CLASSICAL MECHANICS I (PHY 401A) 2020-21 ODD SEMESTER

Tutorial sheet 5

Date: 07. 10. 2020

1. **Mechanical state of a spherical pendulum**

(i) Find the integrals of motion of a spherical pendulum (of mass m) moving on the surface of a sphere of radius ℓ. Can you define an effective potential energy?
(ii) Using the integrals of motion, solve for the mechanical state of such a pendulum.

Solution: (i)The Lagrangian for the spherical pendulum is given by

$$\mathcal{L} = \frac{1}{2}m\ell^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] - mg\ell\cos\theta, \tag{1}$$

since, ϕ is cyclic, $\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\ell^2\sin^2\theta\dot{\phi} = const. = h.$

The energy is

$$\begin{aligned} E &= \frac{1}{2}m\ell^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] + mg\ell\cos\theta \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}\frac{h}{m\ell^2\sin^2\theta} + mg\ell\cos\theta \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2 + U_{eff}. \end{aligned}$$

where, $U_{eff} = \frac{1}{2}\frac{h}{m\ell^2\sin^2\theta} + mg\ell\cos\theta$ is an effective potential energy.

(ii)We have

$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 + U_{eff}, \tag{2}$$

from above we have

$$\dot{\theta}^2 = \frac{2}{m\ell^2}\left[E - U_{eff}\right], \tag{3}$$

or,

$$\dot{\theta} = \sqrt{\frac{2}{m\ell^2}[E - U_{eff}]}, \tag{4}$$

or,

$$\int dt = \int \frac{d\theta}{\sqrt{\frac{2}{m\ell^2}[E - U_{eff}]}} + const. \tag{5}$$

and for ϕ,

$$\phi = \frac{h}{\ell/\sqrt{2m}} \int \frac{d\theta}{\sin^2\theta\sqrt{E - U_{eff}(\theta)}}. \tag{6}$$

leads to elliptic integral of third kind.

2. **Conservation of angular momentum**

   A Lagrangian remains unaltered when the coordinate axes undergo an infinitesimal rotation about an arbitrary direction $\hat{n}$ through an angle ε. Using Noether's theorem, show that the angular momentum is conserved for the system.

   Solution: The general formula for any rotation by an angle ϕ about an axis with direction $\hat{\mathbf{n}}$ is given by

$$\mathbf{R}(\hat{\mathbf{n}}, \phi) = (\cos\phi)\mathbf{r} + (1 - \cos\phi)(\hat{\mathbf{n}} \cdot \mathbf{r})\hat{\mathbf{n}} + \sin\phi(\hat{\mathbf{n}} \times \mathbf{r}), \tag{7}$$

   for $\phi \to 0$, $\cos\phi \approx 1$ and $\sin\phi \approx \phi$.

   So, then $\mathbf{R}(\hat{\mathbf{n}}, \phi) = \mathbf{r} + \phi(\hat{\mathbf{n}} \times \mathbf{r})$.

   So, $\delta\mathbf{r} = \phi(\hat{\mathbf{n}} \times \mathbf{r})$. Let $\phi = \epsilon$, then $\delta\mathbf{r} = \epsilon(\hat{\mathbf{n}} \times \mathbf{r})$.

   with, $\Delta\mathcal{L} = 0$, and $\mathbf{p} = m\dot{\mathbf{r}}$(for most usual case).

   So, $\mathbf{p} \cdot \epsilon(\hat{\mathbf{n}} \times \mathbf{r}) = \epsilon(\hat{\mathbf{n}} \times \mathbf{r}) \cdot \mathbf{p} = \epsilon\hat{\mathbf{n}} \cdot (\mathbf{r} \times \mathbf{p})$.

   As, ϵ is arbitrary and $\hat{\mathbf{n}}$ is an arbitrary direction, we have $(\mathbf{r} \times \mathbf{p})$ as constant of motion.

