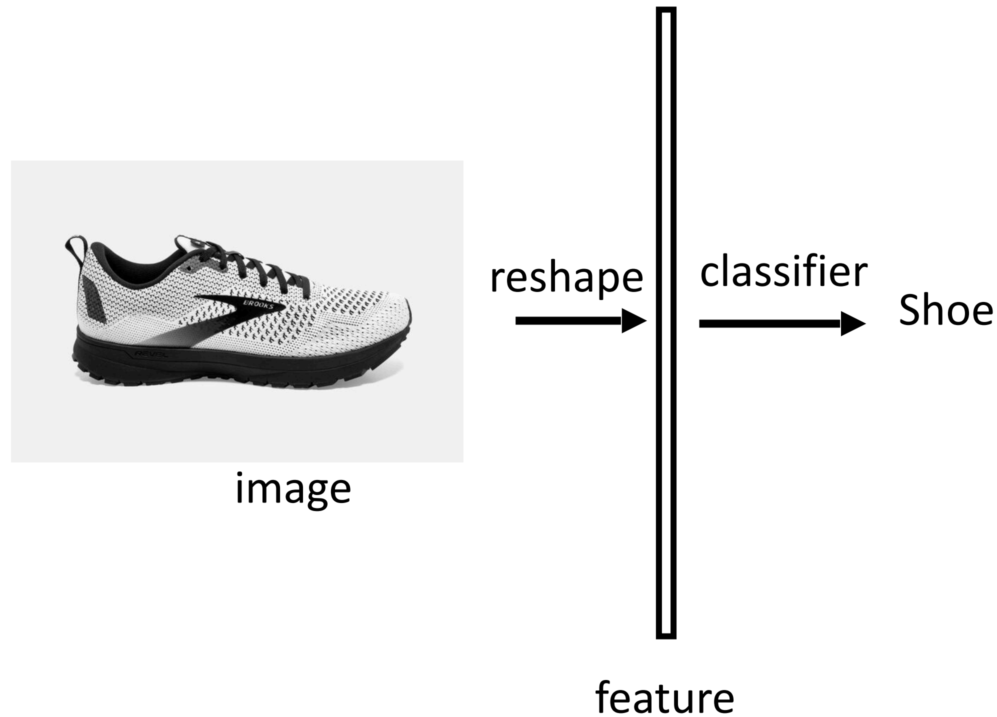
2.2 Matrices

#rows, #cols

- $\mathbb{R}^{m \times n}$ is the set of all real-valued (m, n)-matrices.

Space



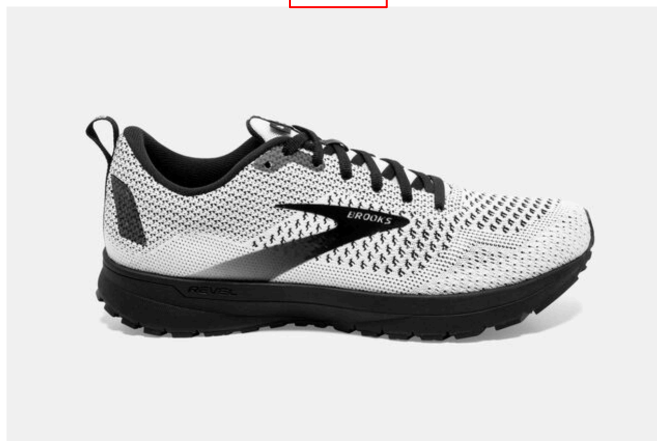


Perhaps the simplest feature representation

Matrix - example

640 Gray scale image

$[0, 255]$



427

427x640

400 Binary image $\{0, 1\}$

255

255x400

Summary statistics table

	TREATMENT					
	A			B		
	N	Mean	SD	N	Mean	SD
LENGTH	50	176.500	5.9083	50	175.640	5.5467
WEIGHT	50	77.680	10.6492	50	76.400	8.4540
Body Mass Index	50	24.918	3.0644	50	24.763	2.4787

3

6

3x6

2.2.1 Matrix Addition and Multiplication

- The sum of two matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ is defined as the element-wise sum,

$$\boldsymbol{A} + \boldsymbol{B} := \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

