QFT ASSIGNMENT I: CLASSICAL FIELD THEORY

DUE DATE: 30.08.2024

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(1) Consider the following action:

$$S = \int dt \int d^3x \; \psi^* \left(i\partial_t \psi + \frac{1}{2m} \overline{\nabla}^2 \psi - V(x)\psi \right).$$

Treat ψ, ψ^* as independent fields.

(a) What is the equation of motion for ψ^*? Does this look familiar?

(b) Manipulate the action so that the Lagrangian becomes of the standard form $\mathcal{L} = \mathcal{L}(\{\varphi_r\}, \{\partial_\mu \varphi_r\})$.

(c) Show that in this form the action has a symmetry

$$\psi(x) \to e^{i\alpha}\psi(x), \quad \psi^*(x) \to e^{-i\alpha}\psi^*(x).$$

(d) Find the conserved current and conserved charge related to this symmetry. Do these remind you of things you have seen in your Quantum Mechanics courses?

(2) A set of fields φ_i $(i = 1, 2 \ldots n)$ transform under Lorentz transformation as

$$\varphi_r(x) = \sum_s S_{rs}(\Lambda)\, \varphi_s(\Lambda x).$$

Suppose the Lagrangian for the theory of these fields falls under the restricted form $\mathcal{L} = \mathcal{L}(\{\varphi_r\}, \{\partial_\mu \varphi_r\})$, which is invariant under Lorentz transformations.

(a) Find the conserved currents and conserved charges under Lorentz symmetry.

(b) Show that the conserved charge under rotation contains a term which corresponds to the orbital angular momentum which is the same as that worked out in class for the scalar fields and another piece which corresponds to the spin of the particle.

(c) If Q^{i0} is the charge associated with boosts, show that

$$\{Q^{i0}, H\}_{PB} = -P^i,$$

where $\{,\}_{PB}$ denotes Poisson brackets and P^i is the momentum.

(3) Consider the following action:

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \phi^2 + \frac{1}{4}\lambda \phi^4 \right).$$

There are different ways of deriving the energy-momentum tensor, which we have seen in class. Calculate $T_{\mu\nu}$ in three different ways, viz.

(a) Use the action and infinitesimal transformations.

(b) Use the expression for the currents derived from restricted form of the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$.

(c) **(Extra credits, but no penalty if not attempted)** Use the definition invoking gravity:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$

(4) Some interesting theories exhibit a phenomenon called scale invariance, which means that they look the same at all length scales. The scale transformations are

$$x^\mu \to x'^\mu = \lambda x^\mu \quad \varphi(x) \to \varphi'(\lambda x) = \lambda^{-\Delta} \varphi(x),$$

where Δ is the scaling dimension of the field φ. Now let us look at the action for a real scalar field

$$S = \int d^4x \, (\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \phi^2 + \lambda \phi^p).$$

(a) Find the scaling dimension of the field such that the kinetic term remains invariant.

(b) Under what values of m and p is scale transformations a symmetry of the action?

(c) Consider the same theory in d dimensions. What is now the scaling dimension of φ which keeps the kinetic term invariant? What are the values of m and p for the invariance of the action?

(d) In $d = 4$, construct the conserved currents associated to scale invariance using Noether's prescription.

(5) Let us consider infinitesimal transformations on an action. Given an infinitesimal set of parameters ϵ_a, one defines the generator G_a of a symmetry transformation as

$$\delta_\epsilon \Phi(x) = \Phi'(x) - \Phi(x) = -i \epsilon_a G_a \Phi(x)$$

The theories that we have considered in class have been translationally invariant as well as Lorentz invariant. Together, the symmetry group is called the Poincare group. We have discussed the infinitesimal forms of the transformations in class. Use these to show the following:

(a) Show that the generators of translations P_μ and Lorentz rotations $M_{\mu\nu}$ are given by

$$P_\mu = -i \partial_\mu, \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu).$$

(b) Find the commutation relations between all the generators.

(c) In Problem 4, you have looked at scale invariance of the action of a real scalar field. Find the generator D of this transformation. Calculate the commutation relations of D with P_μ and $M_{\mu\nu}$.