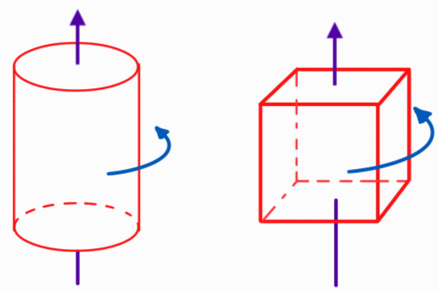
Symmetries and Newton's Laws

A symmetry is the property of **invariance** of a system before and after an operation.

Let us consider a vertical cylinder (See figure). The cylinder (unless we mark with specific symbol or color), if rotated about its vertical axis by an arbitrary amount, looks the same. This signifies the configurational symmetry of a uniform cylinder under rotation about its vertical axis.



Unlike the cylinder, a cube looks identical only when it is rotated about its axis (see figure) through an angle of $\pi/2$, π, $3\pi/2$ etc. So, the cube also has a configuration symmetry under rotation, but not for any arbitrary rotation. As one can easily understand that a uniform cylinder is symmetric (by configuration) under an infinitesimal rotation by an angle $\delta\theta$, which is not true for the cube.

Like the physical objects, we can also find symmetries of physical laws. A physical law is symmetric when its mathematical expression remains the same before and after a transformation of the variables. Such a transformation is called a symmetry transformation.

Here, in the current portion, we shall discuss whether Newton's 2nd law is symmetric under usual space-time transformations:

(a) Space translation: Newton's 2nd law: $m\dfrac{d^2\vec{r}}{dt^2} = \vec{F}$, where $\vec{r}$ is the instantaneous position vector of the particle with respect to an origin $O(0,0,0)$. Now, if a new coordinate system has its origin at $\vec{a} = (a_x, a_y, a_z)$, then the new position vector is $\vec{r}\,' = \vec{r} - \vec{a}$. So,

$$\frac{d^2\vec{r}\,'}{dt^2} = \frac{d^2\vec{r}}{dt^2} \qquad [\because\ \vec{a} \text{ is a constant vector}]$$

and mass is unchanged with the change of origin: $m' = m$. So,

$$m'\frac{d^2\vec{r}\,'}{dt^2} = m\frac{d^2\vec{r}}{dt^2}$$

But in general, $\vec{F}$ can be a function of t, $\vec{r}$ and $\frac{d\vec{r}}{dt}$. Now

$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} \quad \& \quad t' = t, \quad \text{but} \quad \vec{r}\,' = \vec{r} - \vec{a}.$$

So, $\vec{F} \equiv \vec{F}(t, \vec{r}, \vec{v}) = \vec{F}(t', \vec{r}\,' + \vec{a}, \vec{v}\,')$. It therefore seems to have changed under a space translation. However, in reality, for all the usual forces, the space dependence is not the dependence on the current position merely of the particle BUT the dependence is on the relative distance between the point particle and the particle which exerts the force (or rather, creates the force field) i.e. between two massive particles for gravitational force, between two charged particles for electrostatic force etc. For instance,

$$\vec{F}_g = -\frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) \quad ; \quad \vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1)$$

Since under the concerning transformation, $\vec{r}_2 - \vec{r}_1 = (\vec{r}_2 - \vec{a}) - (\vec{r}_1 - \vec{a}) = \vec{r}_2\,' - \vec{r}_1\,'$, we have $\vec{F}\,' = \vec{F}(\vec{r}_2\,' - \vec{r}_1\,', \vec{v}\,', t') = \vec{F}(\vec{r}_2 - \vec{r}_1, \vec{v}, t) = \vec{F}$, and hence

$$m' \frac{d^2\vec{r}\,'}{dt'^2} = \vec{F}\,' \quad \text{is equivalent to} \quad m\frac{d^2\vec{r}}{dt^2} = \vec{F},$$

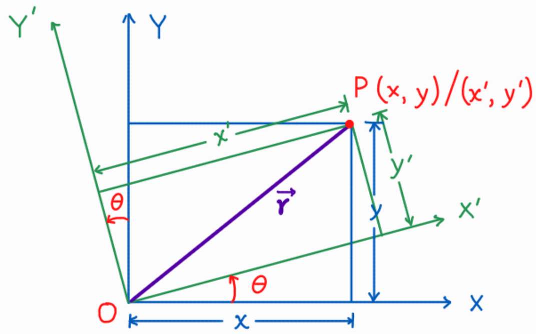
which says that Newton's 2nd law is symmetric under space translations for usual types of forces depending on the relative distance between two interacting particles.

