PHY401A
Classical Mechanics

A branch of physics dedicated to study the motion of physical objects.

If a body does not move as a whole but does get deformed $\Rightarrow$ local movement

$\Rightarrow$ Mechanics

If a body does not move, does not deform macroscopically but responds to the external agent of deformation by microscopic motion $\Rightarrow$ Mechanics

A formal approach to such study is to set up a dynamical theory on that premise

- The state of the system will be well-defined at every instant of time
- The law of evolution in time (the equation of evolution / equation of motion)

Other examples of dynamical theories : Classical Electrodynamics, Quantum Mechanics, etc.

For Classical Mechanics (till now only Newtonian mechanics):

- **State** : Represented by position $\vec{r}(t)$ and velocity $\vec{v}(t) = \dot{\vec{r}}(t) = d\vec{r}/dt$ $(\vec{r}(t), \vec{v}(t))$ in a configuration space.
- **Equation of evolution**: Newton's second law (for a fixed mass):

$$m\frac{d\vec{v}}{dt} = m\dot{\vec{v}} = \underbrace{\vec{F}}_{\text{force}}$$

⚠ Can you say what are the 'states' and the 'evolution equations' for classical electrodynamics and quantum mechanics?

A Quick Recap of Newtonian Mechanics

- **Newton's 2nd Law:** $\frac{d\vec{p}}{dt} \propto \vec{F}_{ext.}$; for non-relativistic case, $\vec{p} = m\vec{v}$ for a particle of mass m.

$$\Rightarrow \frac{d}{dt}(m\vec{v}) = k\vec{F}_{ext.} \longrightarrow \vec{F}_{ext.} = \vec{0} \Rightarrow m\vec{v} \text{ is conserved.}$$

This is the principle of conservation of linear momentum.

$$\Rightarrow m\frac{d\vec{v}}{dt} = k\vec{F}_{ext.} \quad \text{(for constant mass)}$$

- The unit of force is chosen in such a way that $k=1$:

$$\boxed{m\frac{d\vec{v}}{dt} = \vec{F}_{ext.}} \longrightarrow \text{popular form of Newton's 2nd law}$$

- We denote $\frac{d\vec{v}}{dt} \equiv \dot{\vec{v}}$ [This convention will be followed throughout]

$$\boxed{\text{Note that } \frac{\partial\vec{v}}{\partial t} \neq \dot{\vec{v}}}$$

Thus,

$$m\dot{\vec{v}} = \vec{F}_{ext.} = m\frac{d\vec{v}}{d\vec{r}}\cdot\frac{d\vec{r}}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{d\vec{r}}$$

$$\Rightarrow \boxed{m\vec{v}\cdot\frac{d\vec{v}}{d\vec{r}} = \vec{F}_{ext.}} \left(= \vec{F} \ ; \text{ for brevity}\right) \quad \left[\text{another form of Newton's 2nd law}\right.$$

So, one can write

$$m\vec{v}\cdot d\vec{v} = \vec{F}\cdot d\vec{r} = dW \longrightarrow \text{work done mechanically}$$

$$\Rightarrow m\int_{\vec{v}_i}^{\vec{v}_f} \vec{v}\cdot d\vec{v} = \int_0^W dW$$

$$\Rightarrow \boxed{W = \frac{1}{2}m\left(v_f^2 - v_i^2\right)} \longrightarrow \text{Work - Energy Theorem}$$

So, if $v_f > v_i$, $W > 0$ and if $v_f < v_i$, $W < 0$.

Statement: The work done by a net force on a particle is equal to the change in its kinetic energy.

