Cauchy Principal Value:

Occasionally an isolated pole will be directly on the contour of integration, causing the integral to diverge. A simple example is provided by an attempt to evaluate the real integral

$$\int_{-a}^{b} \frac{dx}{x},$$

which is divergent because of the logarithmic singularity at $x = 0$; note that the indefinite integral of x^{-1} is $\ln x$. However, the integral can be given a meaning if we obtain a convergent form when replaced by a limit of the form

$$\lim_{\delta \to 0^+} \left(\int_{-a}^{-\delta} \frac{dx}{x} + \int_{\delta}^{b} \frac{dx}{x} \right).$$

To avoid issues with the logarithm of negative values of x, we can change the variable in the first integral to $y = -x$, and the two integrals are then seen to have the respective values $\ln \delta - \ln a$ and $\ln b - \ln \delta$, with sum $\ln b - \ln a$. What happened is that the increase towards $+\infty$ as $1/x$ approaches zero from the positive values is compensated by a decrease towards $-\infty$ as $1/x$ approaches zero from the negative x. This situation is illustrated graphically in the figure below:

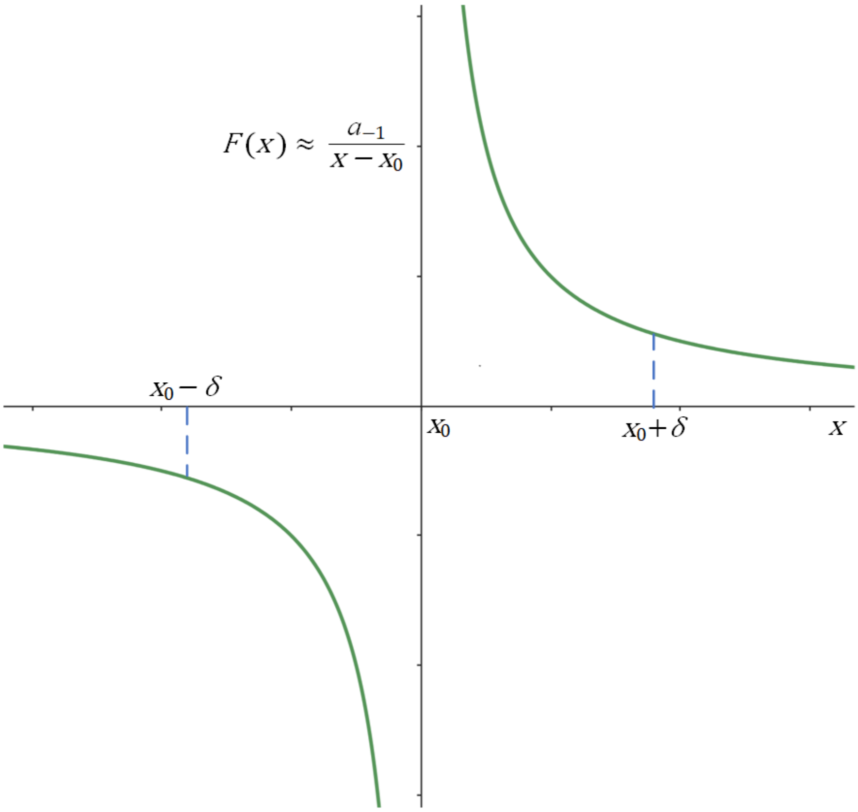

Figure: Cauchy principal value cancellation, integral $\frac{1}{z}$.

Note that the procedure we have described does not make the original integral convergent. In order for the integral to be convergent, it would be necessary that

$$\lim_{\delta_1, \delta_2 \to 0^+} \left[\int_{-a}^{-\delta_1} \frac{dx}{x} + \int_{\delta_2}^{b} \frac{dx}{x} \right]$$

exists (meaning that the limit has a unique value) when δ_1 and δ_2 approach zero **independently**. However, different rates of approach to zero by δ_1 and δ_2 will cause a change in value of the integral. For example, if $\delta_2 = 2\delta_1$, then

$$\lim_{\delta_1,\,\delta_2\to 0^+}\left[\int_{-a}^{-\delta_1}\frac{dx}{x}+\int_{\delta_2}^{b}\frac{dx}{x}\right] = (\ln\delta_1 - \ln a) + (\ln b - \ln\delta_2) = \ln b - \ln a - \ln 2.$$

The limit then has no definite value, confirming our original statement that the integral diverges.

