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Tutorial 2

COMP3670/6670: Introduction to Machine Learning

Question 1 **Matrix Properties**

1. **Uniqueness of inverses**
   Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Assume $\mathbf{A}$ is invertible. Prove that the inverse of $\mathbf{A}$ is unique, (that is, there is only one matrix $\mathbf{B}$ that satisfies $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$)

   Solution. Assume not for contradiction. Then at least two inverses of $\mathbf{A}$ must exist (as $\mathbf{A}$ is invertible.) Let $\mathbf{X}$ and $\mathbf{Y}$ denote distinct inverses of $\mathbf{A}$. (i.e that $\mathbf{X} \neq \mathbf{Y}$). Then by definition,

   $$\mathbf{XA} = \mathbf{AX} = \mathbf{I}$$

   $$\mathbf{YA} = \mathbf{AY} = \mathbf{I}$$

   So then

   $$\mathbf{AY} = \mathbf{AX}$$

   Left multiplying by any inverse of $\mathbf{A}$ (we choose $\mathbf{X}$).

   $$\mathbf{X}(\mathbf{AY}) = \mathbf{X}(\mathbf{AX})$$

   $$(\mathbf{XA})\mathbf{Y} = (\mathbf{XA})\mathbf{X}$$

   $$\mathbf{IY} = \mathbf{IX}$$

   $$\mathbf{Y} = \mathbf{X}$$

   which is a contradiction. Hence inverses are unique.

2. **Inverse of an inverse**
   Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Assume $\mathbf{A}$ is invertible. Prove that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

   Solution. We need to find a matrix $\mathbf{X}$ such that

   $$\mathbf{XA}^{-1} = \mathbf{A}^{-1}\mathbf{X} = \mathbf{I}$$

   Choose $\mathbf{X} = \mathbf{A}$. Note from the definition of the inverse, we have that

   $$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

   Hence by definition, the inverse of $\mathbf{A}^{-1}$ is $\mathbf{A}$, and

   $$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

3. **Distributing the transpose**
   For $\boldsymbol{A} \in R^{m\times n}$, $\boldsymbol{B} \in R^{m\times n}$, prove that $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$

   Solution. We check the i, jth element, and verify they both match.

   $$\begin{aligned}
   &(\mathbf{A} + \mathbf{B})^T_{i,j} \\
   &= (\mathbf{A} + \mathbf{B})_{j,i} \\
   &= \mathbf{A}_{j,i} + \mathbf{B}_{j,i} \\
   &= \mathbf{A}^T_{i,j} + \mathbf{B}^T_{i,j} \\
   &= (\mathbf{A}^T + \mathbf{B}^T)_{i,j}
   \end{aligned}$$

   The above proof works as addition is performed elementwise.

4. **Matrix Cancellation**
   Let $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ all be square matrices of the same dimension. Assume $\mathbf{AB} = \mathbf{AC}$. Does it always follow that $\mathbf{B} = \mathbf{C}$?

   Solution. If $\mathbf{A}$ is invertible, then yes, as

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\mathbf{AC} \\ \mathbf{B} &= \mathbf{C} \end{aligned}$$

   If $\mathbf{A}$ isn't invertible, then it might not hold. (E.g. If $\mathbf{A}$ was the zero matrix, then the equation would hold for any $\mathbf{B}$ and $\mathbf{C}$.)

Question 2 **Moore-Penrose Inverse**

Assuming $\mathbf{A}$ is invertible, prove that the Moore-Penrose inverse $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ equals $\mathbf{A}^{-1}$.

How does this show that the Moore-Penrose inverse is more general than the inverse?

Give an example of a matrix that does not have a Moore-Penrose inverse.

Solution.

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \mathbf{A}^{-1}(\mathbf{A}^T)^{-1}\mathbf{A}^T = \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$$

This is more general than the inverse, as the Moore-Penrose inverse can be defined for non-square matrices, e.g.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The square zero matrix of any dimension $\mathbf{Z}$ has no Moore-Penrose inverse, as $\mathbf{Z}^T\mathbf{Z} = \mathbf{Z}$, and thus $(\mathbf{Z}^T\mathbf{Z})^{-1}$ is undefined.

Question 3 **Linear Equations**

Prove that a system of linear equations $\mathbf{Ax} = \mathbf{b}$ either has no solutions, a unique solution or infinitely many solutions.

(This was done in lecture slides, but try to write the proof in great detail.)

(Hint: If there are at least two solutions $\mathbf{p}$ and $\mathbf{q}$, consider the vector $\mathbf{v}_\lambda = \lambda\mathbf{p} + (1 - \lambda)\mathbf{q}$.)

Solution. If $\mathbf{Ax} = \mathbf{b}$ has no solutions or a unique solution, we are done. So assume not. So there exists at least two distinct solutions $\mathbf{p}$ and $\mathbf{q}$. So we have $\mathbf{Ap} = \mathbf{b}$ and $\mathbf{Aq} = \mathbf{b}$. For some $\lambda \in \mathbb{R}$, let

$$\mathbf{v}_\lambda = \lambda\mathbf{p} + (1 - \lambda)\mathbf{q}$$

Then,

$$\begin{aligned} \mathbf{Av}_\lambda &= \mathbf{A}(\lambda\mathbf{p} + (1 - \lambda)\mathbf{q} \\ &= \lambda\mathbf{Ap} + (1 - \lambda)\mathbf{Aq} \\ &= \lambda\mathbf{b} + (1 - \lambda)\mathbf{b} \\ &= \mathbf{b} \end{aligned}$$

Hence $\mathbf{v}_\lambda$ is a solution for any $\lambda \in \mathbb{R}$, and we have infinitely many solutions.

