Complex Variables and Functions:

Our focus in the present chapter is on functions of a complex variable and on their analytical properties. We know that by defining a complex function $f(z)$ to have the same power series expansion (in z) as the expansion (in x) of the corresponding real function $f(x)$, the real and complex definitions coincide when z is real.

We also know that by use of the polar representation, $z = re^{i\theta}$, we can compute powers and roots of complex quantities. In particular, we noted that roots, viewed as fractional powers, become **multivalued** functions in the complex domain, due to the fact that $\exp(2n\pi i) = 1$ for all positive and negative integers n. We thus found $z^{1/2}$ to have two values (not a surprise, since for positive real x, we have $\pm\sqrt{x}$). But we also noted that $z^{1/m}$ will have m different complex values. We also noted that logarithm becomes multivalued when extended to complex values, with

$$\ln z = \ln\left(re^{i\theta}\right)i = \ln r + i(\theta + 2n\pi),$$

with n any positive or negative integer (including 0).

Cauchy-Riemann Conditions:

Having established complex functions of a complex variable, we now proceed to differentiate them. The derivative of $f(z)$, like that of a real function, is defined by

$$\lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{(z+\delta z) - z} = \lim_{\delta z\to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z) \qquad (1)$$

provided that the limit is independent of the particular approach to the point z. For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$. Now, with z (or z_0) some point in the Argand plane, our requirement that the limit be independent of the direction of approach is very restrictive.

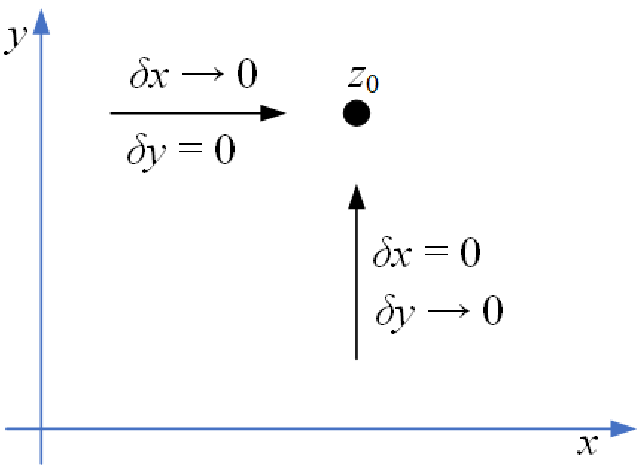
Consider increments δx and δy of the variables x and y, respectively. Then

$$\delta z = \delta x + i\delta y.$$

Also, writing $f = u + iv$, $\delta f = \delta u + i\delta v$, so that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}.$$

Let us take the limit indicated by (1) from two different approaches, as shown in the figure below:



First, with $\delta y = 0$, we let $\delta x \to 0$. Equation (1) yields

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \tag{2}$$

assuming that the partial derivatives exist.

For a second approach, we set $\delta x = 0$ and let $\delta y \to 0$. This leads to

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{3}$$

If we are to have a derivative $\frac{df}{dz}$, equations (2) and (3) must be identical. Equating their corresponding real and imaginary parts (like the components of a vector), we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

These are the famous **Cauchy-Riemann** conditions. The were discovered by Cauchy and used extensively by Riemann in his development of complex variable theory. These Cauchy-Riemann conditions are necessary for the existence of a derivative of $f(z)$. That is in order for $\frac{df}{dz}$ to exist, the Cauchy-Riemann conditions must hold.

Conversely, if the Cauchy-Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative $\frac{df}{dz}$ exists.

To show this, we start by writing

$$\delta f = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + i \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y,$$

where the justification for this expression depends on the continuity of the partial derivatives of u and v. Using the Cauchy-Riemann equations,

$$\delta f = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(-\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right) \delta y = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (\delta x + i \delta y)$$

Replacing $\delta x + i\delta y$ by δz and bringing it to the left-hand side, we reach

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \,,$$

an equation whose right-hand side is independent of the direction of δz (i.e., the relative values of δx and δy). This independence of directionality meets the condition for the existence of the derivative, $\frac{df}{dz}$.

Analytic Functions:

If $f(z)$ is differentiable and single-valued in a region of the complex plane, it is said to be an **analytic** function in that region. Multivalued functions can also be analytic under certain restrictions that make them single-valued in specific regions. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function. If $f'(z)$ does not exist at $z = z_0$, then z_0 is labelled a **singular point**.

Example: Check the analyticity of z^2.

Solution: Let $f(z) = z^2 = x^2 - y^2 + 2ixy \Longrightarrow u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Check:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

We see that $f(z) = z^2$ satisfies the Cauchy-Riemann conditions throughout the complex plane. Since the partial derivatives are clearly continuous, we conclude that $f(z) = z^2$ is analytic and is an entire function.

Example: Check the analyticity of z^*.

Solution: Let $f(z) = z^*$, the complex conjugate of z. Thus, $u = x$ and $v = -y$. Applying the Cauchy-Riemann conditions, we see that

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1.$$

The Cauchy-Riemann conditions are not satisfied for any values of x and y and $f(z) = z^*$ is nowhere an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable in the complex plane.

The derivative of a real function of a real variable is essentially a local characteristic, in that it provides information about the function only in a local neighbourhood, for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far reaching implications, one of which is that the real and imaginary parts of our analytic function must separately satisfy Laplace's equation in two dimensions, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

To verify the above statement, we differentiate the first Cauchy-Riemann equation (4) with respect to x and the second with respect to y, obtaining

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y}, \ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\, \partial x}.$$

Combining these two equations, we easily reach

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5}$$

confirming that $u(x, y)$, the real part of a differentiable complex function, satisfies the Laplace equation. Either by recognizing that $f(z)$ is differentiable, so is $-if(z) = v(x, y) - iu(x, y)$, or by steps similar to those leading to equation (5), we can confirm that $v(x, y)$ also satisfies the two dimensional Laplace equation. Sometimes, u and v are referred to as **harmonic functions**.

The solutions $u(x, y)$ and $v(x, y)$ are complementary in that the curves of constant $u(x, y)$ make orthogonal intersections with the curves of constant $v(x, y)$. To confirm this, note that if (x_0, y_0) is on the curve $u(x, y) = c$, then $(x_0 + dx, y_0 + dy)$ is also on that curve if

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0,$$

meaning that the slope of the curve of constant u at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_u = -\frac{\partial u/\partial x}{\partial u/\partial y},$$

where the derivatives are to be evaluated at (x_0, y_0). Similarly, we can find that the slope of the curve of constant v at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_v = -\frac{\partial v/\,\partial x}{\partial v/\,\partial y} = \frac{\partial u/\,\partial y}{\partial u/\,\partial x}.$$

Comparing these equations, we note that at the same point, the slopes they describe are orthogonal (to check, verify that $dx_u dx_v + dy_u dy_v = 0$).

Finally, the global nature of our analytic function is also illustrated by the fact that it has not only a first derivative, but in addition, derivatives of all higher orders, a property which is not shared by real functions of a real variable.

Derivatives of Analytic Functions:

Working with the real and imaginary parts of an analytic function $f(z)$ is one way to take its derivative; an example of that approach is to use $\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. However, it is usually easier to use the fact that complex differentiation follows the same rules as those for real variables. As a first step in establishing this correspondence, note that if $f(z)$ is analytic, then

$$f'(z) = \frac{\partial f}{\partial x}$$

and that

$$[f(z)g(z)]' = \left(\frac{d}{dz}\right)[f(z)g(z)] = \left(\frac{\partial}{\partial x}\right)[f(z)g(z)]$$

$$= \left(\frac{\partial f}{\partial x}\right)g(z) + f(z)\left(\frac{\partial g}{\partial x}\right) = f'(z)g(z) + f(z)g'(z),$$

the familiar rule for differentiating a product. Also given that $\frac{dz}{dz} = \frac{\partial f}{\partial x} = 1$, we can easily establish that

$\dfrac{d(z^2)}{dz} = 2z$, and, by induction, $\dfrac{d(z^n)}{dz} = nz^{n-1}$.

Functions defined by power series will then have differentiation rules identical to those for the real domain. Functions not ordinarily defined by power series also have the same differentiation rules for the real domain, but that will need to be demonstrated case by case. Here is an example that illustrates the establishment of a derivative formula.

Example: Derivative of logarithm: We want to verify that

$$\frac{d}{dz}(\ln z) = \frac{1}{z}.$$

$$\because \ln z = \ln r + i\theta + 2n\pi i,$$

if we write $\ln z = u + iv$, then $u = \ln r$ and $v = \theta + 2n\pi$. To check whether $\ln z$ satisfies the Cauchy-Riemann conditions, we evaluate

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y} = \frac{y}{r^2},$$
$$\frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

The derivatives of r and θ with respect to x and y are obtained from the equations connecting Cartesian and polar coordinates. Except at $r = 0$, where the derivatives are undefined, the Cauchy-Riemann equations can be confirmed. Then, to obtain the derivative, we can simply apply

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$\Longrightarrow \frac{d}{dz}(\ln z) = \frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{r^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

Because $\ln z$ is multivalued, it will not be analytic except under conditions restricting it to single-valuedness in a specific region.

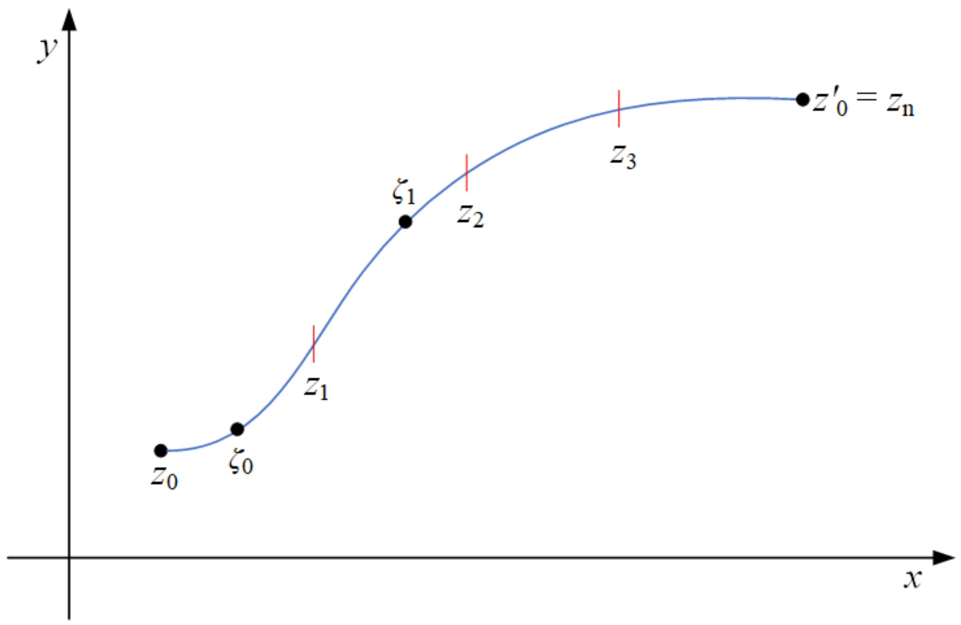
Point at Infinity:

In complex variable theory, infinity is regarded as a single point, and behaviour in its neighbourhood is discussed after making a change of variable from z to $w = \frac{1}{z}$. This transformation has the effect that, for example, $z = -R$, with R large, lies in the w-plane close to $z = +R$, thereby among other things influencing the values computed for derivatives. An elementary consequence is that entire functions, such as z or e^z, have singular points at $z = \infty$. As a trivial example, note that at infinity, the behaviour of z is identified as that of $\frac{1}{w}$ as $w \to 0$, leading to the conclusion that z is singular there.

