COMP2610/COMP6261 – Information Theory

Tutorial 6

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Question 1. Entropy

Let $p = (p_1, p_2, \cdots, p_m)$ be a probability distribution on m elements, i.e, $p_i \geq 0$, and $\sum_{i=1}^m p_i = 1$. Define a new distribution q on $m-1$ elements as $q_1 = p_1$, $q_2 = p_2$, $\cdots$, $q_{m-2} = p_{m-2}$, and $q_{m-1} = p_{m-1} + p_m$, i.e., the distribution q is the same as p on any $i \in \{1, 2, \cdots, m-2\}$, and the probability of the last element in q is the sum of the last two probabilities of p. Show that

$$H(p) = H(q) + (p_{m-1} + p_m)H(\frac{p_{m-1}}{p_{m-1} + p_m}, \frac{p_m}{p_{m-1} + p_m}).$$

Question 2. Mutual Information and Relative Entropy

Let X, Y, Z be three random variables with a joint probability mass function $p(X,Y,Z)$.
(a). Show that

$$I(X,Y;Z) - I(Y,Z;X) = I(Y;Z) - I(X;Y).$$

(b). The relative entropy between the joint distribution and the product of the marginals is $D(p(x,y,z)||p(x)p(y)p(z))$. Show that

$$D(p(x,y,z)||p(x)p(y)p(z)) = I(X;Y) + I(X,Y;Z).$$

Question 3. Markov Chain

Suppose a Markov chain $X_1 \to X_2 \to X_3$, starts in one of n states, i.e., $X_1 \in \{1, 2, \cdots, n\}$. Suppose X_2 will go down to $k < n$ states, i.e., $X_2 \in \{1, 2, \cdots, k\}$. Then X_3 go back to $m > k$ states, i.e., $X_3 \in \{1, 2, \cdots, m\}$.
(a). What is the upper bound of $I(X_1; X_3)$?
(b). Evaluate $I(X1; X3)$ for $k = 1$, and conclude that no dependence can survive.

Question 4. Inequalities

A coin is known to land heads with probability $\frac{1}{5}$. The coin is flipped N times for some even integer N.
(a). Using Markov's inequality, provide a bound on the probability of observing $\frac{N}{2}$ or more heads.
(b). Using Chebyshev's inequality, provide a bound on the probability of observing $\frac{N}{2}$ or more heads. Express your answer in terms of N.