3. **Simplification of integration**

   Using the expressions of the semi-mejor axis a and the eccentricity e of an ellipse, as discussed in lecture 15, show that,

$$\sqrt{\frac{m}{2|E|}} \int \frac{rdr}{\sqrt{-r^2 + \frac{\alpha r}{|E|} - \frac{h^2}{2m|E|}}} = \sqrt{\frac{ma}{\alpha}} \int \frac{rdr}{\sqrt{\left[a^2e^2 - (r-a)^2\right]}}. \tag{8}$$

   Again substituting $r = a(1 - e\cos\xi)$, show that

$$\sqrt{\frac{ma}{\alpha}} \int \frac{rdr}{\sqrt{\left[a^2e^2 - (r-a)^2\right]}} = \sqrt{\frac{ma^3}{\alpha}} \int (1 - e\cos\xi)d\xi \tag{9}$$

   Solution: we know, by definition,

$$a = \frac{\alpha}{2|E|} \quad and \quad e = \sqrt{1 + \frac{2Eh^2}{m\alpha^2}} \tag{10}$$

$$\implies e^2 - 1 = \frac{2Eh^2}{m\alpha^2} = -\frac{2|E|h^2}{m\alpha^2} \tag{11}$$

   now, given integral is

$$\sqrt{\frac{m}{2|E|}} \int \frac{rdr}{\sqrt{-r^2 + \frac{\alpha r}{|E|} - \frac{h^2}{2m|E|}}} \tag{12}$$

we take,

$$-r^2 + \frac{\alpha r}{|E|} - \frac{h^2}{2m|E|} \tag{13}$$

$$= -r^2 + 2ar - a^2e^2 - a^2 \tag{14}$$

$$= a^2e^2 - (r-a)^2 \tag{15}$$

and

$$\sqrt{\frac{m}{2|E|}} = \sqrt{\frac{ma}{\alpha}} \tag{16}$$

from equations (2) and (5),

$$a^2(e^2-1) = \frac{\alpha^2}{4E^2}[\frac{2Eh^2}{m\alpha^2}] \tag{17}$$

$$= -\frac{h^2}{2m|E|} \tag{18}$$

again

$$r = a(1 - e\ cos\xi) \implies dr = ae\ sin\xi \tag{19}$$

then

$$(r-a)^2 = a^2e^2cos^2\xi \tag{20}$$

$$a^2e^2 - (r-a)^2 = a^2e^2sin^2\xi \tag{21}$$

now

$$\sqrt{\frac{ma}{\alpha}} \int \frac{a(1-e\ cos\xi)\ ae\ sin\xi}{ae\ sin\xi} \tag{22}$$

$$= \sqrt{\frac{ma^3}{\alpha}} \int (1 - e\ cos\xi)\ d\xi \qquad [H.P.] \tag{23}$$

4. **Laplace Runge-Lenz vector**

The position vector $\mathbf{r}$ of the Lagrangian of Kepler's problem is translated by an infinitesimal vector $\Delta\mathbf{r}$, where

$$\Delta\mathbf{r} = \boldsymbol{\varepsilon} \times (\mathbf{v} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{v} \times \boldsymbol{\varepsilon}),$$

with ε being an infinitesimal vector.
(i) Show that the Lagrangian undergoes a symmetry transformation as a result of the above transformation.
(ii) Using Noether's theorem, find the corresponding constant of motion.
(iii) Using the constant of motion can you conclude on the nature of the corresponding orbits?

Solution: (i) Variation in vector $\mathbf{r}$ is , $\delta\mathbf{r}$,

$$\delta\mathbf{r} = \boldsymbol{\varepsilon} \times (\mathbf{v} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{v} \times \boldsymbol{\varepsilon}) \tag{24}$$

where $\boldsymbol{\varepsilon}$ is a infinitesimal constant vector. Then,

$$\delta\dot{\mathbf{r}} = \boldsymbol{\varepsilon} \times (\ddot{\mathbf{r}} \times \mathbf{r}) + \dot{\mathbf{r}} \times (\dot{\mathbf{r}} \times \boldsymbol{\varepsilon}) + \mathbf{r} \times (\ddot{\mathbf{r}} \times \boldsymbol{\varepsilon}) \tag{25}$$