- **Note that** all the four fundamental forces (strong, electromagnetic, weak and gravitational) depend on the relative distances between the source and the test particles. Therefore, Newton's 2nd law is applicable at any point/location in space → **Homogeneity of space.**

One should realize that shifting the origin by keeping the combined system of source and test particles fixed is equivalent to keeping the origin fixed and shifting the whole system in the opposite direction.

(b) Spatial rotation: Now we check whether Newton's 2nd law is invariant under an arbitrary rotation of the system in space. Or equivalently, we can check if Newton's 2nd law changes under a rotation of the coordinate axes but keeping the system under study fixed.

We assume that the source particle is at the origin and the test particle is at P (see figure). Without losing generality, we consider a rotation about the Z-axis in the XY-plane. Using elementary algebra and trigonometry, one can show that:



$\vec{r} \to \vec{r}'$, with $\vec{r}' = (x, y)$ and $\vec{r}' = (x', y')$

where

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \rightarrow \textbf{Try at Home}$$

Since the source particle is at origin, the force vector is also emanated from the origin towards an arbitrary direction $\hat{n}$, NOT NECESSARILY along $\vec{OP}$. Then just like $\vec{r}$, $\vec{F}$ also satisfies

$$F'_x = F_x\cos\theta + F_y\sin\theta$$

$$F'_y = -F_x\sin\theta + F_y\cos\theta$$

t and m do not change.

Now, Newton's 2nd law in the rotated frame is to be checked for invariance:

$$F'_x = m\frac{d^2x'}{dt^2} = m\frac{d^2x}{dt^2}\cos\theta + m\frac{d^2y}{dt^2}\sin\theta = F_x\cos\theta + F_y\sin\theta$$

$$F'_y = m\frac{d^2y'}{dt^2} = -m\frac{d^2x}{dt^2}\sin\theta + m\frac{d^2y}{dt^2}\cos\theta = -F_x\sin\theta + F_y\cos\theta$$

Hence, Newton's 2nd law is invariant under spatial rotations. This leads to isotropy of space.

- **Note:** Later we shall see that PCLM is a consequence of homogeneity of space and PCAM is a consequence of the isotropy of space.

(c) Translation in time: It means that we make a shift in the time coordinate: $t \to t' = t + a$ such that $\vec{r} \to \vec{r}' = \vec{r}$. a is an arbitrary constant, and there's no change in spatial coordinates. This leads to PCME, as we will see later.

So, $$\frac{d^2\vec{r}'}{dt'^2} = \frac{d^2\vec{r}}{dt'^2} = \frac{d}{dt'}\left(\frac{d\vec{r}}{dt}\frac{dt}{dt'}\right) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\frac{dt}{dt'}\right)\frac{dt}{dt'}$$

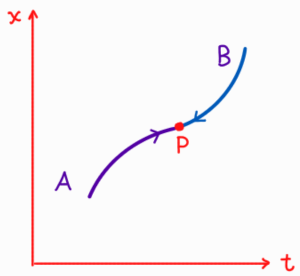
But $\frac{dt'}{dt} = \frac{d}{dt}(t+a) = \frac{dt}{dt} = 1$, so $\frac{dt}{dt'} = \frac{1}{dt'/dt} = 1$. Thus

$$\frac{d^2\vec{r}'}{dt'^2} = \frac{d^2\vec{r}}{dt^2}$$

Thus, Newton's 2nd law is invariant under time translation if the force is also invariant under time translation.

- All the fundamental forces are independent of time and hence they are invariant under time translation $\longrightarrow$ Newton's 2nd law for natural force is symmetric under time translation.
- For a time-dependent force, however (e.g. periodic force of nature $\vec{F} = F_0 \cos(\omega t)\,\hat{n}$), the force is not time-independent, but time translation adds a constant phase $\longrightarrow$ dynamics unchanged.

