CLASSICAL MECHANICS I (PHY 401A) 2020-21 ODD SEMESTER

Tutorial sheet 6

Date: 09 09. 2020.

1. **Oscillator attached to a spring**

   Find the frequency of small oscillations of a particle of mass m which is moving on a (i) horizontal straight line, (ii) vertical circular wire of radius a, and is attached to a massless spring (stays vertical at rest) whose other end is fixed at a distance ℓ to the nearest point of the trajectories. Note that a force F is required to extend the spring to length ℓ.

   Solution:

   (i) Particle moving in a straight line:

   kinetic energy of the particle

$$K.E. = \frac{1}{2}m\dot{x}^2 \tag{1}$$

The potential energy of the spring is equal to the force F multiplied by the extension in the spring. The extension in the spring is

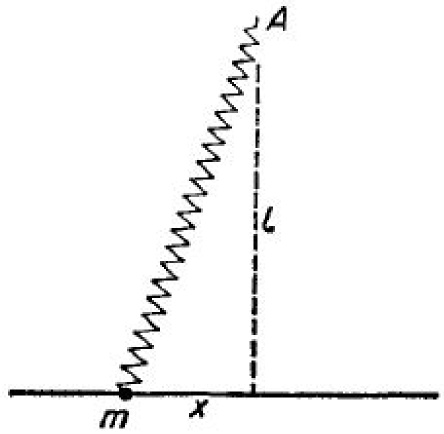

Abbildung 1: particle moving on a line

$$\delta l = \sqrt{l^2 + x^2} - l \tag{2}$$

$$for \;\; x << l, \qquad \delta l = \frac{x^2}{2l} \tag{3}$$

Now, the Lagrangian of the particle can be written as

$$L = \frac{1}{2}m\dot{x}^2 + F\frac{x^2}{2l} \tag{4}$$

from the Lagrangian's equation the equation of motion is

$$\ddot{x} + \frac{Fx}{ml} = 0 \tag{5}$$

and we have

$$\omega = \sqrt{\frac{F}{ml}} \tag{6}$$

(ii) Particle moving in a circle of radius a:

$$K.E. = \frac{1}{2}mr^2\dot{\phi}^2 \tag{7}$$

the potential energy of the spring is equal to the force F multiplied by the extension in the spring. In this case the extension in the spring is

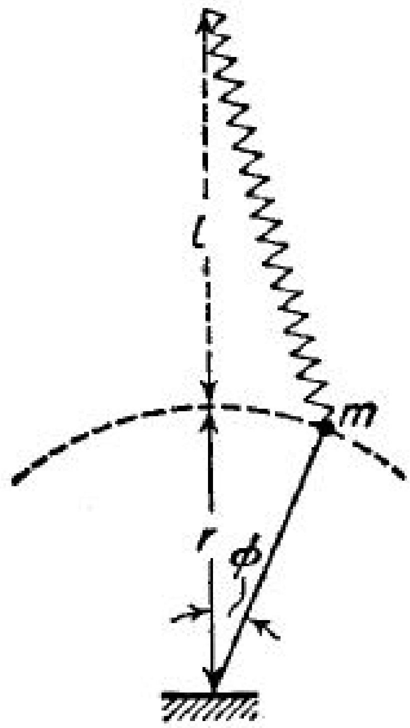

Abbildung 2: particle moving on a ring

$$\delta l = \sqrt{(l + (r - rcos\phi))^2 + (rsin\phi)^2} - l \tag{8}$$

after solving further and putting

$$cos\phi \approx 1 - \frac{\phi^2}{2} \tag{9}$$

$$\delta l = \sqrt{(l^2 + (l + r)^2 - 2r(l + r) - r(l + r)\phi^2} - l \tag{10}$$

Finally

$$\delta l = \frac{r(l + r)\phi^2}{2l} \tag{11}$$

Now, the Lagrangian of the particle can be written as

$$L = \frac{1}{2}mr^2\dot{\phi}^2 + F\frac{r(l+r)\phi^2}{2l} \tag{12}$$

from the Lagrangian's equation the equation of motion is

$$\ddot{\phi} + \phi\frac{F(r+l)}{mrl} = 0 \tag{13}$$

and we have

$$\omega = \sqrt{\frac{F(r+l)}{mrl}} \tag{14}$$

2. **Oscillation under the force of gravity**

   Determine the final amplitude for the oscillations of a system under a force which is (i) 0 for $t < 0$, (ii) $F_0 t/T$ for $0 < t \leq T$ and (iii) F_0 for $t > T$, where F_0 is a constant. Assume that at $t = 0$, the system was at rest and was situated at the position of equilibrium.

