## Integrals with Complex Exponentials:

Consider the definite integral

$$I = \int_{-\infty}^{\infty} f(x)e^{iax}\,dx,$$

with $a$ being real and positive.

We assume the following two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles.
- $\lim\limits_{|z|\to\infty} f(z) = 0,\ 0 \le \arg(z) \le \pi.$

Note that this is a less restrictive condition than the second condition imposed on $f(z)$ for our previous integration of $\int_{-\infty}^{\infty} f(x)dx$.

We again employ the half-circle contour. The application of the calculus of residues is the same as the example just considered. But here we have to work harder to show that the integral over the (infinite) semicircle goes to zero.

This integral becomes, for a semicircle of radius $R$,

$$I_R = \int_0^{\pi} f\left(Re^{i\theta}\right)e^{iaR(\cos\theta + i\sin\theta)}\,iRe^{i\theta}d\theta,$$

where the $\theta$ integration is over the upper half-plane, $0 \le \theta \le \pi$. Let $R$ be sufficiently large that $|f(z)| = \left|f\left(Re^{i\theta}\right)\right| < \varepsilon$ for all $\theta$ within the integration range. Our second assumption on $f(z)$ tells us that as $R \to \infty$, $\varepsilon \to 0$. Then

$$|I_R| \le \varepsilon R \int_0^{\pi} e^{-aR\sin\theta}\,d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta.$$

We now note that in the range $[0, \pi/2]$, $\frac{2}{\pi}\theta \le \sin\theta$, as is easily seen from figure below. Thus,

$$|I_R| \le 2\varepsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\,d\theta = 2\varepsilon R\frac{1 - e^{-aR}}{2aR/\pi} < \frac{\pi}{a}\varepsilon,$$

$\because \frac{2}{\pi}\theta \le \sin\theta$, showing that $\lim\limits_{R\to\infty} I_R = 0$.

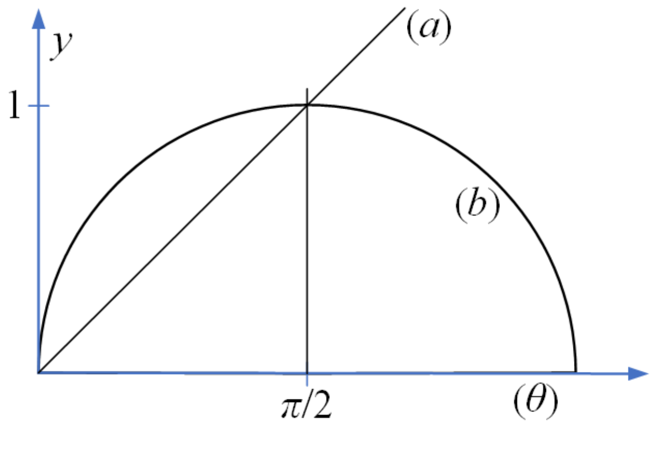


**Figure:** (a) $y = \frac{2}{\pi}\theta$ , (b) $y = \sin\theta$

This result is sometimes known as **Jordan's lemma**. Its formal statement is:

*If* $\lim_{R\to\infty} f(z) = 0$ *for all* $z = Re^{i\theta}$ *in the range* $0 \le \theta \le \pi$, *then*

$$\lim_{R\to\infty} \int_C e^{iaz} f(z) dz = 0,$$

*where* $a > 0$ *and* $C$ *is a semicircle of radius* $R$ *in the upper half-plane with centre at the origin*.

**Note** that for Jordan's lemma, the upper and the lower half-planes are not equivalent, because the condition $a > 0$ causes the exponent $-aR\sin\theta$ only to be negative and yield a negligible result in the upper half-plane. In the lower half-plane, the exponent is positive and the integral on a large semicircle would diverge there. Of course, we could extend the theorem by considering the case $a < 0$, in which event the contour to be used would then be a semicircle in the lower half-plane. Thus, application of the residue theorem yields the general result (for $a > 0$):

$$\int_{-\infty}^{\infty} f(x) e^{iax}\, dx = 2\pi i \sum [\text{residues of } e^{iaz} f(z)]\,, \text{(upper half-plane)}$$

where we have used Jordan's lemma to set to zero the contribution to the contour integral from the large semicircle.