(d) Time reversal: Time reversal means studying the evolution of a dynamical system in the reverse direction of its natural direction. Technically, it means in an equation time reversal can be perceived if $t \to -t$.



Consider the position vs time plot of 1D motion of a particle. The particle starts from A and is moving towards B. Let P be any arbitrary point along the path (curve) $\overset{\frown}{AB}$. We check whether from P if the time runs backwards (conceptual), the particle follows the path of evolution (the same path) backwards and finally gets back to A or not!

In Newton's 2nd law $(m = 1)$,

$$\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

Now, under time reversal, $\vec{r} \to \vec{r}' = \vec{r}$ and $t \to t' = -t$. So $\frac{d\vec{r}'}{dt'} = -\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}'}{dt'^2} = \frac{d^2\vec{r}}{dt^2}$. In fact, if time reversal is done at a time t_0 after the start, then $t' = t_0 - t \Rightarrow dt' = -dt$.

So, in order that Newton's 2nd law be invariant under time reversal, $\vec{F}'$ must be equal to $\vec{F}$.

- Time reversal is always obeyed microscopically for usual gravitational or electromagnetic forces $\longrightarrow$ Reversibility.
- However, including macroscopic effects of dissipation i.e. frictional force or electro-magnetic fields with absorption (in a conductor), one can have violation of time-reversal symmetry in classical systems $\rightarrow$ Irreversibility.

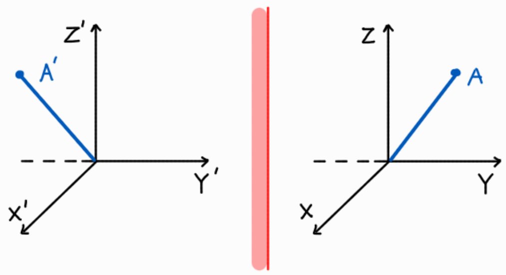
⚠ **What about strong and weak forces? Study.**

Note that, time reversal switches the direction of velocity but not of acceleration. Similarly, electric field is invariant under time reversal but magnetic field changes its sign.

Why? What about Lorentz force and Poynting vector?

(e) Mirror symmetry (special case of parity): In physics, several entities have handed-ness (coil, corkscrew, etc.) while others don't care about handedness (the projectile, a falling body, etc.) when their motion is analyzed. Therefore, it is necessary to check for invariance of physical laws under mirror symmetry.

Let's consider a mirror in the XZ plane (see figure). For measuring the components of the vector, we use the same coordinate axes on either side of the mirror. Hence X-Y-Z $\equiv$ X'-Y'-Z'.



Under the mirror reflection as given in figure, $A'_x = A_x$, $A'_y = -A_y$, $A'_z = A_z$. If $\vec{A} = \vec{r}$, then we have

$$x' = x, \quad y' = -y, \quad z' = z$$

$$v'_x = v_x, \quad v'_y = -v_y, \quad v'_z = v_z$$

$$a'_x = a_x, \quad a'_y = -a_y, \quad a'_z = a_z$$

and $t' = t$, $m' = m$.

So, for Newton's 2nd law to respect mirror reflection symmetry, we must have

$$F'_x = F_x, \quad F'_y = -F_y, \quad F'_z = F_z.$$

This kind of transformation is followed by true or polar vectors → Newton's 2nd law respects mirror symmetry if the force is a true vector.

- If we take the cross product of two true vectors $\vec{A}$ and $\vec{B}$ such that $\vec{C} = \vec{A} \times \vec{B}$, then one can show that

$$C'_x = -C_x, \quad C'_y = C_y, \quad C'_z = -C_z$$

which is the opposite transformation of that of a true vector. $\vec{C}$ is then called an axial vector or a pseudo vector, e.g. angular momentum, magnetic field etc.

