Assignment-2

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Question 1

Question (a):

$Prob(a,b \in S_i, F \cap S_i = \emptyset) = (\frac{1}{k-1})^2(1-\frac{1}{k-1})^{k-1} \geq \frac{1}{4(k-1)^2}$
$Prob(\forall i, S_i$ does not separate a,b from F $) = \prod_{1\leq i\leq r}(1 - Prob(a,b \in S_i, F \cap S_i = \emptyset)) \leq (1-\frac{1}{4(k-1)^2})^r \leq \frac{1}{n^t}$, where r=$4t(k-1)^2\log_e(n)$.
$Prob(\exists((a,b),F)$satisfying (a,b),F are not separated by $S_i) \leq \sum_{((a,b),F)\in V^2\times V^{k-1}} \frac{1}{n^t} \leq \frac{n^{k+1}}{n^t}$
$\therefore$ t should be k+2
$\therefore r = 4(k+2)(k-1)^2\log_e n$

Question (b):

Every spanning forest T_i will have $\leq |V_i|\ edges.$
Probability of a vertex appearing=$\frac{1}{k-1}$.
The expected number of edges in H $\leq \sum_{i=1}^{r}(\frac{1}{k-1})^{|V_i|}|V_i| \leq \sum_{i=1}^{r}\frac{1}{k-1}n$
$= \frac{n}{k-1}4(k+2)(k-1)^2\log_e n \approx O(nk^2 logn)$

$Prob(a \in G, a \notin H) = (1-\frac{1}{k-1})^r \leq \frac{1}{n^{4(k+2)(k-1)}}$
$\implies$ G and H share the same set of vertices with high probability.
Given: G is k-vertex connected.
To prove: H is k-vertex connectivity preserver of G with high probability.
Suppose H is not k-vertex connected.
$\therefore$ $\exists$ a vertex cut of size at most k-1 upon whose removal, H gets disconnected. We call this vertex cut F. Since G is given to be k-vertex connected, removal of F should not disconnect G $\implies$ there must exist a path containing, say vertices a and b, in G-F. With a probability $\geq 1-\frac{1}{n}$, there must exist some S_i that contains a,b but $F \cap S_i = \emptyset$. $\therefore$ No vertex from F is contained in S_i. $\therefore$ There must exist a path connecting a and b in T_i that excludes vertices present in F. Therefore, there must exist a path connecting a and b in H-F as well since G and H share the same set of vertices with high probability. Thus, removal of F does not disconnect H and thus, our earlier claim of H not being k-vertex connected stands false. This proves that H is indeed a k-vertex connectivity preserver of G.

Question 2:

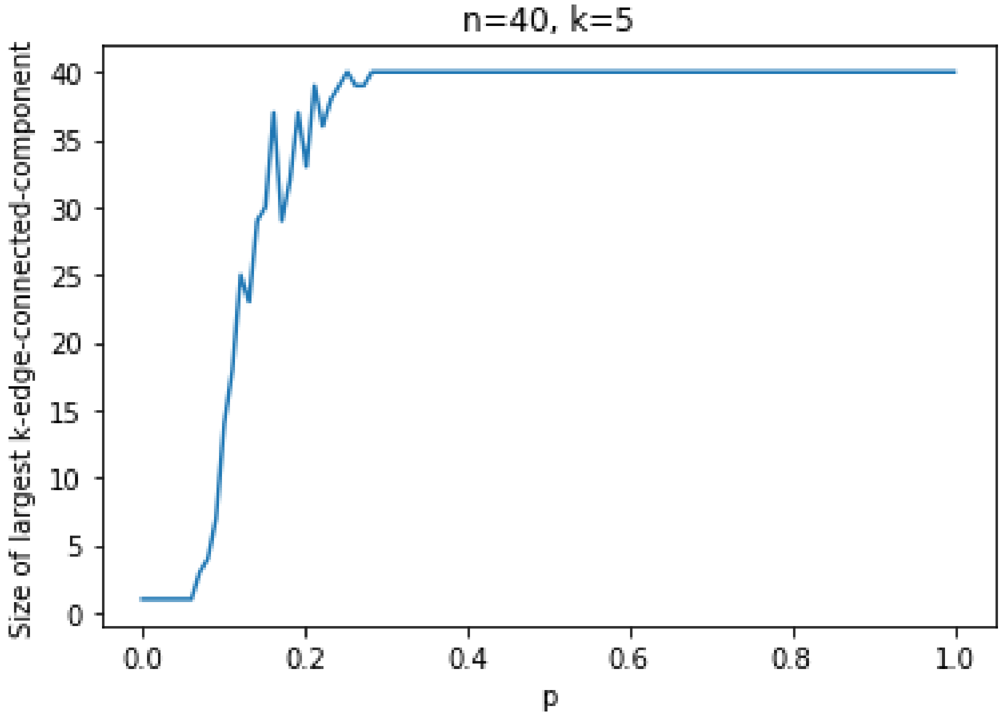
Question a:

