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Mid-Sem PHY401A (Possible solution set)

1.) (a.) $L = \dot{q}^2 - \dot{q}e^{-q} \Rightarrow \frac{\partial L}{\partial \dot{q}} = 2\dot{q} - e^{-q}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 2\ddot{q} + e^{-q}\dot{q} \quad \text{and} \quad \frac{\partial L}{\partial q} = \dot{q}e^{-q}$$

$\therefore$ EL equation $\Rightarrow \ddot{q} = 0 \rightarrow$ Free particle

Alternatively: $-\dot{q}e^{-q} = \frac{d}{dt}(e^{-q})$ which is a gauge over a free particle Lagrangian.

$\rightarrow$ Eq^n of motion will be identical to that of a free particle.

(b.) Since $\ddot{q} = 0 \Rightarrow \dot{q}$ is an integral of motion

Alternative: Since $\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}}\dot{q} - L$ is an integral of motion

$$= (2\dot{q} - e^{-q})\dot{q} - (\dot{q}^2 - \dot{q}e^{-q})$$

$= \dot{q}^2$ is an integral of motion.

(c.) The Lagrangian contains $q \Rightarrow q$ is not a cyclic coordinate $\Rightarrow$ generalized momentum is NOT conserved.

(2.) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - K(x^2 - y^2 - z^2)$

Option 1 $\rightarrow$ $x = x,\ y = \rho\cos\phi,\ z = \rho\sin\phi$ or

Cylindrical polar system $x = x,\ y = \rho\sin\phi,\ z = \rho\cos\phi$

and then the Lagrangian is given by,

$$L=\frac{m}{2}\left(\dot{x}^2+\dot{\rho}^2+\rho^2\dot{\phi}^2\right)-k\left(x^2-\rho^2\right)$$

where ϕ is a cyclic coordinate.

$$p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x},\ p_\phi=\frac{\partial L}{\partial \dot{\phi}}=\underbrace{m\rho^2\dot{\phi}}_{\text{Constant}},\ p_\rho=\frac{\partial L}{\partial \dot{\rho}}=m\dot{\rho}$$

Option 2 → Spherical polar system

$x=r\cos\theta,\ y=r\sin\theta\cos\phi,\ z=r\sin\theta\sin\phi$

or $x=r\cos\theta,\ y=r\sin\theta\sin\phi,\ z=r\sin\theta\cos\phi$

Then the Lagrangian can be given by

$$L=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\phi}^2\right)-kr^2\cos 2\theta$$

where ϕ is cyclic coordinate.

$$p_r=\frac{\partial L}{\partial \dot{r}}=m\dot{r},\ p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta},$$

$$p_\phi=\frac{\partial L}{\partial \dot{\phi}}=\underbrace{mr^2\sin^2\theta\,\dot{\phi}}_{\text{Constant}}$$

Option 3 →

$x=r\sin\theta,\ y=r\cos\theta\cos\phi,\ z=r\cos\theta\sin\phi$

etc.

(3.) $L = \frac{1}{2}\dot{q}\sin^2 q$

ⓐ $E = \left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} - L = \left(\frac{1}{2}\sin^2 q\right)\dot{q} - \frac{1}{2}\dot{q}\sin^2 q = 0$

ⓑ $\frac{\partial L}{\partial \dot{q}} = \frac{1}{2}\sin^2 q \Rightarrow \frac{d}{dt}\left(\frac{1}{2}\sin^2 q\right) = \sin q \cos q \,\dot{q}$

and $\frac{\partial L}{\partial q} = \sin q \cos q \,\dot{q}$

So, EL equation is satisfied for all $q(t)$.

ⓒ $L = \frac{1}{2}\dot{q}\sin^2 q$ is a Gauge function to a zero / null Lagrangian

Alternatively: L is such that the action is extremized for all values of $q(t)$ $\Rightarrow$ all path of evolution possible $\Rightarrow$ Lagrangian does not represent a physical system.

