Example: Another closed loop:

Let's now see what happens to the function $z^{1/2}$ as we pass counterclockwise around a circle of unit radius centred at $z = +2$, starting and ending at $z = +3$. See figure below:

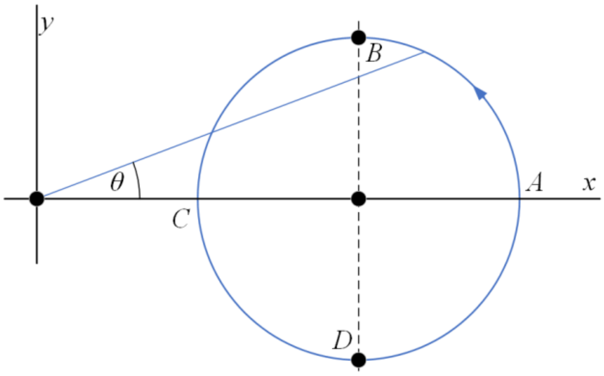

Figure: Path not encircling $z = 0$ for evaluation of $z^{1/2}$.

At $z = 3$, the values of $f(z)$ are $+\sqrt{3}$ and $-\sqrt{3}$; let's start with $f(z_A) = +\sqrt{3}$. As we move from point A through point B to point C, note from the figure that the value of θ first increases (actually, to 30°), and then decreases again to zero; further passage from C to D and back to A causes θ first to decrease (to −30°) and then to return to zero at A. So in this example the closed loop does not bring us to a different value of the multivalued function $z^{1/2}$.

What is special about $z = 0$ is that (from a complex variable viewpoint) it is singular; the function $z^{1/2}$ does not have a derivative there. The lack of a well-defined derivative means that ambiguity in the function value will result from paths that circle such a singular point, which we will call a **branch point**.

The **order of a branch point** is defined as the number of paths around it that must be taken before the function involved returns to its original value; in the case of $z^{1/2}$, we saw that the branch point $z = 0$ is of order 2.

We are now ready to see what must be done to cause a multivalued function to be restricted to single-valuedness on a portion of the complex plane. We simply need to prevent its evaluation on paths that encircle a branch point.

We do so by drawing a line (known as a **branch line**, or more commonly, a **branch cut**) that the evaluation path cannot cross; the branch cut must start from our branch point and continue to infinity (or, if consistent with maintaining single-valuedness, to another finite branch point). The precise path of a branch cut can be chosen freely; what must be chosen appropriately are its endpoints.

Once appropriate branch cut(s) have been drawn, the originally multivalued function has been restricted to being single-valued in the region bounded by the branch cut(s); we call the function made single-valued in this way a branch of the original function. Since we could construct such a branch starting from any one of the values of the original function at a single arbitrary point in our region, we identify our multivalued function as having multiple branches. In the case of $z^{1/2}$, which is double-valued, the number of branches is two.

Note that a function with a branch point and a corresponding branch cut will not be continuous across the cut line. Hence line integrals in opposite directions on the two sides of the branch cut will not generally cancel each other. Branch cuts, therefore, are real boundaries to a region of analyticity, in contrast to the artificial barriers we introduced in extending Cauchy's integral theorem to multiply connected regions.

While from a fundamental viewpoint all branches of a multivalued function $f(z)$ are equally legitimate, it is often convenient to agree on the branch to be used, and such a branch is sometimes called the principal branch, with the value of $f(z)$ on that branch called its principal value. It is common to take the branch of $f(z)$ which is positive for real, positive z as its principal branch.

An observation that is important for complex analysis is that by drawing appropriate branch cut(s), we have restricted a multivalued function to single-valuedness, so that it can be an analytic function within the region bounded by the branch cut(s), and we can therefore apply Cauchy's two theorems to contour integrals within the region of analyticity.

Example: $\ln z$ has an infinite number of branches.

Here we examine the singularity structure of $\ln z$. The logarithm is multivalued, with the polar representation

$$\ln z = \ln\left(re^{i(\theta+2n\pi)}\right) = \ln r + i(\theta + 2n\pi),$$

where n can have **any** positive or negative integer value.

Noting that $\ln z$ is singular at $z = 0$ (it has no derivative there), we now identify $z = 0$ as a branch point. Let's consider what happens if we encircle it by a counterclockwise path on a circle of radius r, starting with the initial value $\ln r$, at $z = r = re^{i\theta}$, with $\theta = 0$. Every passage around the circle will add 2π to θ, and after n complete circuits the value we will have for $\ln z$ would be $\ln r + 2n\pi i$. The branch point of $\ln z$ at $z = 0$ is of infinite order, corresponding to the infinite number of its multiple values. By encircling $z = 0$ repeatedly in the clockwise direction, we can also reach all negative integer values of n.

We can make $\ln z$ single-valued by drawing a branch cut from $z = 0$ to $z = \infty$ in any way (though there is ordinarily no reason to use cuts that are not straight lines). It is typical to identify the branch cut with $n = 0$ as the principal branch of the logarithm. Incidentally, we note that the inverse trigonometric functions, which can be written as logarithms, will also be infinitely multivalued, with principal values that are usually chosen on a branch that will yield real values for real z. Compare the usual choices of the values assigned to the real-variable forms of $\sin^{-1} x = \arcsin x$, etc.

Using the logarithm, we are now in a position to look at the singularity structures of expressions of the form z^p, where both z and p may be complex. To do so, we write

$$z = e^{\ln z} \Longrightarrow z^p = e^{p \ln z},$$

which is single-valued if p is an integer, t-valued if p is a real rational fraction (in lowest terms) of the form s/t, and infinitely multivalued otherwise.