Tutorial sheet 11

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1. **Infinitesimal Canonical Transformation**

   Using infinitesimal CT, prove the Jacobi identity for Poisson brackets.

   Solution:

   Let, $f(p,q)$ and $g(p,q)$ be two dynamical variables depending on phase space coordinates. Let, $h(p,q)$ be the generator of an infinitesimal canonical transformation(ICT). Then for any dynamical variable $X(p,q)$, it's infinitesimal change in phase space can be written as

$$\delta X = \varepsilon[h, X], \tag{1}$$

   where, ε is an infinitesimal parameter and $[h, X]$ is the Poisson bracket of h and X, which in itself is a dynamical variable of phase space coordinates.

   Similarly, we can replace X in Eq.(1) with the Poisson bracket of $f(p,q)$ and $g(p,q)$ which is written as $[f,g]$ and get

$$\delta[f,g] = \varepsilon\left[h, [f,g]\right], \tag{2}$$

   On the other hand as the Poisson bracket between any two variables is independent of the basis it is calculated in, it's change is only due to the change in variables itself, so we can also write

$$\delta[f,g] = [\delta f, g] + [f, \delta g], \tag{3}$$

   again using Eq.(1), we have $\delta f = \varepsilon[h,f]$ and $\delta g = \varepsilon[h,g]$. Hence from Eq.(3) we have

$$\delta[f,g] = \varepsilon[[h,f],g] + \varepsilon[f,[h,g]], \tag{4}$$

   equating Eq.(2) and Eq.(4), we can write

$$\varepsilon\left[h,[f,g]\right] = \varepsilon[[h,f],g] + \varepsilon[f,[h,g]], \tag{5}$$

   from above, eliminating ε, we get

$$[[h,f],g] + [f,[h,g]] - h,[[f,g]] = 0, \tag{6}$$

   again using $[A,B] = -[B,A]$ we can write

$$[[h,f],g] + [[g,h],f] + [[f,g],h] = 0. \tag{7}$$

   which is the required result.

2. **Hamilton-Jacobi method**

Solve Kepler's problem using Hamilton-Jacobi method.

Solution: The Hamiltonian of the Kepler's problem in polar coordinates has the following form

$$H = \frac{1}{2m}[p_r^2 + \frac{p_\theta^2}{r}] + V(r), \tag{8}$$

which is cyclic in θ. The Hamiltonian's characteristic function can be written as

$$W = W_1(r) + \alpha_\theta \theta \tag{9}$$

where α_θ is the constant angular momentum p_θ conjugate to θ.

The Hamilton-Jacobi equation then becomes

$$\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} + 2mV(r) = 2m\alpha_1 \tag{10}$$

where α_1 is constant and can be physically identified as the total energy.

from $equation 3$,

$$\frac{\partial W_1}{\partial r} = \sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\theta^2}{r^2}}, \tag{11}$$

So that W is

$$W = \int dr \sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\theta^2}{r^2}} + \alpha_\theta \theta \tag{12}$$

With this form, the transformation equation for the characteristic function appears as

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \int \frac{m dr}{\sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\theta^2}{r^2}}} \tag{13}$$

and

$$\beta_2 = \frac{\partial W}{\partial \alpha_\theta} = -\int \frac{\alpha_\theta dr}{r^2\sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\theta^2}{r^2}}} + \theta \tag{14}$$

Equation (13) furnishes r as a function of t. At time t = 0, let r have the initial value r_0 then the solution is as following

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(\alpha_1 - V - \frac{\alpha_\theta^2}{2mr^2}\right)}} \tag{15}$$

The remaining transformation equation (14) should provide the orbit equation. If the variable of integration in equation (14) is changed to $u = \frac{1}{r}$, the equation reduces to

$$\theta = \beta_2 - \int \frac{du}{\sqrt{\frac{2m}{\alpha_\theta^2}(\alpha_1 - V) - u^2}} \tag{16}$$

Equation (16), represents the orbit equation.

3. **Action angle variables**

Find the frequency of a one dimensional harmonic oscillator using action- angle variable method.

Solution: Let us take the Hamiltonian for harmonic oscillator as follows -

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \tag{17}$$

Now, for a total energy E of the system,

$$\frac{p^2}{2m} = E - \frac{m\omega^2 x^2}{2} \implies p = \sqrt{2mE - m^2\omega^2 x^2} \tag{18}$$

If the amplitude of the oscillator is x_0 then,

$$E = \frac{m\omega^2 x_0^2}{2} \implies x_0^2 = \frac{2E}{m\omega^2} \tag{19}$$

So the action integral would be,

$$I = \oint p dx = \oint \sqrt{(2mE - m^2\omega^2 x^2)} dx \tag{20}$$

Closed integration in eq (20) is over a time period. So from eqs (18,20) -

$$I = \oint \sqrt{2mE\left(1 - \frac{m\omega^2 x^2}{2E}\right)} dx \implies \int_0^{x_0} 4\sqrt{2mE\left(1 - \frac{x^2}{x_0^2}\right)} dx \tag{21}$$

Let $x/x_0 = t$, then

$$I = \int_0^{x_0} 4\sqrt{2mE\left(1 - \frac{x^2}{x_0^2}\right)} dx = \int_0^{\pi/2} 4x_0\sqrt{2mE\,(1 - t^2)} dt \tag{22}$$

Above equation can be solved by considering $t = sin\theta$. Solving equation (22) one would get,

$$I = \frac{2\pi E}{\omega} \implies E = \frac{I\omega}{2\pi} \tag{23}$$

So the frequency ω_0 conjugate to action variable I is -

$$\omega_0 = \frac{\partial E}{\partial I} = \frac{\omega}{2\pi} \tag{24}$$