Let's now consider the case of a conservative force field, so that

$$\vec{F} = -\nabla V \ ; \quad [V \text{ is a 'potential energy' function}]$$

For this force,

$$m\,\vec{v}\cdot d\vec{v} = \vec{F}\cdot d\vec{r} = -\nabla V\cdot d\vec{r} = -dV$$

$$\Rightarrow \quad d\left(\frac{1}{2}mv^2\right) = -dV$$

$$\Rightarrow \quad d\left(\frac{1}{2}mv^2 + V\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2 + V\right) = 0$$

$\therefore$ $\boxed{\frac{1}{2}mv^2 + V}$ is a constant of the motion. It is often written as

$$E = T + V,$$

where E is the total mechanical energy and T is the kinetic energy of the system, and we have the principle of conservation of mechanical energy (PCME).

- <u>Angular Momentum & Torque</u>

$$\vec{L} \overset{\text{def.}}{=} \vec{r} \times \vec{p} \quad \longrightarrow \text{angular momentum}$$

$$\vec{\tau} \overset{\text{def.}}{=} \frac{d\vec{L}}{dt} = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{=\vec{0}} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} \quad \rightarrow \text{torque}$$

$= \vec{0}$; because $\vec{p} = m\vec{v}$ and $\vec{v} = \frac{d\vec{r}}{dt}$

If $\vec{r} \times \vec{F}$ $\vec{0}$; then $\vec{\tau} = \vec{0}$ and hence

$$\frac{d\vec{L}}{dt} = \vec{0} \; ; \quad \text{so } \vec{L} \text{ will be conserved.}$$

This is the principle of conservation of angular momentum (PCAM).

For instance, for a central force $\vec{F} = F(r)\,\hat{r}$, we have $\vec{r} \times \vec{F} = \vec{0}$ and $\vec{L}$ is conserved.

We shall revisit PCLM, PCAM and PCME later from a different perspective.

- <u>System of Particles</u>

Suppose we have a system of N particles with mass m_i, position $\vec{r}_i$ and velocity $\vec{v}_i$, $i \in \{1, 2, 3, \ldots, N\}$. The total mass moment of the system is defined by

$$\vec{\mu} = \sum_{i=1}^{N} m_i \vec{r}_i$$

Let's find another position (occupied or unoccupied by one of the particles) with position vector $\vec{R}$ such that

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \qquad \text{——— (i)}$$

It is called the center of mass of the system and represents the average position occupied by the particles.

The origin of the center of mass frame is situated at $\vec{R}$. If $\vec{r}_i'$ denotes the position of the i-th particle in the center of mass frame, then we have

$$\vec{r}_i' = \vec{r}_i - \vec{R} \Rightarrow \sum_i m_i \vec{r}_i' = \sum_i m_i \vec{r}_i - \vec{R} \sum_i m_i = \vec{0}$$

$$\Rightarrow \sum_i m_i \vec{r}_i' = \vec{0} \qquad \text{———— (ii)}$$

Thus, the center of mass of a system is such a point about which the mass moment of the system vanishes. One can also obtain

$$M\vec{R} = \sum_i m_i \vec{r}_i$$

$$\Rightarrow \quad M\dot{\vec{R}} = \sum_i m_i \dot{\vec{r}}_i \qquad \text{———— (iii)}$$

So, the total linear momentum of the system of particles is equal to the linear momentum of the center of mass. Also,

$$\sum_i m_i \dot{\vec{r}}_i' = \vec{0}$$

states that the total linear momentum with respect to the center of mass vanishes.

- From (iii), one can see that $M\ddot{\vec{R}} = \sum_i m_i \ddot{\vec{r}}_i \longrightarrow$ total force

So, the resultant force acts on the center of mass of the system of particles.

$$\Rightarrow \quad M\ddot{\vec{R}} = M\dot{\vec{V}} = \sum_i m_i \ddot{\vec{r}}_i = \sum_i \vec{F}_i \qquad [\vec{F}_i : \text{force on the } i\text{-th particle}]$$

