Multiply Connected Regions:

The original statement of Cauchy integral theorem demanded a simply connected region of analyticity. This restriction may be relaxed by the creation of a barrier, a narrow region we choose to exclude from the region identified as analytic. The purpose of the barrier construction is to permit, within a multiply connected region, the identification of curves that can be shrunk to a point within the region that is, the construction of a subregion that is simply connected.

Consider the multiply connected region as shown in figure below in which $f(z)$ is only analytic in the unshaded area labelled R. Cauchy integral theorem is not valid for the contour C.

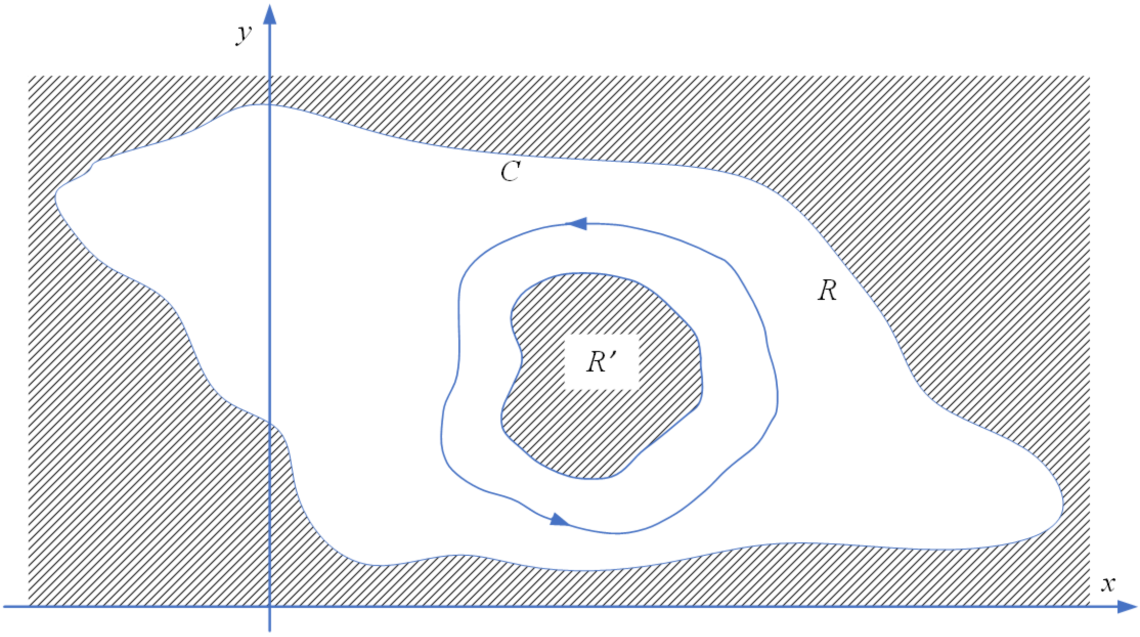

Figure: A closed contour C in a multiply connected region.

But we can construct a contour C' for which the theorem holds. We draw a barrier from the interior forbidden region R', to the forbidden region exterior to R, and then run a new contour C', as shown in figure below.