   Solution:

   The following equation we have to solve to get the amplitude for the oscillations

$$\ddot{x} + \omega^2 x = 0, \qquad t < 0 \tag{15}$$

$$\ddot{x} + \omega^2 x = \frac{F_0 t}{Tm}, \qquad 0 < t < T \tag{16}$$

$$\ddot{x} + \omega^2 x = \frac{F_0}{m}, \qquad t > T \tag{17}$$

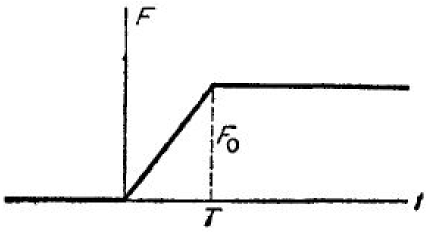

Abbildung 3: Force with time

For $t < 0$, the amplitude of the oscillation is zero because of the initial conditions.

During the interval $0 < t < T$ the oscillations are determined by the initial conditions

$$x, \dot{x} = 0 \tag{18}$$

and the solution can be written as

$$x = \frac{F_0}{mT\omega^3}(\omega t - sin\omega t) \tag{19}$$

the amplitude is

$$a = \frac{F_0}{mT\omega^3} \tag{20}$$

For $t > T$, we seek a solution in the form

$$x = c_1 cos\omega(t-T) + c_2 sin\omega(t-T) + \frac{F_0}{m\omega^2} \tag{21}$$

The continuity of x and first derivative of x at $t = T$ gives

$$c_1 = -\frac{F_0}{mT\omega^3} sin\omega T \tag{22}$$

$$c_2 = \frac{F_0}{mT\omega^3}(1 - cos\omega T) \tag{23}$$

the amplitude is

$$a = \sqrt{c_1^2 + c_2^2} = \frac{2F_0}{mT\omega^3} sin\frac{\omega T}{2} \tag{24}$$

3. **Forced oscillation with a modulated periodic force**

A force $f = (f_0 \exp\alpha t)\cos\gamma t$ (with α and γ being real) is acting on a damped oscillatory system with natural frequency ω_0 and damping coefficient λ. The damping is weak in comparison with the natural frequency. Solve for $x(t)$ both for resonant and non-resonant cases.

Solution: The equation of motion is given by

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f_0}{m}\exp(\alpha t)\cos(\gamma t), \tag{25}$$

The above equation can be written as the real part of the complex equation

$$\ddot{z} + 2\lambda\dot{z} + \omega_0^2 z = \frac{f_0}{m}\exp\left[(\alpha + i\gamma)t\right], \tag{26}$$

we seek for a solution of the form, $z = z_0\exp\left[(\alpha + i\gamma)t\right]$. Putting this in the above equation, we get

$$\left[(\alpha + i\gamma)^2 + 2\lambda(\alpha + i\gamma) + \omega_0^2\right] z_0 = \frac{f_0}{m}, \tag{27}$$

or,

$$z_0 = \frac{(f_0/m)}{(\alpha + i\gamma)^2 + 2\lambda(\alpha + i\gamma) + \omega_0^2} = \frac{(f_0/m)}{(\alpha^2 + \omega_0^2 - \gamma^2 + 2\alpha\lambda) + i2\gamma(\alpha + \lambda)}, \tag{28}$$