Lagrangian for Kepler's problem is (for $\alpha < 0$),

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(\mathbf{r}) \tag{26}$$

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{\alpha}{\mathrm{r}} \tag{27}$$

Note by Euler-Lagrange equation,

$$m\ddot{\mathbf{r}} = -\frac{\alpha\mathbf{r}}{\mathrm{r}^3} \Longrightarrow \ddot{\mathbf{r}} \times \mathbf{r} = \mathbf{0}$$

So,

$$\delta\dot{\mathbf{r}} = \dot{\mathbf{r}} \times (\dot{\mathbf{r}} \times \boldsymbol{\varepsilon}) + \mathbf{r} \times (\ddot{\mathbf{r}} \times \boldsymbol{\varepsilon})$$

Now the variation in Lagrangian is,

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \frac{1}{2}m(\dot{\mathbf{r}} + \delta\dot{\mathbf{r}})^2 - U(\mathbf{r} + \delta\mathbf{r}) - \frac{1}{2}m\dot{\mathbf{r}}^2 + U(\mathbf{r}) \tag{28}$$

$$\delta\mathcal{L} = m\dot{\mathbf{r}}.\delta\dot{\mathbf{r}} - \boldsymbol{\nabla}U.\delta\mathbf{r} \tag{29}$$

Substituting values of $\delta\dot{\mathbf{r}}$ and $\delta\mathbf{r}$ form equations (24) and (4), we get-

$$\delta\mathcal{L} = m\dot{\mathbf{r}}.(\mathbf{r} \times (\ddot{\mathbf{r}} \times \boldsymbol{\varepsilon})) - \frac{\alpha\mathbf{r}}{\mathrm{r}^3}.(\boldsymbol{\varepsilon} \times (\mathbf{v} \times \mathbf{r})) \tag{30}$$

$$= (\dot{\mathbf{r}} \times \mathbf{r}).(-\frac{\alpha\mathbf{r}}{\mathrm{r}^3} \times \boldsymbol{\varepsilon}) - (\frac{\alpha\mathbf{r}}{\mathrm{r}^3} \times \boldsymbol{\varepsilon}).(\mathbf{v} \times \mathbf{r}) \tag{31}$$

$$= 2\alpha\boldsymbol{\varepsilon}.\left(\frac{\mathbf{r}}{\mathrm{r}^3} \times (\mathbf{v} \times \mathbf{r})\right) \tag{32}$$

$$= 2\alpha\boldsymbol{\varepsilon}.\left(\frac{1}{\mathrm{r}}\frac{d\mathbf{r}}{dt} - \frac{\mathbf{r}.\mathbf{v}}{\mathrm{r}^3}\mathbf{r}\right) \tag{33}$$

Note that,

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{\mathrm{r}}\right) = \frac{1}{\mathrm{r}}\frac{d\mathbf{r}}{dt} - \frac{1}{\mathrm{r}^2}\frac{d|\mathbf{r}|}{dt}\mathbf{r} \tag{34}$$

$$= \frac{1}{\mathrm{r}}\frac{d\mathbf{r}}{dt} - \frac{\mathrm{r}}{\mathrm{r}^3}\frac{dr}{dt}\mathbf{r} \tag{35}$$

$$= \frac{1}{\mathrm{r}}\frac{d\mathbf{r}}{dt} - \frac{1}{\mathrm{r}^3}\left(\frac{d\mathbf{r}}{dt}.\mathbf{r}\right)\mathbf{r} \tag{36}$$

From equations (33) and (36)-

$$\delta\mathcal{L} = 2\alpha\boldsymbol{\varepsilon}.\frac{d}{dt}\left(\frac{\mathbf{r}}{\mathrm{r}}\right) \tag{37}$$

$$= \frac{2}{m}\boldsymbol{\varepsilon}.\frac{d}{dt}\left(\frac{m\alpha\mathbf{r}}{\mathrm{r}}\right) \tag{38}$$

$$= \frac{2}{m}\frac{d}{dt}\left(\boldsymbol{\varepsilon}.\frac{m\alpha\mathbf{r}}{\mathrm{r}}\right) = \frac{d}{dt}(\boldsymbol{\varepsilon}.\boldsymbol{f}(\mathbf{r},\mathbf{t})) \tag{39}$$

Where $\boldsymbol{f}(\mathbf{r},t) = \frac{2\alpha\mathbf{r}}{\mathrm{r}}$. Hence we proved that variation in Lagrangian can be written as a total derivative of a function.