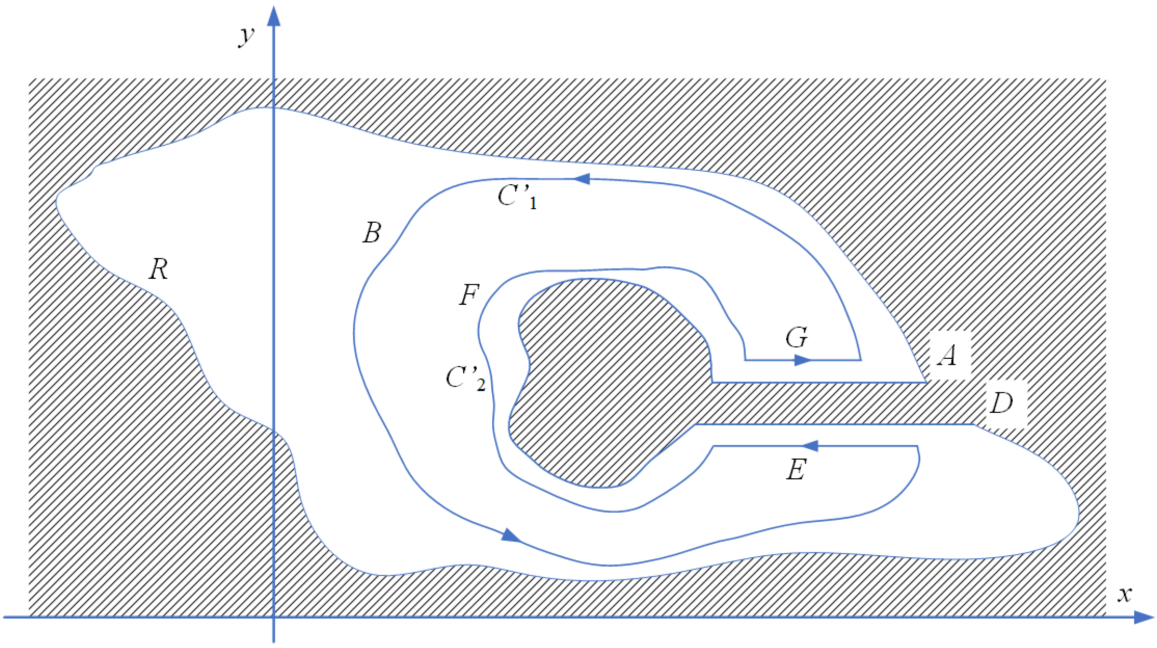

Figure: Conversion of a multiply connected region into a simply connected region.

The new contour C', through ABDEFGA, never crosses the barrier that converts R into simply connected region. Because $f(z)$ is in fact continuous across the barrier dividing DE from GA and the line segments DE and GA can be arbitrarily close together, we have

$$\int_G^A f(z)dz = -\int_D^E f(z)dz$$

Then, invoking Cauchy integral theorem, because the contour is now within a simply connected region,

$$\oint_{C'} f(z)dz = \int_{ABD} f(z)dz + \int_D^E f(z)dz + \int_{EFG} f(z)dz + \int_G^A f(z)dz = 0$$

$$\Rightarrow \int_C f(z)dz = \int_{ABD} f(z)dz + \int_{EFG} f(z)dz = 0$$

$$\left[\because \int_G^A f(z)dz = -\int_D^E f(z)dz\right]$$

Note that A and D are only infinitesimally separated and that $f(z)$ is actually continuous across the barrier. Hence, integration on path ABD will yield the same result a truly closed contour ABDA. Similar remarks apply to the path EFG, which can be replaced by EFGE. Renaming ABDA as C_1' and EFGE as $-C_2'$, we have the simple result:

$$\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz\,,$$

in which C_1' and C_2' are both traversed in the same (counterclockwise, that is, positive) direction.

What we have shown is that the integral of an analytic function over a closed contour surrounding an "island" of non-analyticity can be subjected to any value of the integral. The notion of continuous deformation means that the change in contour must be able to be carried out via a series of small steps, which precludes processes whereby we "jump over" a point or region of non-analyticity.

Since we already know that the integral of an analytic function over a contour in a simply connected region of analyticity has the value zero, we can make the more general statement:

"The integral of an analytic function over a closed path has a value that remains unchanged over all possible continuous deformations of the contour within the region of analyticity."

Looking back at the two examples of this section, we see that the integral of z^2 vanished for both the circular and square contours, as prescribed by Cauchy integral theorem for an analytic function. The integral of z^{-1} did not vanish, and vanishing was not required because there was a point of non-analyticity within the contour. However, the integrals of z^{-1} for the two contours had the same value, as either contour ca be reached by continuous deformation of the other.

We close this section with an extremely important observation:

The integral of $(z - z_0)^n$ around any counterclockwise closed path C that encloses z_0 has, for any integer n, the values

$$\oint_C (z - z_0)dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$