Defining, $z_0 = b\exp(i\delta)$, with $b = \frac{(f_0/m)}{\sqrt{(\alpha^2+\omega_0^2-\gamma^2+2\alpha\lambda)^2+4\gamma^2(\alpha+\lambda)^2}}$ and $\tan\delta = -\frac{2\gamma(\alpha+\lambda)}{(\alpha^2+\omega_0^2-\gamma^2+2\alpha\lambda)}$.
Then the particular integral is given by, $x = b\exp(\alpha t)\cos(\gamma t + \delta)$.
So, the general solution is given for $\lambda < \omega_0$ by

$$x = a\exp(-\lambda t)\cos(\omega_d t + \beta) + b\exp(\alpha t)\cos(\gamma t + \delta), \tag{29}$$

with $\omega_d = \sqrt{\omega_0^2 - \lambda^2}$
At resonance, $b = \frac{(f_0/m)}{\sqrt{(\alpha^2+\omega_0^2-\gamma^2+2\alpha\lambda)^2+4\gamma^2(\alpha+\lambda)^2}}$ does not blow up and the maximum of b is given by the minimum of $\sqrt{(\alpha^2 + \omega_0^2 - \gamma^2 + 2\alpha\lambda)^2 + 4\gamma^2(\alpha + \lambda)^2}$. Which is given by

$$\gamma = \sqrt{\omega_0^2 - \alpha^2 - 2\lambda^2 - 2\alpha\lambda}. \tag{30}$$

4. **Solution of parametric resonance**

In the case of an oscillation with periodic frequency (of period T), if the two independent solutions x_1 and x_2 of the equation of dynamics at time t are so chosen that

$$x_1(t+T) = \mu_1 x_1(t) \tag{31}$$

$$x_2(t+T) = \mu_2 x_2(t) \tag{32}$$

show that the general form of the solution for $x_{1,2}$ will be given by

$$x_{1,2}(t) = \mu_{1,2}^{\frac{t}{T}} \Pi_{1,2}(t) \tag{33}$$

where, $\Pi_{1,2}$ are two periodic functions with period T and $\mu_{1,2}$ are constants.

Solution: First do it for $x_1(t)$. Let us assume following form of $x_1(t)$ -

$$x_1(t) = \mathcal{F}(\mu)\pi_1(t) \tag{34}$$

Where $\pi_1(t)$ is periodic function with time-period T, so that

$$\pi_1(t+T) = \pi_1(t) \tag{35}$$

Our aim is to find the functional form of $\mathcal{F}(\mu)$. Consider the case where, $\mu = 1$, then from equations (31) and (35)-

$$x_1(t+T) = x_1(t) \tag{36}$$

So for μ=1, $\mathcal{F}(\mu)$ must be 1 i.e,

$$\mathcal{F}(\mu) = 1 \;\; for \, \mu = 1 \tag{37}$$

Now assume $\mathcal{F}(\mu)$ is a constant function i.e,

$$\mathcal{F}(\mu) = C \tag{38}$$

Where C is a constant. So from equations (31), (34) and (38), we have following-

$$x_1(t+T) = Cx_1(t) \neq \mu_1 x_1(t) \tag{39}$$

So $\mathcal{F}(\mu)$ can't just be a constant but has to depend on variable t i,e. $\mathcal{F} = \mathcal{F}(\mu, t)$. The only functional form of $\mathcal{F}(\mu, t)$ allowed from equation (37) is -

$$\mathcal{F}(\mu, t) = \mu^{\alpha t} \tag{40}$$

We need to find out α so that equation (31) is satisfied. So

$$x_1(t) = \mu^{\alpha t}\pi_1(t) \tag{41}$$

$$x_1(t+T) = \mu^{\alpha(t+T)}\pi_1(t+T) \tag{42}$$

From equations (31),(35)-

$$\mu x_1(t) = \mu^{\alpha(t+T)}\pi_1(t) \tag{43}$$

$$\mu\mu^{\alpha t}\pi(t) = \mu^{\alpha(t+T)}\pi_1(t) \tag{44}$$

$$\mu = \mu^{\alpha T} \implies \alpha = 1/T \tag{45}$$

So, from equation (41)-

$$x_1(t) = \mu^{t/T}\pi_1(t) \tag{46}$$

Similarly we can show for $x_2(t)$.