Another Integration Technique:

Sometimes we have an integral on the real range $(0, \infty)$ that lacks the symmetry needed to extend the integration range to $(-\infty, \infty)$. However, it may be possible to identify a direction in the complex plane on which the integrand has a value identical to or conveniently related to that of the original integral, thereby permitting the construction of a contour facilitating the evaluation.

Example: Evaluation on a circular sector:

Our problem is to evaluate the integral

$$I = \int_0^\infty \frac{dx}{x^3 + 1},$$

which we cannot convert easily into an integral on the range $(-\infty, \infty)$. However, we note that along a line with argument $\theta = 2\pi/3$, z^3 will have the same values as at corresponding points on the real line; note that $\left(re^{2\pi i/3}\right)^3 = r^3 e^{2\pi i} = r^3$. We therefore consider

$$\oint \frac{dz}{z^3 + 1}$$

on the contour shown in the figure below. The part of the contour along the positive real axis labelled A, simply yields our integral I. The integrand approaches zero sufficiently rapidly for large $|z|$ that the integral on the large circular arc, labelled C in the figure, vanishes.

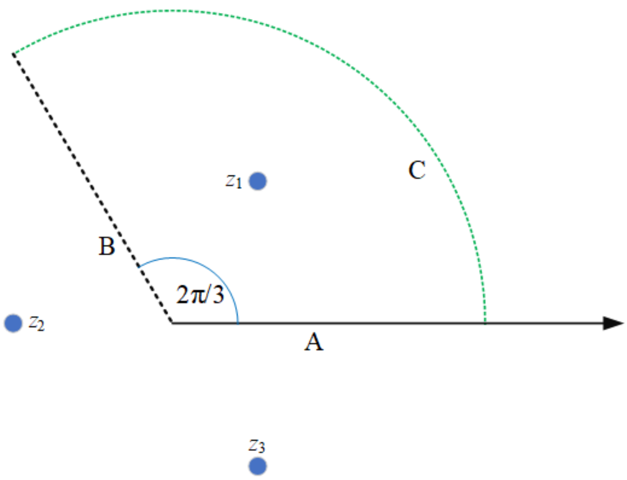

Fig. 1: A suitable contour for evaluation of I

On the remaining segment of the contour, labelled B, we note that $dz = e^{2\pi i/3} dr$, $z^3 = r^3$, and

$$\int_B \frac{dz}{z^3 + 1} = \int_\infty^0 \frac{e^{2\pi i/3}}{r^3 + 1} dr = -e^{2\pi i/3} \int_0^\infty \frac{dr}{r^3 + 1} = -e^{2\pi i/3} I.$$

Therefore,

$$\oint \frac{dz}{z^3 + 1} = \left(1 - e^{2\pi i/3}\right) I.$$

We now need to evaluate our complete contour integral using the residue theorem. The integrand has simple poles at the three roots of the equation $z^3 + 1 = 0$, which are at $z_1 = e^{\pi i/3}$, $z_2 = e^{\pi i}$, and $z_3 = e^{5\pi i/3}$, as marked in the figure. Only the pole at z_1 is enclosed by our contour. The residue at $z = z_1$ is

$$\lim_{z \to z_1} \frac{z - z_1}{z^3 + 1} = \left.\frac{1}{3z^2}\right|_{z=z_1} = \frac{1}{3e^{2\pi i/3}}.$$

Thus,

$$\left(1 - e^{2\pi i/3}\right) I = 2\pi i \left(\frac{1}{3e^{2\pi i/3}}\right).$$

Solution for I is facilitated if we multiply through by $e^{-\pi i/3}$, obtaining initially

$$\left(e^{-\pi i/3} - e^{\pi i/3}\right) I = 2\pi i \left(-\frac{1}{3}\right),$$

which is easily arranged to

$$I = \frac{\pi}{3 \sin \pi/3} = \frac{\pi}{3\sqrt{3}/2} = \frac{2\pi}{3\sqrt{3}}.$$