INDIAN INSTITUTE OF TECHNOLOGY KANPUR Supratik Banerjee

Classical Mechanics I (PHY 401A) 2020-21 Odd semester

Tutorial sheet 10

Date: 25. 11. 2020

1. **Properties of Poisson Brackets**

If $[f, g]$ denotes the Poisson bracket of two functions f and g, then show that

(a) $[f, g] = -[g, f]$

(b) $[f, c] = 0$, for any constant function c

(c) $[f_1 + f_2, g] = [f_1, g] + [f_2, g]$

(d) $[f_1 f_2, g] = f_1 [f_2, g] + f_2 [f_1, g]$

(e) $\frac{\partial}{\partial t}[f, g] = [\frac{\partial f}{\partial t}, g] + [f, \frac{\partial g}{\partial t}]$

(f) $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$ (Jacobi's identity)

Solution:

$$[f, g] = \sum_i [\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}] \quad (definition) \tag{1}$$

(a)

$$[g, f] = \sum_i [\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}] = -[f, g] \tag{2}$$

(b)

$$[f, c] = \sum_i [\frac{\partial f}{\partial p_i}.0 - \frac{\partial f}{\partial q_i}.0] = 0 \tag{3}$$

(c)

$$[f_1 + f_2, g] = \sum_i [\frac{\partial (f_1 + f_2)}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial (f_1 + f_2)}{\partial q_i}\frac{\partial g}{\partial p_i}] \tag{4}$$

$$[f_1 + f_2, g] = \sum_i [\frac{\partial f_1}{\partial p_i}\frac{\partial g}{\partial q_i} + \frac{\partial f_2}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f_1}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f_2}{\partial q_i}\frac{\partial g}{\partial p_i}] \tag{5}$$

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g] \tag{6}$$

(d)

$$[f_1 f_2, g] = \sum_i [\frac{\partial (f_1 f_2)}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial (f_1 f_2)}{\partial q_i}\frac{\partial g}{\partial p_i}] \tag{7}$$

$$= \sum_i [f_1 \frac{\partial f_2}{\partial p_i}\frac{\partial g}{\partial q_i} + f_2 \frac{\partial f_1}{\partial p_i}\frac{\partial g}{\partial q_i} - f_1 \frac{\partial f_2}{\partial q_i}\frac{\partial g}{\partial p_i} - f_2 \frac{\partial f_1}{\partial q_i}\frac{\partial g}{\partial p_i}] \tag{8}$$

$$[f_1 f_2, g] = f_1[f_2, g] + f_2[f_1, g] \tag{9}$$

(e)

$$\frac{\partial}{\partial t}[f, g] = \frac{\partial}{\partial t}\sum_i [\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}] \tag{10}$$

$$\frac{\partial}{\partial t}[f, g] = \sum_i [\frac{\partial}{\partial p_i}(\frac{\partial f}{\partial t})\frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}(\frac{\partial g}{\partial t}) - \frac{\partial}{\partial q_i}(\frac{\partial f}{\partial t})\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}(\frac{\partial g}{\partial t})] \tag{11}$$

$$= [\frac{\partial f}{\partial t}, g] + [f, \frac{\partial g}{\partial t}] \tag{12}$$

(f)

$$[f, [g, h]] + [g, [h, f]] = [f, (\frac{\partial g}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial h}{\partial p})] + [g, (\frac{\partial h}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p})] \tag{13}$$

$$= [f, \frac{\partial g}{\partial p}\frac{\partial h}{\partial q}] - [f, \frac{\partial g}{\partial q}\frac{\partial h}{\partial p}] + [g, \frac{\partial h}{\partial p}\frac{\partial f}{\partial q}] - [g, \frac{\partial h}{\partial q}\frac{\partial f}{\partial p}] \tag{14}$$

$$= [\frac{\partial g}{\partial q}\frac{\partial h}{\partial p}, f] - [\frac{\partial g}{\partial p}\frac{\partial h}{\partial q}, f] - [\frac{\partial h}{\partial p}\frac{\partial f}{\partial q}, g] + [\frac{\partial h}{\partial q}\frac{\partial f}{\partial p}, g] \tag{15}$$

$$= \frac{\partial g}{\partial q}[\frac{\partial h}{\partial p}, f] + \frac{\partial h}{\partial p}[\frac{\partial g}{\partial q}, f] - \frac{\partial g}{\partial p}[\frac{\partial h}{\partial q}, f] - \frac{\partial h}{\partial q}[\frac{\partial g}{\partial p}, f] - \frac{\partial f}{\partial q}[\frac{\partial h}{\partial p}, g] - \frac{\partial h}{\partial p}[\frac{\partial f}{\partial q}, g] + \frac{\partial h}{\partial q}[\frac{\partial f}{\partial p}, g] + \frac{\partial f}{\partial p}[\frac{\partial h}{\partial q}, g] \tag{16}$$

here,

$$\frac{\partial h}{\partial p}([\frac{\partial g}{\partial q}, f] + [g, \frac{\partial f}{\partial q}]) = \frac{\partial h}{\partial p}\frac{\partial}{\partial q}[g, f] \tag{17}$$

and,

$$\frac{\partial h}{\partial q}([\frac{\partial f}{\partial p}, g] + [f, \frac{\partial g}{\partial p}]) = \frac{\partial h}{\partial q}\frac{\partial}{\partial p}[g, f] \tag{18}$$

