Integrals, Range $(-\infty, \infty)$:

Consider now definite integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)dx \,,$$

where it is assumed that

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles. For the moment it will be assumed that *there are no poles on the real axis*. Cases not satisfying this condition will be considered later.
- In the limit $|z| \to \infty$ in the upper half-plane $(0 \leq \arg z \leq \pi)$, $f(z)$ vanishes more strongly than $1/z$.

Note that there is nothing unique about the upper half-plane. The method described here can be applied, with obvious modifications, if $f(z)$ vanishes sufficiently strongly on the lower half-plane.

The second assumption stated above makes it useful to evaluate the contour integral $\oint f(z)dz$ on the contour shown in the figure below, because the integral I is given by the integration along the real axis, while the arc, of radius R, with $R \to \infty$, gives negligible contribution to the contour integral.

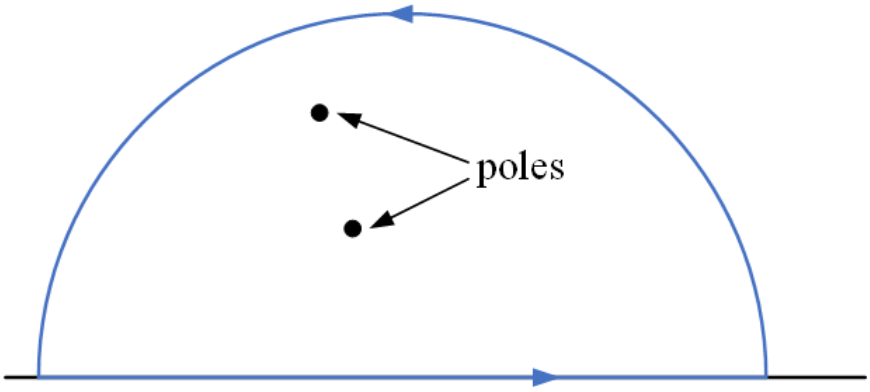

Figure: A contour closed by a large semicircle in the upper half-plane.

Thus,

$$I = \oint f(z)dz \,,$$

and the contour integral can be evaluated by applying the residue theorem.

Situations of this sort are of frequent occurrence and we therefore formalise the conditions under which the integral over a large arc becomes negligible:

If $\lim\limits_{R\to 0} zf(z) = 0$ for all $z = Re^{i\theta}$ with θ in the range $\theta_1 \leq \theta \leq \theta_2$, then

$$\lim_{R\to\infty} \int_C f(z)dz = 0,$$

where C is the arc over which the angular range θ_1 and θ_2 on a circle of radius R with centre at the origin. To prove

$$\lim_{R\to\infty} \int_C f(z)dz = 0,$$

we simply write the integral over C in polar form:

$$\lim_{R\to\infty}\left|\int_C f(z)dz\right| \le \int_{\theta_1}^{\theta_2} \lim_{R\to\infty}\left|f\left(Re^{i\theta}\right)iRe^{i\theta}\right| d\theta \le (\theta_2-\theta_1)\lim_{R\to\infty}\left|f\left(Re^{i\theta}\right)Re^{i\theta}\right| = 0.$$

Now, using the contour, letting C denote the semicircular arc from $\theta = 0$ to $\theta = \pi$.

$$\oint f(z)dz = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_C f(z)dz$$
$$= 2\pi i \sum [\text{residues (Upper half-plane)}],$$

where our second assumption has caused the vanishing of the integral over C.

