Zeroes and Singularities:

Taylor series expansion for the function $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$\because f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' = a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Laurent series expansion for the function $f(z)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Zeroes:

Consider the above Taylor series expansion for the function $f(z)$. Here the function $f(z)$ is analytic within some region. If $f(z)$ vanishes at $z = z_0$, the point z_0 is said to be a zero of $f(z)$. If $a_0 = a_1 = \cdots = a_{m-1} = 0$ but $a_m \neq 0$, then Taylor expansion becomes

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n$$

In this case, $f(z)$ is said to have **a zero of order m** at $z = z_0$. A zero of order 1 ($m = 1$) is called a **simple zero.** Note that for $m = 1$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$
$$= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$

It is clear that $f(z_0) = 0$ and $f'(z_0) \neq 0$, indicate the existence of a simple pole for $f(z)$ at $z = z_0$.

Singularities (Poles):

Points at which the function $f(z)$ is not analytic is called **singular points (singularities).** For example, $z = 0$ is a singular point of the function $f(z) = \frac{1}{z}$.

We define a point z_0 as an **isolated singular point** of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighbouring points. There will therefore be a Laurent expansion about an isolated singular point, and one of the following statements will be true:

1. The most negative power of $z - z_0$ in the Laurent expansion of $f(z)$ about $z = z_0$ will be some finite power, $(z - z_0)^{-n}$, where n is integer, or
2. The Laurent expansion of $f(z)$ about $z = z_0$ will continue to negatively infinite powers of $z - z_0$.

In the first case, the singularity is called a **pole**, and is more specifically identified as a pole of **order $\boldsymbol{n}$**. A pole of order 1 is also called a **simple pole**.

The second case is not referred to as a "pole of infinite order," but is called an **essential singularity**.

One way to identify a pole of $f(z)$ without having available its Laurent expansion is to examine

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for various integers n. The smallest integer n for which this limit exists (i.e., is finite) gives the order of the pole at $z = z_0$. This rule follows directly from the form of the Laurent expansion.

Essential singularities are often identified directly from their Laurent expansions. For example,

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$$

clearly has an essential singularity at $z = 0$.

The behaviour of $f(z)$ as $z \to \infty$ is defined in terms of the behaviour of $f\left(\frac{1}{t}\right)$ as $t \to 0$. Consider the function

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

As $z \to \infty$, we replace z with $\frac{1}{t}$ to obtain

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, t^{2n+1}}.$$

It is clear that $\sin(1/t)$ has an essential singularity at $t = 0$, from which we conclude that $\sin z$ has an essential singularity at $z = \infty$. Note that although $0 \le |\sin x| \le 1 \ \forall \ x \in \mathbb{R}$, while $|\sin(\mathrm{i}y)| = |i \sinh y| = |\sinh \mathrm{y}|$ increases exponentially without limit as y increases.

A function that is analytic throughout the finite complex plane except for isolated poles is called **meromorphic**. Examples are ratios of two polynomials, also $\tan z$ and $\cot z$. As previously mentioned, functions that have no singularities in the finite complex plane are called entire functions. Examples are $\exp z$, $\sin z$, $\cos z$.

In addition to the isolated singularities identified as poles or essential singularities, there are singularities uniquely associated with multivalued functions known as b**ranch points**. This topic will be discussed later in this chapter.

Example: Find the singularities of

(a) $\quad f(z) = \dfrac{1}{z}$ $\qquad$ (b) $\quad f(z) = \dfrac{1}{\sin(1/z)}$.

Solution: (a) $z = 0$ is an isolated singular point for $f(z) = \frac{1}{z}$.

(b) The function $f(z) = \frac{1}{\sin(1/z)}$ has an isolated singularity when $z = 1/n\pi$ for $n = 1, 2,$ .... However, the origin $z = 0$ is not an isolated singular point as a result of $z = 1/n\pi$.

Example: Find the pole and its order for $f(z) = \frac{\sin z}{z^4}$.

Solution: Let us check

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for the smallest integer n such that this limit is finite. Clearly, $z = 0$ is a singular point $f(z)$. Thus,

$$\lim_{z \to 0} z^3 \frac{\sin z}{z^4} = 1$$

Thus, $f(z)$ has a pole of order 3 at $z = 0$.

Also,

$$f(z) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Thus, $f(z)$ has a pole of order 3 at $z = 0$.

Example: Classify the singular point of the function $f(z) = ze^{1/z}$.

Solution:

$$f(z) = ze^{1/z} = z\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right) \Rightarrow f(z) = z + 1 + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} + \cdots$$

There is an essential singularity at $z = 0$.

Example: Find the pole and its order for $f(z) = \frac{1}{z^2 - 1}$.

Solution: Here

$$f(z) = \frac{1}{z^2 - 1} = \frac{1}{(z + 1)(z - 1)}.$$

The poles are at $z = -1$ and $z = +1$.

Let us check

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for the smallest integer n such that this limit is finite and non-zero, where $z_0 = -1$ or $+1$. We get

$$\begin{cases} \lim\limits_{z \to 1} (z - 1)^n \dfrac{1}{(z - 1)(z + 1)} = \dfrac{1}{2} \neq 0 \text{ for } n = 1\,, \\ \lim\limits_{z \to -1} (z + 1)^n \dfrac{1}{(z - 1)(z + 1)} = -\dfrac{1}{2} \neq 0 \text{ for } n = 1\,. \end{cases}$$

Hence, $z = \pm 1$ are simple poles of the given function.

Example: Find the poles and their orders for

$$f(z) = \frac{1}{(z^2 + a^2)^2} \text{ , where } a > 0.$$

Solution: Here

$$f(z) = \frac{1}{(z^2 - a^2)^2} = \frac{1}{(z + ia)^2 (z - ia)^2}.$$

The poles are at $z = -ia$ and $z = +ia$.

Let us check

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for the smallest integer n such that this limit is finite and non-zero, where $z_0 = -ia$ or $+ia$. We get

$$\begin{cases} \lim\limits_{z \to ia} (z - ia)^n \dfrac{1}{(z - ia)^2 (z + ia)^2} = -\dfrac{1}{4a^2} \neq 0 \text{ for } n = 2 \text{ ,} \\ \lim\limits_{z \to -ia} (z + ia)^n \dfrac{1}{(z - ia)^2 (z + ia)^2} = -\dfrac{1}{4a^2} \neq 0 \text{ for } n = 2 \text{ .} \end{cases}$$

Hence, $z = \pm ia$ are poles of second order of the given function.