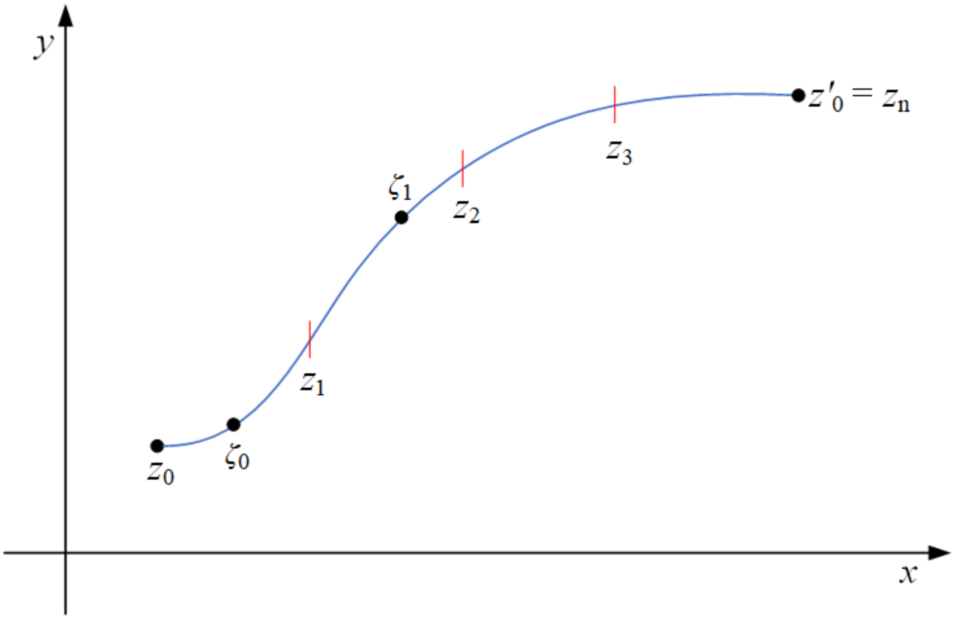
Cauchy's Integral Theorem:

Contour Integrals:

The integral of a complex variable over a path in the complex plane (also known as a **contour**) may be defined in close analogy in the (Riemann) integral of a real function integrated along the real x-axis. We divide the contour, from z_0 to z_0', designated C, into n intervals by picking $n-1$ intermediate points $z_1, z_2, \dots$ on the contour as shown in the figure below:



Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}),$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with $|z_j - z_{j-1}| \to 0,\ \forall\ j$. If $\lim_{n\to\infty} S_n$ exists, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)dz = \int_C f(z)dz.$$

The right-hand side of this equation is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$). As an alternative to the above, the contour integral may be defined by

$$\int_{z_1}^{z_2} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y) + iv(x,y)][dx + idy]$$

$$\Rightarrow \int_{z_1}^{z_2} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y)dx - v(x,y)dy] + i \int_{(x_1,y_1)}^{(x_2,y_2)} [v(x,y)dx + u(x,y)dy],$$

with the path joining (x_1, y_1) to (x_2, y_2) specified. This reduces the complex integral to the complex sum of real integrals. It is somewhat analogous to the replacement of a vector integral by the vector sum of scalar integrals.

Often, we are interested in contours that are **closed**, meaning that the start and the end of the contour are at the same point, so that the contour forms a closed loop. We normally define the region enclosed by a contour as that which lies to the left when the contour is traversed in the indicated direction; thus a contour intended to surround a finite area will normally be deemed to be traversed in the counterclockwise direction. If the origin of a polar coordinate system is within the contour, this convention will cause the normal direction of travel on the contour to be that in which the polar angle θ increases.