- Example

For $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 3 & -2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$, $\boldsymbol{B} = \begin{bmatrix} -5 & 0 \\ 1 & 1 \\ 0 & -4 \end{bmatrix} \in \mathbb{R}^{3\times 2}$, we obtain

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} -5 & 1 \\ 2 & 3 \\ 3 & -6 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

2.2.1 Matrix Addition and Multiplication

- For matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times k}$, the element c_{ij} of the product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} \in \mathbb{R}^{m \times k}$ is defined as

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \ldots, m. \quad j = 1, \ldots, k$$

$$c_{ij} \neq a_{ij} b_{ij}$$

- To compute element c_{ij} we multiply the elements of the ith row of $\boldsymbol{A}$ with the jth column of $\boldsymbol{B}$ and sum them up.

2.2.1 Matrix Addition and Multiplication

- One property that is unique to matrices is the dimension property. This property has two parts:

$$\underbrace{\boldsymbol{A}}_{n \times k} \quad \underbrace{\boldsymbol{B}}_{k \times m} = \underbrace{\boldsymbol{C}}_{n \times m}$$

- Identity Matrix

$$\boxed{\boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\boxed{\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boldsymbol{I}_n := \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

2.2.1 The Properties of Matrix Multiplication

- Associativity

$$\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times q}: \left(\boldsymbol{A}\boldsymbol{B}\right)\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{C})$$

- Distributivity

$$\forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n\times p}: \left(\boldsymbol{A} + \boldsymbol{B}\right)\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{C}$$

$$\boldsymbol{A}\left(\boldsymbol{C} + \boldsymbol{D}\right) = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{A}\boldsymbol{D}$$

- Multiplication with the identity matrix:

$$\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}: \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$$

$\boldsymbol{I}_m \neq \boldsymbol{I}_n$ for $m \neq n$.

2.2.2 Inverse and Transpose

- Inverse: consider a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Let matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ have the property that $\boldsymbol{AB} = \boldsymbol{I}_n = \boldsymbol{BA}$. $\boldsymbol{B}$ is called the inverse of $\boldsymbol{A}$ and denoted by $\boldsymbol{A}^{-1}$.

- Example

$$\boxed{\boldsymbol{AB} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}_2}$$

$$\boxed{\boldsymbol{B} = \boldsymbol{A}^{-1}}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The matrices are inverse to each other, because

$$\boldsymbol{AB} = \boldsymbol{I}_2 = \boldsymbol{BA}$$

2.2.2 Inverse and Transpose

- Transpose: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, the matrix $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ with $b_{ij} = a_{ji}$ is called the transpose of $\boldsymbol{A}$. We write $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -5 & 6 \\ 0 & 1 & 3 \end{bmatrix}, \boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -5 & 1 \\ 2 & 6 & 3 \end{bmatrix}$$

- Important properties of inverses and transposes:

$$(\boldsymbol{A}+\boldsymbol{B})^{-1} \neq \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$$

$$\boxed{\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}}$$

$$(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$$

$$(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$$

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$$

$$(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

$$(\boldsymbol{A}\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$$

$$\boxed{(\boldsymbol{A}\boldsymbol{B}) \bullet (\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}}$$

2.2.2 Inverse and Transpose

- Symmetric: A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} \qquad \boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} \qquad \boxed{\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}}$$

- The sum of symmetric matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ is always symmetric.

$$\boldsymbol{A} + \boldsymbol{B} \stackrel{?}{=} \left(\boldsymbol{A} + \boldsymbol{B}\right)^{\mathrm{T}} \qquad \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

- The product of two symmetric matrices is generally not symmetric

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

2.2.3 Multiplication by a Scalar

- A scalar $\lambda \in \mathbb{R}$
- Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. Then $\lambda \boldsymbol{A} = \boldsymbol{K}$, where $k_{ij} = \lambda a_{ij}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \end{bmatrix} \qquad \lambda = 1.5$$

$$\lambda \boldsymbol{A} = \begin{bmatrix} 1.5 & 0 & 4.5 \\ 3 & 0 & -1.5 \end{bmatrix}$$