P(degree of vertex=d)= $\binom{n-1}{d}p^d(1-p)^{n-1-d}$
$\therefore$ P(node v is isolated) = $(1-p)^{n-1}$
P(i, n-i)= P(a set of nodes is disconnected from n-i nodes where $i \in [1, n/2]$)= $\binom{n}{i}(1-p)^{i(n-i)}$, as i(n-i) edges will have to be absent to disconnect the partition (i, n-i).
$\therefore$ P(G is disconnected) $\leq \sum_{i=1}^{n/2}\binom{n}{i}(1-p)^{i(n-i)} = \sum_{i=1}^{n/2}\binom{n}{i}(1-\frac{5\log n}{n})^{i(n-i)} \leq \sum_{i=1}^{n/2}\binom{n}{i}\frac{1}{n^{\frac{5i(n-i)}{n}}} \leq \sum_{i=1}^{n/2}\frac{n^i}{i!}\frac{1}{n^{\frac{5i(n-i)}{n}}} \leq \sum_{i=1}^{n/2}\frac{n^i}{n^{\frac{5i}{2}}} = \sum_{i=1}^{n/2}\frac{1}{n^{\frac{3i}{2}}} \leq \frac{n}{n^3} = \frac{1}{n^2} \leq \frac{1}{n} \implies$ G is connected with probability at least $1-\frac{1}{n}$.

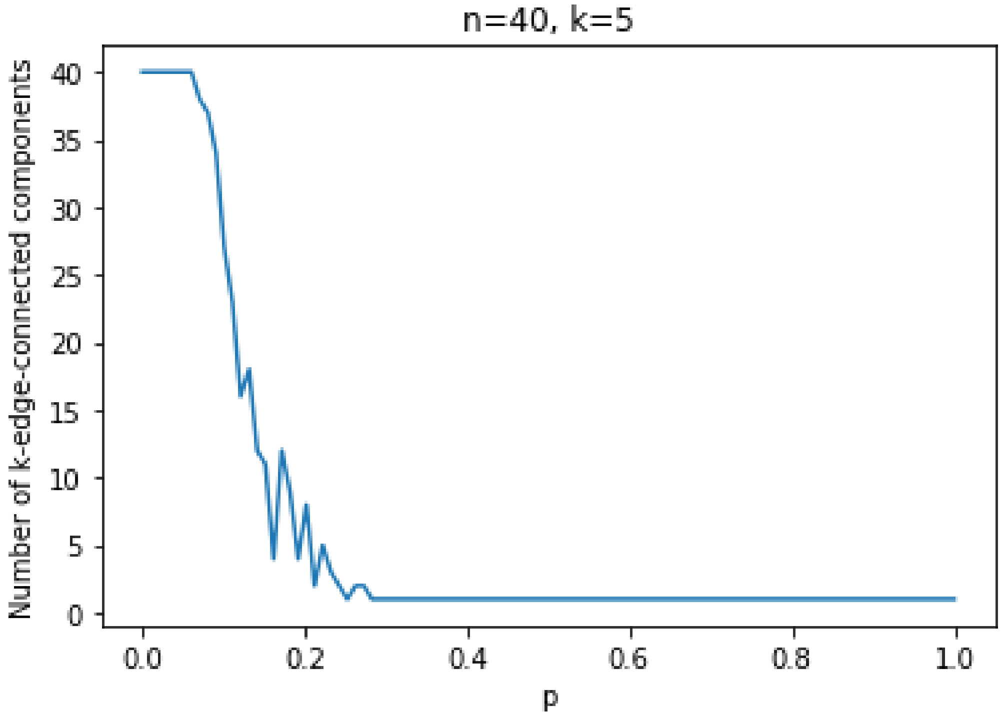
Question b:

Graph G should be a k-edge connected graph. $\therefore$ G should remain connected if k-1 edges are removed but should get disconnected if k edges are removed. Suppose G splits into (i, n-i-k) when k edges are removed but remains connected if k-1 edges are removed instead. Prob(G is a k-edge connected graph) $\leq \sum_{t=k}^{ik}\binom{n}{k}\binom{n-k}{i}\binom{ik}{t}p^t(1-p)^{i(n-i)-t}$, where $\binom{n}{k}$ denotes choosing k edges out of n; $\binom{n-k}{i}$ denotes choosing i nodes out of the remaining n-k nodes upon k edges removal; $\binom{ik}{t}$ denotes choosing edges that keep the graph connected when k-1 edges are removed.
$\therefore$ P(G is k-edge connected) $\leq \sum_{t=k}^{ik}\frac{n^k}{k!}\frac{(n-k)^i}{i!}\frac{(ik)^t}{t!}p^t(1-p)^{i(n-i)-t} \leq \sum_{t=k}^{ik}\frac{n^k}{k!}\frac{(n-k)^i}{i!}\frac{(ik)^t}{t!}p^t(1-p)^{i(n-i)-ik} \leq \sum_{t=k}^{ik}\frac{n^k}{k!}\frac{(n-k)^i}{i!}\frac{(ik)^t}{t!}(\frac{cklogn}{n})^t n^{-(cki(n-i-k))/n} \leq \frac{n^2}{n^c} \leq \frac{1}{n}$.
$\implies$ GH Tree of G has edge weights of at least k with a probability at least $1-\frac{1}{n}$.

Question c:



n=40, k=5
Size of largest k-edge-connected-component
0
5
10
15
20
25
30
35
40
0.0
0.2
0.4
0.6
0.8
1.0
p



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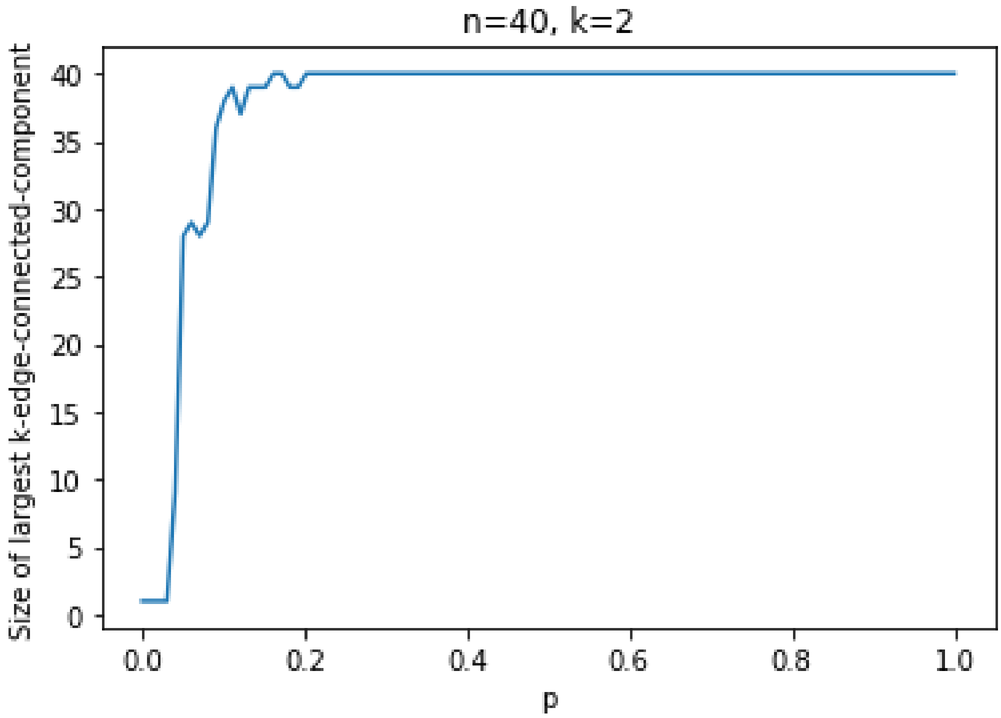