Cauchy's Integral Theorem:

Contour Integrals:

The integral of a complex variable over a path in the complex plane (also known as a **contour**) may be defined in close analogy in the (Riemann) integral of a real function integrated along the real x-axis. We divide the contour, from z_0 to z_0', designated C, into n intervals by picking $n - 1$ intermediate points $z_1, z_2, \ldots$ on the contour as shown in the figure below:



Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}),$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with $|z_j - z_{j-1}| \to 0,\ \forall\ j$. If $\lim_{n\to\infty} S_n$ exists, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)dz = \int_C f(z)dz.$$

The right-hand side of this equation is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$). As an alternative to the above, the contour integral may be defined by

$$\int_{z_1}^{z_2} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y) + iv(x,y)][dx + idy]$$

$$\Rightarrow \int_{z_1}^{z_2} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y)dx - v(x,y)dy] + i \int_{(x_1,y_1)}^{(x_2,y_2)} [v(x,y)dx + u(x,y)dy]\,,$$

with the path joining (x_1, y_1) to (x_2, y_2) specified. This reduces the complex integral to the complex sum of real integrals. It is somewhat analogous to the replacement of a vector integral by the vector sum of scalar integrals.

Often, we are interested in contours that are **closed**, meaning that the start and the end of the contour are at the same point, so that the contour forms a closed loop. We normally define the region enclosed by a contour as that which lies to the left when the contour is traversed in the indicated direction; thus, a contour intended to surround a finite area will normally be deemed to be traversed in the counterclockwise direction. If the origin of a polar coordinate system is within the contour, this convention will cause the normal direction of travel on the contour to be that in which the polar angle θ increases.

Statement of the Theorem:

Cauchy's integral theorem states that:

If $f(z)$ is an analytic function at all points of a simply connected region in the complex plane and if C is a closed contour within that region, then

$$\oint_C f(z)dz = 0.$$

To clarify the above, we need the following definition:

- A region is simply connected if every closed curve within it can be shrunk continuously to a point that is within that region.

In everyday language, a simply connected region is one that has no holes. We also need to explain that the symbol $\oint$ will be used from now on to indicate an integral over a closed contour; a subscript (such as C) is attached when further specification of the contour is desired. Also note that for the theorem to apply, the contour must be within the region of analyticity. That means it cannot be on the boundary of the region.

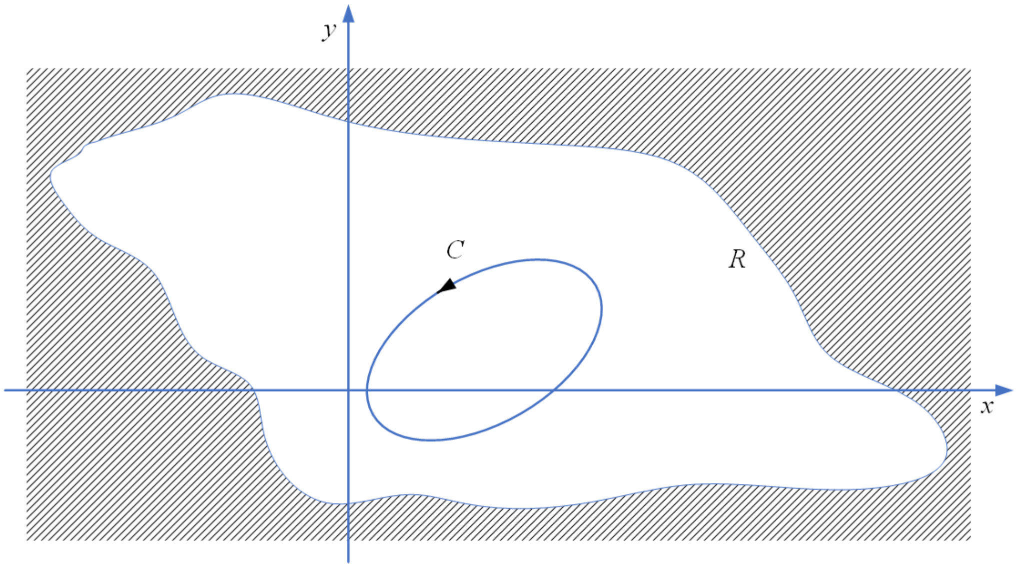


Figure: A closed contour C within a simply connected region R.

Example: z^n on circular contour.

Solution: Let's examine the contour integral $\oint_C z^n dz$, where C is a circle of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise). In polar coordinates, we parametrize the circle as $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. For integers $n \neq -1$, we then obtain

$$\oint_C z^n\, dz = ir^{n+1}\int_0^{2\pi} \exp[i(n+1)\theta]\, d\theta = ir^{n+1}\left.\frac{e^{i(n+1)\theta}}{i(n+1)}\right|_0^{2\pi} = 0 \tag{1}$$

because 2π is a period of $e^{i(n+1)\theta}$. However, for $n = -1$,

$$\oint_C \frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i \tag{2}$$

independent of r but nonzero.

The fact that equation (1) is satisfied for all integers $n \geq 0$ is required by Cauchy's theorem, because for these n values z^n is always analytic for all finite z, and certainly for all points within a circle of radius r.

Cauchy's theorem does not apply for any negative integer n because for these n, z^n is singular at $z = 0$. The theorem therefore does not prescribe any particular values for the integrals of negative n. We see that one such integral (that for $n \neq -1$) has a non-zero value, and that others (for integral $n \neq -1$) do vanish.

Example: z^n on square contour.

Solution: We next examine the integration of z^n for a different contour, a square with vertices at $\pm\frac{1}{2}\pm\frac{1}{2}i$. It is somewhat tedious to perform this integration for general integer n, so we illustrate only with $n = 2$ and $n = -1$.

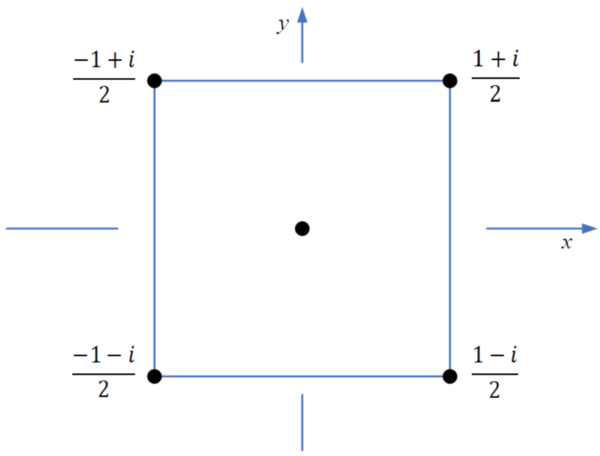


Figure: Square Integration Contour

For $n = 2$, we have $z^2 = x^2 - y^2 + 2ixy$. From the above figure, we identify the contour as consisting of four line segments. On segment 1, $dz = dx$ ($\because y = -1/2, \therefore dy = 0$); on segment 2, $dz = idy, x = 1/2, dx = 0$; on segment 3, $dz = dx, y = 1/2, dy = 0$; and on segment 4, $dz = idy, x = -1/2, dx = 0$. Note that for segments 3 and 4, the integration is in the direction of decreasing values of the integration variable. These segments therefore contribute as following to the integral:

Segment 1: $\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\left(x^2 - \frac{1}{4} - ix\right) = \frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{4} - \frac{i}{2}(0) = -\frac{1}{6}.$

Segment 2: $\int_{-\frac{1}{2}}^{\frac{1}{2}} idy\left(\frac{1}{4} - y^2 + iy\right) = \frac{i}{4} - \frac{i}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{2}(0) = \frac{i}{6}.$

Segment 3: $\int_{\frac{1}{2}}^{-\frac{1}{2}} dx\left(x^2 - \frac{1}{4} + ix\right) = -\frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] + \frac{1}{4} - \frac{i}{2}(0) = \frac{1}{6}.$

Segment 4: $\int_{\frac{1}{2}}^{-\frac{1}{2}} idy\left(\frac{1}{4} - y^2 - iy\right) = -\frac{i}{4} + \frac{i}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{2}(0) = -\frac{i}{6}.$

We find that the integral of z^2 over the square vanishes, just as it did over the circle. This is required by Cauchy's theorem.

For $n = -1$, we have, in Cartesian coordinates,

$$z^{-1} = \frac{x - iy}{x^2 + y^2},$$

and the integral over the four segments of the square takes the form:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x + i/2}{x^2 + 1/4} dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1/2 - iy}{y^2 + 1/4}(idy) + \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{x - i/2}{x^2 + 1/4} dx + \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{1/2 + iy}{y^2 + 1/4}(idy).$$

Several of the terms vanish because they involve the integration of an odd integrand over an even interval, and others simply cancel. All that remains is

$$\int_{\square} z^{-1}\,dz = i\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{x^2+1/4} = 2i\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = 2\pi i\,,$$

the same result as was obtained for the integration of z^{-1} around a circle of any radius. Cauchy's theorem does not apply here, so the non-zero result is not problematic.

