Systems with Velocity Dependent Potentials

In this part, we investigate the possibility of finding the expressions of Lagrangians of the systems which are not conservative in the usual sense, i.e., the force is not derived from a scalar potential function.

For this purpose, let's recall the general form of the Euler - Lagrange equation (equation (iv) in Lecture 5), which is given as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} \; ; \qquad i = 1, 2, \dots, n,$$

where T is the kinetic energy and $\vec{F}$ is the force.

Let's now consider a situation where $\vec{F} \neq -\nabla V$ but $\vec{F} \equiv \vec{F}(\vec{r}, \vec{v})$ is such that one could write

$$\vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right) - \frac{\partial U}{\partial q_i}, \qquad \text{—— (1)}$$

where $U \equiv U(q_1, q_2, \dots, q_n \, ; \, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n)$ is a scalar known as Generalized Potential.

Now one can write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

with

$$L = T - U(q_i, \dot{q}_i)$$

where $U(q_i, \dot{q}_i)$ is a (velocity - dependent) generalized potential.

The best example of such a case is the motion of a charged particle in an electromagnetic field. The force is given by the Lorentz formula

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

and our objective is to find a suitable Lagrangian.

If Φ is the electrostatic scalar potential and $\vec{A}$ is the magnetic vector potential, then

$$\begin{cases} \vec{E} = -\nabla\Phi - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$

which give

$$\vec{F} = q\left[-\nabla\Phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\nabla \times \vec{A})\right] \qquad \text{—— (2)}$$

In the present case, the particle has three degrees of freedom and we can simply choose the Cartesian coordinates to be its generalized coordinates: $\{q\} = (x, y, z)$, and similarly for the generalized velocities: $\{\dot{q}\} = (v_x, v_y, v_z)$. Thus, we can write

$$\nabla \times \vec{v} = \vec{0}\ , \quad (\vec{A}\cdot\nabla)\vec{v} = \vec{0} \qquad [\because\ q \text{ and } \dot{q} \text{ are independent}]$$

Using the vector identity

$$\nabla(\vec{v}\cdot\vec{A}) = \vec{v}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{v}) + (\vec{v}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{v} \qquad \text{— (3)}$$

we have

$$\nabla(\vec{v}\cdot\vec{A}) = \vec{v}\times(\nabla\times\vec{A}) + (\vec{v}\cdot\nabla)\vec{A} \qquad \text{— (4)}$$

Putting the value of $\vec{v}\times(\nabla\times\vec{A})$ in (2), we get

$$\vec{F} = q\left[-\nabla\Phi - \frac{\partial\vec{A}}{\partial t} + \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}\right] \qquad \text{— (5)}$$

Note that $\vec{E}$, $\vec{B}$, Φ and $\vec{A}$ are all functions of $\vec{r}$ and t.

Also,

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}$$

$$\vec{F} = q\left[-\nabla\Phi - \frac{d\vec{A}}{dt} + \nabla(\vec{v}\cdot\vec{A})\right] \qquad \text{— (6)}$$

Again, $\nabla_{\vec{v}}(\vec{A}\cdot\vec{v}) = \vec{A}$ and $\nabla_{\vec{v}}\Phi = \vec{0}$, so

$$\nabla_{\vec{v}}(\vec{A}\cdot\vec{v} - \Phi) = \vec{A} \qquad \text{— (7)}$$

Using (7) in (6), we get

$$\vec{F} = q\left[-\nabla(\Phi - \vec{v}\cdot\vec{A}) + \frac{d}{dt}\nabla_{\vec{v}}(\Phi - \vec{v}\cdot\vec{A})\right]$$

For the x-component, we then have

$$F_x = \vec{F}\cdot\frac{\partial\vec{r}}{\partial x} = q\left[-\frac{\partial}{\partial x}(\Phi - \vec{v}\cdot\vec{A}) + \frac{d}{dt}\left[\frac{\partial}{\partial v_x}(\Phi - \vec{v}\cdot\vec{A})\right]\right]$$

and similarly for the y- and z-components. Thus we have found our U to be $U = q(\Phi - \vec{v}\cdot\vec{A})$ as the generalized potential, such that

$$\boxed{L = T - U = \frac{1}{2}mv^2 - q(\Phi - \vec{v}\cdot\vec{A})}$$

⚠ Can the same formalism be carried out for the Coriolis force?

- **Euler - Lagrange Equations for Non- Potential Forces:**

Let us consider a mechanical system which is evolving under two types of 'body' forces: one is the usual body force and can be derived from a scalar potential function. The other body force cannot be derived from any potential or generalized (velocity dependent) potential function. In this part, we discuss how the Euler - Lagrange equations will be modified to include the effects of such forces.

First of all, let's admit that there is no known general prescription. So, we have to investigate case by case.

A very familiar example of a non-potential force is the force of friction, $\vec{F}_f$.

- proportional to $\vec{v}$ if the particle is moving through a laminar viscous medium
- proportional to v^2 if the particle is moving through a turbulent medium.

Here we just consider the case

$$\vec{F}_f = -k\vec{v}$$

(more generally it can be component-wise proportional to $\vec{v}$: $F_{fx} = -k_x v_x$ etc.)

If we define a function $\mathcal{F}$ such that $\vec{F}_f = -\nabla_{\vec{v}}\mathcal{F}$, then we can write

$$\mathcal{F} = \frac{1}{2}k\left(v_x^2 + v_y^2 + v_z^2\right)$$

$\left[\text{more generally} \quad \mathcal{F} = \frac{1}{2}\left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right)\right]$

$\mathcal{F}$ is known as the **Rayleigh dissipation function**.

Now, recalling the general form of the Euler - Lagrange equations, we can write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

where $Q_j = \vec{F}_f \cdot \dfrac{\partial \vec{r}}{\partial q_j}$ is the generalized force which cannot be derived from any potential function.

Now

$$Q_j = \vec{F}_f \cdot \frac{\partial \vec{r}}{\partial q_j} = -(\nabla_{\vec{v}} \mathcal{F}) \cdot \frac{\partial \vec{r}}{\partial q_j} = -(\nabla_{\vec{v}} \mathcal{F}) \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

So, the modified Euler - Lagrange equations would be

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}}$$

where $L = T - U$ and U is the potential / generalized potential.

- As an example, let's consider a freely falling body under gravity for which $\mathcal{F} = \frac{1}{2} k\dot{y}^2$. Let y be vertical displacement. So,

$$L = \frac{1}{2} m\dot{y}^2 + mgy \qquad [\because V = -mgy]$$

and E-L equation will be

$$\frac{d}{dt}(m\dot{y}) - mg = -k\dot{y}$$

$$\Rightarrow \boxed{m\ddot{y} + k\dot{y} - mg = 0} \longleftarrow \text{equation of motion}$$

For maximum velocity, $\ddot{y} = 0 \Rightarrow k\dot{y} = mg$

$$\Rightarrow \boxed{\dot{y} = \frac{mg}{k}} \longleftarrow \text{Terminal velocity}$$