Computing Residues:

Working Rules:

(a) Obtain an entire Laurent expansion of $f(z)$ about $z = z_0$ to identify a_{-1}, the coefficient of $(z - z_0)^{-1}$ in the expansion.

(b) If $f(z)$ has a simple pole at $z = z_0$, then with a_n the coefficients in the expansion of $f(z)$,
$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots,$$
and, recognizing that $(z - z_0)f(z)$ may not have a form permitting an obvious cancellation of the factor $z - z_0$, we take the limit as $z \to z_0$:
$$a_{-1} = \lim_{z \to z_0}\big((z - z_0)f(z)\big).$$

(c) If there is a pole of order $n > 1$ at $z = z_0$, then $(z - z_0)^n f(z)$ must have the expansion
$$(z - z_0)^n f(z) = a_{-n} + \cdots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + \cdots.$$
We see that a_{-1} is the coefficient of $(z - z_0)^{n-1}$ in the Taylor expansion of $(z - z_0)^n f(z)$ and there we can identify it as satisfying
$$a_{-1} = \frac{1}{(n-1)!}\lim_{z \to z_0}\left[\frac{d^{n-1}}{dz^{n-1}}\big((z - z_0)^n f(z)\big)\right],$$
where a limit is indicated to take account of the fact that the expression involved may be indeterminate.
Note: In general, we can calculate the residue by taking any value of n which is greater than or equal to the order of the pole.

(d) Suppose the function $f(z)$ can be represented by
$$f(z) = \frac{g(z)}{h(z)}$$
where $g(z)$ and $h(z)$ are analytic functions. If $g(z_0) \neq 0$ and $h(z_0) = 0$ but $h'(z_0) \neq 0$, the function $1/h(z)$ has a simple pole at $z = z_0$ and $f(z)$ has a simple pole at $z = z_0$. The residue of $f(z)$ at $z = z_0$ is
$$a_{-1} = \lim_{z \to z_0}\left[\frac{g(z)}{h(z)}(z - z_0)\right] = \lim_{z \to z_0} g(z) \lim_{z \to z_0}\left[\frac{z - z_0}{h(z)}\right] = g(z_0)\left[\frac{1}{h'(z_0)}\right]$$
$$\Longrightarrow \qquad a_{-1} = \frac{g(z_0)}{h'(z_0)}.$$

(e) Sometimes, the general formula is found to be more complicated than the judicious use of power series expansion. See items 4 and 5 in the example below.

(f) Essential singularities will also have well-defined residues but finding them may be more difficult. In principle, one can use
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}}$$
with $n = -1$, but the integral involved may seem intractable. Sometimes the easiest route to the residue is by first finding the Laurent expansion.

Examples: Here are some examples:

1. The residue of $\dfrac{1}{4z+1}$ at $z = -\dfrac{1}{4}$ is $\displaystyle\lim_{z \to -\frac{1}{4}}\left(\frac{z + \frac{1}{4}}{4z+1}\right) = \frac{1}{4}$.

2. Residue of $\dfrac{1}{\sin z}$ at $z = 0$ is $\displaystyle\lim_{z \to 0}\left(\frac{z}{\sin z}\right) = 1$.

3. Residue of $\dfrac{\ln z}{z^2+4}$ at $z = 2i$ is

$$\lim_{z\to 2i}\left(\frac{(z-2i)\ln z}{z^2+4}\right) = \frac{\ln 2 + \pi i/2}{4i} = \frac{\pi}{8} - \frac{i\ln 2}{4}.$$

4. Residue of $\dfrac{z}{\sin^2 z}$ at $z = \pi$; the pole is second order, and the residue is given by

$$\frac{1}{1!}\lim_{z\to\pi}\left(\frac{d}{dz}\frac{z(z-\pi)^2}{\sin^2 z}\right).$$

However, it may be easier to make the substitution $w = z - \pi$, note that $\sin^2 z = \sin^2 w$, and to identify the residue as the coefficient of $1/w$ in the expansion of $(w+\pi)/\sin^2 w$ about $w = 0$. This expansion can be written

$$\frac{w+\pi}{\left(w - \dfrac{w^3}{3!} + \cdots\right)^2} = \frac{w+\pi}{w^2 - \dfrac{w^4}{3} + \cdots}.$$

The denominator expands entirely into even powers of w, so the π in the numerator cannot contribute to the residue. Then from the w in the numerator and the leading term of denominator, we find the residue to be 1.

5. Residue of $f(z) = \dfrac{\cot \pi z}{z(z+2)}$ at $z = 0$.

The pole at $z = 0$ is second-order, and direct application of

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

leads to a complicated indeterminate expression requiring multiple applications of L'Hôpital's rule. Perhaps easier is to introduce the initial terms of the expansion about $z = 0$:

$\cot \pi z = (\pi z)^{-1} + O(z)$, $\dfrac{1}{z+2} = \dfrac{1}{2}\left[1 - \dfrac{z}{2} + O(z^2)\right]$, reaching

$$f(z) = \frac{1}{z}\left[\frac{1}{\pi z} + O(z)\right]\left(\frac{1}{2}\right)\left[1 - \frac{z}{2} + O(z^2)\right],$$

from which we can read out the residue as the coefficient of z^{-1}, namely, $-1/4\pi$.

6. Residue of $e^{-1/z}$ at $z = 0$.

This is an essential singularity; from the Taylor series of e^w with $w = -1/z$, we have

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!}\left(-\frac{1}{z}\right)^2 + \cdots,$$

from which we read out the value of the residue, -1.

Examples: Calculate the residue(s) of the functions:

(a) $f(z) = \dfrac{1}{z^2-1}$

(b) $f(z) = \dfrac{1}{(z^2+a^2)^2}$, where $a > 0$

(c) $f(z) = \dfrac{\sin z}{z^4}$

(d) $f(z) = \dfrac{\sin z}{z^6}$

(e) $f(z) = ze^{1/z}$

(f) $f(z) = z^2 \sin\dfrac{1}{z}$

(g) $f(z) = \dfrac{1 + e^z}{\sin z + z\cos z}$

(h) $f(z) = \dfrac{1}{z(e^z - 1)}$

(i) $f(z) = \dfrac{A(z)}{\sin z}$, where $A(z)$is analytic and contains no zeros.