Gravitational Waves, QNMs and Scalar Field Coupling

PHY502A: MSc Review Project III

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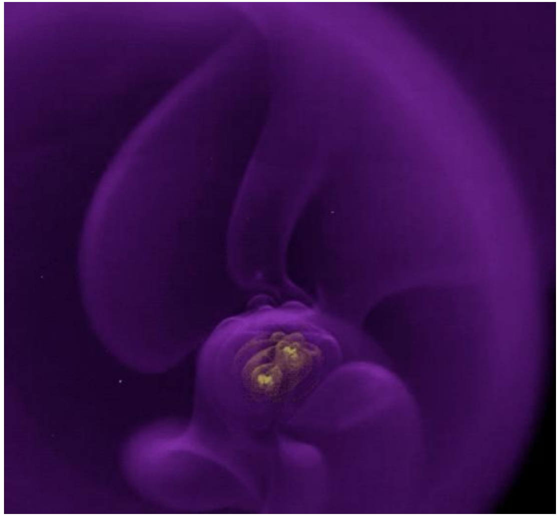
Introduction to Gravitational Waves

- **Definition:** Gravitational waves as perturbations of spacetime caused by massive accelerating objects.

- **Historical Milestones:** Einstein’s prediction and LIGO’s detection in 2015 (GW150914).

- **Importance:** New observational window into phenomena like black hole mergers and neutron star collisions.

https://svs.gsfc.nasa.gov/14132



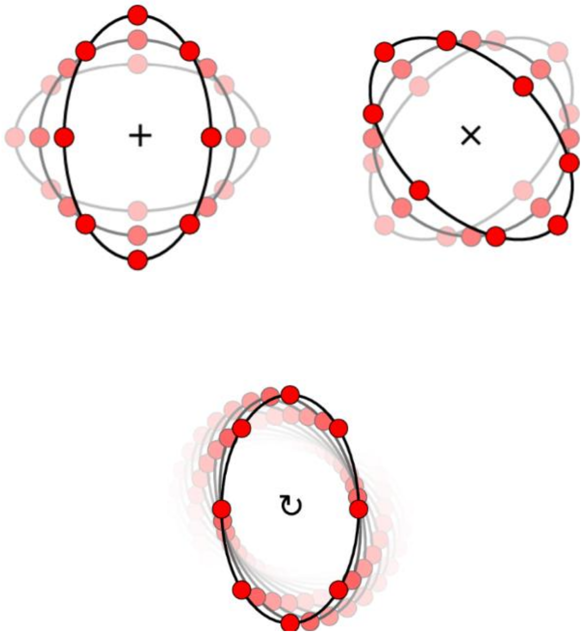
Gravitational waves emitted by two black holes of nearly equal mass as they spiral together and merge.

Linearized Einstein Equations

Linearized Einstein Equations: Start with the Minkowski metric $\eta_{\mu\nu}$, add a small perturbation $h_{\mu\nu}$ such that: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $| h_{\mu\nu} | \ll 1$. Linearizing Einstein's equations under this assumption leads to a wave equation for $h_{\mu\nu}$: $\Box h_{\mu\nu} = 0$.

Gauge Transformations: Using the freedom of coordinate transformations, the perturbations can be expressed in the TT gauge. In this gauge, gravitational waves exhibit two polarizations: h_+ and $h_\times$.

Effect on Matter: Gravitational waves induce oscillations in matter as they pass through, stretching and compressing spacetime alternately in perpendicular directions.



Schwarzschild Metric and Perturbations

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}$$

- Schwarzschild metric describes a static, spherically symmetric black hole.
- Perturbing this metric helps study gravitational wave signals.
- Tensor spherical harmonics decompose the perturbations.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$h_{\mu\nu}^{\text{ax.}} = \begin{bmatrix} 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} \sin\theta\, \partial_\theta P_l(\cos\theta) e^{-i\omega t}$$

$$h_{\mu\nu}^{\text{pol.}} = \begin{bmatrix} H_0\left(1 - \frac{2GM}{r}\right) & H_1 & 0 & 0 \\ H_1 & H_2\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & Kr^2 & 0 \\ 0 & 0 & 0 & Kr^2 \sin^2\theta \end{bmatrix} P_l(\cos\theta) e^{-i\omega t}$$

Regge–Wheeler and Zerilli Equations

- Axial (odd-parity) perturbations lead to the Regge-Wheeler equation.
- Polar (even-parity) perturbations yield the Zerilli equation.
- These equations govern gravitational wave propagation.

$$\frac{d^2\psi}{dr_\star^2} + [\omega^2 - V(r)]\psi = 0$$

$$\psi = \begin{cases} \frac{h_1}{r}\left(1 - \frac{2GM}{r}\right), \\ \frac{\left[l(l+1) + 2 - \frac{4GM}{r}\right] rK + 2r\left(1 - \frac{2GM}{r}\right)^2 \left(H_2 - \frac{dK}{dr}\right)}{l(l+1) - 2 + \frac{6GM}{r}}, \end{cases}$$

for odd and even parity, respectively.