Question 4 **Vector Subspaces**

Prove that the set of solutions to $\mathbf{Ax} = \mathbf{b}$ is a vector subspace [1] if and only if $\mathbf{b} = \mathbf{0}$.

[1]As a reminder, to check if a non-empty set $E \subseteq V$ is a vector subspace of V, we need to check two things:
Closure under addition: For every $\mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} + \mathbf{y} \in U$.
Closure under scalar multiplication: For every $\lambda \in \mathbb{R}, \mathbf{u} \in U$ we have $\lambda\mathbf{u} \in U$.

Solution. Assume $\mathbf{b} = \mathbf{0}$. The set of solutions is not empty, as $\mathbf{A0} = \mathbf{0}$. Let $\mathbf{v}$ and $\mathbf{u}$ denote two solutions. Then the sum $\mathbf{v} + \mathbf{u}$ is also a solution, as

$$\mathbf{A}(\mathbf{v} + \mathbf{u}) = \mathbf{Av} + \mathbf{Au} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

We can scalar multiply any solution $\mathbf{v}$ and still have a solution, as

$$\mathbf{A}(\lambda\mathbf{v}) = \lambda\mathbf{Av} = \lambda\mathbf{0} = \mathbf{0}$$

hence the set of solutions to $\mathbf{Ax} = \mathbf{b}$ is a subspace.

Assume that $\mathbf{b} \neq \mathbf{0}$. Then closure under scalar multiplication fails, as if $\mathbf{v}$ was a solution, then

$$\mathbf{A}(2\mathbf{v}) = 2\mathbf{Av} = 2\mathbf{b} \neq \mathbf{b}$$

and hence, the set of solutions to $\mathbf{Ax} = \mathbf{b}$ is not a subspace.

Question 5 **Linear Independence**

Let $\mathbf{T} \in \mathbb{R}^{n \times m}$ be a matrix. Let $\{\mathbf{u}, \mathbf{v}\}$ be a set of linearly independent vectors in $\mathbb{R}^{m \times 1}$. Assume that $\{\mathbf{Tu}, \mathbf{Tv}\}$ are linearly dependent. Prove there exists non-zero $\mathbf{x} \in \mathbb{R}^{m \times 1}$ such that $\mathbf{Tx} = \mathbf{0}$.

Solution. Linear dependence means there exists scalars c_1 and c_2, at least one of them non-zero, such that

$$c_1\mathbf{Tu} + c_2\mathbf{Tv} = \mathbf{0}$$

Using the fact that matrix multiplication distributes over scalar multiplication and vector addition,

$$\mathbf{T}(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$$

Now since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and it is not the case that both c_1 and c_2 are zero, it follows that $(c_1\mathbf{u} + c_2\mathbf{v}) \neq \mathbf{0}$, hence we have a non-zero solution to $\mathbf{Tx} = \mathbf{0}$.

Question 6 **Combining vector subspaces**

Let V be a vector space. Let $A \subseteq V$ and $B \subseteq V$ be vector subspaces of V.

1. Prove that $A \cap B$ is a vector subspace of V.

   Solution. We need to check the two properties.
   Let $\mathbf{x}, \mathbf{y}$ be in $A \cap B$. Then $\mathbf{x}$ and $\mathbf{y}$ are in A, and so $\mathbf{x} + \mathbf{y} \in A$, since A is a vector subspace, and is closed under addition. By a similar argument, $\mathbf{a} + \mathbf{b} \in B$. Hence, $\mathbf{a} + \mathbf{b} \in A \cap B$, and the set is closed under addition. Let $\lambda \in \mathbb{R}, \mathbf{x} \in A \cap B$. Then $\mathbf{x} \in A$ and $\mathbf{x} \in B$. Since both A and B are vector subspaces, $\lambda\mathbf{x} \in A, \lambda\mathbf{x} \in B$. Thus $\lambda\mathbf{x} \in A \cap B$, and the set is closed under scalar multiplication.

2. **(Tricky)** Prove that $A \cup B$ is a vector subspace of V if and only if A is contained in B, or B is contained in A.

   (This proof is easy in one direction, and tricky the other direction. As a hint, if the sets are not contained in each other, then there must lie a vector in $A \backslash B$ and in $B \backslash A$. Consider the sum of these vectors.)

   Solution. Assume that one of A or B is contained in the other. If $A \subseteq B$, then $A \cup B = B$, and the result immediately follows, as B is a vector subspace. Similar argument for $B \subseteq A$.

   Assume A is not contained in B, and vice versa. Assume for contradiction that $A \cup B$ is a vector subspace. So there must exist an element $\mathbf{a} \in A \backslash B$ and an element $\mathbf{b} \in B \backslash A$. Consider $\mathbf{a} + \mathbf{b}$. Since by assumption $A \cup B$ is a vector subspace, it must be closed under vector addition. So $\mathbf{a} + \mathbf{b}$ lies in A or B (or both.) If $\mathbf{a} + \mathbf{b}$ is in A, then note that $(-1)\mathbf{a} = -\mathbf{a}$ is also in A (by closure under scalar multiplication), but $(\mathbf{a} + \mathbf{b}) + (-\mathbf{a}) = \mathbf{b}$ is not in A, violating the property of closure under addition.
   We can make the same argument if $\mathbf{a} + \mathbf{b}$ is in B.
   In either case we get a contradiction, and $A \cup B$ cannot be a vector subspace.