and,

$$= \frac{\partial g}{\partial q}[\frac{\partial h}{\partial p}, f] - \frac{\partial g}{\partial p}[\frac{\partial h}{\partial q}, f] - \frac{\partial f}{\partial q}[\frac{\partial h}{\partial p}, g] + \frac{\partial f}{\partial p}[\frac{\partial h}{\partial q}, g] = 0 \tag{19}$$

finally, from the addition of the equations 17,18, and 19

$$[h, [g, f]] + 0 = -[h, [f, g]] \tag{20}$$

hence,

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \tag{21}$$

2. **Canonical transformation**

   If $(p, q) \longrightarrow (P, Q)$ is a canonical transformation, then show that

   (a) $[q_j, q_k]_{P,Q} = 0 = [p_j, p_k]_{P,Q}$

   (b) $[p_j, q_k]_{P,Q} = \delta_{jk}$

Again using the above relations, prove that $[f, p_j]_{P,Q} = -\frac{\partial f}{\partial q_j}$ and $[f, q_j]_{P,Q} = \frac{\partial f}{\partial p_j}$.

Solution: (a)

$$[p_j, p_k]_{P,Q} = \sum_i \left[\frac{\partial p_j}{\partial P_i} \frac{\partial p_k}{\partial Q_i} - \frac{\partial p_j}{\partial Q_i} \frac{\partial p_k}{\partial P_i} \right], \tag{22}$$

we introduce generating function $F(q_i, Q_i, t)$ such that $p_i = \frac{\partial F}{\partial q_i}$ and $P_i = -\frac{\partial F}{\partial Q_i}$. Giving $\frac{\partial p_i}{\partial Q_j} = \frac{\partial}{\partial Q_j}\left(-\frac{\partial F}{\partial q_i}\right) = -\frac{\partial P_j}{\partial q_i}$.

Similarly. for $F_2(q_i, p_i, t)$ we have $p_i = \frac{\partial F_2}{\partial q_i}$ and $Q_i = \frac{\partial F_2}{\partial P_i}$ giving us $\frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i}$.

Putting these in Eq.(22)

$$[p_j, p_k]_{P,Q} = -\frac{\partial p_j}{\partial q_k} = 0$$

.

Again, using $F_3(p_i, Q_i, t)$ such that $q_i = -\frac{\partial F_3}{\partial p_i}$ and $P_i = -\frac{\partial F_3}{\partial Q_i}$ which gives us $\frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i}$ and also using $F_4(p_i, P_i)$ such that $q_i = -\frac{\partial F_4}{\partial p_i}$ and $Q_i = \frac{\partial F_4}{\partial P_i}$ which give us $\frac{\partial q_i}{\partial P_j} = -\frac{\partial Q_j}{\partial p_i}$. we can show,

$$[q_j, q_k]_{P,Q} = 0$$

.

(b) We have

$$[p_j, q_k] = \sum_i \left[\frac{\partial p_j}{\partial P_i} \frac{\partial q_k}{\partial Q_i} - \frac{\partial p_j}{\partial Q_i} \frac{\partial q_k}{\partial P_i} \right], \tag{23}$$

Now, using F_3 and F_4 from above we can show $\frac{\partial q_k}{\partial Q_i} = \frac{\partial P_i}{\partial p_k}$ and $\frac{\partial q_k}{\partial P_i} = -\frac{\partial Q_i}{\partial p_k}$. Hence,

$$[p_j, q_k] = \sum_i \left[\frac{\partial p_j}{\partial P_i} \frac{\partial P_i}{\partial p_k} + \frac{\partial p_j}{\partial Q_i} \frac{\partial Q_i}{\partial p_k} \right] = \frac{\partial p_j}{\partial p_k} = \delta_{j,k}$$

.

Again,

$$[f, p_j] = \sum_i \left[\frac{\partial f}{\partial P_i} \frac{\partial p_j}{\partial Q_i} - \frac{\partial f}{\partial Q_i} \frac{\partial p_j}{\partial P_i} \right], \tag{24}$$

using relations for F and F_2 we have

$$[f, p_j] = \sum_i \left[-\frac{\partial f}{\partial P_i} \frac{\partial P_i}{\partial q_j} - \frac{\partial f}{\partial Q_i} \frac{\partial Q_i}{\partial q_j} \right] = -\frac{\partial f}{\partial q_j}$$

.

Similarly, using relations for F_3 and F_4 we can show

$$[f, q_j] = \frac{\partial f}{\partial p_j}$$

3. **Canonically conjugate variables**

Show that the transformation

$$\begin{aligned} p &= \sqrt{2P}\,\cos Q \\ q &= \sqrt{2P}\,\sin Q \end{aligned} \tag{25}$$

is canonical.

Solution:

To show that the given transformation is canonical, we need show that the quantity $PdQ - pdq$ is an exact differential.

From Eq.(25), $dq = \frac{1}{\sqrt{2P}} \sin Q dP + \sqrt{2P} \cos Q dQ$

So.

$$\begin{aligned} PdQ - pdq &= PdQ - (\sqrt{2P} \cos Q) \left[\frac{1}{\sqrt{2P}} \sin Q dP + \sqrt{2P} \cos Q dQ \right] \\ &= PdQ - \sin Q \cos Q dP - 2P \cos^2 Q dQ \\ &= P \left(1 - 2\cos^2 Q\right) dQ - \sin Q \cos Q dP \\ &= -P \cos 2Q dQ - \frac{1}{2} \sin 2Q dP \\ &= -d \left[\frac{1}{2} P \sin 2Q \right]. \end{aligned} \tag{26}$$

Hence, given transformation is canonical.