Branch Points

In addition to the isolated singularities identified as poles or essential singularities, there are *singularities uniquely associated with multivalued functions*. It is useful to work with these functions in ways that to the maximum possible extent remove ambiguity as to the functions values.

Thus, if at a point z_0 (at which $f(z)$ has a derivative), we have chosen a specific value of the multivalued functions $f(z)$, then we can assign to $f(z)$ values at nearby points in a way that causes continuity in $f(z)$. If we think of a succession of closely spaced points in the limit of zero spacing defining a path, our current observation is that a given values of $f(z_0)$ then leads to a unique definition of the value of $f(z)$ to be assigned to each point on the path. This scheme creates no ambiguity so long as the path is entirely open, meaning that the path does not return to any point previously passed. But if the path returns to z_0, thereby forming a **closed loop**, our prescription might lead, upon the return to a different one of the multiple values of $f(z_0)$.

Example: Value of $z^{1/2}$ on a closed loop:

We consider $f(z) = z^{1/2}$ on a path consisting of counterclockwise passage around the unit circle, starting and ending at $z = +1$. At the start point, where $z^{1/2}$ has multiple values $+1$ and -1, let us choose $f(z) = +1$. See figure below.

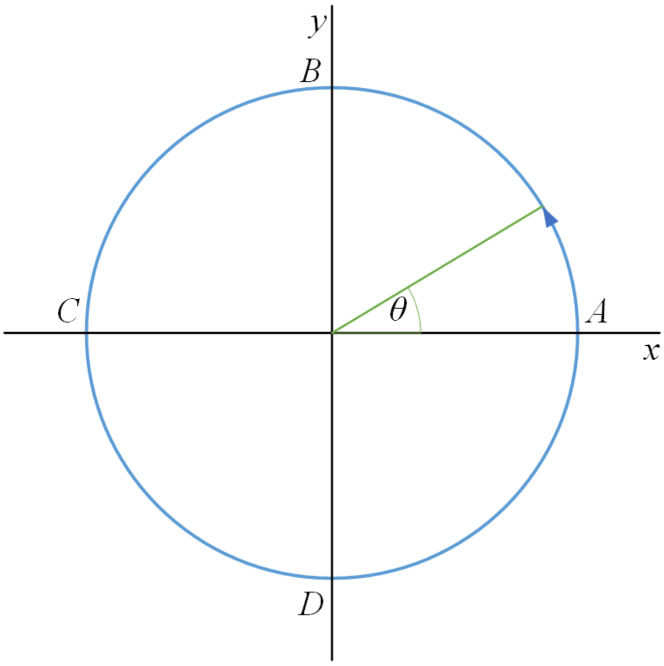

Fig. 1: Path encircling $z = 0$ for evaluation of $z^{1/2}$.

Writing $f(z) = e^{i\theta/2}$, we note that this form with $\theta = 0$ is consistent with the desired starting value $f(z)$, $+1$. In the figure, the start point is labelled *A*. Next, we note that the passage counterclockwise on the unit circle corresponds to an increase in θ, so that at the points marked *B*, *C* and *D* in the figure, the respective values of θ are $\pi/2$, π and $3\pi/2$. Counting further along the path, when we return to point *A*, the value of θ has become 2π (not 0). Then

$$f(z_B) = e^{i\theta_B/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$
$$f(z_C) = e^{i\theta_C/2} = e^{i\pi/2} = +i$$
$$f(z_D) = e^{i\theta_D/2} = e^{3i\pi/4} = \frac{-1+i}{\sqrt{2}}$$

When we return to point *A*, we have $f(+1) = e^{i\pi} = -1$, which is the other value of the multivalued function $z^{1/2}$.

If we continue for a second counterclockwise circuit of the unit circle, the value of θ would continue to increase, from 2π to 4π (reached when we arrive at point *A* after the second loop). We now have $f(+1) = e^{4\pi i/2} = e^{2\pi i} = 1$, so a second circuit has brought us back to the original value. It should now be clear that we are only going to be able to obtain two different value of $z^{1/2}$ for the same point z.