$$\mathrm{V}(r) = \begin{cases} \left(1 - \frac{2GM}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{3\sigma \mathrm{GM}}{r^3}\right], & \text{for odd parity,} \\ \left(1 - \frac{2GM}{r}\right)\frac{2n^2(n+1)r^3 + 6n^2 GMr^2 + 18r(GM)^2 + 18(GM)^3}{r^3(nr + 3GM)^2}, & \text{for even parity,} \end{cases}$$

where $2n = (l-1)(l+2)$

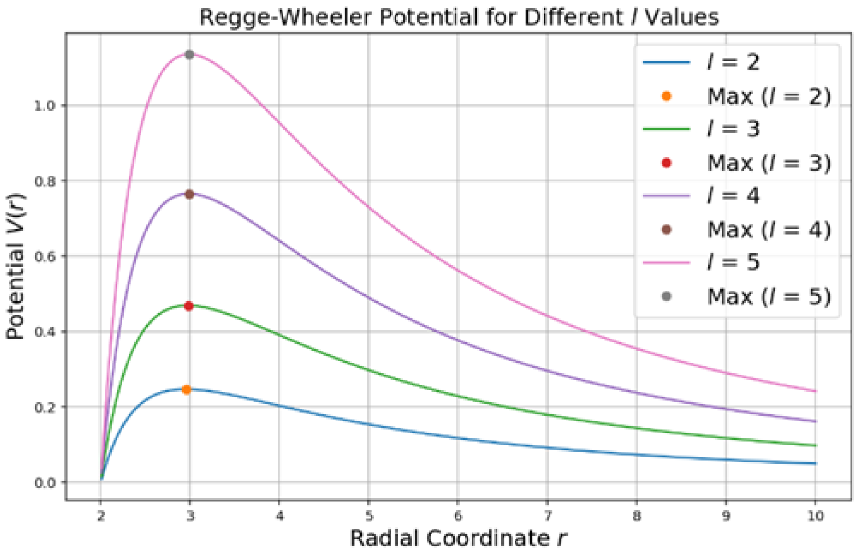
Quasinormal Modes (QNMs)

- QNMs are the “ringing” of black holes after perturbations.
- Described by a complex frequency,

$$\omega = \omega_R + i\omega_I$$

- Damping timescale linked to black hole properties.

ω_R: Encodes properties like mass and spin
ω_I : Reflects decay rates.



Stellar Perturbations

- **Stellar Models:**
  - Assume stars as ideal fluids governed by Tolman-
  - Oppenheimer-Volkov equations.
  - Introduce small perturbations for stability and wave studies.
- **Oscillation Modes:**
  f-modes (fundamental), p-modes (pressure-driven), g-modes (gravity-driven).
- **Connection to Gravitational Waves:**
  Oscillating stars act as gravitational wave sources, providing insights into stellar structures.

$$\frac{dp}{dr} = -\frac{(\rho + p)(Gm + 4\pi r^3 p)}{r^2\left(1 - \frac{2Gm}{r}\right)}$$

$$m(r) = 4\pi \int_0^r \rho r^2 dr$$

$$-\frac{1}{c^2}\frac{\partial^2 X}{\partial t^2} + \frac{\partial^2 X}{\partial r_\star^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6Gm}{r^3} + \rho - P\right] = 0$$

Non-Minimal Coupling to Scalar Fields

- **Modified Action:**

$$S = \int d^4 x \sqrt{-g} \left[\frac{1}{2} F(\phi) R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right]$$

- **Effect on QNMs:** Scalar fields modify the effective potential, altering QNM spectra.

- **Applications:** Relevance to cosmology and astrophysics.

$$F(\phi) G_{\mu\nu} = T_{\mu\nu}^{\text{scalar}} + \nabla_\mu \nabla_\nu F(\phi) - g_{\mu\nu} \Box F(\phi)$$

$$T_{\mu\nu}^{\text{scalar}} =$$

$$\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi$$

$$- g_{\mu\nu} V(\phi)$$

Scalar field equation (modified):

$$\Box \phi - \frac{dF}{d\phi} R + \frac{dV}{d\phi} = 0$$

Challenges and Open Questions

- **Challenges:**

  - Detecting QNM deviations due to scalar fields.
  - Realistic modeling with spin, charge, and dynamic scalar fields.

- **Future Directions:**

  - Numerical studies of modified equations.
  - Probing exotic astrophysical scenarios.

Summary of Key Findings

Summary:

- Linearized Einstein equations reveal the nature of gravitational waves and their polarizations.

- QNMs and stellar perturbations provide insights into black holes and relativistic stars.

- Scalar fields offer potential modifications to gravitational theories.

Thank You!