Lagrangian Mechanics

- **Cyclic coordinates:** The Euler-Lagrange (E-L) equation is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

If the Lagrangian is independent of a coordinate q_k, then

$$\frac{\partial L}{\partial q_k} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

So, $\frac{\partial L}{\partial \dot{q}_k}$ is a constant of the motion. In this case, q_k is called a cyclic coordinate and $p_k = \frac{\partial L}{\partial \dot{q}_k}$, which is the canonical momentum conjugate to q_k, is a constant of the motion.

Example: For a particle moving under a central force

$$\vec{F}(\vec{r}) = -\frac{k}{r^2}\,\hat{r},$$

the particle moves in a plane, and we have

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

But $\vec{F}(\vec{r}) = -\frac{k}{r^2}\hat{r} = -\nabla V(r) = -\frac{dV}{dr}\hat{r} \Rightarrow dV = \frac{k}{r^2} \Rightarrow V(r) = -\frac{k}{r} + C$, and using the boundary condition $\lim_{r\to\infty} V(r) \to 0$ makes $C = 0$. Thus, we have

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}$$

Note that $\partial L/\partial\theta = 0$, so θ is cyclic and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is a constant of the motion.

- **Additivity and Scaling of the Lagrangian:**

Let's go to the E-L equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{— (1)}$$

If we imagine a system consisting of two non-interacting sub-parts A and B, described by two separate sets of generalized coordinates $\{q_A\}$ and $\{q_B\}$, and the corresponding Lagrangians $L_A(q_A, \dot{q}_A, t)$ and $L_B(q_B, \dot{q}_B, t)$. The subparts A and B have the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L_A}{\partial \dot{q}_{Ai}}\right) - \frac{\partial L_A}{\partial q_{Ai}} = 0 \,, \quad \forall \quad i = 1, 2, \ldots, n_A \,; \quad \text{and}$$

$$\frac{d}{dt}\left(\frac{\partial L_B}{\partial \dot{q}_{Bi}}\right) - \frac{\partial L_B}{\partial q_{Bi}} = 0 \,, \quad \forall \quad i = 1, 2, \ldots, n_B \,;$$

where n_A and n_B are the respective degrees of freedom. Also, since q_{Ai} and q_{Bi} are independent and distinct sets of variables, we have

$$\frac{\partial L_A}{\partial \dot{q}_{Bi}} = 0 = \frac{\partial L_A}{\partial q_{Bi}} \quad \text{and} \quad \frac{\partial L_B}{\partial \dot{q}_{Ai}} = 0 = \frac{\partial L_B}{\partial q_{Ai}}$$

Thus, $\boxed{L = L_A + L_B}$ is the composite Lagrangian of the system, and the corresponding Euler - Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \;; \qquad i = 1, 2, \ldots, n_A + n_B.$$

In fact, $q_i \in \{q_{A1}, q_{A2}, \ldots, q_{An_A} ; q_{B1}, q_{B2}, \ldots, q_{Bn_B}\}$.

From (1), we can easily note that the E-L equation is unchanged if L is multiplied by a constant factor.

If a system consists of more than one subparts, then for each subpart, the E- L equations remain invariant under the multiplication of an arbitrary constant (scale factor).

BUT if we recall the additive rule, for having the E-L equations for the composite system satisfied, all the subpart Lagrangians should be multiplied by the same constant. This merely amounts to choosing different units.

- <u>Concept of Energy from Lagrangian:</u>

$L = L(q, \dot{q}, t)$

$$\Rightarrow \quad \frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= \frac{\partial L}{\partial t} + \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\dot{q}_i)$$

$$= \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_i\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left[\sum_i\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i - L\right] = -\frac{\partial L}{\partial t}$$

So, for systems whose Lagrangian L is independent of t (no explicit dependence), we have

$$-\frac{\partial L}{\partial t} = 0 = \frac{d}{dt}\left[\sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i - L\right]$$

$$\Rightarrow \quad \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i - L = \mathcal{E} \quad \text{(say)}$$

where $\mathcal{E}$ is a constant of the motion.

Can we see that $\mathcal{E}$ is nothing but energy for the usual mechanical systems which are conservative?

Previously, we identified, for conservative systems,

$$L = T(q, \dot{q}) - V(q)$$

In cases where T is a quadratic function of velocities (usual case), using Euler's theorem* for homogeneous functions, we get

$$2T = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \qquad [\because V \text{ is assumed to be independent of } \dot{q}_i]$$

$$\Rightarrow \quad \mathcal{E} = 2T - (T - V) = T + V \longrightarrow \text{Total Mechanical Energy}$$

So, if the Lagrangian does not explicitly depend on t, the total mechanical energy is conserved if T is quadratic in velocities and V is a function of the q_i's only.

- **Uniqueness of Lagrangian:**

The question is: "What will be a symmetry transformation of L" so that the E-L equations stay unchanged?

Of course, one transformation is the multiplication of L by an arbitrary constant, so that the E-L equations remain unchanged. But this transformation will not let the action (S) to be at its minimum value. **Is it so?**

$$S' = \alpha \int_{t_1}^{t_2} L\, dt = \alpha S; \quad \delta S = 0 \Rightarrow \delta S' = 0.$$

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* Euler's theorem on homogeneous functions:

$$f(x_1, x_2, \dots, x_n) = \sum_i a_i x_1^p x_2^q \cdots x_n^r \quad ; \quad \text{where} \quad p + q + \cdots + r = k$$

$$\Rightarrow \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} x_j = k f(x_1, x_2, \dots, x_n)$$

However, we can define a transformation

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

Attention! f is a function of q and t.

Under this transformation,

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\, dt = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\, dt + \int_{t_1}^{t_2} \frac{df}{dt} dt$$

$$\Rightarrow \quad S' = S + \int_{t_1}^{t_2} df(q, t) = S' + f(q(t_2), t_2) - f(q(t_1), t_1)$$

For the least action path, we should have $\delta S' = 0$. But here we have

$$\delta S' = \delta S + \delta f(q(t_2), t_2) - \delta f(q(t_1), t_1) = \delta S$$

because both $\delta f(q(t_2), t_2) = 0$ and $\delta f(q(t_1), t_1) = 0$ as at t_1 and t_2, no variation is allowed. Since we already had $\delta S = 0$, we now also have

$$\delta S' = 0$$

What happens if f is a function of $\dot{q}$ too?

So, if the old Lagrangian is following the path of least action, so will do the new Lagrangian L'. It simply means that under this transformation, the form of the equations of motion is unchanged. Hence the Lagrangian is defined only to within an additive total derivative of any function of coordinates and time.