Tutorial sheet 3 solutions

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1. **Constant of motion of a freely falling particle**

   A particle of mass m is falling from rest under gravity with uniform acceleration **g**. After time t from starting, let the position (from the point of start) and the speed of the particle be z and v, respectively. Show that an infinitesimal translation in space will correspond to a symmetry transformation of the Lagrangian. Using Noether's theorem, find the corresponding constant of motion.

   Solution: The Lagrangian for a freely falling particle is given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz, \tag{1}$$

Let us take a infinitesimal space translation as

$$\begin{aligned} x \to x' &= x + \epsilon, \\ y \to y' &= y + \epsilon, \\ z \to z' &= z + \epsilon, \\ \Delta x = \epsilon, &\;\; \Delta \dot{x} = 0, \\ \Delta y = \epsilon, &\;\; \Delta \dot{y} = 0, \\ \Delta z = \epsilon, &\;\; \Delta \dot{z} = 0, \end{aligned} \tag{2}$$

The Lagrangian in the transformed coordinates is given by

$$L' = \frac{1}{2}m(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2) - mgz' = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mg(z+\epsilon) = L - mg\epsilon, \tag{3}$$

So,

$$\delta L = -mg\epsilon = \frac{d}{dt}(\epsilon f), \tag{4}$$

So, $\frac{df}{dt} = -mg$. Hence, L is gauge invariant.

From Noether's theorem the constant of motion is given by

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\Delta z - \epsilon f\right] &= 0, \\ m\dot{z}\epsilon + mgt\epsilon &= C, \\ \epsilon(m\dot{z} + mgt) &= C. \end{aligned} \tag{5}$$

since ϵ is an arbitrary constant, we have $m\dot{z} + mgt = constant$.

2. **Analytical solution of a one dimensional pendulum**

The equation of motion of a one dimensional pendulum is given by

$$\ddot{\theta} + \omega^2 \sin\theta = 0.$$

Find $\theta(t)$ analytically without using small angle approximation.

Solution: The equation of motion of a one dimensional pendulum is given by

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \tag{6}$$

where $\ddot{\theta} = \frac{d^2\theta}{dt^2}$. let, $\frac{d\theta}{dt} = u$, so $\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt} = u\frac{du}{d\theta}$.
with this change of variable Eq.(6) becomes

$$u\frac{du}{d\theta} = -\omega^2 \sin\theta, \tag{7}$$

integrating w.r.t θ we get

$$\frac{u^2}{2} = \omega^2 \cos\theta + C, \tag{8}$$

Now, say at $\theta = \theta_0$, $u = 0$ which gives us $C = -\omega^2 \cos\theta_0$.

$$u^2 = 2\omega^2(\cos\theta - \cos\theta_0) \tag{9}$$

or,

$$\frac{d\theta}{dt} = \pm\omega\sqrt{2}\sqrt{\cos\theta - \cos\theta_0}, \tag{10}$$

restricting our motion from $\theta = \theta_0$ to $\theta = 0$, which is basically one-fourth of a time-period and taking the negative sign in the above

$$\omega t = -\frac{1}{\sqrt{2}}\int_{\theta_0}^{0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{1}{\sqrt{2}}\int_{0}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}, \tag{11}$$

or using, $\cos\theta = 1 - 2\sin^2(\theta/2)$

$$2\omega t = \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}, \tag{12}$$

Now let, $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$
differentiating both sides we get

$$\frac{1}{2}\cos(\theta/2)d\theta = \sin(\theta_0/2)\cos\phi d\phi,$$

calling $k = \sin(\theta_0/2)$

$$d\theta = \frac{2\sin(\theta_0/2)\cos\phi d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

also, when $\theta = 0, \phi = 0$ and $\theta = \theta_0, \phi = \pi/2$.

So,

$$\omega t = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{13}$$

Eq.(13) is an elliptic integral and cannot be solved using elementary functions.

let, $x = -k^2\sin^2\phi$, as $|x| < 1$, we can expand the square root in Eq.(13) in a binomial series

$$\omega t = \int_0^{\pi/2} \left\{ 1 + \frac{1}{2}k^2\sin^2\phi + \frac{1\cdot 3}{2\cdot 4}k^4\sin^4\phi + \cdots \right\} d\phi,$$

$$\omega t = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots \right]. \tag{14}$$

where we have used $\int_0^{\pi/2}\sin^{2n}\phi d\phi = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{\pi}{2}$.

