Applications of Lagrangian Formulation

- **Simple Pendulum:** Although it is sometimes helpful to draw a figure, as said to show the analytical strength of the Lagrangian method, we will not draw any figure. We first have to construct the Lagrangian for a simple pendulum. The string of the pendulum is massless and of a fixed length l. At the end of the string a bob of mass m (modeled as a point mass) is attached. The only generalized coordinate is θ, which is the angle made by the (deflected) string with the vertical (or the mean position of rest). Then, the kinetic energy is given by

$$T = \frac{1}{2} m (l\dot{\theta})^2$$

and the (gravitational) potential energy is

$$V = mgl(1 - \cos\theta),$$

assuming that the mean position of rest has the reference potential energy. Thus, the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta) \qquad \text{— } (i) \end{aligned}$$

The relevant Euler-Lagrange equation for this problem is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\Rightarrow \quad mgl \sin\theta + ml^2\ddot{\theta} = 0$$

$$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

For very small θ, $\sin\theta \approx \theta$, and we have

$$\boxed{\ddot{\theta} + \omega^2\theta = 0},$$

This is the equation of motion for a simple pendulum with very small amplitude, and $\omega = \sqrt{\frac{g}{l}}$ is called the angular frequency.

- **Spherical Pendulum:** This is a pendulum whose bob moves on the surface of a sphere of fixed radius R. Thus, it has two degrees of freedom and it can be described by two generalized coordinates θ and ϕ. The kinetic energy is

$$T = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right)$$

Assuming the reference for the (gravitational) potential energy to be the horizontal plane through the center of the sphere, the potential energy is

$$V = mgR\cos\theta$$

Hence, $$L = T - V = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right) - mgR\cos\theta$$

The relevant Euler-Lagrange equations are given by

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} - \dfrac{\partial L}{\partial \theta} = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\phi}} - \dfrac{\partial L}{\partial \phi} = 0 \end{cases} \quad \Bigg| \quad \begin{array}{l} \dfrac{\partial L}{\partial \theta} = mR^2 \sin\theta\cos\theta\,\dot{\phi}^2 + mgR\sin\theta, \\ \dfrac{\partial L}{\partial \phi} = 0, \quad \dfrac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \quad \dfrac{\partial L}{\partial \dot{\phi}} = mR^2\sin^2\theta\,\dot{\phi} \end{array}$$

$$\Rightarrow \begin{cases} mR^2\ddot{\theta} - mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta = 0 \\ mR^2\sin^2\theta\,\ddot{\phi} + 2mR^2\dot{\phi}\sin\theta\cos\theta\,\dot{\theta} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 - \dfrac{g}{R}\sin\theta = 0 \\ \ddot{\phi} + 2\cot\theta\,\dot{\theta} = 0 \end{cases}$$

But $\dfrac{\partial L}{\partial \phi} = 0 = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\phi}} \Rightarrow \dfrac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}\sin^2\theta$ $= \text{constant}$

$\Rightarrow \dot{\phi} = \dfrac{h}{\sin^2\theta}$; h is a constant.

$$\Rightarrow \begin{cases} \ddot{\theta} - \cot\theta\,\mathrm{cosec}^2\theta\, h^2 - \dfrac{g}{R}\sin\theta = 0 \\ \ddot{\phi} + 2\cot\theta\,\dot{\theta} = 0 \end{cases}$$

In the first equation, we have a term $-\frac{g}{R}\sin\theta$. Notice the $-$ve sign:

$$\boxed{\ddot{\theta} - \cot\theta\,\mathrm{cosec}^2\theta\, h^2 - \frac{g}{R}\sin\theta = 0}$$

If we fix ϕ, then $\dot{\phi} = 0$ and hence $h = 0$.

⚠ How to recover the equation of a simple pendulum? What would we get for $\theta \to 0^\circ$?