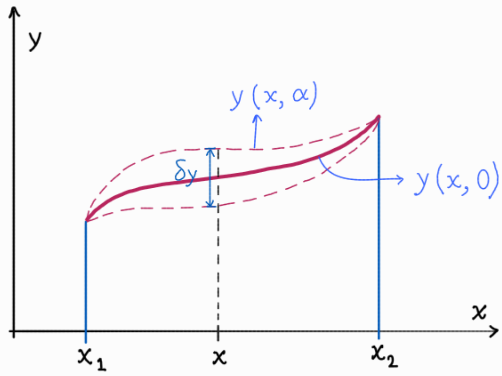
Methods of Calculus of Variations

- **Objective:** To find the curve for which some line integral has a stationary (extremal) value.

Let's consider the general problem. For simplicity we start with a one dimensional problem. We have a function $f(y, y', x)$ defined on a path $y = y(x)$ between two fixed x-values x_1 and x_2, $x_1 < x_2$. Here $y' = dy/dx$. Our main objective is to find a particular path $y(x)$ such that the line integral

$$J = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

has a stationary value relative to paths differing infinitesimally from the correct one. This difference δJ is called the variation of J. δJ must be vanishing with respect to the neighbouring paths $\tilde{y}(x) = y(x, \alpha)$ (where α is a smallness parameter), infinitesimally apart from the correct one $y(x) = y(x, 0)$. Here, $\tilde{y}(x) = y(x, \alpha) = y(x, 0) + \alpha\, \eta(x)$. The function $\eta(x)$ is such that



- it vanishes at x_1 and x_2
- it is well-behaved between x_1 and x_2, i.e. no singularity, and continuous with continuous 1st and 2nd derivatives in the same interval.

So, $y(x, \alpha)$ gives a set of parametric curves in the y-x plot where α is the parameter. Thus,

$$J(\alpha) = \int_{x_1}^{x_2} f\left[y(x, \alpha), y'(x, \alpha), x\right] dx$$

For $J(\alpha)$ to be stationary (either minimal or maximal) we should have

$$\left.\frac{dJ(\alpha)}{d\alpha}\right|_{\alpha = 0} = 0$$

(We evaluate the derivative at $\alpha = 0$ because our assumption is that for $\alpha = 0$, $y(x, \alpha)$ is the stationary / optimal path).

We use Leibniz rule to differentiate under the integral sign:

$$\left.\frac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)dx\Bigg|_{\alpha=0}$$

⚠ **Can you tell why there are no terms like $\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \alpha}$?**

$$\Rightarrow\ 0 = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x\,\partial \alpha}\right)dx\Bigg|_{\alpha=0}$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}\right)dx\Bigg|_{\alpha=0} + \left(\frac{\partial f}{\partial y'}\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0}\right)\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0}dx$$

Recognizing that $\left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0} = \eta(x)$ and using the fact that $\eta(x_1) = 0 = \eta(x_2)$, we get

$$\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx = 0$$

The the above integral vanishes for any arbitrary $\eta(x)$ with the given properties, we must have

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

Now, let's have a look at the quantity

$$\delta y \equiv \left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0} d\alpha$$

It is a variation of y keeping x frozen (if x is time, this δy is known as the virtual displacement). δy is the distance between the optimal path and (any of) its neighboring paths for a given x (see figure on the previous page). In the same way, the infinitesimal variation δJ of $J(\alpha)$ about the correct one is given by

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} d\alpha = \left.\frac{dJ}{d\alpha}\right|_{\alpha=0} d\alpha = \delta J$$

So, an equivalent way of saying that $J(\alpha)$ is stationary, is to say that

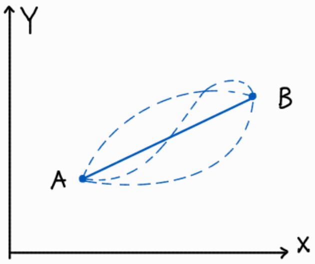
$$\boxed{\delta J = 0}$$

as $d\alpha$ can be arbitrary. We will use this throughout the course.

