Cauchy-Riemann Conditions:

Having established complex functions of a complex variable, we now proceed to differentiate them. The derivative of $f(z)$, like that of a real function, is defined by

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{(z+\delta z) - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z) \tag{1}$$

provided that the limit is independent of the particular approach to the point z. For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$. Now, with z (or z_0) some point in the Argand plane, our requirement that the limit be independent of the direction of approach is very restrictive.

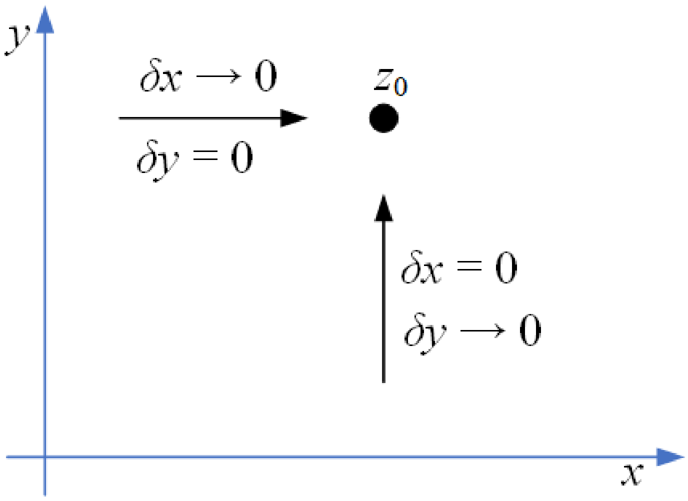
Consider increments δx and δy of the variables x and y, respectively. Then

$$\delta z = \delta x + i\delta y.$$

Also, writing $f = u + iv$, $\delta f = \delta u + i\delta v$, so that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}.$$

Let us take the limit indicated by (1) from two different approaches, as shown in the figure below:



First, with $\delta y = 0$, we let $\delta x \to 0$. Equation (1) yields

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2}$$

assuming that the partial derivatives exist.

For a second approach, we set $\delta x = 0$ and let $\delta y \to 0$. This leads to

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{3}$$

If we are to have a derivative $\frac{df}{dz}$, equations (2) and (3) must be identical. Equating their corresponding real and imaginary parts (like the components of a vector), we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

These are the famous **Cauchy-Riemann** conditions. The were discovered by Cauchy and used extensively by Riemann in his development of complex variable theory. These Cauchy-Riemann conditions are necessary for the existence of a derivative of $f(z)$. That is in order for $\frac{df}{dz}$ to exist, the Cauchy-Riemann conditions must hold.

Conversely, if the Cauchy-Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative $\frac{df}{dz}$ exists.

To show this, we start by writing

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y,$$

where the justification for this expression depends on the continuity of the partial derivatives of u and v. Using the Cauchy-Riemann equations,

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y)$$

Replacing $\delta x + i\delta y$ by δz and bringing it to the left-hand side, we reach

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},$$

an equation whose right-hand side is independent of the direction of δz (i.e., the relative values of δx and δy). This independence of directionality meets the condition for the existence of the derivative, $\frac{df}{dz}$.