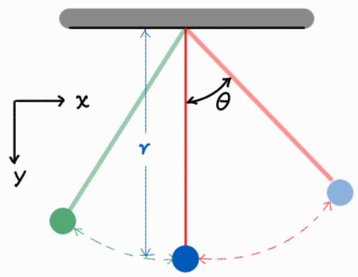
Formulation in terms of Generalized Variables

Unlike Newtonian mechanics, Lagrangian and Hamiltonian formulations employ generalized coordinates. A set of generalized coordinates are the minimal number of variables which can describe the configuration of the system complying with all the constraints of motion imposed.

These are called generalized because they could be anything like distances, angles, Fourier amplitudes, distance along a curve, or even energy and angular momenta etc.! For a system, a judicial choice of generalized coordinates is necessary to easily obtain the equations of motion ⟶ We have to exploit the right symmetries.

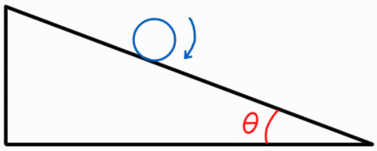
Example - 1: A simple pendulum moving in a vertical plane XY with a string of fixed length. This problem has a plane polar symmetry with coordinates (r, θ). But r is fixed, so only one generalized coordinate θ is present — 1 d.o.f.



Example - 2: To describe the motion of a rigid body, we need 6 generalized coordinates

- 3 position coordinates, e.g. (x, y, z) ⟶ translational d.o.f.
- 3 angles, e.g. Euler angles (θ, ϕ, ψ) ⟶ rotational d.o.f.

⚠ **What will be the set of generalized coordinates for a ball rolling down an inclined plane (without slipping)?**



- Symbols: In standard literatures, generalized coordinates are denoted by q_i, $i = 1, 2, \ldots, n$, where n is the number of degrees of freedom (d.o.f.) of the system. **Note that** the choice of generalized coordinates is not unique. So, between two sets of correctly chosen generalized coordinates, one can be more/less handy than the other.

- In the present context, whenever a constraint is meant, it is a "holonomic constraint." $\longrightarrow$ Check what is that!

After generalized coordinates, we introduce generalized velocities as

$$\dot{q}_j = \frac{dq_j}{dt}$$

If q_j is a position coordinate, $\dot{q}_j$ is a component of linear velocity; if q_j is an angle, then $\dot{q}_j$ is a component of angular velocity, and so on.

For a system, all the q_j's, $\dot{q}_j$'s and time t are mutually independent although one should remember that in reality both q_j and $\dot{q}_j$ are functions of time. But when we use $(q_j, \dot{q}_j, t)$ to describe the state of classical systems the state functional can be written in terms of q_j, $\dot{q}_j$ and t and each one can be varied without disturbing the others.

So, in fact, the state functional may depend explicitly on t or implicitly on t through $q_j(t)$ or $\dot{q}_j(t)$. In the following we try to relate usual coordinates and physical dynamical variables to the generalized coordinates and the generalized velocities.

Suppose we have a system of N particles. If they are free from any constraints, then the system would have $3N$ degrees of freedom. If there are k holonomic constraints, then the system would have $n = 3N - k$ degrees of freedom and it would have n number of generalized coordinates. And we must have

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n; t) \qquad \text{— (1)}$$

for $i = 1, 2, \ldots, N$. Thus,

$$\boxed{\dot{\vec{r}}_i \equiv \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j} \qquad \text{— (2)}$$

If $\frac{\partial \vec{r}_i}{\partial t} \equiv \vec{0}$ and if we write $\dot{\vec{r}}_i = \vec{v}_i$, then

$$\boxed{\vec{v}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j}$$

Note: for $\vec{r}_i = (x_i, y_i, z_i)$ [in Cartesians]

$$\dot{x}_i \equiv v_{xi} = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j$$

Again, the VARIATION or an arbitrary virtual variation of $\vec{r}_i$ will be given by

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \qquad \text{— (3)}$$

with $\delta q_j = \left.\frac{\partial q_j}{\partial \alpha}\right|_{\alpha=0} \cdot d\alpha$

Now we write the acceleration

$$\ddot{\vec{r}}_i = \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right) = \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial t}\right) + \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\dot{q}_j + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j$$

