Tutorial sheet solutions1

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1. **Motion under gravity**

   Two particles move in a uniform gravitational field **g**. Initially they were located at the same point and then moved with velocities $v_1 = 3\ m/s$ and $v_2 = 4\ m/s$ horizontally in opposite directions. Find the distance between the particles when their velocity vectors just come mutually perpendicular.

   Solution: Suppose the two particles moves with velocity $\mathbf{v}_1$, $\mathbf{v}_2$ and became mutually perpendicular after t seconds. From figure 1, you can see that $\mathbf{v}_1$ is the vector sum of $\mathbf{u}_1$ and $\mathbf{g}t$ and $\mathbf{v}_2$ is the vector sum of $\mathbf{u}_2$ and $\mathbf{g}t$. So we can write-

$$\tan\alpha = \frac{gt}{u_1} \tag{1}$$

$$\tan(90-\alpha) = \frac{gt}{u_2} \tag{2}$$

   therefore,

$$\frac{g^2t^2}{u_1u_2} = 1 \tag{3}$$

   which gives

$$t^2 = \frac{u_1u_2}{g^2} = \frac{(3).(4)}{(9.8)^2} \tag{4}$$

$$t = 0.35\ sec \tag{5}$$

   The horizontal separation between the two particles at this time

$$d = (u_1+u_2)t = 2.45\ m \tag{6}$$

2. **Pseudoscalars and pseudovectors**

   Give examples of pseudoscalars and pseudovectors in physics. Find out the rules of transformation of a pseudoscalar and the components of a pseudovector under the reflection of a mirror situated on xz plane.

   Solution: Examples of pseudoscalar are helicity, magnetic flux etc. Examples of pseudovector are magnetic field, angular velocity, vorticity etc.

   Transformation rule for pseudovector:
   A pseudovector can be written as cross product of two true vectors. Say $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is pseudovector, where $\mathbf{A}$ and $\mathbf{B}$ are true vectors. Components of $\mathbf{C}$ are-

$$C_x = A_yB_z - B_yA_z, C_y = A_zB_x - A_xB_z, C_z = A_xB_y - A_yB_x \tag{7}$$

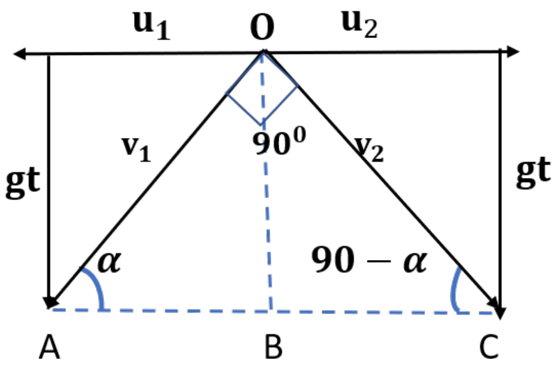


Figure 1

Now mirror transformation of true vectors **A**, **B** about XZ plane are following -

$$A'_x = A_x, A'_y = -A_y, A'_z = A_z \tag{8}$$

$$B'_x = B_x, B'_y = -B_y, B'_z = B_z \tag{9}$$

So mirror transformation of **C** can be written as (from equation(1) and equation(2))-

$$C'_x = -C_x, C'_y = C_y, C'_z = -C_z \tag{10}$$

Transformation rule for pseudoscalar:
Any pseudoscalar can be written as a dot product of a pseudovector and true vector. Say $\mathbf{D} = \mathbf{E}.\mathbf{F}$ is pseudoscalar, where **E** is true-vector and **F** is psuedovector. Components of **D** will be-

$$D = E_x F_x + E_y F_y + E_z F_z \tag{11}$$

Now using the properties of pseudovector (equation (10)) and true-vector (equation (9)), transformation of **D** can be written as-

$$D' = E_x.(-F_x) + (-E_y).F_y + E_z.(-F_z) = -D \tag{12}$$

Note that all prime vectors or scalars are mirror reflections quantities about XZ plane.

3. **Rotational frames of references**

   Let us assume a frame of reference O' is rotating about an inertial frame of reference O with a uniform angular speed ω in an orbit of radius R. Find the corresponding Galilean transformation and hence find the expression of the respective pseudoforces.

   Solution: The frame of reference O' is rotating about another frame of reference O in a circle of fixed radius R with a constant angular speed ω. Since O' does not have any spin motion, the axes of O and O' are always parallel.

   Without losing generality, we can assume that the rotation is about the z (or z') axis. Let us also suppose that, at $t = 0$, the x and x' axes are colinear. The motion of O' around O can be described by the figure below (a view of xy or x'y' plane from the top).

