PHY681(A): End-term Examination

November 21, 2023.

Total Marks: 90.

Answer all questions.

1. Consider the free Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{1}$$

   (a) Show that $\psi \to e^{-i\alpha}\psi$ is a symmetry of the Lagrangian. Find the corresponding conserved current and charge.
   (b) Under what conditions is the transformation ($\psi \to e^{i\alpha\gamma^5}\psi$, $\bar{\psi} \to e^{i\alpha\gamma^5}\bar{\psi}$) a symmetry of the Lagrangian? In this special case, again find the corresponding conserved current and charge.

$5 + 5 = 10$

2. Consider free Dirac theory. Wick contractions in the fermionic theory comes hand in hand with a rule for the sign of a particular contraction.

   (a) State the rule to evaluate the sign.

   (b) Consider the following process:

$$\langle 0|T\{\psi_{a_1}(x_1)\psi_{a_2}(x_2)\psi_{a_3}(x_3)\bar{\psi}_{b_1}(y_1)\bar{\psi}_{b_2}(y_2)\bar{\psi}_{b_3}(y_3)\}|0\rangle$$

   By using fermionic Wick contraction rules, find the expression of the above process.

$2 + 5 = 7$

3. Consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu \partial_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - g\bar{\psi}\phi\psi\,, \tag{2}$$

   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

   (a) Write down the momentum space Feynman rules for the above theory.
   (b) Draw the Feynman diagrams which contribute to the leading corrections of the photon and the scalar propagators.

$9 + 6 = 15$

4. Consider a massive complex scalar field in four spacetime dimensions.

   (a) Write the Lagrangian for the theory and show that this Lagrangian has a global $U(1)$ symmetry.
   (b) Now promote this global $U(1)$ symmetry to a local $U(1)$ symmetry and show how the Lagrangian has to change in order for this to still be a symmetry of the system.
   (c) Add appropriate kinetic terms to any other field that appears. What does this new Lagrangian describe?
   (d) Write down the Feynman rules for this theory.

$3 + 6 + 3 + 6 = 18$

5. Consider a complex scalar field with the following Lagrangian:

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi + m^2 \Phi^\dagger \Phi - \frac{\lambda}{4}\left(\Phi^\dagger \Phi\right)^2 , \ \ \lambda > 0. \tag{3}$$

Notice the "wrong" sign in front of the mass term.

(a) Write the potential V of the Lagrangian in terms of real fields $\Phi(x) = \frac{1}{\sqrt{2}}\big(\sigma(x) + i\chi(x)\big)$. Putting $\chi = 0$, draw the potential as a function of σ.

(b) Consider $V = V(\sigma, \chi)$ and show the extrema of the potential are at $\sigma_c = 0 = \chi_c$ and $\sigma_c^2 + \chi_c^2 = \frac{4m^2}{\lambda}$.

(c) Choose $\chi_c = 0$ in the second solution and show this is the minimum. By computing the value of V at the extrema, verify your answer.

(d) Quantum mechanically, this means giving some vacuum expectation values to the fields, i.e.

$$\langle \sigma \rangle = \langle 0 | \sigma(x) | 0 \rangle = \frac{2m}{\sqrt{\lambda}} , \ \ \langle \chi \rangle = \langle 0 | \chi(x) | 0 \rangle = 0.$$

This means the perturbation theory needs to be done about this minima and hence the fields need to be shifted as

$$\sigma(x) \to \sigma(x) + \frac{2m}{\sqrt{\lambda}} , \ \ \chi(x) \to \chi(x).$$

Plug this back into your initial Lagrangian and expand.

(e) What are the masses of your fields $\sigma(x)$ and $\chi(x)$ now? Does the "wrong sign" still persist?

(f) What is the physics lesson that you derived out of this exercise?

$4 + 2 + 4 + 4 + 4 + 2 = 20$

6. Prove the following identities:

(a) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0$.

(b) $\mathrm{tr}\big(\gamma^\mu \gamma^\nu \gamma^5\big) = 0$.

(c) $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$.

$3 + 3 + 4 = 10$

7. Write short answers to the following questions:

(a) Do fermions live on the spacetime manifold? Explain.

(b) How are gamma matrices related to the Lorentz algebra?

(c) How are spinors of $\mathfrak{su}(2)$ different from spinors of $\mathfrak{sl}(2, \mathbb{C})$?

(d) What is the difference between a global symmetry and a gauge symmetry?

(e) What is the main difficulty in quantizing the electromagnetic theory?

$2 \times 5 = 10$