⚠ **What will happen if $f = f(y_1, y_2, \dots, y_n;\ y_1', y_2', \dots, y_n';\ x)$?**

Applications:

(1) To find the shortest distance between two points in a plane:



The element of an arc of the curve C on a Cartesian plane is given by

$$ds = \sqrt{dx^2 + dy^2}$$

So, the distance S_c along the curve C between the points A and B is given by

$$S_c = \int_c ds = \int_A^B \sqrt{dx^2 + dy^2}\,\Big|_c = \int_A^B dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\Bigg|_c$$

For S_c to be the minimum, we must have

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \;; \quad f(y, y', x) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Thus, $\dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial f}{\partial y'} = \dfrac{y'}{\sqrt{1+(y')^2}}$, so that

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\Rightarrow \quad \frac{y''}{\sqrt{1+(y')^2}} - \frac{(y')^2 y''}{(1+(y')^2)^{3/2}} = 0 \quad \Rightarrow \quad \frac{y''}{\sqrt{1+(y')^2}}\left(1 - \frac{(y')^2}{1+(y')^2}\right) = 0$$

$$\Rightarrow \quad y'' = 0, \quad \text{or} \quad \frac{(y')^2}{1+(y')^2} = 1$$

But the latter equality is never possible and it leads to the absurd result $1 = 0$.

Thus we have

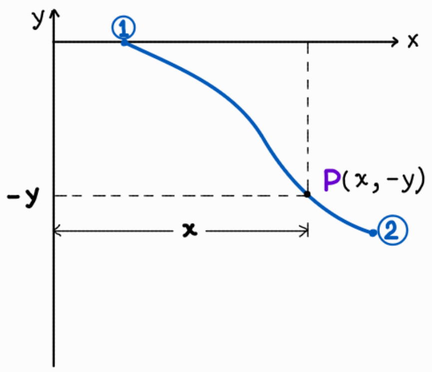
$$\boxed{y'' = 0}$$

$$\Rightarrow \quad \boxed{y = ax + b}\,, \quad \text{for some constants } a \text{ and } b.$$

This is the equation of a straight line, the optimal curve along which the distance between A and B could be the shortest.

Greek for the fastest path.

(2) Brachistochrone problem: Statement – If a point particle is falling from rest under the influence of gravity from a higher point to a lower one, which curve joining the two points would correspond to the least time ?



→ reference of potential

In this problem, we have to minimize the total time spent :

$$t = \int_{①}^{②} dt = \int_{①}^{②} \frac{ds}{v} \ ;$$

ds : distance along the curve, v : speed at any point.

Motion under gravity implies conserved energy in the absence of any dissipative forces such as friction. At ①, $T = 0$, $V = 0$. At ②, $T = \frac{1}{2} mv^2$, $V = -mgy$.

Thus, $T + V = 0 = \frac{1}{2} mv^2 - mgy \Rightarrow \frac{1}{2} mv^2 = mgy \Rightarrow v = \sqrt{2gy} \ ; \ y > 0$

Now

$$t = \int_{①}^{②} \frac{\sqrt{dx^2 + dy^2}}{v} = \int_{①}^{②} \frac{dx\sqrt{1 + (dy/dx)^2}}{\sqrt{2gy}} = \int_{①}^{②} dx \left(\frac{1 + (y')^2}{2gy} \right)^{1/2}$$

$$\Rightarrow \quad f(y, y', x) = \left(\frac{1 + (y')^2}{2gy} \right)^{1/2}$$

The correct curve would satisfy

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0$$

Now $$\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{2gy}{1 + (y')^2} \right)^{1/2} \left(- \frac{1 + (y')^2}{(2gy)^2} \right) \cdot 2g = -\frac{1}{2}\left(\frac{1 + (y')^2}{2g} \right)^{1/2} \frac{1}{y^{3/2}}$$

and $$\frac{\partial f}{\partial y'} = \frac{1}{2}\left(\frac{2gy}{1 + (y')^2} \right)^{1/2} \frac{2y'}{2gy} = \frac{y'}{\sqrt{2gy}\sqrt{1 + (y')^2}}$$

