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Classical Mechanics I (PHY 401A) 2020-21 Odd semester

Tutorial sheet 7

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1. **Dominant balance of terms**

   Consider the quadratic equation

$$x^2 - 2.0004x + 0.9998 = 0. \tag{1}$$

   (a) Apply the regular perturbation theory to solve the equation.

   (b) Can you comment on the problem that one encounters if regular perturbation method is used?

   (c) Can you suggest a way out?

Solution

(a) With $\epsilon = .0002$, we can rewrite the equation as -

$$x^2 - 2(1+\epsilon)x + (1-\epsilon) = 0 \tag{2}$$

In $\epsilon \to 0$, above eqn is -

$$(x-1)^2 = 0 \tag{3}$$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...\infty$, then equation (2) will be-

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ..)^2 - 2(1+\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ..) + (1-\epsilon) = 0 \tag{4}$$

Taking zeroth order term,

$$x_0^2 - 2x_0 + 1 = 0 \implies x_0 = 1, 1 \tag{5}$$

and first order term is -

$$2x_0x_1 - 2x_0 - 2x_1 - 1 = 0 \tag{6}$$

Substituting x_0 -

$$2x_1 - 2 - 2x_1 - 1 = 0 \implies -3 = 0 \tag{7}$$

Equation (7) is absurd. (b) Inconsistency is there, so the assumed series is not ok.

(c)**Mutual balance of different terms-**

One can forget $f(x)$ in front of $g(x)$ as $x \to a$

$$\lim_{x\to a} \frac{|f(x)|}{|g(x)|} = 0 \tag{8}$$

and $f(x) \sim g(x)$ as $x \to a$ if,

$$\lim_{x\to a} \frac{|f(x)|}{|g(x)|} = 1 \tag{9}$$

Since R.H.S of equation (2) is zero, all the terms of identical order should either balance each other or the terms of zeroth order should be balancing each other and all terms with ϵ and higher order should be negligible. Now our equation is -

$$x^2 - 2(1+\epsilon)x + (1-\epsilon) = 0 \tag{10}$$

$$x^2 - 2x + 1 - \epsilon(2x+1) = 0 \tag{11}$$

Let us choose $y = x - 1$, then equation (7) become -

$$y^2 - \epsilon(2(x-1)+3) = 0 \tag{12}$$

$$y^2 - 2y\epsilon - 3\epsilon = 0 \tag{13}$$

We now see, how the are balancing each other.

$$2y\epsilon \sim 3\epsilon \Longrightarrow 2y \sim 3 \Longrightarrow y \sim 1 \tag{14}$$

Again,

$$y^2 = 1 \Longrightarrow 2y\epsilon \sim 3\epsilon \, and \, 3\epsilon \sim \epsilon \tag{15}$$

But $1 \neq 0$ (for $\epsilon \to 0$). Now we take

$$y^2 \sim \epsilon^2, 2y\epsilon \sim \epsilon^2, 3\epsilon \sim \epsilon \tag{16}$$

$\epsilon \to 0$, but the balancing terms are not dominating and finally take-

$$y^2 \sim 3\epsilon \Longrightarrow y \sim \epsilon^{1/2} \tag{17}$$

So terms $y^2\epsilon, 2y\epsilon \sim \epsilon^{3/2}, 3\epsilon \sim \epsilon$ are balancing each other and dominating as $\epsilon << 1$. Now $x - 1 = \sqrt{\epsilon}$ and

$$x = x_0 + \sqrt{\epsilon}x_1 + (\sqrt{\epsilon})^2 x_2 + ... \tag{18}$$

$$x = x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2 \tag{19}$$

So the original equation is-

$$x^2 - 2(1+\epsilon)x + (1-\epsilon) = 0 \tag{20}$$

$$(x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2)^2 - 2(1+\epsilon)(x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2) + (1-\epsilon) = 0 \tag{21}$$

Now zeroth order term gives-

$$x_0^2 - 2x_0 + 1 = 0 \Longrightarrow x_0 = 1 \tag{22}$$

First order $(O(\sqrt{\epsilon}))$term gives-

$$2x_0x_1 - 2x_1 = 0 \Longrightarrow x_0 = 1 \tag{23}$$

second order $(O(\sqrt{\epsilon})^2)$ term gives-

$$x_1^2 - 2x_0 - 1 + 2x_0x_2 - 2x_2 = 0 \Longrightarrow x_1^2 = 3 \Longrightarrow x_1 = \pm\sqrt{3} \tag{24}$$

third order $(O(\sqrt{\epsilon})^3)$ term gives-

$$2x_1x_2 - 2x_1 = 0 \Longrightarrow x_2 = 1 \tag{25}$$

So,

$$x = x_0 + \sqrt{\epsilon}x_1 + \epsilon x_2 \tag{26}$$

$$x = x_0 \pm \sqrt{3\epsilon} + \epsilon. \tag{27}$$

2. **Poincare-Lindstedt method**

Consider the equation

$$\ddot{x} + \epsilon\left(x^2 - 1\right)\dot{x} + x = 0. \tag{28}$$

(a) Apply regular perturbation theory to solve the equation.

(b) Do you encounter any unwanted term? Justify your answer.

(c) Apply Poincare-Lindstedt method to solve the equation. Have you obtained a reasonable solution? Justify.

