Trigonometric Integrals, Range $(0, 2\pi)$:

We consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta,$$

where f is finite for all values of θ. We also require f to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single valued. We make a change of variable to

$$z = e^{i\theta}, dz = ie^{i\theta} d\theta,$$

with the range in θ, namely $(0, 2\pi)$, corresponding to $e^{i\theta}$ moving counterclockwise around the unit circle to form a closed contour. The now make the substitutions

$$d\theta = -i\frac{dz}{z}, \ \sin\theta = \frac{z - z^{-1}}{2i}, \ \cos\theta = \frac{z + z^{-1}}{2}.$$

Our integral then becomes

$$I = -i \oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z},$$

with the path of integration, the unit circle. By the residue theorem,

$$I = (-i)2\pi i \sum (\text{residues within the unit circle}).$$

Note that we must use residues of f/z. Here are two preliminary examples.

Example 1: Integrand has $\cos\theta$ in the denominator:

Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1.$$

Thus,

$$I = -i \oint_{\text{unit circle}} \frac{dz}{z\left[1 + \left(\frac{a}{2}\right)(z + z^{-1})\right]} = -i\frac{2}{a}\oint \frac{dz}{z^2 + \left(\frac{2}{a}\right)z + 1}.$$

The denominator has roots

$$z_1 = -\frac{1 + \sqrt{1 - a^2}}{a} \text{ and } z_2 = -\frac{1 - \sqrt{1 - a^2}}{a}.$$

Noting that $z_1 z_2 = 1$, it is easy to see that z_2 is withing the unit circle and $z_1$is outside. Writing the integral in the form

$$\oint \frac{dz}{(z - z_1)(z - z_2)},$$

we see that the residue of the integrand at $z = z_2$ is $1/(z_2 - z_1)$, so an application of the residue theorem yields

$$I = -i\frac{2}{a}2\pi i\frac{1}{z_2 - z_1}.$$

Inserting the values of z_1 and z_2, we obtain the final result

$$\int_0^{2\pi}\frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}}, \qquad |a| < 1.$$

Example2: Consider $\quad I = \displaystyle\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{5 - 4\cos\theta}.$

Thus,

$$I = \oint\frac{\frac{1}{2}(z^2 + z^{-2})}{5 - 2(z + z^{-1})}\left(\frac{-idz}{z}\right) = \frac{i}{4}\oint\frac{(z^4 + 1)dz}{z^2\left(z - \frac{1}{2}\right)(z - 2)}$$

where the integration is around the unit circle.

We see that the integrand has poles at $z = 0$ (of order 2) and (simple poles at) $z = 1/2$ and $z = 2$. Only the poles at $z = 0$ and $z = 1/2$ are within the contour.

The residue at $z = 0$:

$$\frac{d}{dz}\left[\frac{z^4 + 1}{\left(z - \frac{1}{2}\right)(z - 2)}\right]\Bigg|_{z=0} = \frac{5}{2},$$

while its residue at $z = 1/2$ is

$$\frac{z^4 + 1}{z^2(z - 2)}\Bigg|_{z=1/2} = -\frac{17}{6}.$$

Applying the residue theorem, we have

$$I = \frac{i}{4}(2\pi i)\left[\frac{5}{2} - \frac{17}{6}\right] = \frac{\pi}{6}.$$