Taylor Expansion and Laurent Expansion:

Taylor Expansion:

The Cauchy integral formula of the preceding section opens up the way for another derivation of Taylor's series for functions of complex variables. Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the Argand diagram for which $f(z)$ is not analytic. We construct a circle C centred at $z = z_0$ with radius less than $|z_1 - z_0|$. Since z_1 is assumed to be the nearest point at which $f(z)$ is not analytic, $f(z)$ is necessarily analytic on and within C.

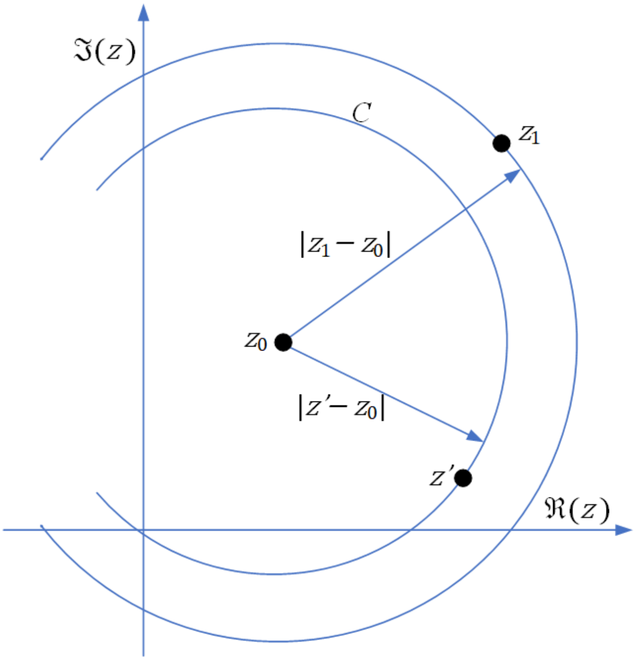

Figure: Circular domains for Taylor expansion.

From the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{z'-z} = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)-(z-z_0)}$$

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}.$$

Here z' is a point on the contour C and z is any point interior to C. It is not legal yet to expand the denominator of the integrand by the binomial theorem, for we have not yet proved the binomial theorem for complex variables. Instead, we note the identity

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n,$$

which may easily be verified by multiplying both sides by $1 - t$. This infinite series is convergent for $|t| < 1$.

Now, for a point z interior to C, $|z - z_0| < |z' - z_0|$, and so

$$f(z) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')dz'}{(z'-z_0)^{n+1}}.$$

Interchanging the order of integration and summation, which is valid because $\frac{1}{1-t}$ is uniformly convergent for $|t| < 1 - \varepsilon$, with $0 < \varepsilon < 1$, we obtain

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}}$$

$$\Longrightarrow \qquad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n ,$$

$$\left[\because f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}\right]$$

which is our desired Taylor expansion.

When $z_0 = 0$, the Taylor series becomes **McLaurin series**.

It is important to note that our derivation not only produces the expansion given in $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$; but also shows that this expansion converges when $|z - z_0| < |z_1 - z_0|$. For this reason, the circle defined by $|z - z_0| = |z_1 - z_0|$ is called the **circle of convergence** of our Taylor series.

Alternatively, the distance $|z_1 - z_0|$ is sometimes referred to as the **radius of convergence** of the Taylor series. In view of the earlier definition of z_1, we can say that:

The Taylor series of a function $f(z)$ about any interior point z_0 of a region in which $f(z)$ is analytic is a unique expansion that will have a radius of convergence equal to the distance from z_0 to the singularity of $f(z)$ closest to z_0, meaning that the Taylor series will converge **within** this circle of convergence. The Taylor series may or may not converge at individual points on the circle of convergence.

Example: Expand $1/(1 - z)$ in a Taylor series about $z_0 = i$.

Solution: We have

$$f(z) = (1 - z)^{-1}$$

$$\Longrightarrow f^{(n)}(z) = n!\,(1 - z)^{-(n+1)}$$

Hence, the required expansion is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(i)}{n!} (z - i)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z - i)^n}{(1 - i)^{n+1}}$$

Example: Expand $f(z) = \ln(1 + z)$ in a Taylor series about $z_0 = 0$.

Solution: We have $f(0) = \ln 1 = 0$. Also, $f^{(n)}(z) = (-1)^{n-1}(n - 1)!/(1 + z)^n$, for $n \geq 1$. Hence, we have

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots .$$