Solution:

(a) We take

$$x = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots, \quad (\epsilon > 0, \epsilon \ll 1) \tag{29}$$

putting this into the original equation

$$(\ddot{x_0} + \ddot{x_1}\epsilon + \ddot{x_2}\epsilon^2) + \epsilon\left[(x_0 + x_1\epsilon + x_2\epsilon^2)^2 - 1\right](\dot{x_0} + \dot{x_1}\epsilon + \dot{x_2}\epsilon^2) + (x_0 + x_1\epsilon + x_2\epsilon^2) = 0, \tag{30}$$

0-th order: $\ddot{x_0} + x_0 = 0 \Rightarrow x_0 = A\cos t + B\sin t$

let us take $x_0 = \cos t$ as the initial condition.

1-st order: $\ddot{x_1} + x_1 = (1 - x_0^2)\dot{x_0} = -\sin^2 t(\sin t) = -\sin^3 t = \frac{1}{4}\sin 3t - \frac{3}{4}\sin t$.

(b) So, there is a resonant term due to $\frac{3}{4}\sin t$ part of the force, secular terms will be there.

Now, from

$$\begin{aligned}\ddot{x} + \epsilon\left(x^2 - 1\right)\dot{x} + x &= 0,\\ \dot{x}\ddot{x} + \varepsilon\left(x^2 - 1\right)\dot{x}^2 + x\dot{x} &= 0,\\ \frac{d}{dt}(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2) &= \epsilon(1 - x^2)\dot{x}^2,\end{aligned} \tag{31}$$

atleast for $x^2 > 1$, the solution should be bounded. So, we use Poincare-Lindstedt method to get rid of the secular equation.

(c) Let us define $\tau = \omega t$. So, $\frac{dx}{dt} = \omega\frac{dx}{d\tau}$ and $\frac{d^2x}{dt^2} = \omega^2\frac{d^2x}{d\tau^2}$.

So, Eq(28) takes up the form

$$\omega^2\frac{d^2x}{d\tau^2} + \epsilon(x^2 - 1)\omega\frac{dx}{d\tau} + x = 0, \tag{32}$$

Now,

$$x = x_0(\tau) + x_1(\tau)\epsilon + x_2(\tau)\epsilon^2 + \cdots\infty, \tag{33}$$

and

$$\omega = \omega_0 + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots\infty, \tag{34}$$

taking $\omega_0 = 1$ we can write

$$
\begin{aligned}
&(1+\omega_1\epsilon+\omega_2\epsilon^2+\cdots\infty)^2\frac{d^2}{d\tau^2}\left[x_0(\tau)+x_1(\tau)\epsilon+x_2(\tau)\epsilon^2+\cdots\infty\right] \\
&+\epsilon(1+\omega_1\epsilon+\omega_2\epsilon^2+\cdots\infty)\left[(x_0(\tau)+x_1(\tau)\epsilon+x_2(\tau)\epsilon^2+\cdots\infty)^2-1\right]\frac{d}{d\tau}\left[x_0(\tau)+x_1(\tau)\epsilon+\cdots\right] \\
&+\left[x_0(\tau)+x_1(\tau)\epsilon+x_2(\tau)\epsilon^2+\cdots\infty\right]=0, \qquad (35)
\end{aligned}
$$

0-th order: $\frac{d^2x_0}{d\tau^2} + x_0 = 0,$

1-st order: $\frac{d^2x_1}{d\tau^2} + 2\omega_1\frac{d^2x_0}{d\tau^2} + (x_0^2-1)\frac{dx_0}{d\tau} + x_1 = 0,$

For the 0-th order we have, $x_0(\tau) = A\cos\tau + B\sin\tau$. With initial conditions $x_0(0) = a, \dot{x_0}(0) = 0$, giving us $x_0 = \cos\tau$.

So,

$$
\begin{aligned}
\frac{d^2x_1}{d\tau^2} + x_1 &= -2\omega_1(-a\cos\tau) + (1-a^2\cos^2\tau)(-a\sin\tau) \\
&= 2\omega_1 a\cos\tau - \left[(a\sin\tau)(1-a^2+a^2\sin^2\tau)\right] \\
&= 2\omega_1 a\cos\tau - \left[a\sin\tau - a^3\sin\tau + a^3\left(\frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau\right)\right] \\
&= 2\omega_1 a\cos\tau - \left[\left(a-\frac{a^3}{4}\right)\sin\tau - \frac{a^3}{4}\sin 3\tau\right], \qquad (36)
\end{aligned}
$$

To get rid of the secular equation: $\omega_1 = 0$ and $a\left(1-\frac{a^2}{4}\right) = 0 \Rightarrow a = \pm 2$.

So then, $\frac{d^2x_1}{d\tau^2} + x_1 = 2\sin 3\tau$. Let us take $x_1 = X_1\sin 3\tau$.

which gives, $-9X_1 + X_1 = 2 \Rightarrow X_1 = -\frac{1}{4}$.

The general solution is given by: $x_1 = C\cos\tau + D\sin\tau - \frac{1}{4}\sin 3\tau$.

We also have, $x_1(0) = 0 = \dot{x_1}(0)$. Giving $C = 0$ and $D = \frac{3}{4}$.

So, $x_1(\tau) = \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau = \sin^3\tau$.

Hence the total solution: $x(\tau) = \pm 2\cos\tau + \epsilon\sin^3\tau$.