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School of Computing Tutorial 1
Thang Bui & Jo Ciucă

COMP3670/6670: Introduction to Machine Learning

Question 1 **Systems of Linear Equations**

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

for some constants $b_1, \ldots, b_5 \in \mathbb{R}$.

1. Show that $\mathbf{A}$ is non-invertible.

   Solution. We row reduce $\mathbf{A}$ as follows,

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow (R_3 = R_3 + 1/3 \cdot R_1)$$

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow (R_5 = R_5 + R_3)$$

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Since we can row reduce the matrix to one with a zero row, this means that the matrix does not have a pivot in each column, and thus is non-invertible.

2. Find the set of solutions $\{\mathbf{x} : \mathbf{Ax} = \mathbf{b}\}$.

   Solution. We form the augmented matrix, and row reduce as above.

$$\left[\begin{array}{ccccc|c} 0 & 3 & 0 & 0 & 0 & b_1 \\ -2 & 0 & 1 & 0 & 0 & b_2 \\ 0 & -1 & 0 & -1 & 0 & b_3 \\ 0 & 0 & -1 & 0 & 4 & b_4 \\ 0 & 0 & 0 & 1 & 0 & b_5 \end{array}\right]$$

$$\Big\downarrow (R_3 = R_3 + 1/3 \cdot R_1)$$

$$\left[\begin{array}{ccccc|c} 0 & 3 & 0 & 0 & 0 & b_1 \\ -2 & 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -1 & 0 & b_3 + 1/3 \cdot b_1 \\ 0 & 0 & -1 & 0 & 4 & b_4 \\ 0 & 0 & 0 & 1 & 0 & b_5 \end{array}\right]$$

$$\Big\downarrow (R_5 = R_5 + R_3)$$

$$\left[\begin{array}{ccccc|c} 0 & 3 & 0 & 0 & 0 & b_1 \\ -2 & 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -1 & 0 & b_3 + 1/3 \cdot b_1 \\ 0 & 0 & -1 & 0 & 4 & b_4 \\ 0 & 0 & 0 & 0 & 0 & b_5 + b_3 + 1/3 \cdot b_1 \end{array}\right]$$

Now, if $b_5 + b_3 + 1/3 \cdot b_1 \neq 0$, then we have a contradiction, and no solutions exist.
If $b_5 + b_3 + 1/3 \cdot b_1 = 0$, then we obtain

$$\begin{aligned} x_2 &= 1/3 \cdot b_1 \\ -2x_1 + x_3 &= b_2 \\ x_4 &= -b_3 - 1/3 \cdot b_1 \\ -x_3 + 4x_5 &= b_4 \end{aligned}$$

We know there is one free variable from the last row of the reduced matrix. We can choose any one of x_i to be the free variable. For example, if we choose x_3 to be the free variable, we'll then need to represent other variable(s) using the free variable(s). In this case, the solution can be rearranged as

$$\begin{aligned} x_2 &= 1/3 \cdot b_1 + 0 \cdot x_3 \\ x_1 &= -1/2 \cdot b_2 + 1/2 \cdot x_3 \\ x_4 &= -b_3 - 1/3 \cdot b_1 + 0 \cdot x_3 \\ x_5 &= 1/4 \cdot b_4 + 1/4 \cdot x_3 \\ x_3 &= 0 + x_3 \end{aligned}$$

Note how we separate constants and variables. Now, write this in the vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1/2 \cdot b_2 \\ 1/3 \cdot b_1 \\ 0 \\ -b_3 - 1/3 \cdot b_1 \\ 1/4 \cdot b_4 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 0 \\ 1/4 \end{bmatrix}, x_3 \in \mathbb{R}$$

so the solution space can be written as

$$\begin{cases} \left\{ \begin{bmatrix} -1/2 \cdot b_2 \\ 1/3 \cdot b_1 \\ 0 \\ -b_3 - 1/3 \cdot b_1 \\ 1/4 \cdot b_4 \end{bmatrix} + \alpha \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 0 \\ 1/4 \end{bmatrix}, : \alpha \in \mathbb{R} \right\} & \text{if } b_5 + b_3 + 1/3 \cdot b_1 = 0 \\ \varnothing & \text{if } b_5 + b_3 + 1/3 \cdot b_1 \neq 0 \end{cases}$$

3. Hence, or otherwise, find a non-zero value for $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$.

Solution. Simply let all the b_i be zero, and use the same solution set as before

$$\left\{ \alpha \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 0 \\ 1/4 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

We can then obtain the required value of $\mathbf{x}$ by choosing α to be any non-zero constant, say, 1. Hence, choosing

$$\mathbf{x} = \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 0 \\ 1/4 \end{bmatrix}$$

satisfies $\mathbf{Ax} = \mathbf{0}$.

Question 2 — Matrix Inverses

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for some constants $a, b, c \in \mathbb{R}$.

