Analytic Functions:

If $f(z)$ is differentiable and single-valued in a region of the complex plane, it is said to be an **analytic** function in that region. Multivalued functions can also be analytic under certain restrictions that make them single-valued in specific regions. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function. If $f'(z)$ does not exist at $z = z_0$, then z_0 is labelled a **singular point**.

Example: Check the analyticity of z^2.

Solution: Let $f(z) = z^2 = x^2 - y^2 + 2ixy \Longrightarrow u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Check:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}\,,\ \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

We see that $f(z) = z^2$ satisfies the Cauchy-Riemann conditions throughout the complex plane. Since the partial derivatives are clearly continuous, we conclude that $f(z) = z^2$ is analytic, and is an entire function.

Example: Check the analyticity of z^*.

Solution: Let $f(z) = z^*$, the complex conjugate of z. Thus, $u = x$ and $v = -y$. Applying the Cauchy-Riemann conditions, we see that

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1.$$

The Cauchy-Riemann conditions are not satisfied for any values of x and y and $f(z) = z^*$ is nowhere an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable in the complex plane.

The derivative of a real function of a real variable is essentially a local characteristic, in that it provides information about the function only in a local neighbourhood, for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far reaching implications, one of which is that the real and imaginary parts of our analytic function must separately satisfy Laplace's equation in two dimensions, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

To verify the above statement, we differentiate the first Cauchy-Riemann equation (4) with respect to x and the second with respect to y, obtaining

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y},\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}.$$

Combining these two equations, we easily reach

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (5)$$

confirming that $u(x, y)$, the real part of a differentiable complex function, satisfies the Laplace equation. Either by recognizing that $f(z)$ is differentiable, so is $-if(z) = v(x, y) - iu(x, y)$, or by steps similar to those leading to equation (5), we can confirm that $v(x, y)$ also satisfies the two dimensional Laplace equation. Sometimes, u and v are referred to as **harmonic functions**.

The solutions $u(x,y)$ and $v(x,y)$ are complementary in that the curves of constant $u(x,y)$ make orthogonal intersections with the curves of constant $v(x,y)$. To confirm this, note that if (x_0,y_0) is on the curve $u(x,y) = c$, then $(x_0 + dx, y_0 + dy)$ is also on that curve if

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0,$$

meaning that the slope of the curve of constant u at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_u = -\frac{\partial u/\partial x}{\partial u/\partial y},$$

where the derivatives are to be evaluated at (x_0, y_0). Similarly, we can find that the slope of the curve of constant v at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_v = -\frac{\partial v/\partial x}{\partial v/\partial y} = \frac{\partial u/\partial y}{\partial u/\partial x}.$$

Comparing these equations, we note that at the same point, the slopes they describe are orthogonal (to check, verify that $dx_u dx_v + dy_u dy_v = 0$).

Finally, the global nature of our analytic function is also illustrated by the fact that it has not only a first derivative, but in addition, derivatives of all higher orders, a property which is not shared by real functions of a real variable.