Example: Multiple Branch Points:

Consider the function

$$f(z) = (z^2 - 1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}.$$

The first factor on the right-hand side, $(z+1)^{1/2}$, has a branch point at $z = -1$. The second factor has a branch point at $z = +1$. At infinity, $f(z)$ has simple pole. This is best seen by substituting $z = 1/t$ and making a binomial expansion at $t = 0$:

$$(z^2 - 1)^{1/2} = \frac{1}{t}(1 - t^2)^{1/2} = \frac{1}{t}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n t^{2n} = \frac{1}{t} - \frac{1}{2}t - \frac{1}{8}t^3 + \cdots.$$

We want to make $f(z)$ single-valued by making appropriate branch cut(s). There are many ways to accomplish this, but one we wish to investigate is the possibility of making a branch cut from $z = -1$ to $z = +1$, as shown in figure below:

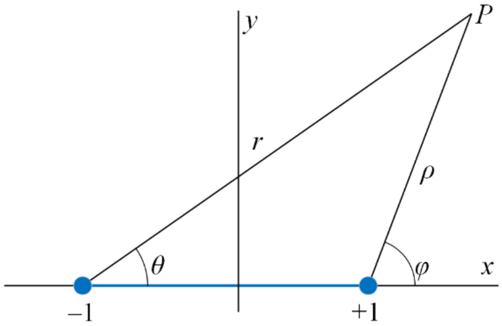

Figure: Possible branch cut and the quantities relating a point *P* as the branch points.

To determine whether this branch cut makes our $f(z)$ single-valued, we need to see what happens to each of the multivalent factors in $f(z)$ as we move around on its Argand diagram. Figure also identifies the quantities that are relevant for this purpose, namely those that relate a point *P* to the branch points. In particular, we have written the position relative to the branch point at $z = 1$ as $z - 1 = \rho e^{i\varphi}$ with the position relative to $z = -1$ denoted $z + 1 = re^{i\theta}$.

With these definitions, we have

$$f(z) = r^{1/2}\rho^{1/2}e^{(\theta+\varphi)/2}.$$

Our mission is to note how φ and θ change as we move along the path, so that we can use the correct value of each for evaluating $f(z)$.

We consider a closed path starting at point *A* in figure below, proceeding via points *B* through *F*, then back to *A*. At the start point, we choose $\theta = \varphi = 0$, thereby causing the multivalued $f(z_A)$ to have the specific value $+\sqrt{3}$. As we pass **above** $z = +1$ on the way to point *B*, θ remains essentially zero, but φ increases from 0 to π. These angles do not change as we pass from *B* to *C*, but on going to point *D*, θ increases to π, and then, passing **below** $z = -1$ on the on the way to point *E*, it further increases to 2π (not zero!). Meanwhile, φ remains essentially at π. Finally, returning to point *A* **below** $z = +1$, φ increases to 2π so that upon the return to point *A* both φ and θ have become 2π.

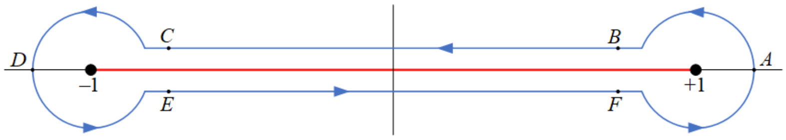

Figure: Path around the branch cut

The behaviour of these angles and the values of $(\theta + \varphi)/2$ (the argument of $f(z)$) are tabulated in Table given below:

Table:

Points	θ	φ	$(\theta + \varphi)/2$
A	0	0	0
B	0	π	$\pi/2$
C	0	π	$\pi/2$
D	π	π	π
E	2π	π	$3\pi/2$
F	2π	π	$3\pi/2$
A	2π	2π	2π

Two features emerge from this analysis:

1. The phase of $f(z)$ at points *B* and *C* is not the same as that at points *E* and *F*. This behaviour can be expected at a branch cut.
2. The phase of $f(z)$ at point *A'* (the return to *A*) exceeds that at point *A* by 2π, meaning that the function $f(z) = (z^2 - 1)^{1/2}$ is **single-valued** for the contour shown, encircling both branch points.

What actually happened is that each of the two multivalued factors contributed a sign change upon passage around the closed loop, so the two factors together restored the original sign of $f(z)$.