Cauchy Integral Formula:

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. This means that the contour C is to be traversed in the **counterclockwise** direction. We seek to prove the following result, known as **Cauchy integral formula:**

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

in which z_0 is any point in the interior region bounded by C. Note that since z is on the contour while z_0 is in the interior, $z-z_0 \neq 0$ and the integral is well-defined. Although $f(z)$ is assumed analytic, the integrand $f(z)/(z-z_0)$ is not analytic at $z=z_0$ unless $f(z_0)=0$. We now deform the contour, to make it a circle of small radius r about $z=z_0$, traversed, like the original contour, in the counterclockwise direction. As shown in the preceding section, this does not change the value of the integral. We therefore write $z=z_0+re^{i\theta}$, so $dz = ire^{i\theta}d\theta$, the integration is from $\theta=0$ to $\theta=2\pi$, and

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f\left(z_0+re^{i\theta}\right)}{re^{i\theta}}\,ire^{i\theta}\,d\theta.$$

Taking the limit when $r\to 0$, we obtain

$$\oint_C \frac{f(z)}{z-z_0}\,dz = if(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0),$$

where we have replaced $f(z)$ by its limit $f(z_0)$ because it is analytic and therefore continuous at $z=z_0$. This proves the Cauchy integral formula.

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an **arbitrary interior point** $z=z_0$ once the values on the boundary C are specified.

It has been emphasized that z_0 is an interior point. What happens if z_0 is exterior to C? In this case the entire integrand is analytic on and within C, Cauchy integral theorem applies, and the integral vanishes. Summarizing, we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \begin{cases} f(z_0), & z_0 \text{ within the contour,} \\ 0, & z_0 \text{ is exterior to the contour.}\end{cases}$$

Example: Find

$$I = \oint_C \frac{dz}{z(z+2)}$$

where the integration is counterclockwise over the unit circle.

Solution: The factor $\frac{1}{(z+2)}$ is analytic within the region enclosed by the contour, so this is a case of Cauchy integral formula with $f(z) = \frac{1}{z+2}$ and $z_0 = 0$. The result is immediate:

$$I = 2\pi i \left[\frac{1}{z+2}\right]_{z=0} = \pi i.$$

Example: Find

$$I = \oint_C \frac{dz}{4z^2 - 1},$$

where the integration is counterclockwise over the unit circle.

Solution: The denominator factors into $4z^2 - 1 = 4(z - 1/2)(z + 1/2)$, and it is apparent that the region of integration contains two singular factors. However, we may still use Cauchy integral formula if we make the partial fraction expansion:

$$\frac{1}{4z^2 - 1} = \frac{1}{4}\left[\frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}}\right],$$

after which we integrate the two terms separately. We have

$$I = \frac{1}{4}\left[\oint_C \frac{dz}{z + \frac{1}{2}} - \oint_C \frac{dz}{z + \frac{1}{2}}\right]$$

Each integral is a case of Cauchy integral formula with $f(z) = 1$, and for both integrals, the point $z_0 = \pm 1/2$ is within the contour, so each evaluates to $2\pi i$, and their sum is zero. So $I = 0$.

Multiply Connected Regions:

Derivatives:

Cauchy's integral formula can be used to obtain an expression for the derivative of $f(z)$. Differentiating

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0} dz = f(z_0)$$

with respect to z_0, and interchanging the differentiation and the z integration,

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^2} dz.$$

Differentiating again,

$$f''(z_0) = \frac{2}{2\pi i}\oint \frac{f(z)}{(z - z_0)^3} dz.$$

Continuing, we get

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}} dz\,;$$

that is, the requirement that $f(z)$ be analytic guarantees not only a first derivative, but derivatives of all orders as well. The derivatives of $f(z)$ are automatically analytic.

Example: Find $I = \oint_C \frac{\sin^2 z}{(z-a)^4} dz$, where the integral is counterclockwise on a contour that encircles the point $z = a$.

Solution: This is a case of

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz\,;$$

with $n = 3$ and $f(z) = \sin^2 z$. Therefore,

$$I = \frac{2\pi i}{3} \left[\frac{d^3}{dz^3} \sin^2 z \right]_{z=a} = \frac{\pi i}{3} [-8 \sin z \cos z]_{z=a} = -\frac{8\pi i}{3} \sin a \cos a.$$

Morera's Theorem:

A further application of Cauchy's integral formula is in the proof of **Morera's theorem**, which is the converse of the Cauchy integral theorem. The theorem states the following:

If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.

Taylor Expansion:

Laurent Expansion (Laurent Series):

Sometimes we encounter functions that are analytic in an annular region, say between circles of inner radius r and outer radius R, about a point z_0, as shown in figure below:

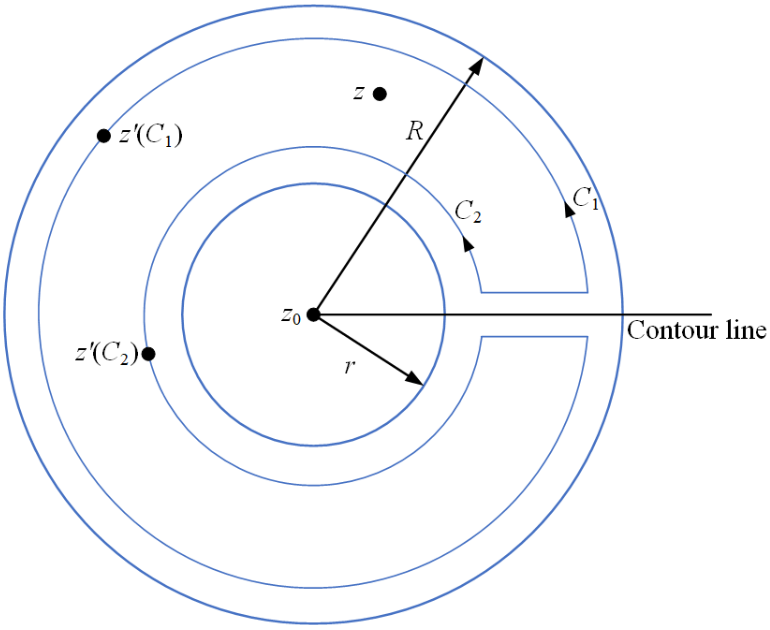


Figure: Annular region for Laurent series.
$|z' - z_0|_{C_1} > |z - z_0|$; $|z' - z_0|_{C_2} < |z - z_0|$.

We assume $f(z)$ to be such a function, with z a typical point in the annular region. Drawing an imaginary barrier to convert our region into a simply connected region, we apply Cauchy's integral formula to evaluate $f(z)$, using the contour shown in the figure. Note that the contour consists of the two circles centred at z_0, labelled C_1 and C_2 (which can be considered closed since the barrier is fictitious), plus segments on either side of the barrier whose contributions will cancel. We assign C_2 and C_1 the radii r_2 and r_1, respectively, where $r < r_2 < r_1 < R$. Then, from Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z' - z} dz' ,$$

Note that an explicit minus sign has been introduced so that the contour C_2 (like C_1) is to be traversed in the positive (counterclockwise) sense.

The treatment of the above equation now proceeds exactly like that of

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)\left[1 - \dfrac{(z - z_0)}{(z' - z_0)}\right]} dz'$$

in the development of the Taylor series. Each denominator is written as $(z' - z_0) - (z - z_0)$ expanded by the binomial theorem. Noting that for C_1, $|z' - z_0| > |z - z_0|$ while for C_2, $|z' - z_0| < |z - z_0|$, we find

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')}{(z' - z_0)^{n+1}} dz' + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z') dz'.$$

The minus sign has been absorbed by the binomial expansion. Labelling the first series by S_1 and the second series by S_2, we have

$$S_1 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')}{(z' - z_0)^{n+1}} dz' ,$$

which has the same form as the regular Taylor expansion, convergent for $|z - z_0| < |z' - z_0| = r_1$, that is, for all z **interior** to the larger circle C_1.

For the second series, we have

$$S_2 = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z') dz',$$

convergent for $|z - z_0| > |z' - z_0| = r_2$, that is, for all z **exterior** to the smaller circle C_2. Remember, C_2 now goes counterclockwise.

These two series are combined into one series, known as Laurent series, of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n , \qquad \text{where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'.$$

Since convergence of a binomial expansion is not relevant to the evaluation of a_n, C in that equation may be any contour within the annular region $r < |z - z_0| < R$ that encircles z_0 once in a counterclockwise sense. If such annular region of analyticity does exist, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent series, or Laurent expansion of $f(z)$.

The Laurent series differs from the Taylor series by the obvious feature of negative powers of $(z - z_0)$. For this reason, the Laurent series will always diverge at least $z = z_0$ and perhaps as far out as some distance r. In addition, note that Laurent series coefficients need not come from evaluation of contour integrals (which may be very intractable). Other techniques, such as ordinary series expansions, may provide the coefficients.

Example: Let $f(z) = [z(z-1)]^{-1}$. Find the Laurent expansion about $z_0 = 0$.

Solution: Here, $r > 0$ and $R < 1$. These limitations arise because $f(z)$ diverges both at $z = 0$ and $z = 1$. A partial fraction expansion, followed by the binomial expansion of $(1-z)^{-1}$, yields the Laurent series

$$\frac{1}{z(z-1)} = -\frac{1}{1-z} - \frac{1}{z} = -\frac{1}{z} - 1 - z - z^2 - \cdots = -\sum_{n=-1}^{\infty} z^n.$$

We have

$$a_n = \frac{1}{2\pi i} \oint \frac{dz'}{(z')^{n+2}(z'-1)} = \begin{cases} -1 & \text{for } n \geq -1, \\ 0 & \text{for } n < -1, \end{cases}$$

$$\because a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_0)^{n+1}} dz',$$

where the contour is counterclockwise in the annular region between $z' = 0$ and $|z'| = 1$.

The required expansion is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-1}^{\infty} a_n z^n = -\sum_{n=-1}^{\infty} z^n.$$

Example: Find the Laurent expansions for $f(z)$ in the regions

(a) $0 < |z| < 2$, (b) $2 < |z| < \infty$.

Solution:

(a)

$$f(z) = \frac{1}{z(z-2)} = -\frac{1}{2}\frac{1}{z}\left(\frac{1}{1-z/2}\right) = -\frac{1}{2}\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \qquad \left[\because |z| < 2; \left|\frac{z}{2}\right| < 1\right]$$

$$\Rightarrow f(z) = -\frac{1}{2z}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right) \Rightarrow f(z) = -\frac{1}{2z} - \frac{1}{2} - \frac{z}{2^3} - \frac{z^2}{2^4} - \cdots$$

(b)

$$f(z) = \frac{1}{z(z-2)} = \frac{1}{z^2(1-2/z)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n, \qquad \left[\because |z| > 2; \left|\frac{2}{z}\right| < 1\right]$$

$$\Rightarrow f(z) = \frac{1}{z^2}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right) \Rightarrow f(z) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{2^2}{z^4} + \cdots$$

Example: Find the Laurent expansion for $f(z) = \frac{1}{z(z-1)}$ in the region $|z| > 1$.