Since $\frac{d}{dt}$ commutes with $\frac{\partial}{\partial q_j}$ and $\frac{\partial}{\partial t}$, we have

$$\ddot{\vec{r}}_i = \frac{\partial \dot{\vec{r}}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j$$

Substituting the value of $\dot{\vec{r}}_i$, we get

$$\ddot{\vec{r}}_i = \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right) + \sum_{k=1}^{n} \dot{q}_k \frac{\partial}{\partial q_k}\left(\frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)$$
$$+ \sum_{k=1}^{n} \ddot{q}_k \frac{\partial \vec{r}_i}{\partial q_k}$$

$$\Rightarrow \boxed{\ddot{\vec{r}}_i = \frac{\partial^2 \vec{r}}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial t\, \partial q_j}\dot{q}_j + \sum_{k=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial t\, \partial q_k}\dot{q}_k + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_j\, \partial q_k}\dot{q}_j \dot{q}_k + \sum_{k=1}^{n} \ddot{q}_k \frac{\partial \vec{r}_i}{\partial q_k}}$$

— (4)

Kinetic Energy:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial t} + \sum_{j'=1}^{n} \frac{\partial \vec{r}_i}{\partial q_{j'}}\dot{q}_{j'}\right)$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 + \sum_{i=1}^{N}\sum_{j=1}^{n} m_i \frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{j'=1}^{n} m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_{j'}}\dot{q}_j \dot{q}_{j'}$$

— (5)

If $\vec{r}_i$'s do not depend on t explicitly (most common case), we then have $\frac{\partial \vec{r}_i}{\partial t} = \vec{0}$, and so

$$\boxed{T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n} m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_j \dot{q}_k} \qquad \text{— (6)}$$

Now when the generalized coordinates constitute an orthogonal system, the cross quadratic terms will vanish. Then the kinetic energy will be given by

$$\boxed{T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n} m_i \frac{\partial \vec{\mathbf{r}}_i}{\partial q_j}\cdot\frac{\partial \vec{\mathbf{r}}_i}{\partial q_k}\,\dot{q}_j\,\dot{q}_k\,\delta_{jk}}$$

Generalized Momentum: In normal case, the kinetic energy is given by $T = \frac{1}{2} m_i \dot{\vec{\mathbf{r}}}_i^2$ and the linear momentum is given by $\vec{\mathbf{p}}_i = m_i \dot{\vec{\mathbf{r}}}_i$ for the i-th particle. So, one can easily express $\vec{\mathbf{p}} = \nabla_{\dot{\vec{\mathbf{r}}}} T$, where $\nabla_{\dot{\vec{\mathbf{r}}}} \equiv \hat{\mathbf{x}}\frac{\partial}{\partial \dot{x}_i} + \hat{\mathbf{y}}\frac{\partial}{\partial \dot{y}_i} + \hat{\mathbf{z}}\frac{\partial}{\partial \dot{z}_i}$ is the gradient operator in the velocity space. We can generalize this definition by defining

$$p_k = \frac{\partial T}{\partial \dot{q}_k}$$

which we call the k-th component $(k = 1, 2, \cdots, n)$ of the generalized momentum.

Example: For a particle undergoing 2D motion under a central force,

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Now $q_1 \equiv r$, $q_2 \equiv \theta$, $\dot{q}_1 \equiv \dot{r}$, $\dot{q}_2 \equiv \dot{\theta}$, so

- $p_1 = p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \longrightarrow$ usual linear momentum component
- $p_2 = p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \longrightarrow$ angular momentum component

- We define the virtual work done by the force $\vec{\mathbf{F}}_i$ as

$$\begin{aligned}\delta W_i &= \vec{\mathbf{F}}_i \cdot \delta\vec{\mathbf{r}}_i \\ &= \vec{\mathbf{F}}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{\mathbf{r}}_i}{\partial q_j}\,\delta q_j \\ &= \sum_j Q_j\,\delta q_j\end{aligned}$$

with $Q_j = \vec{\mathbf{F}}_i \cdot \frac{\partial \vec{\mathbf{r}}_i}{\partial q_j}$ known as the generalized force.