CLASSICAL MECHANICS I (PHY 401A) 2020-21 ODD SEMESTER

Tutorial sheet solutions2

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1. **Euler-Lagrange equation under point transformation**

   Let us consider a mechanical system with n degrees of freedom. Let $\{q_1, q_2, ..., q_n\}$ be one set of generalized coordinates and $L(q, \dot{q}, t)$ be the Lagrangian of the system. If we transform to another set of generalized coordinates $\{s_1, s_2, ..., s_n\}$ such that

$$q_j = q_j(s_1, s_2, ..., s_n, t), \quad j = 1, 2, ..., n$$

   show that L will also satisfy the Euler-Lagrange equation with respect to the new set of generalized coordinates.

   Solution: Lagrangian $L(q, \dot{q}, t)$ follows Euler-Lagrange equations with generalized coordinates $\{q_i\}$. Point transformation of $\{q_i\}$ to $\{s_i\}$ is

$$q_j = q_j(s_1, s_2..s_n, t) \quad j = 1, 2, 3....n \tag{1}$$

   Then one can also write that

$$\dot{q}_j = \sum_k \frac{\partial q_j}{\partial s_k}\dot{s_k} + \frac{\partial q_j}{\partial t} = f(s_1, ..., s_n, \dot{s_1}, ..., \dot{s_n}, t) \tag{2}$$

   We have to show the following-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s_k}}\right) - \frac{\partial L}{\partial s_k} = 0 \tag{3}$$

   Now

$$\frac{\partial L}{\partial \dot{s_k}} = \sum_l \left[\frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial \dot{s_k}} + \frac{\partial L}{\partial q_l}\frac{\partial q_l}{\partial \dot{s_k}} + \frac{\partial L}{\partial t}\frac{\partial t}{\partial \dot{s_k}}\right] \tag{4}$$

   $\partial q_l/\partial \dot{s_k}$ will be zero as q_j doesn't depend on $\dot{s_k}$ (see equation (1)). $\partial t/\partial \dot{s_k}$ will be zero since time is the intrinsic independent variable. Also

$$\frac{\partial \dot{q}_l}{\partial \dot{s_k}} = \frac{\partial q_l}{\partial s_k} \tag{5}$$

   Equation (4) is shown in class. Now equation (3) will be -

$$\frac{\partial L}{\partial \dot{s_k}} = \sum_l \frac{\partial L}{\partial \dot{q}_l}\frac{\partial q_l}{\partial s_k} \tag{6}$$

   Again,

$$\frac{\partial L}{\partial s_k} = \sum_l \left[\frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k} + \frac{\partial L}{\partial q_l}\frac{\partial q_l}{\partial s_k} + \frac{\partial L}{\partial t}\frac{\partial t}{\partial s_k}\right] \tag{7}$$

$\partial \dot{q}_l/\partial s_k$ will be non-zero as derivatives of q_l will in general depend on s_k. Again $\partial t/\partial s_k$ will be zero. So equation (6) will be-

$$\frac{\partial L}{\partial s_k} = \sum_l \left[\frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k} + \frac{\partial L}{\partial q_l}\frac{\partial q_l}{\partial s_k}\right] \tag{8}$$

Substituting equations (5), (7) in L.H.S of equation (2)-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s_k}}\right) - \frac{\partial L}{\partial s_k} = \sum_l \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_l}}\right)\frac{\partial q_l}{\partial s_k} + \frac{\partial L}{\partial \dot{q_l}}\frac{d}{dt}\left(\frac{\partial q_l}{\partial s_k}\right) - \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k} - \frac{\partial L}{\partial q_l}\frac{\partial q_l}{\partial s_k}\right] \tag{9}$$

Using the following identity (done in class)-

$$\frac{d}{dt}\left(\frac{\partial q_l}{\partial s_k}\right) = \frac{\partial \dot{q}_l}{\partial s_k} \tag{10}$$