2.2.3 Multiplication by a Scalar

- For $\lambda, \varphi \in \mathbb{R}$, there following properties hold:
- Associativity

$$(\lambda\varphi)\boldsymbol{C} = \lambda(\varphi\boldsymbol{C}), \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

$$\lambda\left(\boldsymbol{BC}\right) = (\lambda\boldsymbol{B})\boldsymbol{C} = \boldsymbol{B}(\lambda\boldsymbol{C}) = \left(\boldsymbol{BC}\right)\lambda, \quad \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{n\times k}$$

- Transpose

$$(\lambda\boldsymbol{C})^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\lambda^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\lambda = \lambda\boldsymbol{C}^{\mathrm{T}}, \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

- Distributivity

$$(\lambda + \varphi)\boldsymbol{C} = \lambda\boldsymbol{C} + \varphi\boldsymbol{C}, \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

$$\lambda\left(\boldsymbol{B} + \boldsymbol{C}\right) = \lambda\boldsymbol{B} + \lambda\boldsymbol{C}, \quad \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

Matrix Algebra	Number Algebra
1) $A(B+C)=AB+AC$	$a(b+c)=ab+ac$
2) $AB \neq BA$ in general	$ab=ba$
3) $(AB)C=A(BC)$	$a(bc)=(ab)c$
4) $AI=IA=A$	$1.a=a.1=a$
5) $AO=OA=O$	$0.a=a.0=0$
6) AB may not be possible	ab is always possible
7) $AB=0$ does not imply $A=0$ or $B=0$	$ab=0 \longrightarrow a=0$ or $b=0$

2.2.4 Compact Representations of Systems of Linear Equations

- Consider the system of linear equations,

$$2x_1 + 3x_2 + 5x_3 = 1$$
$$4x_1 - 2x_2 - 7x_3 = 8$$
$$9x_1 + 5x_2 - 3x_3 = 2$$

- Using matrix multiplication, we can write it into a compact form

$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & -7 \\ 9 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

2.3 Solving systems of linear equations

2.3 Solving Systems of Linear Equations

- Now we have a general form of an equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_w \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned},$$

- 2.3.1 Particular and General Solution

Step 1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$

Step 2. Find all solutions to $\boldsymbol{Ax} = \boldsymbol{0}$

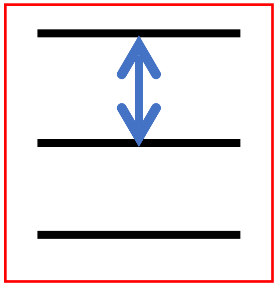
Step 3. Combine the solutions from steps 1. and 2. to get the general solution.

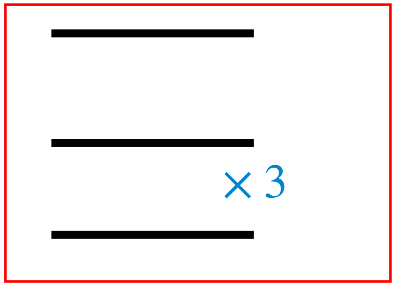
We use Gaussian elimination to solve the equation system.

2.3.2 Elementary Transformations

- Elementary transformations keep the solution set the same but transform the equation system into a simpler form.

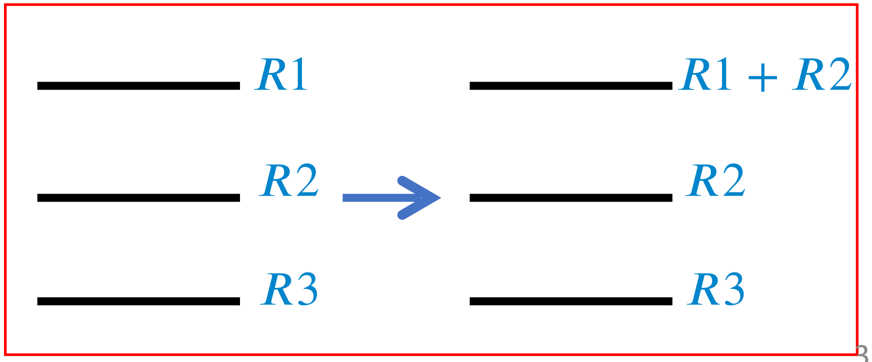
- Elementary transformations include:
- Exchange of two equations