Now, $\vec{F}_i = \vec{F}_{i,\text{ext.}} + \underbrace{\vec{F}_{i,\text{int.}}}_{\text{force due to all the other particles}}$

$$= \vec{F}_{i,\text{ext.}} + \sum_{j \neq i} \vec{F}_{i,j}$$

$$\Rightarrow \quad M\dot{\vec{V}} = \sum_i \vec{F}_{i,\text{ext.}} + \sum_i \sum_{j\neq i} \vec{F}_{i,j}$$

By Newton's 3rd law, $\vec{F}_{i,j} = -\vec{F}_{j,i}$, so we get

$$M\dot{\vec{V}} = \sum_i \vec{F}_{i,\text{ext.}} = \vec{F}_{\text{ext.}} \qquad \text{———— (iv)}$$

So, the net force which acts on the center of mass is simply given by the total external force. If $\vec{F}_{\text{ext.}} = \vec{0}$, the total linear momentum $\vec{P}$ $(= M\vec{V})$ of the system is conserved $\longrightarrow$ PCLM for a system of particles.

- The total angular momentum of a system of particles is the vector sum of the angular momenta of all individual particles. So, about the origin $(0,0,0)$, the total angular momentum of the system is given by

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

Substituting $\vec{r}_i = \vec{r}_i' + \vec{R}$, we get

$$\vec{L} = \sum_i (\vec{r}_i' + \vec{R}) \times m_i(\vec{v}_i' + \vec{V})$$

$$= \sum_i \vec{r}_i' \times m_i(\vec{v}_i' + \vec{V}) + \vec{R} \times \sum_i m_i(\vec{v}_i' + \vec{V})$$

$$= \sum_i \vec{r}_i' \times m_i \vec{v}_i' + \underbrace{\left(\sum_i m_i \vec{r}_i'\right)}_{=\vec{0}} \times \vec{V} + \vec{R} \times \underbrace{\sum_i m_i \vec{v}_i'}_{=\vec{0}} + \vec{R} \times M\vec{V}$$

$$\Rightarrow \boxed{\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r}_i' \times m_i \vec{v}_i'} \quad \text{———} \quad (iv)$$

Hence, the total angular momentum is the vector sum of the angular momentum of the center of mass and the angular momenta of all the individual particles about the center of mass.

⚠ Can you express, similar to the above method, the kinetic energy of a system of particles?

Now, the rate of change of $\vec{L}$ is given by

$$\frac{d\vec{L}}{dt} = \frac{d}{dt} \sum_i (\vec{r}_i \times m_i \vec{v}_i) = \sum_i \vec{r}_i \times m_i \dot{\vec{v}}_i$$

$$= \sum_i \vec{r}_i \times (\vec{F}_{i,ext.} + \vec{F}_{i,int.})$$

$$= \sum_i \vec{r}_i \times \left(\vec{F}_{i,ext.} + \sum_{j \neq i} \vec{F}_{i,j}\right)$$

$$= \sum_i \vec{r}_i \times \vec{F}_{i,ext.} + \sum_i \vec{r}_i \times \sum_{j \neq i} \vec{F}_{i,j}$$

$$= \vec{\tau}_{ext.} + \sum_{\substack{i,j \\ j<i}} (\vec{r}_i \times \vec{F}_{i,j}) + \sum_{\substack{i,j \\ j>i}} (\vec{r}_i \times \vec{F}_{i,j})$$

$$= \vec{\tau}_{ext.} + \sum_{\substack{i,j \\ j<i}} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{i,j}$$

If $\vec{F}_{i,j} \parallel \vec{r}_i - \vec{r}_j$, then the second term vanishes, and we finally have (for internal forces that are central):

$$\boxed{\frac{d\vec{L}}{dt} = \tau_{ext.}} \quad \text{———} \quad (v)$$

So, the rate of change of total angular momentum about the origin is equal to the net external torque acting on the system about the origin. If $\vec{\tau}_{ext.} = \vec{0}$, then $\vec{L}$ is a constant of the motion. This constitutes the principle of conservation of angular momentum (PCAM) for a system of particles.