   The corresponding Galilean transformation is given by,

$$\begin{aligned} x &= x' + R\cos\omega t, \\ y &= y' + R\sin\omega t, \\ z &= z', \\ t &= t' \end{aligned}$$

Differentiating twice with respect to time t, we get in xy plane,

$$\ddot{x} = \ddot{x}' - \omega^2 R \cos \omega t,$$
$$\ddot{y} = \ddot{y}' - \omega^2 R \sin \omega t,$$

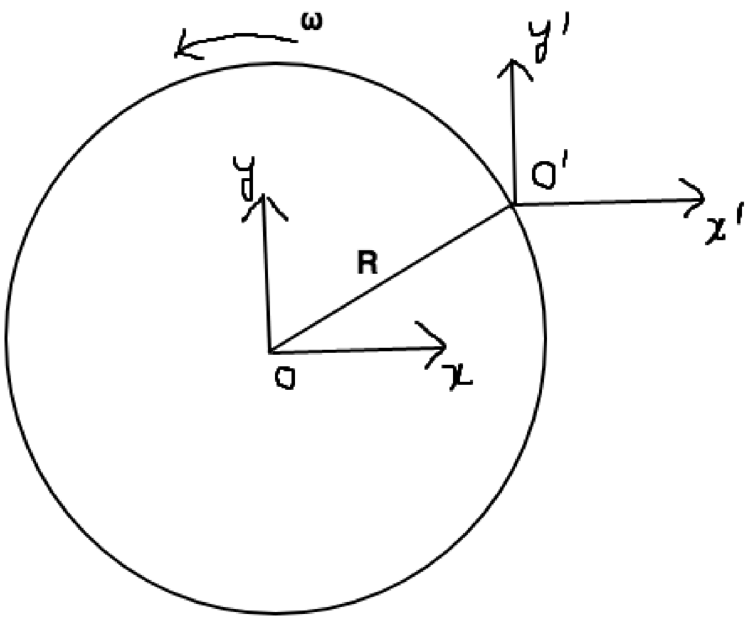
Since we assume that Newton's second law is valid in O, then we can write

$$\mathbf{F} = m(\ddot{x} + \ddot{y} + \ddot{z}) = m(\ddot{x}' + \ddot{y}' + \ddot{z}') - m\omega^2 \mathbf{R}$$

Note that, usual force laws depnd only on the relative distance. So the actual force expression does not change which implies $\mathbf{F} = \mathbf{F}'$. So in order that Newton's second law be valid in O' as well, we need to have,

$$m\ddot{\mathbf{r}} = \mathbf{F}' + m\omega^2 \mathbf{R}$$

$m\omega^2 \mathbf{R}$ is the pseudo force, celled the centrifugal force.



4. **Minimum surface of revolution**

Let us consider a simple and smooth curve on xy plane between (x_1, y_1) and (x_2, y_2). We make a surface of revolution by revolving the curve about y axis. Using the calculus of variation, find the curve $y(x)$ which gives the minimum area on the surface of revolution (draw a figure).

Solution: A elementary area (shaded region) on the surface of revolution is given by, $dA = 2\pi x ds$. Where, $ds = \sqrt{dx^2 + dy^2}$. Hence, $dA = 2\pi x dx \sqrt{1 + \dot{y}^2}$ where $\dot{y} = dy/dx$. The total surface area between (x_1, y_1) and (x_2, y_2) is thus given by

$$A = \int_1^2 2\pi x \sqrt{1 + \dot{y}^2} dx, \tag{13}$$

We need to minimise A, and the necessary condition for minimisation is

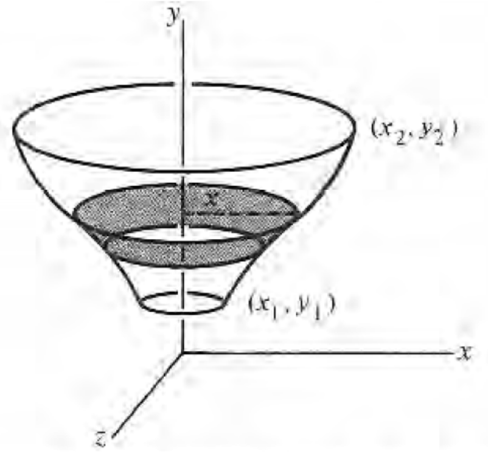


Figure 3

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0, \tag{14}$$

where in our case, $f = x\sqrt{1+\dot{y}^2}$.
calculating the derivatives give us, $\frac{\partial f}{\partial y} = 0$; $\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$.

So, we need to solve

$$\frac{d}{dx}\left[\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right] = 0, \tag{15}$$

or

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a, \tag{16}$$

a is a constant. Squaring the above equation and grouping terms

$$\dot{y}^2(x^2 - a^2) = a^2, \tag{17}$$

or solving

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}. \tag{18}$$

The general solution of the above equation is given by

$$y = a\int \frac{dx}{\sqrt{x^2 - a^2}} + b = a\cosh^{-1}\left(\frac{x}{a}\right) + b, \tag{19}$$

or

$$x = a\cosh\left(\frac{y-b}{a}\right). \tag{20}$$

where b is a constant of integration.