Solution:

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2(1-1/z)} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

Example: Find the Laurent expansion for $f(z) = \frac{e^z}{(z-2)^3}$ about $z = 2$.

Solution: Put $t = z - 2$. Then

$$f(z) = \frac{e^{t+2}}{t^3} = \frac{e^2}{t^3}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right] = e^2\left[\frac{1}{t^3} + \frac{1}{t^2} + \frac{1}{2!\,t} + \frac{1}{3!} + \cdots\right]$$

$$\Rightarrow f(z) = e^2\left[\frac{1}{(z-2)^3} + \frac{1}{(z-2)^2} + \frac{1}{2!\,(z-2)} + \frac{1}{3!} + \cdots\right].$$

Zeroes and Singularities:

Taylor series expansion for the function $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$\because f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z' - z_0)^{n+1}}\,dz' = a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Laurent series expansion for the function $f(z)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Zeroes:

Consider the above Taylor series expansion for the function $f(z)$. Here the function $f(z)$ is analytic within some region. If $f(z)$ vanishes at $z = z_0$, the point z_0 is said to be a zero of $f(z)$. If $a_0 = a_1 = \cdots = a_{m-1} = 0$ but $a_m \neq 0$, then Taylor expansion becomes

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n$$

In this case, $f(z)$ is said to have **a zero of order $\boldsymbol{m}$** at $z = z_0$. A zero of order 1 ($m = 1$) is called a **simple zero.** Note that for $m = 1$,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \\ &= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \end{aligned}$$

It is clear that $f(z_0) = 0$ and $f'(z_0) \neq 0$, indicate the existence of a simple pole for $f(z)$ at $z = z_0$.

Singularities (Poles):

Points at which the function $f(z)$ is not analytic is called **singular points (singularities).** For example, $z = 0$ is a singular point of the function $f(z) = \frac{1}{z}$.

We define a point z_0 as an **isolated singular point** of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighbouring points. There will therefore be a Laurent expansion about an isolated singular point, and one of the following statements will be true:

1. The most negative power of $z - z_0$ in the Laurent expansion of $f(z)$ about $z = z_0$ will be some finite power, $(z - z_0)^{-n}$, where n is integer, or
2. The Laurent expansion of $f(z)$ about $z = z_0$ will continue to negatively infinite powers of $z - z_0$.

In the first case, the singularity is called a **pole**, and is more specifically identified as a pole of **order $\boldsymbol{n}$**. A pole of order 1 is also called a **simple pole**.

The second case is not referred to as a "pole of infinite order," but is called an **essential singularity**.

One way to identify a pole of $f(z)$ without having available its Laurent expansion is to examine

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for various integers n. The smallest integer n for which this limit exists (i.e., is finite) gives the order of the pole at $z = z_0$. This rule follows directly from the form of the Laurent expansion.

Essential singularities are often identified directly from their Laurent expansions. For example,

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$$

clearly has an essential singularity at $z = 0$.

The behaviour of $f(z)$ as $z \to \infty$ is defined in terms of the behaviour of $f\left(\frac{1}{t}\right)$ as $t \to 0$. Consider the function

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

As $z \to \infty$, we replace z with $\frac{1}{t}$ to obtain

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, t^{2n+1}}.$$

It is clear that $\sin(1/t)$ has an essential singularity at $t = 0$, from which we conclude that $\sin z$ has an essential singularity at $z = \infty$. Note that although $0 \leq |\sin x| \leq 1 \;\; \forall \;\; x \in \mathbb{R}$, while $|\sin(\mathrm{iy})| = |i \sinh y| = |\sinh \mathrm{y}|$ increases exponentially without limit as y increases.

A function that is analytic throughout the finite complex plane except for isolated poles is called **meromorphic**. Examples are ratios of two polynomials, also $\tan z$ and $\cot z$. As previously mentioned, functions that have no singularities in the finite complex plane are called entire functions. Examples are $\exp z$, $\sin z$, $\cos z$.

In addition to the isolated singularities identified as poles or essential singularities, there are singularities uniquely associated with multivalued functions known as **branch points**. This topic will be discussed later in this chapter.

Example: Find the singularities of

(a) $\quad f(z) = \dfrac{1}{z} \qquad\qquad$ (b) $\quad f(z) = \dfrac{1}{\sin(1/z)}$.

Solution: (a) $z = 0$ is an isolated singular point for $f(z) = \frac{1}{z}$.

(b) The function $f(z) = \frac{1}{\sin(1/z)}$ has an isolated singularity when $z = 1/n\pi$ for $n = 1, 2, \ldots$. However, the origin $z = 0$ is not an isolated singular point as a result of $z = 1/n\pi$.

Example: Find the pole and its order for $f(z) = \frac{\sin z}{z^4}$.

Solution: Let us check

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for the smallest integer n such that this limit is finite. Clearly, $z = 0$ is a singular point $f(z)$. Thus,

$$\lim_{z \to 0} z^3 \frac{\sin z}{z^4} = 1$$

Thus, $f(z)$ has a pole of order 3 at $z = 0$.

Also,

$$f(z) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Thus, $f(z)$ has a pole of order 3 at $z = 0$.

Example: Classify the singular point of the function $f(z) = ze^{1/z}$.

Solution:

$$f(z) = ze^{1/z} = z\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right) \Longrightarrow f(z) = z + 1 + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots$$

There is an essential singularity at $z = 0$.

Example: Find the pole and its order for $f(z) = \frac{1}{z^2-1}$.

Solution: Here

$$f(z) = \frac{1}{z^2 - 1} = \frac{1}{(z+1)(z-1)}.$$

The poles are at $z = -1$ and $z = +1$.

Let us check

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for the smallest integer n such that this limit is finite and non-zero, where $z_0 = -1$ or $+1$. We get

$$\begin{cases} \lim\limits_{z \to 1} (z-1)^n \dfrac{1}{(z-1)(z+1)} = \dfrac{1}{2} \neq 0 \text{ for } n = 1\,, \\ \lim\limits_{z \to -1} (z+1)^n \dfrac{1}{(z-1)(z+1)} = -\dfrac{1}{2} \neq 0 \text{ for } n = 1\,. \end{cases}$$

Hence, $z = \pm 1$ are simple poles of the given function.

Example: Find the poles and their orders for

$$f(z) = \frac{1}{(z^2 + a^2)^2} \text{ , where } a > 0.$$

Solution: Here

$$f(z) = \frac{1}{(z^2 - a^2)^2} = \frac{1}{(z+ia)^2(z-ia)^2}.$$

The poles are at $z = -ia$ and $z = +ia$.

Let us check

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for the smallest integer n such that this limit is finite and non-zero, where $z_0 = -ia$ or $+ia$. We get

$$\begin{cases} \lim\limits_{z \to ia} (z-ia)^n \dfrac{1}{(z-ia)^2(z+ia)^2} = -\dfrac{1}{4a^2} \neq 0 \text{ for } n = 2\,, \\ \lim\limits_{z \to -ia} (z+ia)^n \dfrac{1}{(z-ia)^2(z+ia)^2} = -\dfrac{1}{4a^2} \neq 0 \text{ for } n = 2\,. \end{cases}$$

Hence, $z = \pm ia$ are poles of second order of the given function.

Residue Theorem:

If the Laurent expansion of a function,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is integrated term by term using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain

$$a_n \oint (z-z_0)^n \, dz = 0, n \neq -1,$$

$$a_{-1} \oint (z-z_0)^{-1} \, dz = 2\pi i a_{-1}$$

The integral of $(z-z_0)^n$ around any counterclockwise closed path C that encloses z_0 has, for any integer n, the values

$$\oint_C (z-z_0)^n \, dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

Thus, we have

$$\oint f(z)dz = 2\pi i a_{-1}.$$

The constant a_{-1}, the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion, is called the **residue** of $f(z)$ at $z = z_0$.

Now consider the evaluation of the integral, over a closed contour C, of a function that has isolated singularities at points $z_1, z_2, \dots$. We can handle this integral by deforming our contour as shown in the figure below. Cauchy's integral theorem then leads to

$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots = 0,$$

where C is in the positive, counterclockwise direction, but the contours C_1, C_2, …, that, respectively, encircle $z_1, z_2, \dots$, are clockwise. Thus, referring to $\oint f(z)dz = 2\pi i a_{-1}$, the integrals C_i about the individual isolated singularities have the values

$$\oint_{C_i} f(z)dz = -2\pi i a_{-1,\,i} \,.$$

where $a_{-1,\,i}$ is the residue obtained from the Laurent expansion about the singular point $z = z_i$. The negative sign comes from the clockwise integration. We have

$$\oint_C \mathrm{f(z)dz} = 2\pi \mathrm{i}\left(\mathrm{a}_{-1,\,1} + \mathrm{a}_{-1,\,2} + \cdots\right)$$
$$= 2\pi i(\text{sum of the enclosed residues})$$

This is the **residue theorem**. The problem of evaluating a set of contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points.

Computing Residues:

Cauchy's Principal Value:

Evaluation of Definite Integrals

We start with applications to integrals containing trigonometric functions, which we can often convert to forms in which the variable of integration (originally an angle) is converted into a complex variable z, with the integration integrals becoming a contour integral over the unit circle.

Trigonometric Integrals, Range $(\mathbf{0}, \mathbf{2\pi})$:

We consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta ,$$

where f is finite for all values of θ. We also require f to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single valued. We make a change of variable to

$$z = e^{i\theta}, dz = ie^{i\theta} d\theta,$$

with the range in θ, namely $(0, 2\pi)$, corresponding to $e^{i\theta}$ moving counterclockwise around the unit circle to form a closed contour. The new make the substitutions

$$d\theta = -i\frac{dz}{z}, \ \sin\theta = \frac{z - z^{-1}}{2i}, \ \cos\theta = \frac{z + z^{-1}}{2}.$$

Our integral then becomes

$$I = -i \oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{z},$$

with the path of integration, the unit circle. By the residue theorem,

$$I = (-i) 2\pi i \sum \text{(residues within the unit circle)}.$$

Note that we must use residues of f/z. Here are two preliminary examples.