- **Total Mechanical Energy of a System of Particles:**

Finally, we compute the total mechanical energy E for a system of particles. So,

$$E = \sum_i \frac{1}{2} m_i v_i^2 + \underbrace{\sum_i V_i^{ext.}}_{\text{external pot. energy}} + \underbrace{\sum_i V_i^{int.}}_{\text{internal pot. energy}}$$

The above expression is a näive (but intuitive) generalization of the energy expression for one single particle. As, one can easily understand that, here we have already considered the cases where both the external and internal forces are conservative in nature, i.e., both are derived from some scalar potential functions.

Next, we try to investigate the time rate of change of E, starting from Newton's 2nd law for a system of particles:

$$\sum_i m_i \ddot{\vec{r}}_i = \sum_i \vec{F}_i^{ext.} + \sum_i \vec{F}_i^{int.}$$

$$= \sum_i \vec{F}_i^{ext.} + \sum_i \sum_{j \neq i} \vec{F}_{i,j} \quad \text{———} \quad (vi)$$

Note that, $m_i \ddot{\vec{r}}_i \cdot \dot{\vec{r}}_i = m_i \frac{d\vec{v}_i}{dt} \cdot \vec{v}_i = \frac{d}{dt}\left(\frac{1}{2} m_i v_i^2\right)$ — (a)

and

$$\vec{F}_i^{ext.} = -\nabla V_i^{ext} \Rightarrow \vec{F}_i^{ext.} \cdot \dot{\vec{r}}_i = -\nabla V_i^{ext} \cdot \frac{d\vec{r}_i}{dt} = -\frac{dV_i^{ext}}{dt} \quad - (b)$$

We assume that $\vec{F}_i^{int.}$ is a central force and

$$V_{i,j} \equiv V_{i,j}(r_{ij}) = V_{j,i} \equiv V_{j,i}(r_{ij}) \; ; \quad r_{ij} = |\vec{r}_i - \vec{r}_j| .$$

Next, $\sum_i \sum_{j\neq i} \vec{F}_{i,j} \cdot d\dot{\vec{r}}_i = \frac{1}{2} \sum_i \sum_j \left(\vec{F}_{i,j} \cdot d\dot{\vec{r}}_i + \vec{F}_{j,i} \cdot d\dot{\vec{r}}_j \right)$

$$= \frac{1}{2} \sum_{\substack{i \quad j \\ (i \neq j)}} \sum \left(-\nabla_i V_{ij} \cdot d\dot{\vec{r}}_i - \nabla_j V_{ji} \cdot d\dot{\vec{r}}_j \right) ; \quad \nabla_i \equiv \frac{\partial}{\partial \vec{r}_i}$$

$$= -\frac{1}{2} \sum_i \sum_j \frac{dV_{ij}}{dt} = -\frac{d}{dt}\left(V^{int.}\right) \quad \text{— (c)}$$

with $V^{int.} = \frac{1}{2} \sum_i \sum_j V_{ij}$

Now, combining (c) with summed up (a) and (b) over i, we obtain by (vi)

$$\frac{d}{dt} \sum_i \left(\frac{1}{2} m_i v_i^2 \right) = -\frac{d}{dt} \sum_i V_i^{ext.} - \frac{d}{dt} \sum_i V_i^{int.}$$

where $V_i^{int.} = \frac{1}{2} \sum_j V_{ij}$

Thus, we finally have

$$\frac{d}{dt} \left[\sum_i \frac{1}{2} m_i v_i^2 + \sum_i V_i^{ext.} + \sum_i V_i^{int.} \right] = 0$$

$$\Rightarrow \boxed{\frac{dE}{dt} = 0}$$

This is the principle of conservation of mechanical energy (PCME) for a system of N particles. Remember, this conservation holds good on several assumptions.

⚠ Think what will happen if the internal forces are **not central** in nature?

♣ In the next section, we will come back to revisit these dynamical properties in the framework of Lagrangian formulation.