Equation (8) can be simplified

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s_k}}\right) - \frac{\partial L}{\partial s_k} = \sum_l \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_l}}\right)\frac{\partial q_l}{\partial s_k} + \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k} - \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k} - \frac{\partial L}{\partial q_l}\frac{\partial q_l}{\partial s_k}\right] \tag{11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s_k}}\right) - \frac{\partial L}{\partial s_k} = \sum_l \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_l}}\right) - \frac{\partial L}{\partial q_l}\right]\frac{\partial q_l}{\partial s_k} \tag{12}$$

Term in square bracket in equation (11) is zero because Lagrangian satisfies Euler-Lagrange equation with coordinates $\{q_i\}$. Hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s_k}}\right) - \frac{\partial L}{\partial s_k} = 0 \tag{13}$$

2. **Double Pendulum**

A double pendulum consists of two point masses m_1 and m_2. m_1 suspends from a rigid ceiling through a rigid massless rod of length ℓ_1 whereas m_2 suspends from m_1 through another rigid but massless rod of length ℓ_2. The angular displacements of m_1 and m_2, with respect to the vertical, are given by θ_1 and θ_2 respectively.

(a) Draw a figure of the system.

(b) Suggest a set of generalized coordinates appropriate for the system.

(c) Obtain the expression of the Lagrangian of the system.

(d) Write down the relevant Euler-Lagrange equations and solve them to get the equation of motion of the double pendulum.

(e) Find the simplified equations of motion when the angular displacements are very small.

(a) The figure for the double pendulum is given below

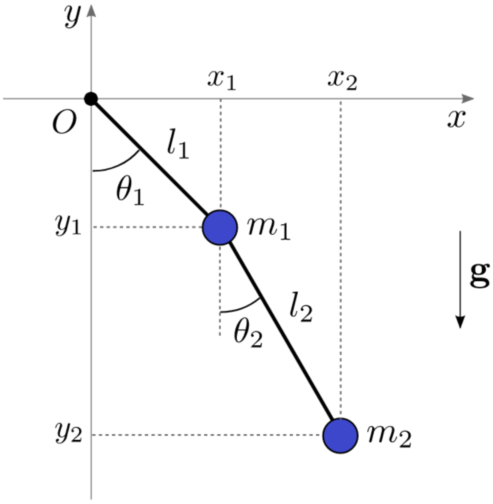

Figure 1: Double Pendulum

(b) A appropriate set of generalised coordinates for this problem would be θ_1 and θ_2. Note that here, from the very beginning we assumed that the pendulum's motion is confined in vertical plane although their is no hard and fast rule to suupose that. So generally speaking, the pendulum should also have a degree of freedom perpendicular to the vertical plane and hence additional generalized coordinate. However, we are particularly interested to see what happens in the vertical plane. One can easily check what is happening in the direction perpendicular to the vertical plane.

(c) The coordinates of the pendulum bobs are given by

$$x_1 = l_1 \sin\theta_1, \;\; x2 = l_1 \sin\theta_1 + l_2 \sin\theta_2, \;\; y_1 = -l_1 \cos\theta_1, \;\; y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2, \tag{14}$$

The expression for kinetic energy is given by

$$T = \frac{1}{2} m_1 \left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2} m_2 \left(\dot{x_2}^2 + \dot{y_2}^2\right), \tag{15}$$

using (14) the above expression can be written as

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right], \tag{16}$$

The potential energy of the system is given by

$$V = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2, \tag{17}$$

The Lagrangian of the system is given by

$$\begin{aligned} L = T - V = \frac{1}{2} m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 &\left[l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right] \\ &+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2. \end{aligned} \tag{18}$$

(d) The Euler-Lagrange equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \left(\frac{\partial L}{\partial \theta_i}\right) = 0, \tag{19}$$

for $i = 1, 2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0, \tag{20}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0, \tag{21}$$

The partial derivatives are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2), \tag{22}$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 g(m_1 + m_2)\sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{23}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_1^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2), \tag{24}$$

$$\frac{\partial L}{\partial \theta_2} = -l_2 g m_2 \sin\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{25}$$

Using (22-25) we get the governing equations for double pendulum

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 = 0, \tag{26}$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0. \tag{27}$$

(e) In the limit of small angular displacements we can write

$$\sin\theta_1 \approx \theta_1, \;\; \sin\theta_2 \approx \theta_2, \;\; \cos(\theta_1 - \theta_2) \approx 1, \;\; \sin(\theta_1 - \theta_2) \approx (\theta_1 - \theta_2)$$