3. **Galilean transformation**

Show that Galilean transformation is a symmetry transformation for the Lagrangian of a non relativistic freely moving particle.

Solution: To show that the Galilean transformation is a symmetry transformation, we have to show that the new Lagrangian differs from the original one by a total time derivative of a function F(x,t).

The Lagrangian of a non relativistic freely moving particle is given by,

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 \tag{15}$$

from Galilean transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t, \quad \dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{V} \tag{16}$$

now the Lagrange can be written as

$$L' = \frac{1}{2}m\dot{\mathbf{r}}'^2 = \frac{1}{2}m(\dot{\mathbf{r}} - \mathbf{V})^2 = \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{d}{dt}\left(m\mathbf{r}\cdot\mathbf{V} - \frac{1}{2}mV^2t\right) \tag{17}$$

Hence, the Lagrangian of a free particle undergoes a symmetry transformation under Galilean transformation.

4. **Lagrangian with second order time derivative**

If the Lagrangian L of a mechanical system is given by $L = L(t, q, \dot{q}, \ddot{q})$ and the Hamilton's principle is valid with zero variation of both q and $\dot{q}$ at the initial and final time instants, derive the corresponding Euler-Lagrange equation. Using that, find also the equation of motion of the Lagrangian

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2.$$

Can you identify the system?

Solution: Given Lagrangian is $L = L(t, q, \dot{q}, \ddot{q})$. According to Hamilton's principle of least action between two given time instants t_1 and t_2, system chooses the path $q(t)$ for which action $S = \int_{t_1}^{t_2} Ldt$ is stationary i,e $\delta S = 0$. Now let us take another path $q'(t)$ virtually varied from actual path $q(t)$ such that -

$$q'(t) = q(t) + \delta q(t) \tag{18}$$

$$\dot{q}'(t) = \dot{q}(t) + \delta\dot{q}(t) \tag{19}$$

with zero variation at end points

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{20}$$

$$\delta\dot{q}(t_1) = \delta\dot{q}(t_2) = 0 \tag{21}$$

Now

$$\delta S = \int_{t_1}^{t_2} \delta L dt \tag{22}$$

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}\right] dt \tag{23}$$

Again $\delta\dot{q} = \frac{d}{dt}(\delta q)$ and $\delta\ddot{q} = \frac{d}{dt}(\delta\dot{q})$ (d/dt and δ commute). Integrating by parts last two terms of Eq. (26), we get -

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q dt + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q dt + \left[\frac{\partial L}{\partial \ddot{q}}\delta\dot{q}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta\dot{q} dt \tag{24}$$

From Eqs. (23,24), second and fourth term of Eq. (27) will be zero. Integrating by parts last term of Eq. (27)-

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q dt - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta q dt \tag{25}$$

Again third term of Eq. (28) will be zero.

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right]\delta q dt \tag{26}$$

Now setting $\delta S = 0$. From Eq. (25) and Eq. (29), we get -

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right]\delta q dt = 0 \tag{27}$$

As Eq. (30) is valid for arbitrary δq and dt, so the term inside bracket must vanish.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0 \tag{28}$$

Above is Euler-Lagrange equation for the given Lagrangian. Now for Lagrangian,

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2 \tag{29}$$

So,

$$\frac{\partial L}{\partial q} = -\frac{m}{2}\ddot{q} - kq, \frac{\partial L}{\partial \dot{q}} = 0, \frac{\partial L}{\partial \ddot{q}} = -\frac{m}{2}q \tag{30}$$

Substituting above in Eq. (31), we get -

$$-\frac{m}{2}\ddot{q} - kq - \frac{m}{2}\ddot{q} = 0 \tag{31}$$

$$m\ddot{q} + kq = 0 \tag{32}$$

Above is a equation of motion for a simple harmonic oscillator.