(4.) $L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 + kx$

$$= \frac{m}{2}\dot{x}^2 - \frac{k}{2}\left[x^2 - 2x + 1\right] + \frac{k}{2}$$

So, an equivalent L is

$$L' = \frac{m}{2}\dot{y}^2 - \frac{k}{2}y^2 \quad [\text{as } y = x - 1]$$

$\longrightarrow$ Simple Harmonic Oscillator

OR, L signifies a Simple Harmonic Oscillator under a Linear potential (e.g. gravity)

(5.) $L = \frac{1}{12} m^2 \dot{x}^4 + m\dot{x}^2 V(x) - V^2(x)$

(a) Since, $\frac{\partial L}{\partial t} = 0$, then $\frac{\partial L}{\partial \dot{x}}\dot{x} - L$ is an integral of motion.

Now, $\frac{\partial L}{\partial \dot{x}}\dot{x} - L$

$$= \dot{x}\left[\frac{1}{3} m^2 \dot{x}^3 + 2m\dot{x} V(x)\right] - \left[\frac{1}{12} m^2 \dot{x}^4 + m\dot{x}^2 V(x) - V^2(x)\right]$$

$$= \frac{1}{4} m^2 \dot{x}^4 + m\dot{x}^2 V(x) + V^2(x)$$

$$= \left[\frac{1}{2} m\dot{x}^2 + V(x)\right]^2$$

(b) Since the Kinetic energy is not a quadratic function of $\dot{x}$, the energy function cannot be expressed as $T + U$

(6.) $\vec{F} = \frac{mk}{r^3}(x\hat{r} + y\hat{\theta}) = -\vec{\nabla}V$

So, $\frac{\partial V}{\partial r} = \frac{mkx}{r^3} = \frac{mk\cos\theta}{r^2}$ &

$$\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{mky}{r^3} \Rightarrow \frac{\partial V}{\partial \theta} = \frac{mk\sin\theta}{r}$$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial \theta}d\theta$$

$$= \frac{mk\cos\theta}{r^2}dr + \frac{mk\sin\theta}{r}d\theta$$

$$= mk\left[\frac{\cos\theta\, dr - r\, d(\cos\theta)}{r^2}\right]$$

$$= d\left(\frac{mk\cos\theta}{r}\right) \Rightarrow V = \frac{mk\cos\theta}{r} + K$$

K constant

$$\therefore L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mk\cos\theta}{r}$$

[setting $K=0$]

(7.) $L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{k}{2}(q_1^2 + q_2^2)$

Lagrangian of two independent simple harmonic oscillators.

The system can be written in plane polar coordinates

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2}r^2 \qquad \begin{bmatrix} q_1 = r\cos\theta \\ q_2 = r\sin\theta \end{bmatrix}$$

Since θ is cyclic, $p_\theta = mr^2\dot{\theta}$ is a constant of motion.

As $\vec{L} = m\vec{r}\times\dot{\vec{r}} = m\, r\hat{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$

$= mr^2\dot{\theta}\,\hat{z}$ $\therefore$ z component of angular momentum.

$\downarrow$

$mr^2\dot{\theta}$

Now, in terms of q_1 & q_2,

$$m\vec{r}\times\dot{\vec{r}} = m(q_1\hat{x} + q_2\hat{y})\times(\dot{q}_1\hat{x} + \dot{q}_2\hat{y})$$

$$= \underbrace{m(q_1\dot{q}_2 - q_2\dot{q}_1)}_{= mr^2\dot{\theta}}\hat{z}$$

So, $q_1\dot{q}_2 - q_2\dot{q}_1$ is a constant of motion.

Alternative: $\quad \ddot{q}_1 = -q_1\,,\ \ddot{q}_2 = -q_2$

Credit is given to

Now, $\frac{d}{dt}\left(q_1\dot{q}_2 - q_2\dot{q}_1\right)$

$$= q_1\ddot{q}_2 + \cancel{\dot{q}_1\dot{q}_2} - \cancel{\dot{q}_2\dot{q}_1} - q_2\ddot{q}_1$$

$$= -q_1q_2 + q_2q_1 = 0$$

(8) An inertial frame is a frame of reference for which space is homogeneous and isotropic and time is homogeneous.