The governing equations in the limit of small displacements are therefore give by

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + m_2 l_2 \dot{\theta}_2^2 (\theta_1 - \theta_2) + g(m_1 + m_2)\theta_1 = 0, \tag{28}$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 - m_2 l_1 \dot{\theta}_1^2 (\theta_1 - \theta_2) + m_2 g \theta_2 = 0. \tag{29}$$

3. **Motion under central force field**

A particle is moving under an attractive central force with inverse square law. Construct the Lagrangian for the motion of the particle and solve the appropriate Euler-Lagrange equations to find the equations of motion of the particle. (Assume that the particle moves in a plane)

Solution: A particle is moving under an attractive central force with inverse square law. We suppose that the motion of the particle is confined in a plane (which comes from our previous knowledge)So the potential energy will very as $1/r$,

$$V(r) = -k/r \tag{30}$$

where k is the proportionality constant. let us work in plane polar coordinates (r, θ). The kinetic energy of the system can be written as

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{31}$$

Now from equations 1 and 2, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{32}$$

We observe that L is independent of θ. So from Lagrange's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \tag{33}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = constant \tag{34}$$

which is the angular momentum and it is conserved. Turning now to the r-component of Lagrange's equations, we obtain

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - V'(r), \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{35}$$

now the radial equation of motion

$$m\ddot{r} - mr(\dot{\theta}^2) = -V'(r) = -k/r^2. \tag{36}$$

The equations (34) and (36) describe the motion under central force field.

4. **Motion of a projectile**

A particle of mass m is projected with initial velocity u at an angle α with the horizontal. Use Lagrange's method to describe the motion of the projectile. The air resistance is neglected here.

Solution: We are considering a particle of mass m is projected with initial velocity u at an angle α with the horizontal as shown in the figure 1. The air resistance is neglected here. The Lagrangian of the considered system can be written as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + +\dot{z}^2) - mgy \tag{37}$$

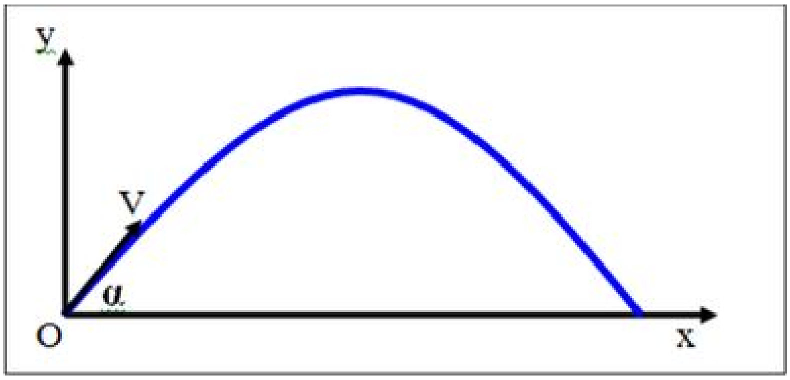

Figure 2: Schematic of the projectile

The Newtonian motion equations using the Euler-Lagrange equations have the simple form

$$m\ddot{x} = 0, \;\; m\ddot{y} = -mg, \;\; m\ddot{z} = 0 \tag{38}$$

with respect to the initial conditions $x(0) = 0$, $y(0) = 0$ and $z(0) = 0$; and also $\dot{x}(0) = u\ cos\alpha$, $\dot{y}(0) = u\ sin\alpha$ and $\dot{z}(0) = 0$, we get the solutions

$$x(t) = u\ cos\alpha\ .\ t, \;\; y(t) = u\ sin\alpha\ .\ t - \frac{gt^2}{2}, z(t) = 0 \tag{39}$$

which represents the parametric expression of the projectile motion. So the motion of the particle takes place in xy-plane. By the eliminating of the time parameter t, $t = x/u\ cos\alpha$, we obtain

$$y = x\ tan\alpha - \frac{gx^2}{2u^2cos^2\alpha} \tag{40}$$

which represents the equation of a parabola. The trajectory of the projectile will be along this parabola.