Residue Theorem:

If the Laurent expansion of a function,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is integrated term by term using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain

$$a_n \oint (z - z_0)^n \, dz = 0, n \neq -1,$$

$$a_{-1} \oint (z - z_0)^{-1} \, dz = 2\pi i a_{-1}$$

The integral of $(z - z_0)^n$ around any counterclockwise closed path C that encloses z_0 has, for any integer n, the values

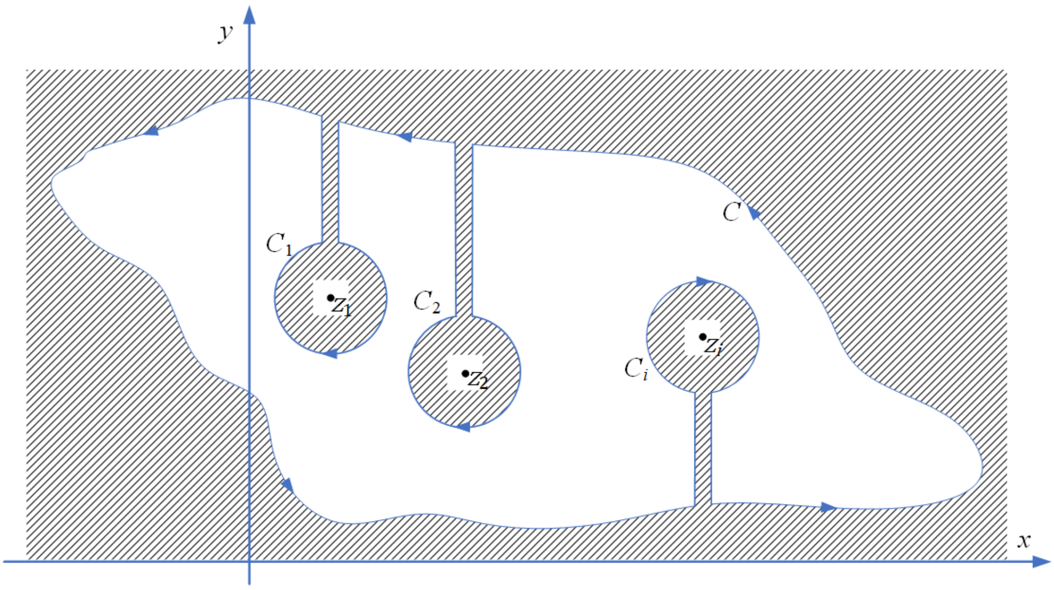
$$\oint_C (z - z_0)^n \, dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

Thus, we have

$$\oint f(z) dz = 2\pi i a_{-1}.$$

The constant a_{-1}, the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion, is called the **residue** of $f(z)$ at $z = z_0$.

Now consider the evaluation of the integral, over a closed contour C, of a function that has isolated singularities at points $z_1, z_2, \ldots$. We can handle this integral by deforming our contour as shown in the figure below.



Cauchy integral theorem then leads to

$$\oint_C f(z) dz + \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \cdots = 0,$$

where C is in the positive, counterclockwise direction, but the contours C_1, C_2, …, that, respectively, encircle $z_1, z_2, \ldots$, are clockwise. Thus, referring to $\oint f(z)dz = 2\pi i a_{-1}$, the integrals C_i about the individual isolated singularities have the values

$$\oint_{C_i} f(z)dz = -2\pi i a_{-1,\, i}\,.$$

where $a_{-1,\, i}$ is the residue obtained from the Laurent expansion about the singular point $z = z_i$. The negative sign comes from the clockwise integration. We have

$$\begin{aligned}\oint_{C} \mathrm{f(z)dz} &= 2\pi \mathrm{i}\left(\mathrm{a}_{-1,\,1} + \mathrm{a}_{-1,\,2} + \cdots\right) \\ &= 2\pi i(\text{sum of the enclosed residues})\end{aligned}$$

This is the **residue theorem**. The problem of evaluating a set of contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points.