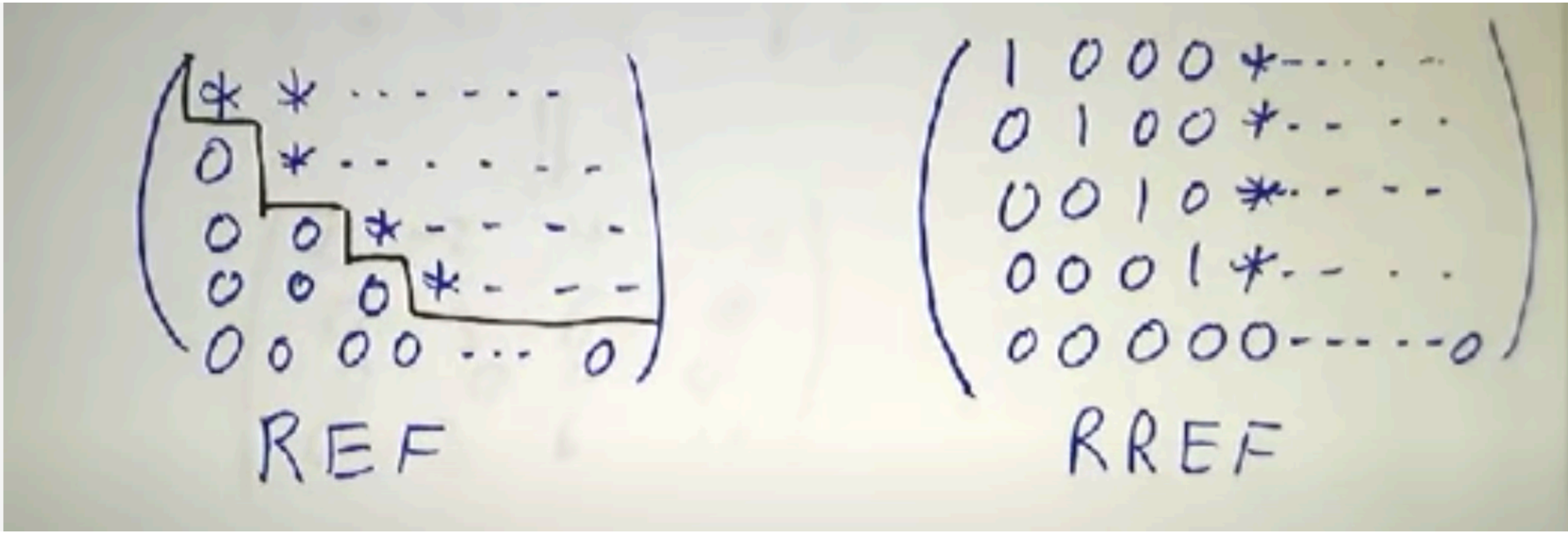


- Multiplication of an equation (row) with a constant $\lambda \in \mathbb{R} \backslash \{0\}$

- Addition of two equations (rows)



Row-echelon form (REF) and reduced row-echelon form (RREF)



From Lorenzo A. Sadun's teaching video

- Row Echelon Form
- All rows with 0s must be at the bottom
- You don't have to have all 0 rows, but if they are they must be at the bottom
- Staircase pattern of the 1st non-zero entries of each row (pivots)
- A pivot is always strictly to the right of the pivot of the row above it

- Reduced Row Echelon Form
- Every pivot is 1
- The pivot is the only non-zero entry in its column

Gaussian Elimination - example

$$\begin{aligned} x_1 + x_2 - x_3 &= 9 \\ x_2 + 3x_3 &= 3 \\ -x_1 - 2x_3 &= 2 \end{aligned}$$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ -1 & 0 & -2 & 2 \end{array}\right]$$

$$\Big\downarrow \text{R1+R3 -> R3}$$

$$\begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -3 & 11 \end{bmatrix} \xrightarrow[\text{R3-R2 -> R3}]{} \begin{bmatrix} 1 & 1 & -1 & 9 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & 8 \end{bmatrix}$$

