## Laurent Expansion (Laurent Series):

Sometimes we encounter functions that are analytic in an annular region, say between circles of inner radius $r$ and outer radius $R$, about a point $z_0$, as shown in figure below:

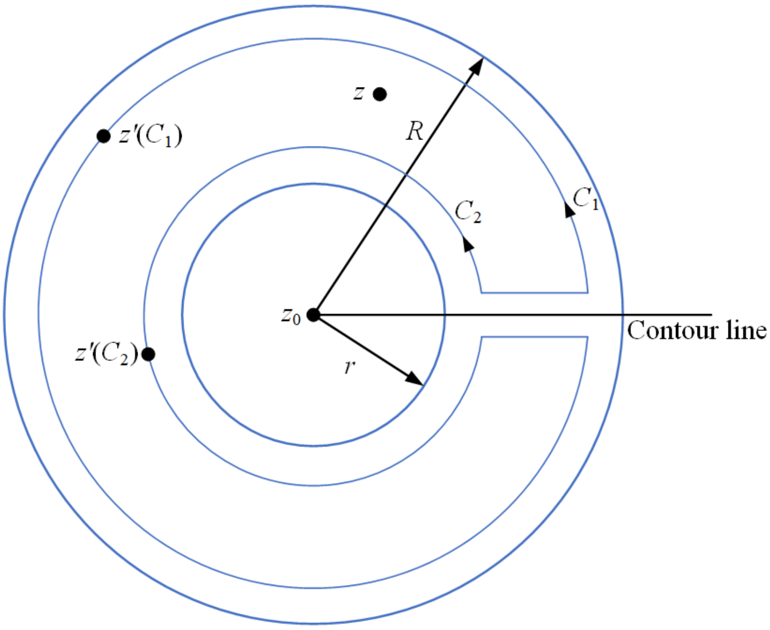


**Figure:** Annular region for Laurent series.
$|z' - z_0|_{C_1} > |z - z_0|$; $|z' - z_0|_{C_2} < |z - z_0|$.

We assume $f(z)$ to be such a function, with $z$ a typical point in the annular region. Drawing an imaginary barrier to convert our region into a simply connected region, we apply Cauchy's integral formula to evaluate $f(z)$, using the contour shown in the figure. Note that the contour consists of the two circles centred at $z_0$, labelled $C_1$ and $C_2$ (which can be considered closed since the barrier is fictitious), plus segments on either side of the barrier whose contributions will cancel. We assign $C_2$ and $C_1$ the radii $r_2$ and $r_1$, respectively, where $r < r_2 < r_1 < R$. Then, from Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z' - z} dz',$$

Note that an explicit minus sign has been introduced so that the contour $C_2$ (like $C_1$) is to be traversed in the positive (counterclockwise) sense.

The treatment of the above equation now proceeds exactly like that of

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z' - z_0)\left[1 - \frac{(z - z_0)}{(z' - z_0)}\right]} dz'$$

in the development of the Taylor series. Each denominator is written as $(z' - z_0) - (z - z_0)$ expanded by the binomial theorem. Noting that for $C_1$, $|z' - z_0| > |z - z_0|$ while for $C_2$, $|z' - z_0| < |z - z_0|$, we find

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1} \frac{f(z')}{(z' - z_0)^{n+1}} dz' + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z') dz'.$$

The minus sign has been absorbed by the binomial expansion. Labelling the first series by $S_1$ and the second series by $S_2$, we have

$$S_1 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}}dz',$$

which has the same form as the regular Taylor expansion, convergent for $|z-z_0| < |z'-z_0| = r_1$, that is, for all $z$ **interior** to the larger circle $C_1$.

For the second series, we have

$$S_2 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n} \oint_{C_2} (z'-z_0)^{n-1} f(z')dz',$$

convergent for $|z-z_0| > |z'-z_0| = r_2$, that is, for all $z$ **exterior** to the smaller circle $C_2$. Remember, $C_2$ now goes counterclockwise.

These two series are combined into one series, known as Laurent series, of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \qquad \text{where } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'.$$

Since convergence of a binomial expansion is not relevant to the evaluation of $a_n$, $C$ in that equation may be any contour within the annular region $r < |z-z_0| < R$ that encircles $z_0$ once in a counterclockwise sense. If such annular region of analyticity does exist, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

is the Laurent series, or Laurent expansion of $f(z)$.

The Laurent series differs from the Taylor series by the obvious feature of negative powers of $(z-z_0)$. For this reason the Laurent series will always diverge at least $z = z_0$ and perhaps as far out as some distance $r$. In addition, note that Laurent series coefficients need not come from evaluation of contour integrals (which may be very intractable). Other techniques, such as ordinary series expansions, may provide the coefficients.

**Example:** Let $f(z) = [z(z-1)]^{-1}$. Find the Laurent expansion about $z_0 = 0$.

**Solution:** Here, $r > 0$ and $R < 1$. These limitations arise because $f(z)$ diverges both at $z = 0$ and $z = 1$. A partial fraction expansion, followed by the binomial expansion of $(1-z)^{-1}$, yields the Laurent series

$$\frac{1}{z(z-1)} = -\frac{1}{1-z} - \frac{1}{z} = -\frac{1}{z} - 1 - z - z^2 - \cdots = -\sum_{n=-1}^{\infty} z^n.$$

We have

$$a_n = \frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)} = \begin{cases} -1 & \text{for } n \geq -1, \\ 0 & \text{for } n < -1, \end{cases}$$

$$\because a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz',$$

where the contour is counterclockwise in the annular region between $z' = 0$ and $|z'| = 1$.

The required expansion is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-1}^{\infty} a_n z^n = -\sum_{n=-1}^{\infty} z^n.$$

**Example:** Find the Laurent expansions for $f(z) = [z(z-2)]^{-1}$ in the regions

(a) $0 < |z| < 2$, (b) $2 < |z| < \infty$.

**Solution:**

(a)

$$f(z) = \frac{1}{z(z-2)} = -\frac{1}{2}\frac{1}{z}\left(\frac{1}{1 - z/2}\right) = -\frac{1}{2}\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \qquad \left[\because |z| < 2;\ \left|\frac{z}{2}\right| < 1\right]$$

$$\Rightarrow f(z) = -\frac{1}{2z}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right) \Rightarrow f(z) = -\frac{1}{2z} - \frac{1}{2} - \frac{z}{2^3} - \frac{z^2}{2^4} - \cdots$$

(b)

$$f(z) = \frac{1}{z(z-2)} = \frac{1}{z^2(1 - 2/z)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n, \qquad \left[\because |z| > 2;\ \left|\frac{2}{z}\right| < 1\right]$$

$$\Rightarrow f(z) = \frac{1}{z^2}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right) \Rightarrow f(z) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{2^2}{z^4} + \cdots$$

**Example:** Find the Laurent expansion for $f(z) = \frac{1}{z(z-1)}$ in the region $|z| > 1$.

**Solution:**

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2(1 - 1/z)} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

**Example:** Find the Laurent expansion for $f(z) = \frac{e^z}{(z-2)^3}$ about $z = 2$.

**Solution:** Put $t = z - 2$. Then

$$f(z) = \frac{e^{t+2}}{t^3} = \frac{e^2}{t^3}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right] = e^2\left[\frac{1}{t^3} + \frac{1}{t^2} + \frac{1}{2!\,t} + \frac{1}{3!} + \cdots\right]$$

$$\Rightarrow f(z) = e^2\left[\frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + \frac{1}{2!\,(z-2)} + \frac{1}{3!} + \cdots\right].$$