Hamilton's Principle

- Also called the **Principle of Least Action**, although 'Stationary Action' is the correct term.

- Statement: Every mechanical system is characterized by a function $L = L(q_1, q_2, \ldots, q_s;\ \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s;\ t) \equiv L(q, \dot{q}, t)$ [for brevity] and the motion / mechanical evolution of that system will be governed by the following rule:

  Between two given time instants (initial and final) t_1 and t_2 and with two given sets of generalized coordinates $q^{(1)}$ and $q^{(2)}$, the system always chooses the path of least action to evolve, where the action S is defined as

$$S = \int_{t_1}^{t_2} L\, dt .$$

  The function L is called the Lagrangian of the system.

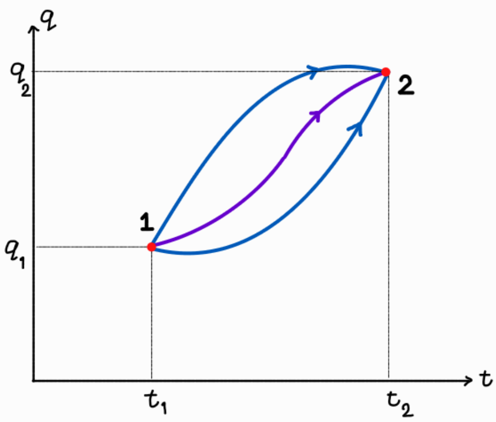
⚠ **Why L contains only q and $\dot{q}$ (besides t), and not $\ddot{q}$, $\dddot{q}$, etc.?**

- Derivation of Euler-Lagrange Equation:

  For simplicity, we assume that our system has only one degree of freedom. Thus,

$$L = L(q, \dot{q}, t)$$

  The objective is to determine $q(t)$ using the extremum condition. Then only we can predict the position of the particle at any t.



Let's consider another path of evolution, slightly different from the real path. For the new path, the generalized coordinates will be given by

$$q'(t) = q(t) + \delta q(t)$$

where $\delta q(t)$ is known as variation; time is frozen here. It gives infinitesimal shift from one trajectory to the other. At the initial and the final instants, the coordinates are given — thus

$$\delta q(t_1) = 0 = \delta q(t_2) \qquad \text{— (A)}$$

The principle of least action says that if the path of evolution is slightly varied with respect to the real one, then $\delta S = 0$. This translates to

$$\int_{t_1}^{t_2} L(q+\delta q,\ \dot{q}+\delta\dot{q},\ t)\,dt - \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt = 0$$

The operators δ and $\frac{d}{dt}$ commute because they're independent of each other. Thus,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt = 0 \qquad \text{— (1)}$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) \right) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right) \delta q\, dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q \right) dt = 0 \qquad \text{— (2)}$$

But
$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q \right) dt = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}(t)}\delta q(t) \Bigg|_{t_2}^{t_1}$$

$$= \frac{\partial L(q,\dot{q},t_2)}{\partial \dot{q}(t_2)}\delta q(t_2) - \frac{\partial L(q,\dot{q},t_1)}{\partial \dot{q}(t_1)}\delta q(t_1)$$

$$= 0 \quad [\text{by using } (A)]$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right) \delta q\, dt = 0 \qquad \text{— (3)}$$

The above integral is identically zero for all variations δq of q. Hence

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$

The above equation inside the box is known as the Euler-Lagrange equation. For systems with more than one degree of freedom, we have identical equations for each q_i, $i = 1, 2, \ldots, s$:

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0}$$

Have you understood how?

Note: We said our goal was to find $q(t)$ for which $S = \int_{t_1}^{t_2} L\,dt$ is minimum. But we derived an equation for $L(q,\dot{q},t)$. Can we reconcile?

• Form of the Lagrangian for a Mechanical System:

In the previous derivation, we did not use/ did not need to use the explicit form of the Lagrangian. But to derive the equations of motion, one cannot but use the explicit dependence of L on q, $\dot{q}$, and t. A simple approach for the usual classical mechanical systems is given here.