Gaussian Elimination - example

- Seek all solutions to the following system of equations

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & - & 2x_3 & + & 5x_4 & = & 1 \\ x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 2 \\ -x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 4 \end{array}$$

$$\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 1 & 2 & -1 & 3 & 2 \\ -1 & -2 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{\text{Swap R1 and R2}} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & 3 & -2 & 5 & 1 \\ -1 & -2 & 1 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow[\text{R1+R3 -> R3}]{\text{R2-2R1 -> R2}} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

row-echelon form (REF)

How to find the general solution to $\boldsymbol{Ax} = \boldsymbol{b}$

$$
\begin{array}{rcrcrcrcl}
2x_1 & + & 3x_2 & - & 2x_3 & + & 5x_4 & = & 1 \\
x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 2 \\
-x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 4
\end{array}
$$

Gaussian elimination $\rightsquigarrow$

$$
\begin{array}{c}
\begin{bmatrix}
1 & 2 & -1 & 3 & 2 \\
0 & -1 & 0 & -1 & -3 \\
0 & 0 & 0 & 2 & 6
\end{bmatrix} \\
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & \hat{\boldsymbol{b}}
\end{array}
\end{array}
$$

Step 1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$

Step 2. Find all solutions to $\boldsymbol{Ax} = \boldsymbol{0}$

Step 3. Combine the solutions from steps 1. and 2. to the general solution

Step 1: Finding a particular solution to $\boldsymbol{Ax = b}$

Let free variables be 0, calculate the value of basic variables

$$\begin{array}{ccccc} & & & & \\ \left[\begin{array}{ccccc} 1 & 2 & -1 & 3 & 2 \\ 0 & -1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right] \\ x_1 \quad x_2 \quad x_3 \quad x_4 \quad \hat{\boldsymbol{b}} \end{array}$$

x_3: free variable

$x_1\ x_2\ x_4$: basic variables

$$\boxed{0 + 0 + 0 + 2x_4 = 6} \longrightarrow \boxed{x_4 = 3}$$

$$\boxed{0 - x_2 + 0 - x_4 = -3} \longrightarrow \boxed{x_2 = 0}$$

$$\boxed{\begin{array}{l} x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ x_1 + 0 - 0 + 9 = 2 \end{array}} \longrightarrow \boxed{x_1 = -7}$$

A particular solution:

$$\begin{bmatrix} -7 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Step 2: Find all solutions to $\boldsymbol{Ax} = \mathbf{0}$

Let one free variables be 1, and the rest free variables be 0, calculate the value of basic variables

$$\begin{array}{ccccc} \left[\begin{matrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{matrix}\right] \\ x_1 \quad x_2 \quad x_3 \quad \boxed{x_4} \quad \mathbf{0} \end{array}$$

We first immediately get $x_4 = 0$ from Row 3.

After setting $x_3 = 1$, we have
$0 - x_2 + 0 - x_4 = 0$,
$x_1 + 2x_2 - 1 + 3x_4 = 0$

$\longrightarrow$ all solutions to $\boldsymbol{Ax} = \mathbf{0}$: $\left\{ \boldsymbol{x} \in R^4 : \boldsymbol{x} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \lambda \in R \right\}$

Step 3: Combine the solutions from steps 1. and 2. to the general solution

all solutions to $\boldsymbol{Ax} = \boldsymbol{b}$: $\left\{ \boldsymbol{x} \in R^4 : \boldsymbol{x} = \begin{bmatrix} -7 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \lambda \in R \right\}$

