

# Hamilton's Principle

- Also called the **Principle of Least Action**, although 'Stationary Action' is the correct term.
- Statement:** Every mechanical system is characterized by a function  $L = L(q_1, q_2, \dots, q_s; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s; t) \equiv L(q, \dot{q}, t)$  [for brevity] and the motion/mechanical evolution of that system will be governed by the following rule:

Between two given time instants (initial and final)  $t_1$  and  $t_2$  and with two given sets of generalized coordinates  $q^{(1)}$  and  $q^{(2)}$ , the system always chooses the path of least action to evolve, where the action  $S$  is defined as

$$S = \int_{t_1}^{t_2} L dt.$$

The function  $L$  is called the Lagrangian of the system.

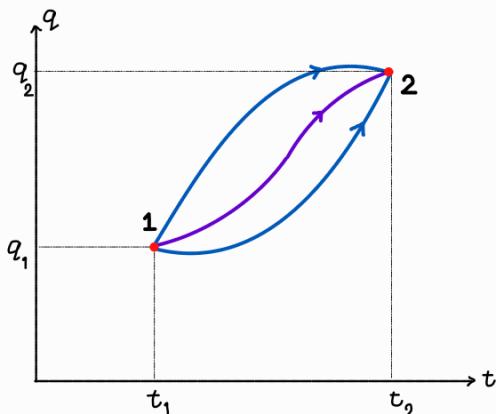
⚠ Why  $L$  contains only  $q$  and  $\dot{q}$  (besides  $t$ ), and not  $\ddot{q}$ ,  $\ddot{q}$ , etc.?

- Derivation of Euler-Lagrange Equation:**

For simplicity, we assume that our system has only one degree of freedom. Thus,

$$L = L(q, \dot{q}, t)$$

The objective is to determine  $q(t)$  using the extremum condition. Then only we can predict the position of the particle at any  $t$ .



Let's consider another path of evolution, slightly different from the real path. For the new path, the generalized coordinates will be given by

$$q'(t) = q(t) + \delta q(t)$$

where  $\delta q(t)$  is known as variation; time is frozen here. It gives infinitesimal shift from one trajectory to the other. At the initial and the final instants, the coordinates are given — thus

$$\delta q(t_1) = 0 = \delta q(t_2) \quad \text{--- (A)}$$

The principle of least action says that if the path of evolution is slightly varied with respect to the real one, then  $\delta S = 0$ . This translates to

$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$$

The operators  $\delta$  and  $\frac{d}{dt}$  commute because they're independent of each other. Thus,

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0 \quad \text{--- (1)}$$

$$\Rightarrow \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt + \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) dt = 0 \quad \text{--- (2)}$$

$$\begin{aligned} \text{But } \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) dt &= \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}(t)} \delta q(t) \Big|_{t_1}^{t_2} \\ &= \frac{\partial L(q, \dot{q}, t_2)}{\partial \dot{q}(t_2)} \delta q(t_2) - \frac{\partial L(q, \dot{q}, t_1)}{\partial \dot{q}(t_1)} \delta q(t_1) \\ &= 0 \quad [\text{by using (A)}] \end{aligned}$$

$$\Rightarrow \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt = 0 \quad \text{--- (3)}$$

The above integral is identically zero for all variations  $\delta q$  of  $q$ . Hence

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0}$$

The above equation inside the box is known as the Euler-Lagrange equation. For systems with more than one degree of freedom, we have identical equations for each  $q_i$ ,  $i = 1, 2, \dots, s$ :

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0}$$

 Have you understood how?

Note: We said our goal was to find  $q(t)$  for which  $S = \int_{t_1}^{t_2} L dt$  is minimum. But we derived an equation for  $L(q, \dot{q}, t)$ . Can we reconcile?

- Form of the Lagrangian for a Mechanical System:

In the previous derivation, we did not use / did not need to use the explicit form of the Lagrangian. But to derive the equations of motion, one cannot but use the explicit dependence of  $L$  on  $q$ ,  $\dot{q}$ , and  $t$ . A simple approach for the usual classical mechanical systems is given here.

