PHY681(A) Quantum Field Theory: Mid-term Exam

September 22, 2023.

Total Marks: 60.

Answer all questions.

1. Consider Maxwell's theory of Electromagnetism. This is given by a Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

(a) What are the equations of motion of the system? Express these in terms of Electric and Magnetic fields. Are these all the Maxwell's equations? If not, how do you get the rest of the equations?

(b) Now add the following term to the Lagrangian above

$$\mathcal{L} = \mathcal{L}_0 + g\mathbf{E}\cdot\mathbf{B}, \tag{2}$$

where g is a constant. Can you express this covariantly (i.e., in terms of A_μ)? How does this change the equations of motion?

(c) Now let's go back to the original Maxwell Lagrangian. Find the energy momentum tensor $T_{\mu\nu}$ of the theory by looking at translation invariance of the action. You can use the formulation for the restricted Lagrangians done in the class.

(d) Remember the energy momentum tensor of the above theory can be computed also by looking at the gravitational prescription, i.e.,

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}.$$

Without calculations, can you say whether $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are equal? Is there a prescription to make these equal if they are not? (Again, you need to indicate the method and not work this out.)

4+4+7+5=20

Sol: (a) We have the Lagrangian

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu).\end{aligned}$$

Now,

$$\frac{\partial\mathcal{L}_0}{\partial A^\rho} = 0 \text{ and } \frac{\partial\mathcal{L}_0}{\partial\,\partial_\rho A^\sigma} = -\frac{1}{2}\left[\delta^\rho_\mu g_{\nu\sigma}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial_\mu A_\nu\left(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma\right)\right]$$

$$= -\frac{1}{2}[2(\partial^\rho A_\sigma - \partial_\sigma A^\rho)] = -(\partial^\rho A_\sigma - \partial_\sigma A^\rho)$$

Hence, the equation of motion implies (with the Lorenz gauge $\partial_\rho A^\rho = 0$) Maxwell equations

$$\partial_\rho \partial^\rho A^\sigma = 0$$

in a source free region ($J^\mu = 0$). In terms of $\mathbf{E}$ and $\mathbf{B}$ fields, the above equation becomes

$$\nabla \cdot \mathbf{E} = 0 \text{ and } \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

These are two of the four Maxwell equations. The rest two are embedded in the definition of the tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which can be written as

$$\nabla \cdot \mathbf{B} = 0 \text{ and } \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

(b) Let's add another term to the above Lagrangian, so that we now have

$$\mathcal{L} = \mathcal{L}_0 + g\mathbf{E} \cdot \mathbf{B}$$

2. Consider a real scalar field in 3 spacetime dimensions (2 spatial and 1 temporal direction). The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \sum_{n\geq 3}\lambda_n\varphi^n\,. \tag{3}$$

(a) Classify the coupling constants λ_n according to their behaviour in terms of energies.
(b) What does the low-energy effective theory of the above scalar field look like?
(c) Now suppose we are in D dimensions. For what value of D does φ^3 term become marginal?

6+4+2=12

3. Consider the above Lagrangian (3) but now in usual 4 (3+1) dimensions.

(a) Under what circumstances is $\varphi \to -\varphi$ a symmetry of this theory?
(b) State Noether's theorem.
(c) Suppose you restrict yourself to $n = 3, 4$ in the sum of the Lagrangian (3). When $\varphi \to -\varphi$ is a symmetry of the above theory, use Noether's theorem and prescription to calculate the conserved currents and the conserved charge of the theory. Explain your answer.

3+2+5=10

4. Consider a real scalar field in usual 4 spacetime dimensions with a φ^4 interaction. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4. \tag{4}$$

Consider processes with four external points at order λ in perturbation theory.

(a) Draw all the diagrams that contribute to this process at this order.
(b) Evaluate all amplitudes together with the proper combinatorial factors. Use Wick's theorem.

4+9=13

5. Consider the free theory of a complex scalar field ψ given by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \, \partial^\mu \psi - M^2 \psi^* \psi. \tag{5}$$

(a) Expand the fields in terms of two sets of creation and annihilation operators (b, $b^\dagger$, c, $c^\dagger$).

(b) The theory has a symmetry under phase rotations: $\psi \to e^{i\alpha}\psi$.
Find the conserved charge corresponding to this symmetry and using the expressions for the fields in terms of the creation and annihilation operators, re-express this in terms of (b, $b^\dagger$, c, $c^\dagger$).
Use normal ordering to simplify your answer so that this becomes a difference of operators which counts the number of particles and anti-particles.

1+4=5