Another example

$$\begin{aligned} -2x_1 &+ 4x_2 - 2x_3 - x_4 + 4x_5 = -3 \\ 4x_1 &- 8x_2 + 3x_3 - 3x_4 + x_5 = 2 \\ x_1 &- 2x_2 + x_3 - x_4 + x_5 = 0 \\ x_1 &- 2x_2 \qquad\quad - 3x_4 + 4x_5 = a \end{aligned}$$

$$\begin{bmatrix} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & a \end{bmatrix} \xrightarrow{\text{Swap R1 and R3}} \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{bmatrix}$$

$$\xrightarrow[\text{R4-R1 -> R4}]{\text{R2-4R1 -> R2, R3+2R1 -> R3}} \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{bmatrix} \xrightarrow{\text{R4-R2 -> R4}} \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & -3 & 6 & a-2 \end{bmatrix}$$

$$\xrightarrow{\text{R4-R3 -> R4}} \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{bmatrix}$$

a must equal -1 for this equation system to have solutions

Finding a particular solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

Let free variables be 0, calculate the value of basic variables

$$
\begin{array}{c}
\left[\begin{array}{ccccc|c}
1 & -2 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 3 & -2 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}
\end{array}
$$

It’s already in the REF. We let x_2 and x_5 be 0.

$$x_4 - 2x_5 = 1 \longrightarrow x_4 = 1$$

$$x_3 - x_4 + 3x_5 = -2 \longrightarrow x_3 = -1$$

$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0 \longrightarrow x_1 = 2$$

A particular solution:

$$\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Find all solutions to $\boldsymbol{Ax} = \mathbf{0}$

- Let one free variables be 1, and the rest free variables be 0, calculate the value of basic variables

$$
\begin{array}{cccccc}
\multicolumn{6}{c}{\begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \\
x_1 & x_2 & x_3 & x_4 & x_5 &
\end{array}
$$

Let x_2 be 1 and x_5 be 0. We get $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Let x_2 be 0 and x_5 be 1. We get $\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

all solutions to

$$\boldsymbol{Ax} = \mathbf{0}: \left\{ \boldsymbol{x} \in R^5 : \lambda_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \lambda_1, \lambda_2 \in R \right\}$$

How to find the general solution to $\boldsymbol{Ax} = \boldsymbol{b}$

- Step 3. Combine the solutions from steps 1. and 2. to the general solution

Step 1: $\boxed{\boldsymbol{Ax} = \boldsymbol{b}}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \#$$

Step 2: $\boxed{\boldsymbol{Ax} = \boldsymbol{0}}$

$$\left\{ \boldsymbol{x} \in R^5 : \boldsymbol{x} = \lambda_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \lambda_1, \lambda_2 \in R \right\}$$

General solution:

$$\left\{ \boldsymbol{x} \in R^5 : \boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \lambda_1, \lambda_2 \in R \right\}$$

Proof

- Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with $m < n$, then $\boldsymbol{Ax} = \boldsymbol{0}$ has infinitely many solutions
- Proof
- This system always has at least one solution since homogeneous
  - Consider $\boldsymbol{A}\boldsymbol{0} = \boldsymbol{0}$ always holds
- Matrix $\boldsymbol{A}$ brought in row echelon form contains at most m pivots.

For example, $\begin{bmatrix} \mathbf{1} & -2 & 0 & 0 & -2 \\ 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 5}$

- There will have $n - m \geq 1$ non-pivot columns, or free variables. It means we can find at least one solution $\boldsymbol{x}^* \neq \boldsymbol{0}$. Then, $\lambda \boldsymbol{x}^*$, $\lambda \in \mathbb{R}$ are solutions to $\boldsymbol{Ax} = \boldsymbol{0}$.

Proof

- A system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ either has no solutions, a unique solution or infinitely many solutions
- Proof

proof by contradiction

- Let's assume the system $\boldsymbol{Ax} = \boldsymbol{b}$ has two solutions $\boldsymbol{p}$ and $\boldsymbol{q}$.
- We have

a form of proof that establishes the truth or the validity of a proposition, by showing that assuming the proposition to be false leads to a contradiction.

$$\boldsymbol{Ap} = \boldsymbol{b} \qquad \boldsymbol{Aq} = \boldsymbol{b}$$

- Consider

$$\boldsymbol{v} = \boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p}), t \in \mathbb{R}$$

- We have

$$\boldsymbol{Av} = \boldsymbol{A}(\boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p})) = \boldsymbol{Ap} + t(\boldsymbol{Aq} - \boldsymbol{Ap}) = \boldsymbol{b} + t(\boldsymbol{b} - \boldsymbol{b}) = \boldsymbol{b}$$