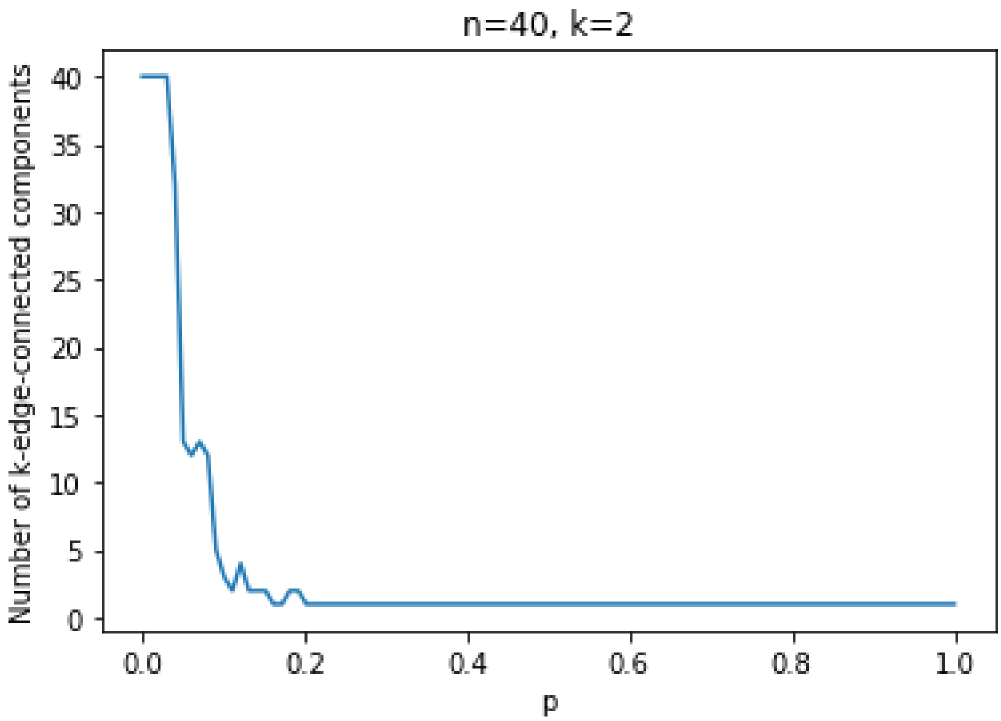

Adding an edge will reduce the number of connected components to create a larger connected component, as evident in the graphs. Upon increasing the probability of an edge getting added, the number of connected component decreases, while the size of largest connected component increases. When p is of the order O($\frac{\lambda logn}{n}$), where $\lambda > 1$, the graph G(n,p) is connected with a probability $\geq 1-\frac{1}{n}$, as shown in Q2 part a and b. As evident in the plots, the size and number of connected components change dramatically when p reaches order $O(\frac{logn}{n}) \approx log(40)/40 = 0.04$. Once p crosses this threshold, the size of largest connected component increases while the number of connected components decreases as the graph begins to get connected.