From Newton's laws, we know for one particle,

$$\begin{aligned}\vec{F} &= m\ddot{\vec{r}} \\ \Rightarrow \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} &= m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} = m \frac{d}{dt}(\dot{\vec{r}}) \cdot \frac{\partial \vec{r}}{\partial q_i} \\ &= m \frac{d}{dt} \left( \dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} \right) - m\dot{\vec{r}} \cdot \frac{d}{dt} \left( \frac{\partial \vec{r}}{\partial q_i} \right) \\ &= \frac{d}{dt} \left( m\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} \right) - m\dot{\vec{r}} \frac{d}{dt} \left( \frac{\partial \vec{r}}{\partial q_i} \right) \quad — (i)\end{aligned}$$

Now  $\frac{\partial \vec{r}}{\partial q_i} = \frac{d\vec{r}}{dq_i} - \sum_{j \neq i} \frac{\partial \vec{r}}{\partial q_j} \frac{dq_j}{dq_i}$

Also,  $\dot{\vec{r}} = \frac{\partial \vec{r}}{\partial q_i} \dot{q}_i + \sum_{j \neq i} \frac{\partial \vec{r}}{\partial q_j} \dot{q}_j \Rightarrow \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial \vec{r}}{\partial q_i} + \sum_{j \neq i} \frac{\partial \vec{r}}{\partial q_j} \frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}}{\partial q_i} + \sum_{j \neq i} \frac{\partial \vec{r}}{\partial q_j} \delta_{ij} = \frac{\partial \vec{r}}{\partial q_i}$   
 $\left[ \because \frac{\partial \vec{r}}{\partial q_j} \text{ are functions of } q \text{'s, not } \dot{q} \text{'s} \right]$

$$\Rightarrow \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial \vec{r}}{\partial q_i} \quad — (ii)$$

Putting back in (i), we get

$$m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} = \frac{d}{dt} \left( m\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial \dot{q}_i} \right) - m\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} \quad — (iii)$$

But  $\frac{d}{dt} \left( \frac{\partial \vec{r}}{\partial q_i} \right) = \frac{\partial}{\partial t} \left( \frac{\partial \vec{r}}{\partial q_i} \right)^0 + \sum_j \frac{\partial}{\partial q_j} \left( \frac{\partial \vec{r}}{\partial q_i} \right) \dot{q}_j + \sum_j \frac{\partial}{\partial \dot{q}_j} \left( \frac{\partial \vec{r}}{\partial q_i} \right) \ddot{q}_j$   
 $= \sum_j \frac{\partial}{\partial q_j} \left( \frac{\partial \vec{r}}{\partial q_i} \right) \dot{q}_j = \frac{\partial}{\partial q_i} \left[ \sum_j \frac{\partial \vec{r}}{\partial q_j} \dot{q}_j \right] ; \quad [\because \vec{r} \text{ is not a function of } t \text{ & } \dot{q}_j \text{'s}]$

So finally using (i) and (iii), we write,

$$\vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} = \frac{d}{dt} \left( m\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial \dot{q}_i} \right) - m\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i}$$

$$\Rightarrow \boxed{\vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}} \quad — (iv)$$

This is valid for any mechanical system. For the case of conservative systems, however,

$\vec{F} = -\nabla V$ , where  $V$  is the potential energy. So,

$$\vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} = -\nabla V \cdot \frac{\partial \vec{r}}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

If we further assume that  $\frac{\partial V}{\partial \dot{q}_i} = 0 \quad \forall i$ , then we can write

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} + \frac{d}{dt} \left( \frac{\partial V}{\partial \dot{q}_i} \right)$$

$$\Rightarrow \frac{d}{dt} \left( \frac{\partial \chi}{\partial \dot{q}_i} \right) - \frac{\partial \chi}{\partial q_i} = 0, \text{ where } \chi = T - V$$

So, basically  $\chi$  is something behaving like the Lagrangian  $L$ . Hence, one can identify that for a simple particle under a conservative force,

$$\boxed{L = T - V}$$

- Remember, the minus sign has appeared as a result of the convention, i.e.,

$$\vec{F} = -\nabla V.$$

⚠ What will be the Lagrangians of a free particle and a particle falling under gravity?