Example 1: Integrand has $\cos\theta$ in the denominator:

Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1.$$

Thus,

$$I = -i \oint_{\text{unit circle}} \frac{dz}{z\left[1 + \left(\frac{a}{2}\right)(z + z^{-1})\right]} = -i\frac{2}{a} \oint \frac{dz}{z^2 + \left(\frac{2}{a}\right) z + 1}.$$

The denominator has roots

$$z_1 = -\frac{1 + \sqrt{1 - a^2}}{a} \text{ and } z_2 = -\frac{1 - \sqrt{1 - a^2}}{a}.$$

Noting that $z_1 z_2 = 1$, it is easy to see that z_2 is withing the unit circle and $z_1$is outside. Writing the integral in the form

$$\oint \frac{dz}{(z - z_1)(z - z_2)},$$

we see that the residue of the integral at $z = z_2$ is $1/(z_2 - z_1)$, so application of the residue theorem yields

$$I = -i\frac{2}{a} 2\pi i \frac{1}{z_2 - z_1}.$$

Inserting the values of z_1 and z_2, we obtain the final result

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \qquad |a| < 1.$$

Example2: Consider $\quad I = \int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 - 4\cos\theta}.$

Thus,

$$I = \oint \frac{\frac{1}{2}(z^2 + z^{-2})}{5 - 2(z + z^{-1})} \left(\frac{-idz}{z}\right) = \frac{i}{4} \oint \frac{(z^4 + 1)dz}{z^2 \left(z - \frac{1}{2}\right)(z - 2)}$$

where the integration is around the unit circle.

We see that the integrand has poles at $z = 0$ (of order 2) and (simple poles at) $z = 1/2$ and $z = 2$. Only the poles at $z = 0$ and $z = 1/2$ are within the contour.

The residue at $z = 0$:

$$\frac{d}{dz}\left[\frac{z^4 + 1}{\left(z - \frac{1}{2}\right)(z - 2)}\right]\Bigg|_{z=0} = \frac{5}{2},$$

while its residue at $z = 1/2$ is

$$\frac{z^4 + 1}{z^2(z - 2)}\bigg|_{z=1/2} = -\frac{17}{6}.$$

Applying the residue theorem, we have

$$I = \frac{i}{4}(2\pi i)\left[\frac{5}{2} - \frac{17}{6}\right] = \frac{\pi}{6}.$$

Integrals, Range $(-\infty, \infty)$:

Consider now definite integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)dx,$$

where it is assumed that

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles. For the moment it will be assumed that *there are no poles on the real axis*. Cases not satisfying this condition will be considered later.
- In the limit $|z| \to \infty$ in the upper half-plane ($0 \le \arg z \le \pi$), $f(z)$ vanishes more strongly than $1/z$.

Note that there is nothing unique about the upper half-plane. The method described here can be applied, with obvious modifications, if $f(z)$ vanishes sufficiently strongly on the lower half-plane.

The second assumption stated above makes it useful to evaluate the contour integral $\oint f(z)dz$ on the contour shown in the figure below, because the integral I is given by the integration along the real axis, while the arc, of radius R, with $R \to \infty$, gives negligible contribution to the contour integral.

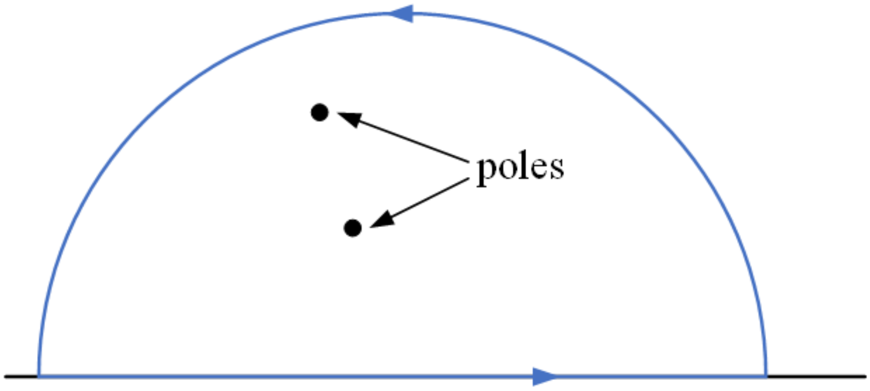


Figure: A contour closed by a large semicircle in the upper half-plane.

Thus,

$$I = \oint f(z)dz\,,$$

and the contour integral can be evaluated by applying the residue theorem.

Situations of this sort are of frequent occurrence, and we therefore formalise the conditions under which the integral over a large arc becomes negligible:

If $\lim\limits_{R\to 0} zf(z) = 0$ for all $z = Re^{i\theta}$ with θ in the range $\theta_1 \le \theta \le \theta_2$, then

$$\lim_{R\to\infty} \int_C f(z)dz = 0,$$

where C is the arc over which the angular range θ_1 and θ_2 on a circle of radius R with centre at the origin. To prove

$$\lim_{R\to\infty} \int_C f(z)dz = 0,$$

we simply write the integral over C in polar form:

$$\lim_{R\to\infty} \left|\int_C f(z)dz\right| \le \int_{\theta_1}^{\theta_2} \lim_{R\to\infty} \left|f\left(Re^{i\theta}\right)iRe^{i\theta}\right| d\theta \le (\theta_2 - \theta_1) \lim_{R\to\infty} \left|f\left(Re^{i\theta}\right)Re^{i\theta}\right| = 0.$$

Now, using the contour, letting C denote the semicircular arc from $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned}\oint f(z)dz &= \lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_C f(z)dz \\ &= 2\pi i \sum [\text{residues (Upper half-plane)}],\end{aligned}$$

where our second assumption has caused the vanishing of the integral over C.

Note that we have equally well have closed the contour with a semicircle in the lower half-plane, as $zf(z)$ vanishes on that arc as well as that in the upper half-plane. Then, taking the contour so that real axis is traversed from $-\infty$ to ∞, the path would be clockwise as shown below:

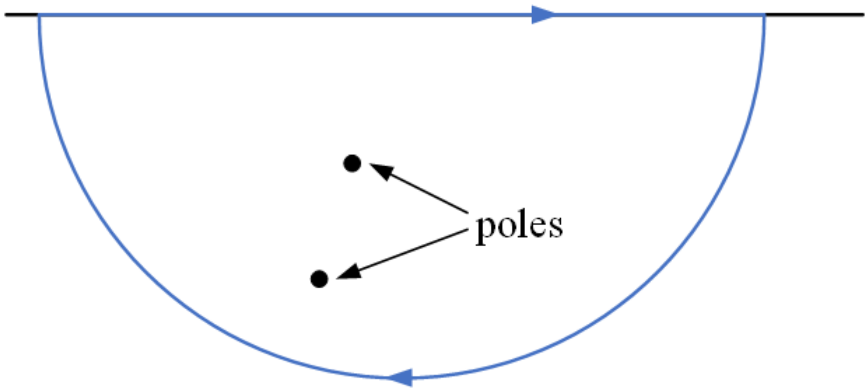


Figure: A contour closed by a large semicircle in the lower half-plane.

Example: Evaluate

$$I = \int_0^\infty \frac{dx}{1+x^2}.$$

This is not in the form we require, but it can be made so by noting that the integrand is even, and we can write

$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

We note that $f(z) = 1/(1+z^2)$ is meromorphic, all its singularities for finite z are poles, and it also has the property that $zf(z)$ vanishes in the limit of large $|z|$. Therefore, we may write

$$\frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{1}{2}(2\pi i)\sum\left[\text{residues of } \frac{1}{z+z^2} \text{ (upper half-plane)}\right].$$

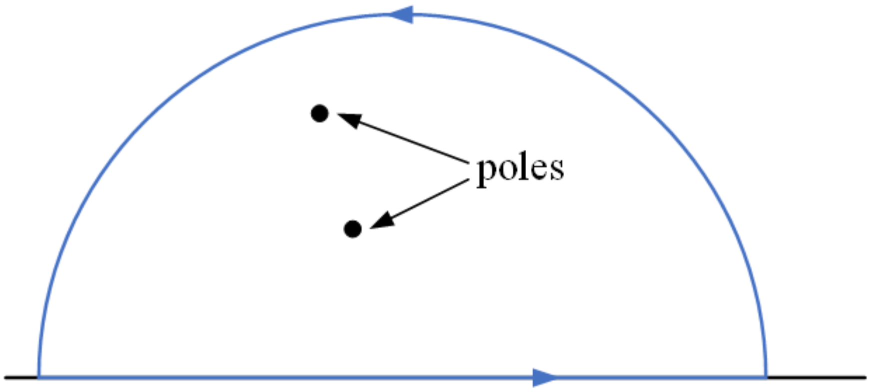


Figure: A contour closed by a large semicircle in the upper half-plane.

Here and in every other similar problem, we have the question: Where are the poles? Rewriting the integrand as

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)},$$

we see that there are simple poles (order 1) at $z = i$ and $z = -i$. The residues are:

At $z = i$:

$$\frac{1}{(z+i)(z-i)}(z-i)\bigg|_{z=i} = \frac{1}{2i},$$

and at $z = -i$:

$$\frac{1}{(z+i)(z-i)}(z+i)\bigg|_{z=-i} = -\frac{1}{2i}.$$

However, only the pole at $z = i$ is enclosed by the contour, so our result is

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}(2\pi i)\frac{1}{2i} = \frac{\pi}{2}.$$

Note that we could equally well have closed the contour with a semicircle in the lower half-plane, as $zf(z)$ vanishes on that arc as well as that in the upper half-plane. Then, taking the contour so the real axis is traversed from $-\infty$ to ∞, the path would be clockwise. So, we would need to take $-2\pi i$ times the residue of the pole that is now encircled at ($z = -i$). Thus, we have $I = -\frac{1}{2}(2\pi i)(-1/2i)$, which (as it must) evaluates to the same result we obtained previously, namely $\pi/2$.

Integrals with Complex Exponentials:

Consider the definite integral

$$I = \int_{-\infty}^{\infty} f(x)e^{iax}\,dx,$$

with a being real and positive.

We assume the following two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles.
- $\lim_{|z|\to\infty} f(z) = 0,\ 0 \le \arg(z) \le \pi.$

Note that this is a less restrictive condition than the second condition imposed on $f(z)$ for our previous integration of $\int_{-\infty}^{\infty} f(x)dx$.

