Derivatives of Analytic Functions:

Working with the real and imaginary parts of an analytic function $f(z)$ is one way to take its derivative; an example of that approach is to use $\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. However, it is usually easier to use the fact that complex differentiation follows the same rules as those for real variables. As a first step in establishing this correspondence, note that if $f(z)$ is analytic, then

$$f'(z) = \frac{\partial f}{\partial x}$$

and that

$$[f(z)g(z)]' = \left(\frac{d}{dz}\right)[f(z)g(z)] = \left(\frac{\partial}{\partial x}\right)[f(z)g(z)]$$
$$= \left(\frac{\partial f}{\partial x}\right)g(z) + f(z)\left(\frac{\partial g}{\partial x}\right) = f'(z)g(z) + f(z)g'(z),$$

the familiar rule for differentiating a product. Also given that $\frac{dz}{dz} = \frac{\partial f}{\partial x} = 1$, we can easily establish that

$\dfrac{d(z^2)}{dz} = 2z$, and, by induction, $\dfrac{d(z^n)}{dz} = nz^{n-1}$.

Functions defined by power series will then have differentiation rules identical to those for the real domain. Functions not ordinarily defined by power series also have the same differentiation rules for the real domain, but that will need to be demonstrated case by case. Here is an example that illustrates the establishment of a derivative formula.

Example: Derivative of logarithm: We want to verify that

$$\frac{d}{dz}(\ln z) = \frac{1}{z}.$$

$$\because \ln z = \ln r + i\theta + 2n\pi i,$$

if we write $\ln z = u + iv$, then $u = \ln r$ and $v = \theta + 2n\pi$. To check whether $\ln z$ satisfies the Cauchy-Riemann conditions, we evaluate

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y} = \frac{y}{r^2},$$
$$\frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

The derivatives of r and θ with respect to x and y are obtained from the equations connecting Cartesian and polar coordinates. Except at $r = 0$, where the derivatives are undefined, the Cauchy-Riemann equations can be confirmed. Then, to obtain the derivative, we can simply apply

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$\Rightarrow \frac{d}{dz}(\ln z) = \frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{r^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

Because $\ln z$ is multivalued, it will not be analytic except under conditions restricting it to single-valuedness in a specific region.