Statement of the Theorem:

Cauchy's integral theorem states that:

If $f(z)$ is an analytic function at all points of a simply connected region in the complex plane and if C is a closed contour within that region, then

$$\oint_C f(z)dz = 0.$$

To clarify the above, we need the following definition:

- A region is simply connected if every closed curve within it can be shrunk continuously to a point that is within that region.

In everyday language, a simply connected region is one that has no holes. We also need to explain that the symbol $\oint$ will be used from now on to indicate an integral over a closed contour; a subscript (such as C) is attached when further specification of the contour is desired. Also note that for the theorem to apply, the contour must be <u>within</u> the region of analyticity. That means it cannot be on the boundary of the region.

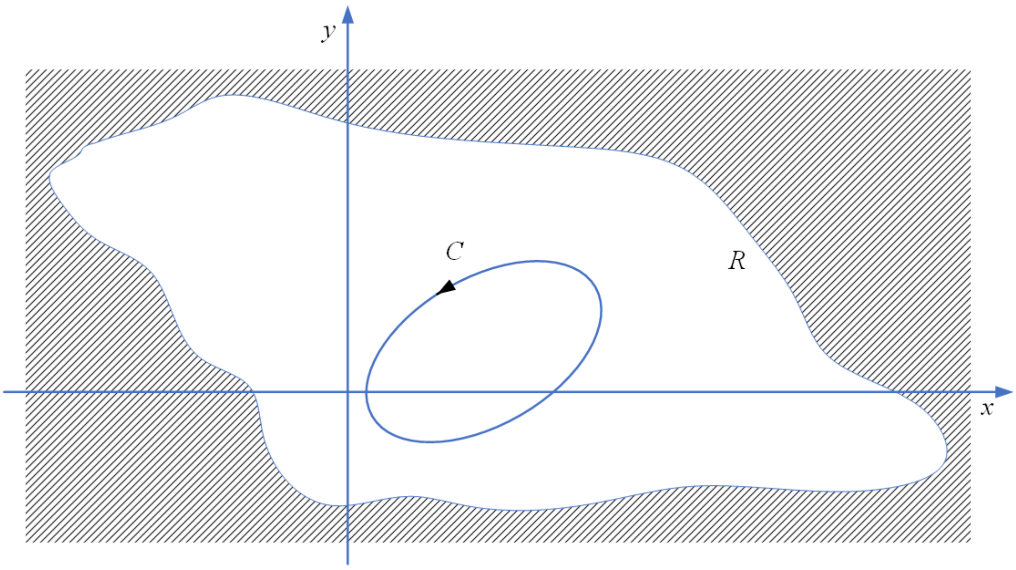

Figure: A closed contour C within a simply connected region R.

Example: z^n on circular contour.

Solution: Let's examine the contour integral $\oint_C z^n dz$, where C is a circle of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise). In polar coordinates, we parametrize the circle as $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. For integers $n \neq -1$, we then obtain

$$\oint_C z^n\, dz = ir^{n+1}\int_0^{2\pi} \exp[i(n+1)\theta]\, d\theta = ir^{n+1}\left.\frac{e^{i(n+1)\theta}}{i(n+1)}\right|_0^{2\pi} = 0 \tag{1}$$

because 2π is a period of $e^{i(n+1)\theta}$. However, for $n = -1$,

$$\oint_C \frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i \tag{2}$$

independent of r but nonzero.

The fact that equation (1) is satisfied for all integers $n \geq 0$ is required by Cauchy's theorem, because for these n values z^n is always analytic for all finite z, and certainly for all points within a circle of radius r.

Cauchy's theorem does not apply for any negative integer n because for these n, z^n is singular at $z = 0$. The theorem therefore does not prescribe any particular values for the integrals of negative n. We see that one such integral (that for $n \neq -1$) has a non-zero value, and that others (for integral $n \neq -1$) do vanish.

Example: z^n on square contour.

Solution: We next examine the integration of z^n for a different contour, a square with vertices at $\pm\frac{1}{2} \pm \frac{1}{2}i$. It is somewhat tedious to perform this integration for general integer n, so we illustrate only with $n = 2$ and $n = -1$.

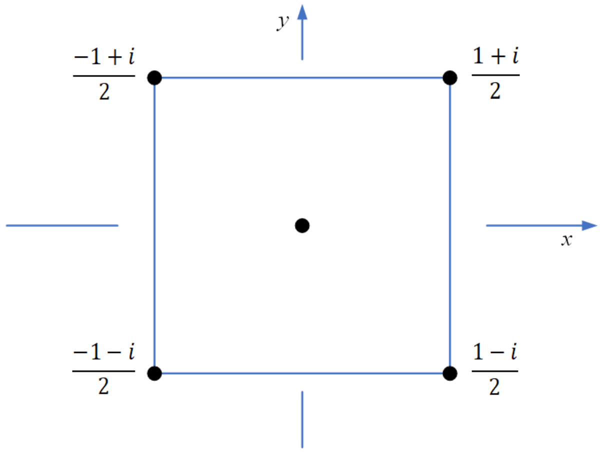

Figure: Square Integration Contour

For $n = 2$, we have $z^2 = x^2 - y^2 + 2ixy$. From the above figure, we identify the contour as consisting of four line segments. On segment 1, $dz = dx$ ($\because y = -1/2, \therefore dy = 0$); on segment 2, $dz = idy, x = 1/2, dx = 0$; on segment 3, $dz = dx, y = 1/2, dy = 0$; and on segment 4, $dz = idy, x = -1/2, dx = 0$. Note that for segments 3 and 4, the integration is in the direction of decreasing values of the integration variable. These segments therefore contribute as following to the integral:

Segment 1: $$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\left(x^2 - \frac{1}{4} - ix\right) = \frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{4} - \frac{i}{2}(0) = -\frac{1}{6}.$$

Segment 2: $$\int_{-\frac{1}{2}}^{\frac{1}{2}} idy\left(\frac{1}{4} - y^2 + iy\right) = \frac{i}{4} - \frac{i}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{2}(0) = \frac{i}{6}.$$

Segment 3: $$\int_{\frac{1}{2}}^{-\frac{1}{2}} dx\left(x^2 - \frac{1}{4} + ix\right) = -\frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] + \frac{1}{4} - \frac{i}{2}(0) = \frac{1}{6}.$$

Segment 4: $$\int_{\frac{1}{2}}^{-\frac{1}{2}} idy\left(\frac{1}{4} - y^2 - iy\right) = -\frac{i}{4} + \frac{i}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{2}(0) = -\frac{i}{6}.$$

We find that the integral of z^2 over the square vanishes, just as it did over the circle. This is required by Cauchy's theorem.

For $n = -1$, we have, in Cartesian coordinates,

$$z^{-1} = \frac{x - iy}{x^2 + y^2} \,,$$

and the integral over the four segments of the square takes the form:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x + i/2}{x^2 + 1/4} dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1/2 - iy}{y^2 + 1/4} (idy) + \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{x - i/2}{x^2 + 1/4} dx + \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{1/2 + iy}{y^2 + 1/4} (idy).$$

Several of the terms vanish because they involve the integration of an odd integrand over an even interval, and others simply cancel. All that remains is

$$\int_{\square} z^{-1}\, dz = i \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{x^2 + 1/4} = 2i \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 2\pi i \,,$$

the same result as was obtained for the integration of z^{-1} around a circle of any radius. Cauchy's theorem does not apply here, so the non-zero result is not problematic.