We again employ the half-circle contour. The application of the calculus of residues is the same as the example just considered. But here we have to work harder to show that the integral over the (infinite) semicircle goes to zero.

This integral becomes, for a semicircle of radius R,

$$I_R = \int_0^\pi f\left(Re^{i\theta}\right)e^{iaR(\cos\theta + i\sin\theta)}\, iRe^{i\theta}d\theta,$$

where the θ integration is over the upper half-plane, $0 \le \theta \le \pi$. Let R be sufficiently large that $|f(z)| = \left|f\left(Re^{i\theta}\right)\right| < \varepsilon$ for all θ within the integration range. Our second assumption on $f(z)$ tells us that as $R \to \infty$, $\varepsilon \to 0$. Then

$$|I_R| \le \varepsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta.$$

We now note that in the range $[0, \pi/2]$, $\frac{2}{\pi}\theta \le \sin\theta$, as is easily seen from figure below. Thus,

$$|I_R| \le 2\varepsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta = 2\varepsilon R \frac{1 - e^{-aR}}{2aR/\pi} < \frac{\pi}{a}\varepsilon,$$

$\because \frac{2}{\pi}\theta \le \sin\theta$, showing that $\lim_{R\to\infty} I_R = 0$.

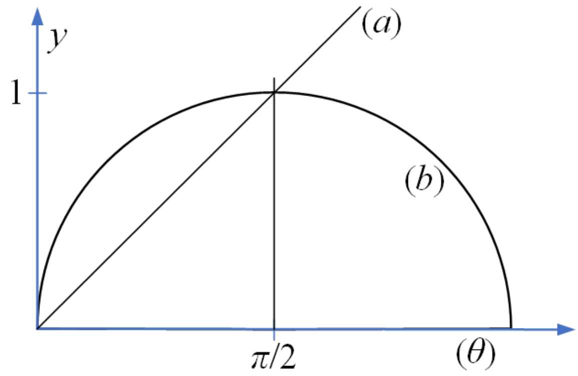


Figure: (a) $y = \frac{2}{\pi}\theta$, (b) $y = sin\,\theta$

This result is sometimes known as **Jordan's lemma**. Its formal statement is:

If $\lim_{R\to\infty} f(z) = 0$ *for all* $z = Re^{i\theta}$ *in the range* $0 \le \theta \le \pi$, *then*

$$\lim_{R\to\infty} \int_C e^{iaz} f(z) dz = 0,$$

where $a > 0$ *and* C *is a semicircle of radius* R *in the upper half-plane with centre at the origin*.

Note that for Jordan's lemma, the upper and the lower half-planes are not equivalent, because the condition $a > 0$ causes the exponent $-aR\sin\theta$ only to be negative and yield a negligible result in the upper half-plane. In the lower half-plane, the exponent is positive and the integral on a large semicircle would diverge there. Of course, we could extend the theorem by considering the case $a > 0$, in which event the contour to be used would then be a semicircle in the lower half-plane. Thus, application of the residue theorem yields the general result (for $a > 0$):

$$\int_{-\infty}^{\infty} f(x)e^{iax}\, dx = 2\pi i \sum \text{[residues of } e^{iaz} f(z)\text{]}, \text{ (upper half-plane)}$$

where we have used Jordan's lemma to set to zero the contribution to the contour integral from the large semicircle.

Example: Consider the integral

$$I = \int_0^{\infty} \frac{\cos x}{x^2 + 1} dx\,,$$

which we initially manipulate using $\cos x = \left(e^{ix} + e^{-ix}\right)/2$, as follows:

$$I = \frac{1}{2}\int_0^\infty \frac{e^{ix}}{x^2+1}dx + \frac{1}{2}\int_0^\infty \frac{e^{-ix}}{x^2+1}dx = \frac{1}{2}\int_0^\infty \frac{e^{ix}}{x^2+1}dx + \frac{1}{2}\int_0^{-\infty} \frac{e^{ix}}{(-x)^2+1}d(-x)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx\,,$$

thereby bringing I to the form presently under discussion.

We now note that in this problem, $f(z) = 1/(z^2+1)$, which certainly approaches zero for large $|z|$, and the exponential factor is of the form e^{iaz}, with $a = +1$. The quantity whose residues are needed is

$$\frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)},$$

and we note that the exponential, an entire function, contributes no singularities. So, our singularities are simple poles at $z = \pm i$. Only the pole at $z = +i$ lies within the contour, and its residue is $e^{i^2}/2i$, which reduces to $1/2ie$. Our integral therefore has the value

$$I = \frac{1}{2}(2\pi i)\frac{1}{2ie} = \frac{\pi}{2e}.$$

Our next example is an important integral, the evaluation of which involves the principal-value concept and a contour that apparently needs to go through a pole.

Example: Singularities on the contour of integration: We now consider the evaluation of

$$I = \int_0^\infty \frac{\sin x}{x}dx\,.$$

Writing the integrand as$\left(e^{iz} - e^{-iz}\right)/2iz$, an attempt to do as we did in previous example leads to the problem that each of the two integrals into which I can be separated is individually divergent. This is a problem we have already encountered in discussing the Cauchy principal value of this integral. We write I as

$$I = P\int_{-\infty}^{\infty} \frac{e^{ix}dx}{2ix},$$

suggesting that we consider the integration of $e^{iz}/2iz$ over a suitable closed contour.

We now note that although the gap at $x = 0$ is infinitesimal, that point is a pole of $e^{iz}/2iz$, and we must draw a contour which avoids it, using a small semicircle to connect the points at $-\delta$ and $+\delta$. Choosing the small semicircle **above** the pole, as in figure below, we then have a contour that encloses **no** singularities.

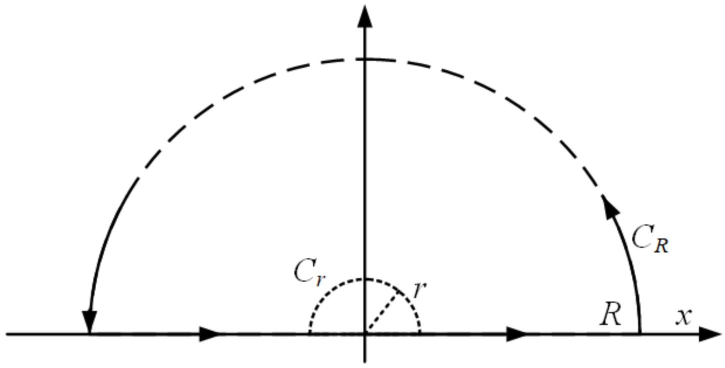


Figure: A contour including a Cauchy principal value integral.

The integral around this contour can now be identified as consisting of

(1) the two semi-infinite segments constituting the principal value integral,
(2) the large semicircle C_R of radius R ($R \to \infty$), and
(3) a semicircle C_r, of radius r ($r \to 0$), traversed **clockwise**, so

$$\oint \frac{e^{iz}}{2iz} dz = I + \int_{C_r} \frac{e^{iz}}{2iz} dz + \int_{C_R} \frac{e^{iz}}{2iz} dz = 0.$$

By Jordan's lemma, the integral over C_R vanishes. As discussed, the clockwise path C_r halfway around the pole at $z = 0$, contributes half the value of a full circuit namely (allowing for the clockwise direction of travel) $-\pi i$ times the residue of $e^{iz}/2iz$ at $z = 0$. This residue has value $1/2i$, so

$$\int_{C_r} \frac{e^{iz}}{2iz} dz = -\pi/2.$$

Thus,

$$\oint \frac{e^{iz}}{2iz} dz = I + \int_{C_r} \frac{e^{iz}}{2iz} dz + \int_{C_R} \frac{e^{iz}}{2iz} dz = 0 \Longrightarrow I = -\int_{C_r} \frac{e^{iz}}{2iz} dz = \frac{\pi}{2}.$$

For I, we then obtain

$$I = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Note that it was necessary to close the contour in the upper half plane. On a large circle in the lower half-plane, e^{iz} becomes infinite and Jordan's lemma cannot be applied.

Another Integration Technique:

Sometimes we have an integral on the real range $(0, \infty)$ that lacks the symmetry needed to extend the integration range to $(-\infty, \infty)$. However, it may be possible to identify a direction in the complex plane on which the integrand has a value identical to or conveniently related to that of the original integral, thereby permitting the construction of a contour facilitating the evaluation.

Example: Evaluation on a circular sector:

Our problem is to evaluate the integral

$$I = \int_0^\infty \frac{dx}{x^3 + 1},$$

which we cannot convert easily into an integral on the range $(-\infty, \infty)$. However, we note that along a line with argument $\theta = 2\pi/3$, z^3 will have the same values as at corresponding points on the real line; note that $\left(re^{2\pi i/3}\right)^3 = r^3 e^{2\pi i} = r^3$. We therefore consider

$$\oint \frac{dz}{z^3 + 1}$$

on the contour shown in the figure below. The part of the contour along the positive real axis labelled A, simply yields our integral I. The integrand approaches zero sufficiently rapidly for large $|z|$ that the integral on the large circular arc, labelled C in the figure, vanishes.

