## Cauchy Integral Formula:

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour $C$ and within the interior region bounded by $C$. This means that the contour $C$ is to be traversed in the **counterclockwise** direction. We seek to prove the following result, known as **Cauchy integral formula:**

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)$$

in which $z_0$ is any point in the interior region bounded by $C$. Note that since $z$ is on the contour while $z_0$ is in the interior, $z - z_0 \neq 0$ and the integral is well-defined. Although $f(z)$ is assumed analytic, the integrand $f(z)/(z-z_0)$ is not analytic at $z = z_0$ unless $f(z_0) = 0$. We now deform the contour, to make it a circle of small radius $r$ about $z = z_0$, traversed, like the original contour, in the counterclockwise direction. As shown in the preceding section, this does not change the value of the integral. We therefore write $z = z_0 + re^{i\theta}$, so $dz = ire^{i\theta}d\theta$, the integration is from $\theta = 0$ to $\theta = 2\pi$, and

$$\oint_C \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right)}{re^{i\theta}} ire^{i\theta}d\theta.$$

Taking the limit when $r \to 0$, we obtain

$$\oint_C \frac{f(z)}{z-z_0}dz = if(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0),$$

where we have replaced $f(z)$ by its limit $f(z_0)$ because it is analytic and therefore continuous at $z = z_0$. This proves the Cauchy integral formula.

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an **arbitrary interior point** $z = z_0$ once the values on the boundary $C$ are specified.

It has been emphasized that $z_0$ is an interior point. What happens if $z_0$ is exterior to $C$? In this case the entire integrand is analytic on and within $C$, Cauchy integral theorem applies, and the integral vanishes. Summarizing, we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), & z_0 \text{ within the contour,} \\ 0, & z_0 \text{ is exterior to the contour.} \end{cases}$$

**Example:** Find

$$I = \oint_C \frac{dz}{z(z+2)}$$

where the integration is counterclockwise over the unit circle.

**Solution:** The factor $\frac{1}{(z+2)}$ is analytic within the region enclosed by the contour, so this is a case of Cauchy integral formula with $f(z) = \frac{1}{z+2}$ and $z_0 = 0$. The result is immediate:

$$I = 2\pi i\left[\frac{1}{z+2}\right]_{z=0} = \pi i.$$

**Example:** Find

$$I = \oint_C \frac{dz}{4z^2 - 1},$$

where the integration is counterclockwise over the unit circle.

**Solution:** The denominator factors into $4z^2 - 1 = 4(z - 1/2)(z + 1/2)$, and it is apparent that the region of integration contains two singular factors. However, we may still use Cauchy integral formula if we make the partial fraction expansion:

$$\frac{1}{4z^2 - 1} = \frac{1}{4}\left[\frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}}\right],$$

after which we integrate the two terms separately. We have

$$I = \frac{1}{4}\left[\oint_C \frac{dz}{z + \frac{1}{2}} - \oint_C \frac{dz}{z + \frac{1}{2}}\right]$$

Each integral is a case of Cauchy integral formula with $f(z) = 1$, and for both integrals, the point $z_0 = \pm 1/2$ is within the contour, so each evaluates to $2\pi i$, and their sum is zero. So $I = 0$.