

## Trigonometric Integrals, Range $(0, 2\pi)$ :

We consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin \theta, \cos \theta) d\theta,$$

where  $f$  is finite for all values of  $\theta$ . We also require  $f$  to be a rational function of  $\sin \theta$  and  $\cos \theta$  so that it will be single valued. We make a change of variable to

$$z = e^{i\theta}, dz = ie^{i\theta} d\theta,$$

with the range in  $\theta$ , namely  $(0, 2\pi)$ , corresponding to  $e^{i\theta}$  moving counterclockwise around the unit circle to form a closed contour. The new make the substitutions

$$d\theta = -i \frac{dz}{z}, \sin \theta = \frac{z - z^{-1}}{2i}, \cos \theta = \frac{z + z^{-1}}{2}.$$

Our integral then becomes

$$I = -i \oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{z},$$

with the path of integration the unit circle. By the residue theorem,

$$I = (-i)2\pi i \sum (\text{residues within the unit circle}).$$

Note that we must use residues of  $f/z$ . Here are two preliminary examples.

**Example 1:** Integrand has  $\cos \theta$  in the denominator:

Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta}, \quad |a| < 1.$$

Thus,

$$I = -i \oint_{\text{unit circle}} \frac{dz}{z \left[ 1 + \left(\frac{a}{2}\right)(z + z^{-1}) \right]} = -i \frac{2}{a} \oint \frac{dz}{z^2 + \left(\frac{2}{a}\right)z + 1}.$$

The denominator has roots

$$z_1 = -\frac{1 + \sqrt{1 - a^2}}{a} \quad \text{and} \quad z_2 = -\frac{1 - \sqrt{1 - a^2}}{a}.$$

Noting that  $z_1 z_2 = 1$ , it is easy to see that  $z_2$  is within the unit circle and  $z_1$  is outside. Writing the integral in the form

$$\oint \frac{dz}{(z - z_1)(z - z_2)},$$

we see that the residue of the integral at  $z = z_2$  is  $1/(z_2 - z_1)$ , so application of the residue theorem yields

$$I = -i \frac{2}{a} 2\pi i \frac{1}{z_2 - z_1}.$$

Inserting the values of  $z_1$  and  $z_2$ , we obtain the final result

$$\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad |a| < 1.$$

**Example2:** Consider 
$$I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 - 4 \cos \theta}.$$

Thus,

$$I = \oint \frac{\frac{1}{2}(z^2 + z^{-2})}{5 - 2(z + z^{-1})} \left( \frac{-idz}{z} \right) = \frac{i}{4} \oint \frac{(z^4 + 1)dz}{z^2 \left( z - \frac{1}{2} \right) (z - 2)}$$

where the integration is around the unit circle.

We see that the integrand has poles at  $z = 0$  (of order 2) and (simple poles at)  $z = 1/2$  and  $z = 2$ . Only the poles at  $z = 0$  and  $z = 1/2$  are within the contour.

The residue at  $z = 0$ :

$$\frac{d}{dz} \left[ \frac{z^4 + 1}{\left( z - \frac{1}{2} \right) (z - 2)} \right] \bigg|_{z=0} = \frac{5}{2},$$

while its residue at  $z = 1/2$  is

$$\frac{z^4 + 1}{z^2(z - 2)} \bigg|_{z=1/2} = -\frac{17}{6}.$$

Applying the residue theorem, we have

$$I = \frac{i}{4} (2\pi i) \left[ \frac{5}{2} - \frac{17}{6} \right] = \frac{\pi}{6}.$$