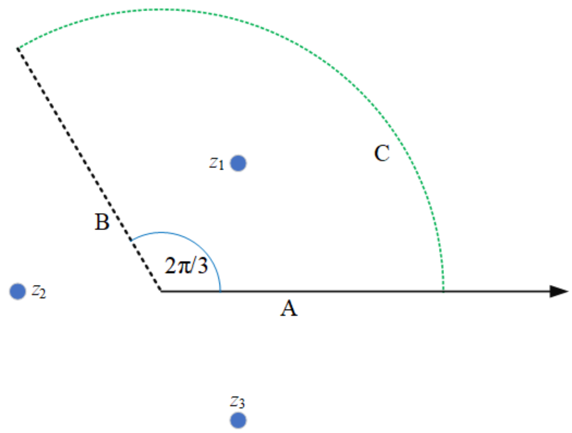


Fig. 1: A suitable contour for evaluation of I

On the remaining segment of the contour, labelled B, we note that $dz = e^{2\pi i/3} dr$, $z^3 = r^3$, and

$$\int_B \frac{dz}{z^3 + 1} = \int_\infty^0 \frac{e^{2\pi i/3}}{r^3 + 1} dr = -e^{2\pi i/3} \int_0^\infty \frac{dr}{r^3 + 1} = -e^{2\pi i/3} I.$$

Therefore,

$$\oint \frac{dz}{z^3 + 1} = \left(1 - e^{2\pi i/3}\right) I.$$

We now need to evaluate our complete contour integral using the residue theorem. The integrand has simple poles at the three roots of the equation $z^3 + 1 = 0$, which are at $z_1 = e^{\pi i/3}$, $z_2 = e^{\pi i}$, and $z_3 = e^{5\pi i/3}$, as marked in the figure. Only the pole at z_1 is enclosed by our contour. The residue at $z = z_1$ is

$$\lim_{z \to z_1} \frac{z - z_1}{z^3 + 1} = \left.\frac{1}{3z^2}\right|_{z = z_1} = \frac{1}{3e^{2\pi i/3}}.$$

Thus,

$$\left(1 - e^{2\pi i/3}\right) I = 2\pi i \left(\frac{1}{3e^{2\pi i/3}}\right).$$

Solution for I is facilitated if we multiply through by $e^{-\pi i/3}$, obtaining initially

$$\left(e^{-\pi i/3} - e^{\pi i/3}\right) I = 2\pi i \left(-\frac{1}{3}\right),$$

which is easily arranged to

$$I = \frac{\pi}{3\sin\pi/3} = \frac{\pi}{3\sqrt{3}/2} = \frac{2\pi}{3\sqrt{3}}.$$

Branch Points:

In addition to the isolated singularities identified as poles or essential singularities, there are *singularities uniquely associated with multivalued functions*. It is useful to work with these functions in ways that to the maximum possible extent remove ambiguity as to the function's values.

Thus, if at a point z_0 (at which $f(z)$ has a derivative), we have chosen a specific value of the multivalued functions $f(z)$, then we can assign to $f(z)$ values at nearby points in a way that causes continuity in $f(z)$. If we think of a succession of closely spaced points in the limit of zero spacing defining a path, our current observation is that a given values of $f(z_0)$ then leads to a unique definition of the value of $f(z)$ to be assigned to each point on the path. This scheme creates no ambiguity so long as the path is entirely open, meaning that the path does not return to any point previously passed. But if the path returns to z_0, thereby forming a **closed loop**, our prescription might lead, upon the return to a different one of the multiple values of $f(z_0)$.

Example: Value of $z^{1/2}$ on a closed loop:

We consider $f(z) = z^{1/2}$ on a path consisting of counterclockwise passage around the unit circle, starting and ending at $z = +1$. At the start point, where $z^{1/2}$ has multiple values $+1$ and -1, let us choose $f(z) = +1$. See figure below.

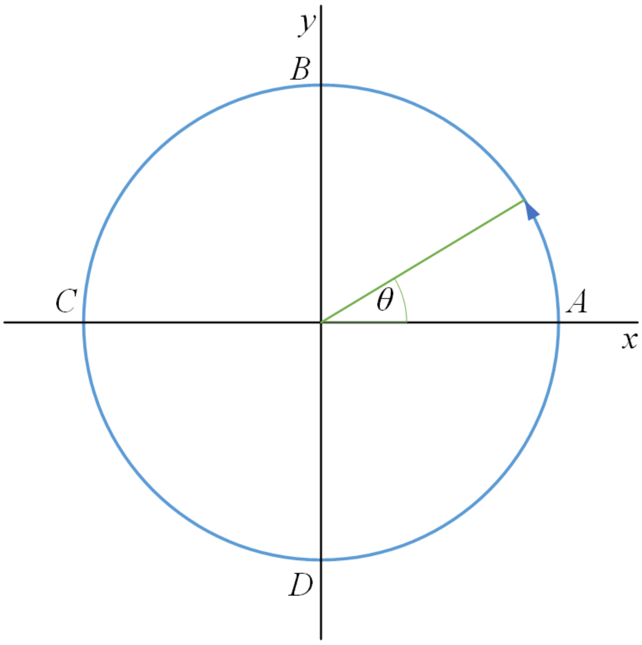


Fig. 1: Path encircling $z = 0$ for evaluation of $z^{1/2}$.

Writing $f(z) = e^{i\theta/2}$, we note that this form with $\theta = 0$ is consistent with the desired starting value $f(z)$, $+1$. In the figure, the start point is labelled A. Next, we note that the passage counterclockwise on the unit circle corresponds to an increase in θ, so that at the points marked B, C and D in the figure, the respective values of θ are $\pi/2$, π and $3\pi/2$. Counting further along the path, when we return to point A, the value of θ has become 2π (not 0). Then

$$f(z_B) = e^{i\theta_B/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$
$$f(z_C) = e^{i\theta_C/2} = e^{i\pi/2} = +i$$
$$f(z_D) = e^{i\theta_D/2} = e^{3i\pi/4} = \frac{-1+i}{\sqrt{2}}$$

When we return to point A, we have $f(+1) = e^{i\pi} = -1$, which is the other value of the multivalued function $z^{1/2}$.

If we continue for a second counterclockwise circuit of the unit circle, the value of θ would continue to increase, from 2π to 4π (reached when we arrive at point A after the second loop). We now have $f(+1) = e^{4\pi i/2} = e^{2\pi i} = 1$, so a second circuit has brought us back to the original value. It should now be clear that we are only going to be able to obtain two different values of $z^{1/2}$ for the same point z.

Example: Another closed loop:

Let us now see what happens to the function $z^{1/2}$ as we pass counterclockwise around a circle of unit radius centred at $z = +2$, starting and ending at $z = +3$. See figure below:

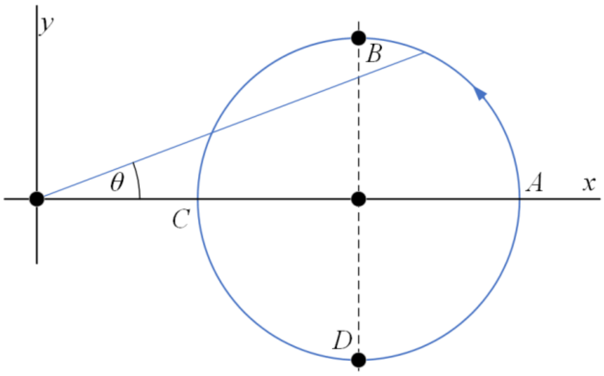


Figure: Path not encircling $z = 0$ for evaluation of $z^{1/2}$.

At $z = 3$, the values of $f(z)$ are $+\sqrt{3}$ and $-\sqrt{3}$; let us start with $f(z_A) = +\sqrt{3}$. As we move from point A through point B to point C, note from the figure that the value of θ first increases (actually, to 30°), and then decreases again to zero; further passage from C to D and back to A causes θ first to decrease (to –30°) and then to return to zero at A. So, in this example the closed loop does not bring us to a different value of the multivalued function $z^{1/2}$.

What is special about $z = 0$ is that (from a complex variable viewpoint) it is singular; the function $z^{1/2}$ does not have a derivative there. The lack of a well-defined derivative means that ambiguity in the function value will result from paths that circle such a singular point, which we will call a **branch point**.

The **order of a branch point** is defined as the number of paths around it that must be taken before the function involved returns to its original value; in the case of $z^{1/2}$, we saw that the branch point $z = 0$ is of order 2.

We are now ready to see what must be done to cause a multivalued function to be restricted to single-valuedness on a portion of the complex plane. We simply need to prevent its evaluation on paths that encircle a branch point.

We do so by drawing a line (known as a **branch line**, or more commonly, a **branch cut**) that the evaluation path cannot cross; the branch cut must start from our branch point and continue to infinity (or, if consistent with maintaining single-valuedness, to another finite branch point). The precise path of a branch cut can be chosen freely; what must be chosen appropriately are its endpoints.

Once appropriate branch cut(s) have been drawn, the originally multivalued function has been restricted to being single-valued in the region bounded by the branch cut(s); we call the function made single-valued in this way a branch of the original function. Since we could construct such a branch starting from any one of the values of the original function at a single arbitrary point in our region, we identify our multivalued function as having multiple branches. In the case of $z^{1/2}$, which is double-valued, the number of branches is two.

Note that a function with a branch point and a corresponding branch cut will not be continuous across the cut line. Hence line integrals in opposite directions on the two sides of the branch cut will not generally cancel each other. Branch cuts, therefore, are real boundaries to a region of analyticity, in contrast to the artificial barriers we introduced in extending Cauchy's integral theorem to multiply connected regions.

While from a fundamental viewpoint all branches of a multivalued function $f(z)$ are equally legitimate, it is often convenient to agree on the branch to be used, and such a branch is sometimes called the principal branch, with the value of $f(z)$ on that branch called its principal value. It is common to take the branch of $f(z)$ which is positive for real, positive z as its principal branch.

An observation that is important for complex analysis is that by drawing appropriate branch cut(s), we have restricted a multivalued function to single-valuedness, so that it can be an analytic function within the region bounded by the branch cut(s), and we can therefore apply Cauchy's two theorems to contour integrals within the region of analyticity.

Example: $\ln z$ has an infinite number of branches.

Here we examine the singularity structure of $\ln z$. The logarithm is multivalued, with the polar representation

$$\ln z = \ln\left(re^{i(\theta+2n\pi)}\right) = \ln r + i(\theta + 2n\pi),$$

where n can have **any** positive or negative integer value.

Noting that $\ln z$ is singular at $z = 0$ (it has no derivative there), we now identify $z = 0$ as a branch point. Let us consider what happens if we encircle it by a counterclockwise path on a circle of radius r, starting with the initial value $\ln r$, at $z = r = re^{i\theta}$, with $\theta = 0$. Every passage around the circle will add 2π to θ, and after n complete circuits the value we will have for $\ln z$ would be $\ln r + 2n\pi i$. The branch point of $\ln z$ at $z = 0$ is of infinite order, corresponding to the infinite number of its multiple values. By encircling $z = 0$ repeatedly in the clockwise direction, we can also reach all negative integer values of n.

We can make $\ln z$ single-valued by drawing a branch cut from $z = 0$ to $z = \infty$ in any way (though there is ordinarily no reason to use cuts that are not straight lines). It is typical to identify the branch cut with $n = 0$ as the principal branch of the logarithm. Incidentally, we note that the inverse trigonometric functions, which can be written as logarithms, will also be infinitely multivalued, with principal values that are usually chosen on a branch that will yield real values for real z. Compare the usual choices of the values assigned to the real-variable forms of $\sin^{-1} x = \arcsin x$, etc.

