# Evaluation of Definite Integrals with Branch Points:

## Avoidance of branch points:

Sometimes we must deal with integrals whose integrands have branch points. In order to use contour integration methods of such integrals we must choose contours that avoid the branch points, enclosing only point singularities.

**Example:** Integral containing logarithm:

We now look at

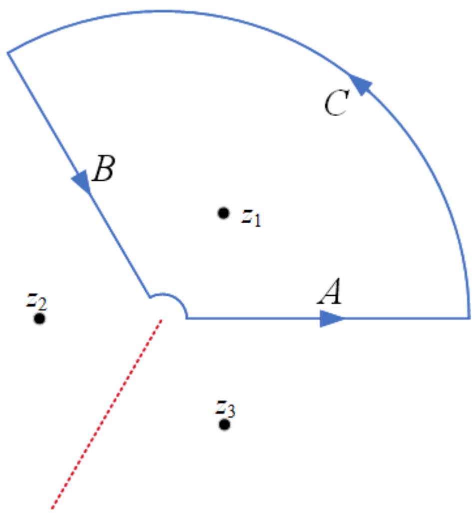
$$I = \int_0^\infty \frac{\ln x}{x^3+1}\,dx.$$

The integrand is singular at $x = 0$, but the integration converges (the indefinite integration of $\ln x$ is $x \ln x - x$). However, in the complex plane, this singularity manifests itself as a branch point, so if we are to recase this problem in a way involving a contour integral, we must avoid $z = 0$ and a branch cut from that point to $z = \infty$.

It turns out to be convenient to use a contour similar to that for previous example except that we must make a small circular detour about $z = 0$ and then draw the branch cut in a direction that remains outside our chosen contour. The integrand has simple poles at the three roots of $z^3 + 1$, which are at $z_1 = e^{\pi i/3}$, $z_2 = e^{\pi i}$ and $z_3 = e^{5\pi i/3}$, as marked in the figure. We consider a contour integral

$$\oint \frac{\ln z}{z^3+1}\,dz\,,$$

where the contour and the locations of the singularities of the integrand are as illustrated in the figure below:




The integral over the large circular arc, labelled $C$, vanishes, as the factor $z^3$ in the denominator dominates over the weakly divergent factor $\ln z$ in the numerator (which diverges more weakly than any positive power of $z$). We also get no contribution to the contour integral from the arc at small $r$, since we have there

$$\lim_{r\to 0}\int_0^{2\pi/3} \frac{\ln\left(re^{i\theta}\right)}{1+r^3e^{3i\theta}}\, ire^{i\theta}\,d\theta\,,$$

which vanishes because $r \ln r \to 0$.

The integrals over the segment labelled $A$ and $B$ do not vanish. To evaluate the integral over these segments, we need to make an appropriate choice of the multivalued function $\ln z$. It is natural to choose the branch so that on the real axis we have $\ln z = \ln x$ (and not $\ln x + 2n\pi i$ for some nonzero $n$). Then the integral over the segment labelled $A$ will have the value $I$ (Note: Because the integral converges at $x = 0$, the value is not affected by the fact that this segment terminates infinitesimally before reaching that point).

To compute the integral over $B$, we note that on this segment $\theta = 2\pi/3$, $z = re^{i\theta} \Longrightarrow z^3 = r^3$ and $dz = e^{2\pi i/3}dr$, also but note that $\ln z = \ln r + 2\pi i/3$. There is little temptation here to use a different one of the multiple values of the logarithm, but for future reference note that we **must** use the value that is reached continuously from the value we already chose on the positive real axis, moving in a way that does not cross the branch cut. Thus, we cannot reach segment $A$ by clockwise travel from the positive real axis (thereby getting $\ln z = \ln r - 4\pi i/3$) or any other value that would require multiple circuits around the branch point $z = 0$.

Based on the foregoing, we have

$$\int_B \frac{\ln z}{z^3+1}dz = \int_\infty^0 \frac{\ln r + 2\pi i/3}{r^3+1} e^{2\pi i/3}\, dr = -e^{\pi i/3} I - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{dr}{r^3+1}.$$

$$\Longrightarrow \quad \oint \frac{\ln z}{z^3+1}dz = \left(1 - e^{2\pi i/3}\right) I - \frac{2\pi i}{3} e^{2\pi i/3} \left(\frac{2\pi}{3\sqrt{3}}\right). \qquad \left[\because \int_0^\infty \frac{dr}{r^3+1} = \frac{2\pi}{3\sqrt{3}}\right]$$

Our next step is to use the residue theorem to evaluate the contour integral. Only the pole at $z = z_1$ lies withing the contour. The residue we must compute is

$$\lim_{z\to z_1} \frac{(z-z_1)\ln z}{z^3+1} = \left.\frac{\ln z}{3z^2}\right|_{z=z_1} = \frac{\pi i/3}{3e^{2\pi i/3}} = \frac{\pi i}{9} e^{-2\pi i/3},$$

and application of the residue theorem yields

$$\left(1 - e^{2\pi i/3}\right) I - \frac{2\pi i}{3} e^{2\pi i/3} \left(\frac{2\pi}{3\sqrt{3}}\right) = 2\pi i \left(\frac{\pi i}{9}\right) e^{-2\pi i/3}.$$

Solving for $I$, we get

$$I = \frac{2\pi^2}{27}.$$