The Australian National University Semester 2, 2023
School of Computing Tutorial 1
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COMP3670/6670: Introduction to Machine Learning

Question 1 **Systems of Linear Equations**

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

for some constants $b_1, \ldots, b_5 \in \mathbb{R}$.

1. Show that $\mathbf{A}$ is non-invertible.
2. Find the set of solutions $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$.
3. Hence, or otherwise, find a non-zero value for $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Question 2 **Matrix Inverses**

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for some constants $a, b, c \in \mathbb{R}$.

1. For what values of a, b, c is the inverse of $\mathbf{A}$ defined?
2. Find $\mathbf{A}^{-1}$ assuming the properties on a, b, c to ensure the inverse exists.

Question 3 **Which matricies commute?**

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Find all matrices $\mathbf{B} \in \mathbb{R}^{2\times 2}$ such that $\mathbf{AB} = \mathbf{BA}$.

Question 4 **Proving Properties of Matrix Operations**

For each of the following statements, if it is true, prove it. If it is false, give a counter-example.

1. Let $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Assume that both $\mathbf{A}$ and $\mathbf{B}$ are invertible.
   Does $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ hold?
2. Let $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Assume that both $\mathbf{A}$ and $\mathbf{B}$ are invertible.
   Does $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$ hold?
   Note: in general, $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1}\mathbf{A}^{-1}$ [1]

[1] a special case of Woodbury matrix identity

3. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Both $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ are well-defined[2] and symmetric[3] matrices.

4. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. If $\mathbf{A}$ is non-invertible, then there must exist two different vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{Au} = \mathbf{Av}$.

5. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. If there exists two different vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{Au} = \mathbf{Av}$, then $\mathbf{A}$ is non-invertible.

6. If there exists two different vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{Au} = \mathbf{Av}$, then there exists a non-zero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$.

[2] as in, the matrix product is defined

[3] A *symmetric* matrix is one equal to its transpose.