Generalizing from the above example, we define the **Cauchy principal value** of the real integral of a function $f(x)$ with an isolated singularity on the integration path at the point x_0 as the limit

$$\lim_{\delta\to 0^+}\int^{x_0-\delta} f(x)dx + \int_{x_0+\delta} f(x)dx.$$

The Cauchy principal value is sometimes indicated by preceding the integral sign by **P**, as in

$$P\int f(x)dx.$$

This notation, of course, presumes that the location of the singularity is known.

Example: A Cauchy principal value:

Consider the integral

$$I = \int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}dx$$

We would like to separate this expression for I into two terms, but if we do so, each will become a logarithmically divergent integral. However, if we change the integration range in the above expression from $(0,\infty)$ to (δ,∞), then the integral remains unchanged in the limit of small δ, and the integrals in the second expression for I remain convergent so long as δ is not precisely zero. Rewriting the second term of the second integral,

$$-\int_\delta^\infty \frac{e^{-ix}}{2ix}dx = -\int_{-\delta}^{-\infty}\frac{e^{ix}}{2ix}dx = \int_{-\infty}^{-\delta}\frac{e^{ix}}{2ix}dx\,,$$

we see that the two integrals which together form I can be written (in the limit when $\delta \to 0^+$) as the Cauchy principal value integral

$$I = P\int_{-\infty}^{\infty}\frac{e^{ix}}{2ix}dx.$$

The Cauchy principal value has implications for complex variable theory.

Suppose now that, instead of having a break in the integration path from $x_0-\delta$ to $x_0+\delta$, we connect the two parts of the path by a circular arc passing in the complex plane either above or below the singularity at x_0. Let's continue the discussion in the conventional complex variable notation, denoting the singular point as z_0, so our arc will be a half circle (of radius δ) passing either counterclockwise **below** the singularity at z_0 or clockwise **above** z_0.

We restrict further analysis to singularities no stronger than $1/(z-z_0)$, so we are dealing with a simple pole. Looking at the Laurent expansion of the function $f(z)$ to be integrated, it will have initial terms

$$\frac{a_{-1}}{z-z_0} + a_0 + \cdots,$$

and the integration over a semicircle of radius δ will take (in the limit $\delta \to 0^+$) one of the two forms (in the polar representation $z - z_0 = re^{i\theta}$, with $dz = ire^{i\theta}d\theta$ and $r = \delta$):

$$I_{\text{over}} = \int_{\pi}^{0} d\theta i\delta e^{i\theta}\left[\frac{a_{-1}}{\delta e^{i\theta}} + a_0 + \cdots\right] = \int_{\pi}^{0} \left(ia_{-1} + i\delta e^{i\theta}a_0 + \cdots\right)d\theta \to -i\pi a_{-1},$$

or

$$I_{\text{under}} = \int_{\pi}^{2\pi} d\theta i\delta e^{i\theta}\left[\frac{a_{-1}}{\delta e^{i\theta}} + a_0 + \cdots\right] = \int_{\pi}^{2\pi} \left(ia_{-1} + i\delta e^{i\theta}a_0 + \cdots\right)d\theta \to i\pi a_{-1}.$$

Note that all but the first term in each of the two equations vanishes in the limit $\delta \to 0^+$, and that each of these equations yields a result that is in magnitude half the value that would have been obtained by a null circuit around the pole. The signs associated with semicircles correspond as expected to the direction of travel, and the two semicircular integrals average to zero.

We will occasionally want to evaluate a contour integral of a function $f(z)$ on a closed path that includes the two pieces of a Cauchy principal value integral

$$P\int f(z)dz$$

with a simple pole at z_0, a semicircular arc connecting them at the singularity, and whatever other curve C is needed to close the contour (see figure below).