- We thus have infinitely many solutions (by varying t)

Calculating the Inverse with Gaussian Elimination

- To compute the inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$,
- We need to find a matrix $\boldsymbol{X}$ that satisfies $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}_n$.
- Then, $\boldsymbol{X} = \boldsymbol{A}^{-1}$.
- We can write this down as a set of simultaneous linear equations $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}_n$, where we solve for $\boldsymbol{X} = [\boldsymbol{x}_1 | \cdots | \boldsymbol{x}_n]$
- We use the augmented matrix notation and use Gaussian Elimination.

$$[\boldsymbol{A} \mid \boldsymbol{I}_n] \rightsquigarrow \cdots \rightsquigarrow [\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}]$$

Calculating the Inverse with Gaussian Elimination

$$\left[\boldsymbol{A} \mid \boldsymbol{I}_n\right] \rightsquigarrow \cdots \rightsquigarrow \left[\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}\right]$$

Calculating the Inverse with Gaussian Elimination

- Example: determine the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in R^4$$

- First, write down the augmented matrix

$$\boldsymbol{A} = \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\qquad\qquad\qquad \boldsymbol{A} \qquad\qquad\qquad\qquad \boldsymbol{I_n}$$

Calculating the Inverse with Gaussian Elimination

- Use Gaussian elimination to bring it into reduced row-echelon form (RREF)

$$\boldsymbol{A} = \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \underset{\text{REF}}{\cdots} \rightsquigarrow \boldsymbol{A} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right]$$

$\boldsymbol{A}^{-1}$

- The desired inverse is given as its right-hand side

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Checking invertibility by calculating Reduced Row-echelon form - example

$$\begin{bmatrix} 2 & -2 & 4 & -2 \\ 2 & 1 & 10 & 7 \\ -4 & 4 & -8 & 4 \\ 4 & -1 & 14 & 6 \end{bmatrix} \xrightarrow[\text{R4-2R1 -> R4}]{\text{R2-R1 -> R2, R3+2R1 -> R3}} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 \end{bmatrix} \xrightarrow[\text{R3 and R4}]{\text{Swap}} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{R3-R2 -> R3}} \underbrace{\begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boxed{\text{REF}}} \xrightarrow[\text{by scalar}]{\text{Multiplication}} \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is not invertible

$$\xrightarrow[\text{R2-3R3 -> R2}]{\text{R1+R3 -> R1}} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{R1+R2 -> R1}} \underbrace{\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boxed{\text{RREF}}} \neq I_4$$

Moore-Penrose pseudo-inverse

- We can calculate $\boldsymbol{A}^{-1}$ only when $\boldsymbol{A}$ is a square matrix and is invertible
- Otherwise, under mild conditions, we can use the following pseudo-inverse:

$$\boldsymbol{A}x = \boldsymbol{b} \Leftrightarrow \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}x = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} \Leftrightarrow x = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$$

- $\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}$ is the Moore-Penrose pseudo-inverse of $\boldsymbol{A}$

https://en.wikipedia.org/wiki/Moore%E2%80%93Penrose_inverse

Check your understanding

- Which of the following are correct?

(A) A vector, when multiplied by a scale, is still a vector.

(B) For a system of linear equations with n variables, it is possible that none of them are free variables.

(C) For a system of linear equations with n variables, the maximum number of pivots in the REF is $n - 1$.

(D) A matrix, when added by an identity matrix, stays as is.

(E) We can use matrix transpose in Gaussian Elimination.

(F) Two arbitrary matrices can be multiplied.

(G) Two arbitrary matrices can be added.

(H) A greyscale image with black borders is not a matrix.

Check your understanding

- Let A, B, C be 2x2 matrices.
- Which of the following are equivalent to A(B+C)?
  - AB+AC
  - BA+CA
  - A(C+B)
  - (B+C)A

- Which of the following expressions are equivalent to $I_2(AB)$?
  - AB
  - BA
  - $(AB)I_2$
  - $(BA)I_2$

Next lecture: Textbook Sec. 2.4 - 2.7