Using the logarithm, we are now in a position to look at the singularity structures of expressions of the form z^p, where both z and p may be complex. To do so, we write

$$z = e^{\ln z} \Longrightarrow z^p = e^{p \ln z},$$

which is single-valued if p is an integer, t-valued if p is a real rational fraction (in lowest terms) of the form s/t, and infinitely multivalued otherwise.

Note:

(a) If $f(z) = \ln(z - z_0)$, then z_0 is the branch point.

(b) If $f(z)$ is of the form $z^a(z-1)^b$, then:

(i) $z = 0$ is a branch point if a is not an integer.

(ii) $z = 1$ is a branch point if b is not an integer.

(iii) $z = \infty$ is a branch point if $a + b$ is not an integer.

(iv) There are no other branch points.

Example: Multiple Branch Points:

Consider the function

$$f(z) = (z^2 - 1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}.$$

The first factor on the right-hand side, $(z+1)^{1/2}$, has a branch point at $z = -1$. The second factor has a branch point at $z = +1$. At infinity, $f(z)$ has simple pole. This is best seen by substituting $z = 1/t$ and making a binomial expansion at $t = 0$:

$$(z^2-1)^{1/2} = \frac{1}{t}(1-t^2)^{1/2} = \frac{1}{t}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n\, t^{2n} = \frac{1}{t} - \frac{1}{2}t - \frac{1}{8}t^3 + \cdots.$$

We want to make $f(z)$ single-valued by making appropriate branch cut(s). There are many ways to accomplish this, but one we wish to investigate is the possibility of making a branch cut from $z = -1$ to $z = +1$, as shown in figure below:

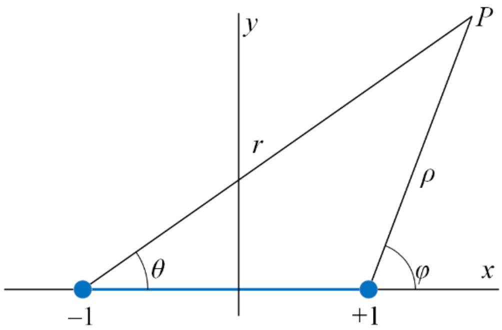


Figure: Possible branch cut and the quantities relating a point *P* as the branch points.

To determine whether this branch cut makes our $f(z)$ single-valued, we need to see what happens to each of the multivalent factors in $f(z)$ as we move around on its Argand diagram. Figure also identifies the quantities that are relevant for this purpose, namely those that relate a point P to the branch points. In particular, we have written the position relative to the branch point at $z = 1$ as $z - 1 = \rho e^{i\varphi}$ with the position relative to $z = -1$ denoted $z + 1 = re^{i\theta}$.

With these definitions, we have

$$f(z) = r^{1/2}\rho^{1/2}e^{(\theta+\varphi)/2}.$$

Our mission is to note how φ and θ change as we move along the path, so that we can use the correct value of each for evaluating $f(z)$.

We consider a closed path starting at point *A* in figure below, proceeding via points *B* through *F*, then back to *A*. At the start point, we choose $\theta = \varphi = 0$, thereby causing the multivalued $f(z_A)$ to have the specific value $+\sqrt{3}$. As we pass **above** $z = +1$ on the way to point *B*, θ remains essentially zero, but φ increases from 0 to π. These angles do not change as we pass from *B* to *C*, but on going to point *D*, θ increases to π, and then, passing **below** $z = -1$ on the on the way to point *E*, it further increases to 2π (not zero!). Meanwhile, φ remains essentially at π. Finally, returning to point *A* **below** $z = +1$, φ increases to 2π so that upon the return to point *A* both φ and θ have become 2π.

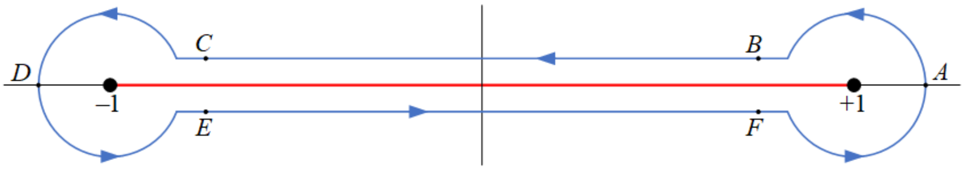


Figure: Path around the branch cut

The behaviour of these angles and the values of $(\theta + \varphi)/2$ (the argument of $f(z)$) are tabulated in Table given below:

Table:

Points	θ	φ	$(\theta + \varphi)/2$
A	0	0	0
B	0	π	$\pi/2$
C	0	π	$\pi/2$
D	π	π	π
E	2π	π	$3\pi/2$
F	2π	π	$3\pi/2$
A	2π	2π	2π

Two features emerge from this analysis:

1. The phase of $f(z)$ at points *B* and *C* is not the same as that at points *E* and *F*. This behaviour can be expected at a branch cut.
2. The phase of $f(z)$ at point *A'* (the return to *A*) exceeds that at point *A* by 2π, meaning that the function $f(z) = (z^2 - 1)^{1/2}$ is **single-valued** for the contour shown, encircling both branch points.

What actually happened is that each of the two multivalued factors contributed a sign change upon passage around the closed loop, so the two factors together restored the original sign of $f(z)$.

Evaluation of Definite Integrals with Branch Points:

Avoidance of branch points:

Sometimes we must deal with integrals whose integrands have branch points. In order to use contour integration methods of such integrals we must choose contours that avoid the branch points, enclosing only point singularities.

Example: Integral containing logarithm:

We now look at

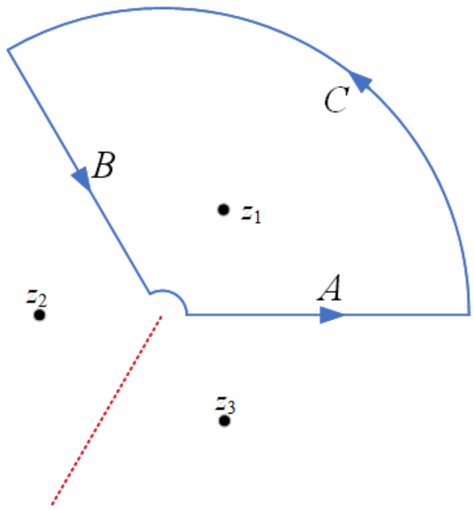
$$I = \int_0^\infty \frac{\ln x}{x^3 + 1} dx.$$

The integrand is singular at $x = 0$, but the integration converges (the indefinite integration of $\ln x$ is $x \ln x - x$). However, in the complex plane, this singularity manifests itself as a branch point, so if we are to recase this problem in a way involving a contour integral, we must avoid $z = 0$ and a branch cut from that point to $z = \infty$.

It turns out to be convenient to use a contour similar to that for previous example except that we must make a small circular detour about $z = 0$ and then draw the branch cut in a direction that remains outside our chosen contour. The integrand has simple poles at the three roots of $z^3 + 1$, which are at $z_1 = e^{\pi i/3}$, $z_2 = e^{\pi i}$ and $z_3 = e^{5\pi i/3}$, as marked in the figure. We consider a contour integral

$$\oint \frac{\ln z}{z^3 + 1} dz\,,$$

where the contour and the locations of the singularities of the integrand are as illustrated in the figure below:



The integral over the large circular arc, labelled C, vanishes, as the factor z^3 in the denominator dominates over the weakly divergent factor $\ln z$ in the numerator (which diverges more weakly than any positive power of z). We also get no contribution to the contour integral from the arc at small r, since we have there

$$\lim_{r\to 0} \int_0^{2\pi/3} \frac{\ln(re^{i\theta})}{1 + r^3 e^{3i\theta}} ire^{i\theta}\, d\theta\,,$$

which vanishes because $r \ln r \to 0$.

The integrals over the segment labelled A and B do not vanish. To evaluate the integral over these segments, we need to make an appropriate choice of the multivalued function $\ln z$. It is natural to choose the branch so that on the real axis we have $\ln z = \ln x$ (and not $\ln x + 2n\pi i$ for some nonzero n). Then the integral over the segment labelled A will have the value I (Note: Because the integral converges at $x = 0$, the value is not affected by the fact that this segment terminates infinitesimally before reaching that point).

To compute the integral over B, we note that on this segment $\theta = 2\pi/3$, $z = re^{i\theta} \Longrightarrow z^3 = r^3$ and $dz = e^{2\pi i/3}dr$, also but note that $\ln z = \ln r + 2\pi i/3$. There is little temptation here to use a different one of the multiple values of the logarithm, but for future reference note that we **must** use the value that is reached continuously from the value we already chose on the positive real axis, moving in a way that does not cross the branch cut. Thus, we cannot reach segment A by clockwise travel from the positive real axis (thereby getting $\ln z = \ln r - 4\pi i/3$) or any other value that would require multiple circuits around the branch point $z = 0$.

Based on the foregoing, we have

$$\int_B \frac{\ln z}{z^3+1}dz = \int_\infty^0 \frac{\ln r + 2\pi i/3}{r^3+1} e^{2\pi i/3}\, dr = -e^{\pi i/3}I - \frac{2\pi i}{3}e^{2\pi i/3}\int_0^\infty \frac{dr}{r^3+1}.$$

$$\Rightarrow \quad \oint \frac{\ln z}{z^3+1}dz = \left(1 - e^{2\pi i/3}\right)I - \frac{2\pi i}{3}e^{2\pi i/3}\left(\frac{2\pi}{3\sqrt{3}}\right). \qquad \left[\because \int_0^\infty \frac{dr}{r^3+1} = \frac{2\pi}{3\sqrt{3}}\right]$$

Our next step is to use the residue theorem to evaluate the contour integral. Only the pole at $z = z_1$ lies withing the contour. The residue we must compute is

$$\lim_{z\to z_1} \frac{(z-z_1)\ln z}{z^3+1} = \left.\frac{\ln z}{3z^2}\right|_{z=z_1} = \frac{\pi i/3}{3e^{2\pi i/3}} = \frac{\pi i}{9}e^{-2\pi i/3},$$

and application of the residue theorem yields

$$\left(1 - e^{2\pi i/3}\right)I - \frac{2\pi i}{3}e^{2\pi i/3}\left(\frac{2\pi}{3\sqrt{3}}\right) = 2\pi i\left(\frac{\pi i}{9}\right)e^{-2\pi i/3}.$$

Solving for I, we get

$$I = \frac{2\pi^2}{27}.$$