Thus, for the correct curve,

$$-\frac{1}{2}\left(\frac{1 + (y')^2}{y^3} \right)^{1/2} = \frac{d}{dx}\left(\frac{y'}{\sqrt{y(1 + (y')^2)}} \right)$$

$$= \frac{y''}{\sqrt{y(1 + (y')^2)}} - \frac{1}{2} \frac{y'}{[y(1 + (y')^2)]^{3/2}} [y'[1 + (y')^2] + y \cdot 2y'y'']$$

$$= \frac{y''}{\sqrt{y(1 + (y')^2)}} - \frac{1}{2} \frac{(y')^2 [1 + (y')^2 + 2y\, y'']}{[y(1 + (y')^2)]^{3/2}}$$

$$= \frac{y''}{\sqrt{y(1 + (y')^2)}} - \frac{1}{2} \frac{(y')^2}{[y^3(1 + (y')^2)]^{1/2}} - \frac{(y')^2 y''}{(y[1 + (y')^2]^3)^{1/2}}$$

$$\Rightarrow -\frac{1}{2}\frac{1+(y')^2}{y} = y'' - \frac{1}{2}\frac{(y')^2}{y} - \frac{(y')^2 y''}{1+(y')^2}$$

$$\Rightarrow -\frac{1+(y')^2}{2y} + \frac{(y')^2}{2y} = \frac{y''}{1+(y')^2}$$

$$\Rightarrow \frac{y''}{1+(y')^2} = -\frac{1}{2y}$$

$$\Rightarrow 2yy'' = -1-(y')^2 \Rightarrow 2yy'' + (y')^2 + 1 = 0$$

$$\Rightarrow 2yy'y'' + (y')^3 + y' = 0$$

$$\Rightarrow y\frac{d}{dx}(y')^2 + (y')^2\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}\left[y(y')^2\right] + \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}\left[y + y(y')^2\right] = 0 \Rightarrow y\left[1+(y')^2\right] = C > 0;$$ C is a constant.

$$\Rightarrow y' = \sqrt{\frac{C-y}{y}} \Rightarrow dx = \sqrt{\frac{y}{C-y}}\,dy$$

Substituting $y = C\sin^2 t$, we get

$$\int dx = 2C\int \sin^2 t\,dt = C\int(1-\cos 2t)\,dt$$

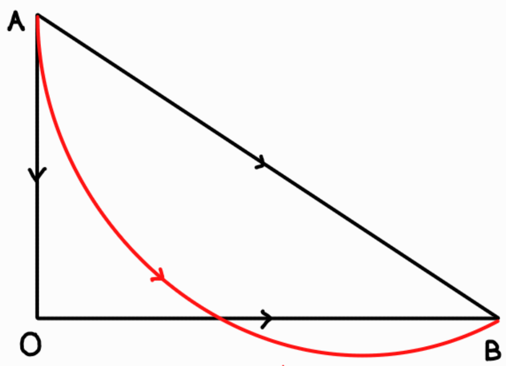
$$\Rightarrow x(t) = Ct - \tfrac{1}{2}C\sin(2t) + D$$

with $$y(t) = \frac{C}{2}\left[1-\cos(2t)\right]$$

If we suppose that this path passes through the origin, then $D = 0$, and we get a family of curves parametrized by t:

$$x(t) = Ct - \tfrac{1}{2}C\sin(2t)$$

$$y(t) = \frac{C}{2}\left[1-\cos(2t)\right]$$



Then we take the curve which also passes through the point 2 (x_2, y_2). This curve is part of a cycloid.

Historically, this problem led John Bernoulli to the formal foundation of the calculus of variations.