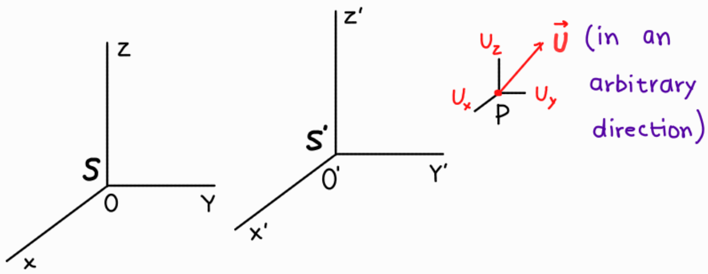
- Similar to vectors, we also have true and pseudo scalars. A true scalar does not change sign under mirror reflection whereas a pseudoscalar e.g. helicity (check on the Internet) does.

N.B.: In weak nuclear reactions, time reversal symmetry breaks down since the potential function of weak nuclear force contains a non-zero pseudoscalar part.

⚠ **How do the vectors/pseudovectors change under parity?**

(f) Galilean transformation (GT): This relates the laws of physics in two inertial frames moving with a uniform speed $\vec{U}$ with respect to each other (≡ verification of Newton's 1st law). Clearly, under this transformation, we have

$$\left.\begin{aligned} x' &= x - U_x t \\ y' &= y - U_y t \\ z' &= z - U_z t \\ t' &= t \end{aligned}\right\} \quad \text{and} \quad m' = m.$$



This states a situation where the reference frame S' is moving with a uniform velocity $\vec{U}$ with respect to S. At $t = 0 = t'$, both the origins O and O' coincide and the above transformation is written for any time instant $t = t$ after the initial instant.

Galilean transformation is a type of spatial translation (not by a constant amount) which is linearly proportional to time. Hence the particle velocities are related by $\vec{v}' = \vec{v} - \vec{U}$ (translation of velocity by a constant amount). Whence clearly we have

$$\frac{d\vec{v}'}{dt'} = \frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} \qquad \left[\because \vec{U} \text{ is a constant}\right]$$

Assuming $m' = m$ (just an assumption, verified experimentally), we get

$$m'\frac{d\vec{v}'}{dt'} = m\frac{d\vec{v}}{dt}$$

In order that Newton's 2nd law be invariant under GT, $\vec{F}' = \vec{F}$. Note that, like we discussed in (a), usual forces always depend on the relative distance between two spatial points. For those, we can say

$$\vec{F}' = \vec{F}(\vec{r}_2' - \vec{r}_1') = \vec{F}\left[(\vec{r}_2 - \vec{U}t) - (\vec{r}_1 - \vec{U}t)\right] = \vec{F}(\vec{r}_2 - \vec{r}_1) = \vec{F}$$

So, for natural forces, we always have Newton's 2nd law valid under GT.

- Note that Maxwell's equations are not invariant under GT $\rightarrow$ a more general transformation – the Lorentz transformation (LT) is required. (Relativistic LT reduces to GT for $v \ll c$.)

- <u>Non-inertial frames and pseudo forces:</u>

As we all know that Newton's 1st law says that all the inertial frames are equivalent and previously we showed that Newton's 2nd law is symmetric under GT. Here we investigate how Newton's 2nd law is modified if one of the two frames is a non-inertial frame.

For simplicity, we assume that S' is accelerating with respect to S with a uniform acceleration $\vec{a}$. Then the transformation will be given by

$$\left.\begin{aligned} x' &= x - \tfrac{1}{2}a_x t^2 \\ y' &= y - \tfrac{1}{2}a_y t^2 \\ z' &= z - \tfrac{1}{2}a_z t^2 \end{aligned}\right\}$$

Think what happens if $\vec{a}$ is not constant. How to handle this?

$$t' = t; \quad m' = m; \quad \text{also } \underbrace{\vec{F}' = \vec{F}}.$$

↳ have you understood why?

and finally, we get

$$\vec{v}' = \vec{v} - \vec{a}t \Rightarrow m'\frac{d\vec{v}'}{dt} = m\frac{d\vec{v}}{dt} - m\vec{a}$$

$\therefore \quad \vec{F}' = \vec{F} - m\vec{a}$. So in S, $m\frac{d\vec{v}}{dt} = \vec{F}$; in S', we have $\boxed{m'\frac{d\vec{v}'}{dt'} = \vec{F}' = \vec{F} - m\vec{a}}$

(Case of rotational frame will be done in tutorial)

Pseudo force