<u>Mid Sem PHY690M. 19th February, 2024. Time : 2 Hours.</u>

Please show all workings.

1) An example of a two dimensional black hole is given by the line element

$$ds^2 = -\left(1 - \frac{M}{r}\right) dt^2 + \frac{k dr^2}{4r^2(1 - M/r)}.$$

a) Calculate all non-zero components of the Christoffel connection. [2]

b) Consider a timelike geodesic. By writing down an appropriate Lagrangian, compute $\dot{t}$. Hence compute $\dot{r}$ for this geodesic. Write your answer for $\dot{r}$ in terms of the conserved energy E. [2]

c) If $u^\mu = (\dot{t}, \dot{r})$ is the tangent to the geodesic, make an appropriate ansatz for the normal n^μ and solve for the components of n^μ. [2]

d) Calculate the non-zero components of the Riemann tensor $R_{\mu\nu\lambda\rho}$. You may use the formula

$R^\mu{}_{\nu\lambda\rho} = \Gamma^\mu{}_{\nu\rho,\,\lambda} - \Gamma^\mu{}_{\nu\lambda,\,\rho} + \Gamma^\mu{}_{\sigma\lambda}\Gamma^\sigma{}_{\nu\rho} - \Gamma^\mu{}_{\sigma\rho}\Gamma^\sigma{}_{\nu\lambda}.$ [2]

e) Here, $n^\mu{}_{;\alpha} u^\alpha = 0$ (no need to show this). Using this information, write the geodesic deviation equation in terms of x where the deviation vector $\xi^\mu = x n^\mu$. [2]

f) For the special case of $E = 4$, show that a general solution for x is $x = C_1 \tau^2 + C_2 \frac{1}{\tau}$, where τ is the proper time and C_1 and C_2 are constants. [2]

g) Consider the case $C_1 = 0$. Calculate $\frac{d^2 x}{d\tau^2}$ in terms of k, M and r. [2]

2) Consider the two dimensional Euclidean metric given by

$$ds^2 = \frac{dx^2}{y(1 - x^2)} + \frac{dy^2}{y(1 + y)^2}.$$

a) Identify an appropriate cyclic coordinate. After this, write the first integrals of the geodesic equations using an appropriate Lagrangian. [2]

b) Compute the non-zero components of the Christoffel connection. [2]

c) Calculate the expansion scalar. [2]