**Example:** Consider the integral

$$I = \int_0^{\infty} \frac{\cos x}{x^2+1} dx\,,$$

which we initially manipulate using $\cos x = \left(e^{ix} + e^{-ix}\right)/2$ , as follows:

$$I = \frac{1}{2}\int_0^{\infty} \frac{e^{ix}}{x^2+1} dx + \frac{1}{2}\int_0^{\infty} \frac{e^{-ix}}{x^2+1} dx = \frac{1}{2}\int_0^{\infty} \frac{e^{ix}}{x^2+1} dx + \frac{1}{2}\int_0^{-\infty} \frac{e^{ix}}{(-x)^2+1} d(-x)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx\,,$$

thereby bringing $I$ to the form presently under discussion.

We now note that in this problem, $f(z) = 1/(z^2+1)$, which certainly approaches zero for large $|z|$, and the exponential factor is of the form $e^{iaz}$, with $a = +1$. The quantity whose residues are needed is

$$\frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)},$$

and we note that the exponential, an entire function, contributes no singularities. So, our singularities are simple poles at $z = \pm i$. Only the pole at $z = +i$ lies within the contour, and its residue is $e^{i^2}/2i$, which reduces to $1/2ie$. Our integral therefore has the value

$$I = \frac{1}{2}(2\pi i)\frac{1}{2ie} = \frac{\pi}{2e}.$$

Our next example is an important integral, the evaluation of which involves the principal-value concept and a contour that apparently needs to go through a pole.

**Example:** Singularities on the contour of integration: We now consider the evaluation of

$$I = \int_0^{\infty} \frac{\sin x}{x} dx\,.$$

Writing the integrand as$\left(e^{iz} - e^{-iz}\right)/2iz$, an attempt to do as we did in previous example leads to the problem that each of the two integrals into which $I$ can be separated is individually divergent. This is a problem we have already encountered in discussing the Cauchy principal value of this integral. We write $I$ as

$$I = P\int_{-\infty}^{\infty} \frac{e^{ix}dx}{2ix},$$

suggesting that we consider the integration of $e^{iz}/2iz$ over a suitable closed contour.

We now note that although the gap at $x = 0$ is infinitesimal, that point is a pole of $e^{iz}/2iz$, and we must draw a contour which avoids it, using a small semicircle to connect the points at $-\delta$ and $+\delta$. Choosing the small semicircle **above** the pole, as in figure below, we then have a contour that encloses **no** singularities.

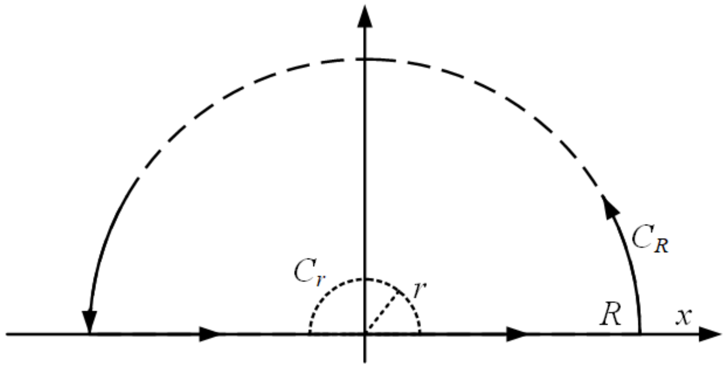


**Figure:** A contour including a Cauchy principal value integral.

The integral around this contour can now be identified as consisting of

(1) the two semi-infinite segments constituting the principal value integral,
(2) the large semicircle $C_R$ of radius $R$ ($R \to \infty$), and
(3) a semicircle $C_r$, of radius $r$ ($r \to 0$), traversed **clockwise**, so

$$\oint \frac{e^{iz}}{2iz}dz = I + \int_{C_r} \frac{e^{iz}}{2iz}dz + \int_{C_R} \frac{e^{iz}}{2iz}dz = 0.$$

By Jordan's lemma, the integral over $C_R$ vanishes. As discussed, the clockwise path $C_r$ halfway around the pole at $z = 0$, contributes half the value of a full circuit namely (allowing for the clockwise direction of travel) $-\pi i$ times the residue of $e^{iz}/2iz$ at $z = 0$. This residue has value $1/2i$, so

$$\int_{C_r} \frac{e^{iz}}{2iz}dz = -\pi/2.$$

Thus,

$$\oint \frac{e^{iz}}{2iz}dz = I + \int_{C_r} \frac{e^{iz}}{2iz}dz + \int_{C_R} \frac{e^{iz}}{2iz}dz = 0 \Longrightarrow I = -\int_{C_r} \frac{e^{iz}}{2iz}dz = \frac{\pi}{2}.$$

For $I$, we then obtain

$$I = \int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Note that it was necessary to close the contour in the upper half plane. On a large circle in the lower half-plane, $e^{iz}$ becomes infinite and Jordan's lemma cannot be applied.