Note that we have equally well have closed the contour with a semicircle in the lower half-plane, as $zf(z)$ vanishes on that arc as well as that in the upper half-plane. Then, taking the contour so that real axis is traversed from $-\infty$ to ∞, the path would be clockwise as shown below:

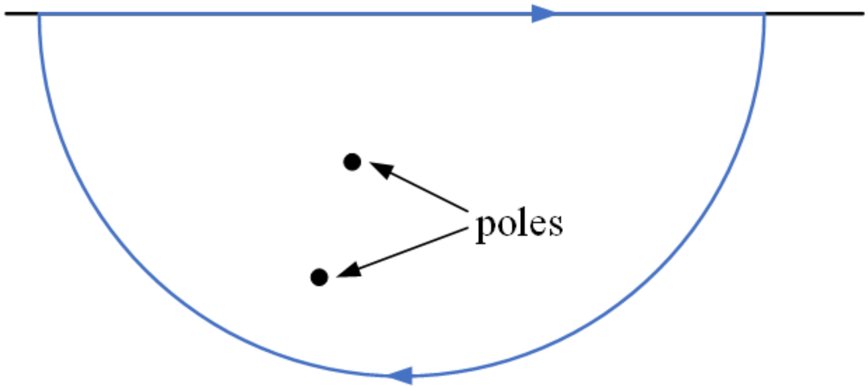

Figure: A contour closed by a large semicircle in the lower half-plane.

Example: Evaluate

$$I = \int_0^\infty \frac{dx}{1+x^2}.$$

This is not in the form we require, but it can be made so by noting that the integrand is even and we can write

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

We note that $f(z) = 1/(1+z^2)$ is meromorphic, all its singularities for finite z are poles, and it also has the property that $zf(z)$ vanishes in the limit of large $|z|$. Therefore, we may write

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}(2\pi i)\sum\left[\text{residues of } \frac{1}{z+z^2} \text{ (upper half-plane)}\right].$$

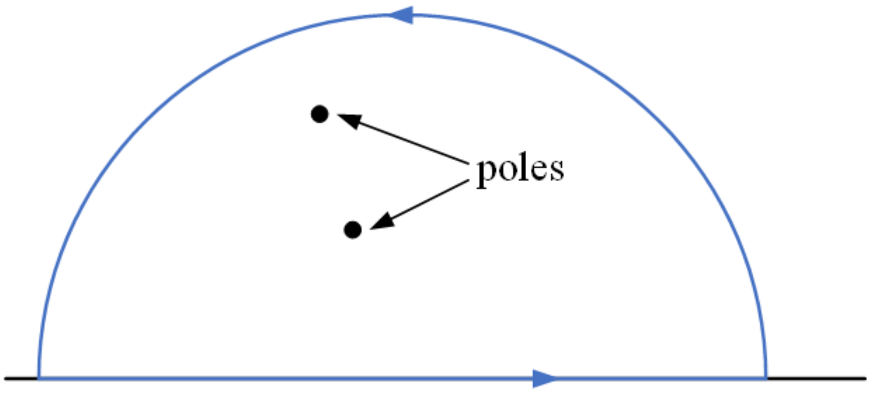

Figure: A contour closed by a large semicircle in the upper half-plane.

Here and in every other similar problem, we have the question: Where are the poles? Rewriting the integrand as

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)},$$

we see that there are simple poles (order 1) at $z = i$ and $z = -i$. The residues are:

At $z = i$:

$$\frac{1}{(z+i)(z-i)}(z-i)\bigg|_{z=i} = \frac{1}{2i},$$

and at $z = -i$:

$$\frac{1}{(z+i)(z-i)}(z+i)\bigg|_{z=-i} = -\frac{1}{2i}.$$

However, only the pole at $z = i$ is enclosed by the contour, so our result is

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}(2\pi i)\frac{1}{2i} = \frac{\pi}{2}.$$

Note that we could equally well have closed the contour with a semicircle in the lower half-plane, as $zf(z)$ vanishes on that arc as well as that in the upper half-plane. Then, taking the contour so the real axis is traversed from $-\infty$ to ∞, the path would be clockwise. So we would need to take $-2\pi i$ times the residue of the pole that is now encircled at $(z = -i)$. Thus, we have $I = -\frac{1}{2}(2\pi i)(-1/2i)$, which (as it must) evaluates to the same result we obtained previously, namely $\pi/2$.