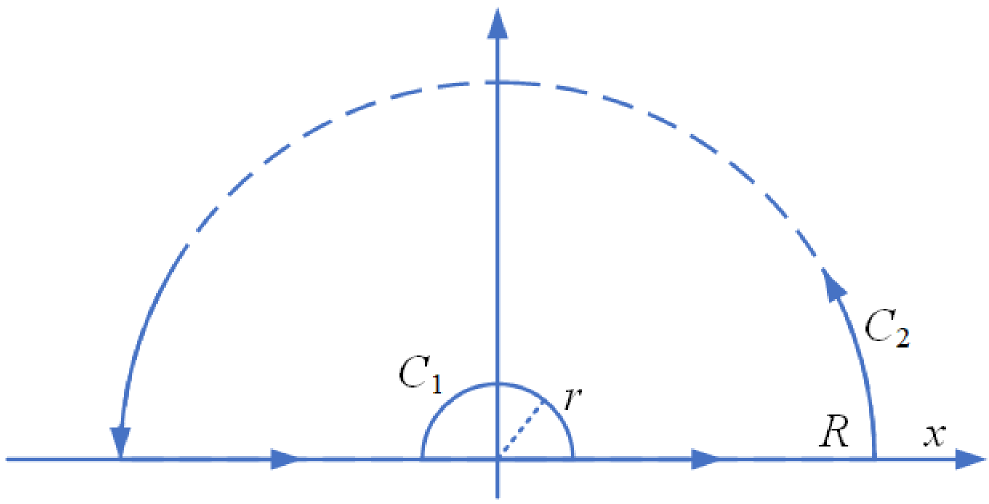

Figure: A contour including a Cauchy principal value integral.

These contributions combine as follows, noting that in the figure the contour passes over the point z_0:

$$P\int f(z)dz + I_{\text{over}} + \int_{C_2} f(z)dz = 2\pi i \sum \text{Residues (other than at } z_0\text{)},$$

which rearrange to give

$$P\int f(z)dz = -I_{\text{over}} - \int_{C_2} f(z)dz + 2\pi i \sum \text{Residues (other than at } z_0).$$

On the other hand, we could have chosen the contour to pass **under** z_0, in which case we would get

$$P\int f(z)dz = -I_{\text{under}} - \int_{C_2} f(z)dz + 2\pi i \sum \text{Residues(other than at } z_0) + 2\pi i a_{-1}.$$

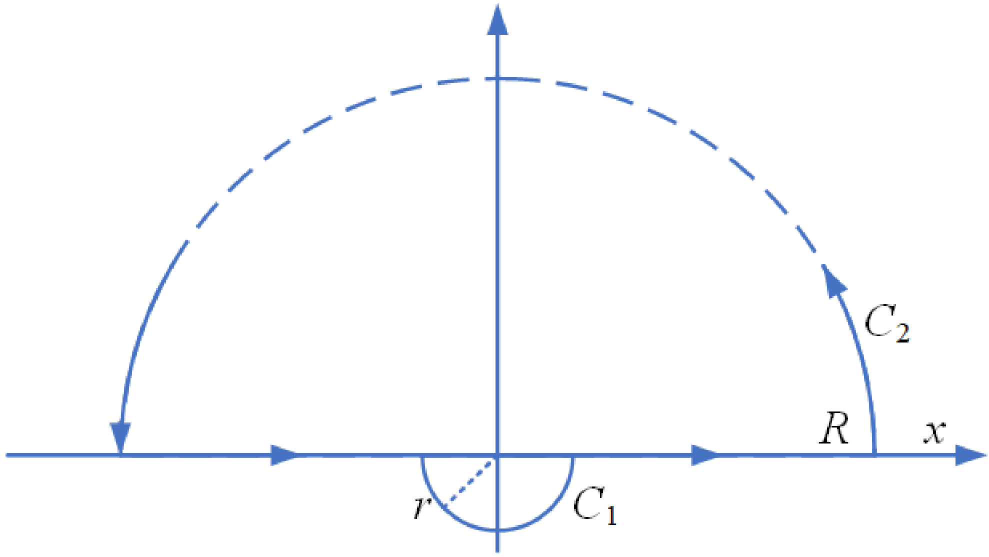

Figure: A contour including a Cauchy principal value integral.

The above equations are in agreement because $2\pi i a_{-1} - I_{\text{under}} = -I_{\text{over}}$, so for the purpose of evaluating the Cauchy principal value integral, it makes no difference whether we go below or above the singularity on the original integration path.