Derivatives:

Cauchy's integral formula can be used to obtain an expression for the derivative of $f(z)$. Differentiating

$$\frac{1}{2\pi \mathrm{i}}\oint_C \frac{f(z)}{z - z_0}dz = \mathrm{f}(z_0)$$

with respect to z_0, and interchanging the differentiation and the z integration,

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^2}dz.$$

Differentiating again,

$$f''(z_0) = \frac{2}{2\pi i}\oint \frac{f(z)}{(z - z_0)^3}dz.$$

Continuing, we get

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}}dz\,;$$

that is, the requirement that $f(z)$ be analytic guarantees not only a first derivative, but derivatives of all orders as well. The derivatives of $f(z)$ are automatically analytic.

Example: Find $I = \oint_C \frac{\sin^2 z}{(z-a)^4}dz$, where the integral is counterclockwise on a contour that encircles the point $z = a$.

Solution: This is a case of

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}}dz\,;$$

with $n = 3$ and $f(z) = \sin^2 z$. Therefore,

$$I = \frac{2\pi i}{3}\left[\frac{d^3}{dz^3}\sin^2 z\right]_{z=a} = \frac{\pi i}{3}[-8\sin z\cos z]_{z=a} = -\frac{8\pi i}{3}\sin a\cos a.$$

Morera's Theorem:

A further application of Cauchy's integral formula is in the proof of **Morera's theorem**, which is the converse of the Cauchy integral theorem. The theorem states the following:

If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.