(ii) According to Noether's theorem: constant of motion is $\sum_i p_i\Delta q_i - \boldsymbol{\varepsilon}.\boldsymbol{f}$,where $\boldsymbol{p} = m\dot{\mathbf{r}}$. So -

$$\sum_i p_i\Delta q_i = \boldsymbol{p}.\delta\mathbf{r} \tag{40}$$

$$= \frac{\partial L}{\partial\dot{\mathbf{r}}}.\delta\mathbf{r} \tag{41}$$

$$= m\dot{\mathbf{r}}.(\boldsymbol{\varepsilon} \times (\mathbf{v} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{v} \times \boldsymbol{\varepsilon})) \tag{42}$$

$$= (\dot{\mathbf{r}} \times \boldsymbol{\varepsilon}).\,(\mathbf{p} \times \mathbf{r}) + (\mathbf{p} \times \mathbf{r})\,.(\mathbf{v} \times \boldsymbol{\varepsilon}) \tag{43}$$

$$= 2(\dot{\mathbf{r}} \times \boldsymbol{\varepsilon}).\,(\mathbf{p} \times \mathbf{r}) \tag{44}$$

$$= \frac{2}{m}(\boldsymbol{\varepsilon} \times \boldsymbol{p}).\,(\boldsymbol{r} \times \mathbf{p}) \tag{45}$$

$$= \frac{2}{m}(\boldsymbol{\varepsilon} \times \boldsymbol{p}).\boldsymbol{L} \tag{46}$$

Where $\mathbf{L}$ is angular momentum. So the constant of motion is (from equations (39) and (46))-

$$= \frac{2}{m}(\boldsymbol{\varepsilon} \times \boldsymbol{p}).\boldsymbol{L} - \frac{2}{m}\boldsymbol{\varepsilon}.\frac{m\alpha\mathbf{r}}{\mathrm{r}} \tag{47}$$

$$\frac{2}{m}\boldsymbol{\varepsilon}.\left((\boldsymbol{p} \times \boldsymbol{L}) - \frac{m\alpha\mathbf{r}}{\mathrm{r}}\right) \tag{48}$$

Since $\boldsymbol{\varepsilon}$ is a constant vector so, constant of motion is -

$$= (\boldsymbol{p} \times \boldsymbol{L}) - \frac{m\alpha\mathbf{r}}{\mathrm{r}} \tag{49}$$

Quantity in equation (49) is Laplace-Runge Lenz vector.

(iii) Let,

$$\boldsymbol{A} = (\boldsymbol{p} \times \boldsymbol{L}) - \frac{m\alpha\mathbf{r}}{\mathrm{r}} \tag{50}$$

So,

$$\mathbf{r}.\boldsymbol{A} = \mathbf{r}.(\boldsymbol{p} \times \boldsymbol{L}) - m\alpha\mathrm{r} \tag{51}$$

$$= (\mathbf{r} \times \boldsymbol{p}).\boldsymbol{L} - m\alpha\mathrm{r} \tag{52}$$

$$= L^2 - m\alpha\mathrm{r} \tag{53}$$

Now if $\mathbf{r}.\boldsymbol{A} = A\mathrm{r}cos\phi$, where ϕ is the angle between $\boldsymbol{A}$ and $\mathbf{r}$. Then,

$$A\mathrm{r}cos\phi = L^2 - m\alpha\mathrm{r} \tag{54}$$

So,

$$(m\alpha + Acos\phi)\mathrm{r} = L^2 \tag{55}$$

$$\mathrm{r} = \frac{L^2/m\alpha}{1 + \frac{A}{m\alpha}cos\phi} \equiv \frac{p}{1 + ecos\phi} \tag{56}$$

Where p is semi-latus rectum and e is eccentricity.