OR, An inertial frame is one with respect to which, if an isolated particle is at rest, will remain at rest.

(9) The explicit time dependent term in the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - at^4$ does NOT represent the interaction with external agent as this is nothing but a gauge to a free particle Lagrangian.

* When the potential depends both on the particles position and time, then that would represent the interaction with an external agent.

(10.) $L = \dot{q} + \dot{q}^2 + \dot{q}^3$

(a) Properties of L:

(i) L is a fn of $\dot{q}$ only.

(ii) q is cyclic $\Rightarrow \dfrac{\partial L}{\partial \dot{q}} = 1 + 2\dot{q} + 3\dot{q}^2$ is constant of motion

$\Rightarrow 2\dot{q} + 3\dot{q}^2$ is constant of motion $= K_1$

(iii) $\dfrac{\partial L}{\partial t} = 0 \Rightarrow \dot{q}\dfrac{\partial L}{\partial \dot{q}} - L$

$= \dot{q}(1 + 2\dot{q} + 3\dot{q}^2) - (\dot{q} + \dot{q}^2 + \dot{q}^3)$

$= \dot{q}^2 + 2\dot{q}^3$ is constant of motion.

$= K_2$

(b) $\dfrac{\partial L}{\partial \dot{q}} = 1 + 2\dot{q} + 3\dot{q}^2$

$\Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = 2\ddot{q} + 6\dot{q}\ddot{q} = 0$

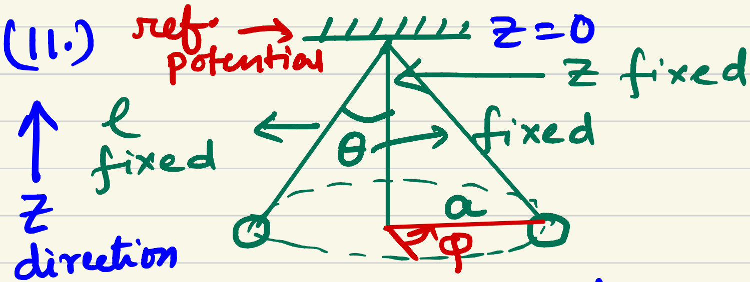
$\Rightarrow \ddot{q}(1 + 3\dot{q}) = 0 \Rightarrow$ Free particle

(c) Since $L = L(\ddot{q})$ only, this can be of a free particle

<u>OR</u>: $L = \dot{q} + \dot{q}^2 + \dot{q}^3 = \dot{q}^2 + \dfrac{K_2 - \dot{q}^2}{2} + \dot{q}$

$= m\dot{q}^2 + \underbrace{\dfrac{d}{dt}(q + K)}_{\text{gauge}}$

(11.)



For conical pendulum, φ is the only generalized coordinate

$$L = \frac{1}{2} m a^2 \dot{\varphi}^2$$

[No need to write the constant potential term]

$$\therefore \frac{\partial L}{\partial \dot{\varphi}} = \underbrace{m a^2 \dot{\varphi}}_{\text{Constant}} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m a^2 \ddot{\varphi}$$

$\therefore$ EL equation $\Rightarrow \ddot{\varphi} = 0$

$\Rightarrow \dot{\varphi}$ = constant. (integral of motion)

and $\frac{\partial L}{\partial t} = 0 \Rightarrow \dot{\varphi}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - L$

$$= m a^2 \dot{\varphi}^2 - \frac{1}{2} m a^2 \dot{\varphi}^2$$

$$= m a^2 \dot{\varphi}^2 \rightarrow \text{integral of motion}$$

* When the string length varies:

then the z also varies keeping θ constant so that the bob moves on the cone surface.

$\Rightarrow$ a varies as $r/z = \tan\theta = \beta$

$$\therefore L = \frac{1}{2} m \left[\beta^2 \dot{z}^2 + \beta^2 z^2 \dot{\varphi}^2 + \dot{z}^2 \right] + mgz$$

$$= \frac{1}{2} m \left[(1+\beta^2) \dot{z}^2 + \beta^2 z^2 \dot{\varphi}^2 \right] + mgz$$

$\rightarrow \varphi$ is Cyclic.