From Newton's laws, we know for one particle,

$$\vec{F} = m\ddot{\vec{r}}$$

$$\Rightarrow \quad \vec{F}\cdot\frac{\partial \vec{r}}{\partial q_i} = m\ddot{\vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_i} = m\frac{d}{dt}(\dot{\vec{r}})\cdot\frac{\partial \vec{r}}{\partial q_i}$$

$$= m\frac{d}{dt}\left(\dot{\vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_i}\right) - m\dot{\vec{r}}\cdot\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_i}\right)$$

$$= \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_i}\right) - m\dot{\vec{r}}\,\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_i}\right) \quad \text{— (i)}$$

Now $\quad \dfrac{\partial \vec{r}}{\partial q_i} = \dfrac{d\vec{r}}{dq_i} - \displaystyle\sum_{j\neq i}\frac{\partial \vec{r}}{\partial q_j}\frac{dq_j}{dq_i}$

Also, $\quad \dot{\vec{r}} = \dfrac{\partial \vec{r}}{\partial q_i}\dot{q}_i + \displaystyle\sum_{j\neq i}\frac{\partial \vec{r}}{\partial q_j}\dot{q}_j \Rightarrow \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial \vec{r}}{\partial q_i} + \sum_{j\neq i}\frac{\partial \vec{r}}{\partial q_j}\frac{\partial \dot{q}_j}{\partial \dot{q}_i} = \frac{\partial \vec{r}}{\partial q_i} + \sum_{j\neq i}\frac{\partial \vec{r}}{\partial q_j}\delta_{ij} = \frac{\partial \vec{r}}{\partial q_i}$

$\left[\because \dfrac{\partial \vec{r}}{\partial q_j} \text{ are functions of } q\text{'s, not } \dot{q}\text{'s}\right]$

$$\Rightarrow \quad \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial \vec{r}}{\partial q_i} \quad \text{— (ii)}$$

Putting back in (i), we get

$$m\ddot{\vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_i} = \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i}\right) - m\dot{\vec{r}}\cdot\frac{\partial \dot{\vec{r}}}{\partial q_i} \quad \text{— (iii)}$$

But $\quad \dfrac{d}{dt}\left(\dfrac{\partial \vec{r}}{\partial q_i}\right) = \cancelto{0}{\dfrac{\partial}{\partial t}\left(\dfrac{\partial \vec{r}}{\partial q_i}\right)} + \displaystyle\sum_j \frac{\partial}{\partial q_j}\left(\frac{\partial \vec{r}}{\partial q_i}\right)\dot{q}_j + \sum_j \cancelto{0}{\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial \vec{r}}{\partial q_i}\right)}\ddot{q}_j$

$$= \sum_j \frac{\partial}{\partial q_j}\left(\frac{\partial \vec{r}}{\partial q_i}\right)\dot{q}_j = \frac{\partial}{\partial q_i}\left[\sum_j \frac{\partial \vec{r}}{\partial q_j}\dot{q}_j\right] ; \quad \left[\because \vec{r} \text{ is not a function of } t \text{ \& } \dot{q}_j\text{'s}\right]$$

So finally using (i) and (iii), we write,

$$\vec{F}\cdot\frac{\partial \vec{r}}{\partial q_i} = \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i}\right) - m\dot{\vec{r}}\cdot\frac{\partial \dot{\vec{r}}}{\partial q_i}$$

$$\Rightarrow \quad \boxed{\vec{F}\cdot\frac{\partial \vec{r}}{\partial q_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}} \qquad \text{--- (iv)}$$

This is valid for any mechanical system. For the case of conservative systems, however, $\vec{F} = -\nabla V$, where V is the potential energy. So,

$$\vec{F}\cdot\frac{\partial \vec{r}}{\partial q_i} = -\nabla V\cdot\frac{\partial \vec{r}}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

If we further assume that $\frac{\partial V}{\partial \dot{q}_i} = 0 \;\forall\; i$, then we can write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \chi}{\partial \dot{q}_i}\right) - \frac{\partial \chi}{\partial q_i} = 0, \quad \text{where } \chi = T - V$$

So, basically χ is something behaving like the Lagrangian L. Hence, one can identify that for a simple particle under a conservative force,

$$\boxed{L = T - V}$$

- Remember, the minus sign has appeared as a result of the convention, i.e., $$\vec{F} = -\nabla V.$$

⚠ What will be the Lagrangians of a free particle and a particle falling under gravity?