1. For what values of a, b, c is the inverse of $\mathbf{A}$ defined?

Solution. We directly compute the inverse via row reduction, and we don't require any assumptions on a, b, c for the inverse to exist.
Row reduce $[\boldsymbol{A}\ I]$ to get $[I\ \boldsymbol{A}^{-1}]$ as follows (here R_i stands for ith row):

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} (R_1 = R_1 + aR_2) \\ (R_2 = R_2 + bR_2) \\ (R_3 = R_3 + cR_4) \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -ab & 0 & 1 & a & 0 & 0 \\ 0 & 1 & 0 & -bc & 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} (R_1 = R_1 + abR_3) \\ (R_2 = R_2 + bcR_4) \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

2. Find $\mathbf{A}^{-1}$ assuming the properties on a, b, c to ensure the inverse exists.

   Solution. We found $\mathbf{A}^{-1}$ above.

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 3 **Which matricies commute?**

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Find all matrices $\mathbf{B} \in \mathbb{R}^{2\times 2}$ such that $\mathbf{AB} = \mathbf{BA}$.

Solution. We write $\mathbf{B}$ as an arbitrary 2×2 matrix, form the equation $\mathbf{AB} = \mathbf{BA}$, and then find what constraints are required on $\mathbf{B}$. So, we can write $\mathbf{B}$ as

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and then expand out $\mathbf{AB} = \mathbf{BA}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a-c & b-d \\ c-a & d-b \end{bmatrix} = \begin{bmatrix} a-b & b-a \\ c-d & d-c \end{bmatrix}$$

This gives us the 4 constraints

$$a - c = a - b$$
$$b - d = b - a$$
$$c - a = c - d$$
$$d - b = d - c$$

which, when rearranged (and removing redundant equations) gives

$$a = d \quad b = c$$

when means that $\mathbf{AB} = \mathbf{BA}$ if and only if

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for some $a, b \in \mathbb{R}$.

Question 4 **Proving Properties of Matrix Operations**

For each of the following statements, if it is true, prove it. If it is false, give a counter-example.

1. Let $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Assume that both $\mathbf{A}$ and $\mathbf{B}$ are invertible.
   Does $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ hold?

   Solution. True, we merely need to verify that $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is an inverse of $\mathbf{AB}$, by left multiplying to see if we obtain the identity, and the same with right multiplication.

$$\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{AB} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$
$$\mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

2. Let $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Assume that both $\mathbf{A}$ and $\mathbf{B}$ are invertible.
   Does $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$ hold?
   Note: in general, $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{A}^{-1}$ [1]

   Solution. False, we choose $\mathbf{A} = \mathbf{I}$ and $\mathbf{B} = -\mathbf{I}$. Then, $\mathbf{A}^{-1} + \mathbf{B}^{-1} = \mathbf{I} + -\mathbf{I} = \mathbf{0}$, but $(\mathbf{A}+\mathbf{B})^{-1} = (\mathbf{I} + -\mathbf{I})^{-1} = \mathbf{0}^{-1}$, which is undefined.

3. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Both $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ are well-defined[2] and symmetric[3] matrices.

   Solution. True, as shown
   $$(\mathbf{A}\mathbf{A}^T)^T = (\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T$$
   $$(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A}$$
   Also note that $\mathbf{A} \in \mathbb{R}^{m\times n}$, so the products $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ are well-defined.

4. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. If $\mathbf{A}$ is non-invertible, then there must exist two different vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$.

   Solution. False, consider the matrix
   $$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
   which is clearly non-invertible, as it isn't even square.
   Then, taking two arbitrary vectors
   $$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$
   we evaluate the equation $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$ which gives
   $$\begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$$
   which is true iff $\mathbf{u} = \mathbf{v}$.

5. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. If there exists two different vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$, then $\mathbf{A}$ is non-invertible.

   Solution. True, assume for a contradiction that $\mathbf{A}$ is invertible. Then,
   $$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{A}\mathbf{v} \\ \mathbf{A}^{-1}\mathbf{A}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{A}\mathbf{v} \\ \mathbf{I}\mathbf{u} &= \mathbf{I}\mathbf{v} \\ \mathbf{u} &= \mathbf{v} \end{aligned}$$
   a contradiction, as we have that $\mathbf{u}$ and $\mathbf{v}$ are different.

6. If there exists two different vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$, then there exists a non-zero vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$.

   Solution. True, as shown,
   $$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{A}\mathbf{v} \\ \mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{u} &= \mathbf{A}\mathbf{v} - \mathbf{A}\mathbf{u} \\ \mathbf{0} &= \mathbf{A}(\mathbf{v} - \mathbf{u}) \end{aligned}$$
   Since $\mathbf{v} \neq \mathbf{u}$ we have that $\mathbf{v} - \mathbf{u} \neq \mathbf{0}$, as required.

[1] a special case of Woodbury matrix identity
[2] as in, the matrix product is defined
[3] A *symmetric* matrix is one equal to its transpose.