Lecture - 2

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PHY781: Quantum Field Theory - II

2. Dirac Bracket

- ## The Dirac Method and the Dirac Bracket:

We now wish to focus on the non-invertible case, i.e., when

$$\det.\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0$$

The $N \times N$ matrix $b = \dfrac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ has rank $R < N$. Let

$$\dot{q}_a = f_a(q_i, p^b); \quad a, b = 1, 2, \ldots, R.$$

[Originally we had a system of equations:

$$p^i = b^{ij}(q)\dot{q}_j + \tilde{b}^{ij}(q)q_j \quad ; \quad i, j = 1, 2, \ldots, N.$$

In a constrained system, only R out of the N momenta are independent — let's call them p^b. Similarly, let's call the R independent velocities as $\dot{q}_a$. Thus we've split the equations in this system into two groups — one in which all the velocities are independent and also all the momenta are independent, and these R equations can be inverted to express the velocities in terms of N coordinates and R momenta:

$$\dot{q}_a = f_a(q_i, p^b); \quad i = 1, 2, \ldots, N; \; a, b = 1, 2, \ldots, R.$$

In the second group of equations, the velocities, and hence the momenta are not all independent. Hence, we cannot express these $N-R$ velocities in terms of N coordinates and $N-R$ momenta. So we fall back to the relations:

$$p^a = g^a(q_i, \dot{q}_b) \; ; \quad a = 1, 2, \ldots, R \; ; \quad i = 1, 2, \ldots, N \; ;$$

$$p^\alpha = g^\alpha(q_i, p^a) ; \quad \alpha = R+1, R+2, \ldots, N.$$

Recall the equations:

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tilde{b} & b \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

This is split as:

$$
\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \\ \hline p^1 \\ p^2 \\ \vdots \\ p^R \\ \hline p^{R+1} \\ \vdots \\ p^N \end{pmatrix}
=
\begin{array}{cc}
 & \begin{array}{cccccccccccc} 1 & 2 & \cdots & N & N+1 & N+2 & \cdots & N+R & N+R+1 & & \cdots & 2N \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ N \\ N+1 \\ N+2 \\ \vdots \\ N+R \\ N+R+1 \\ \vdots \\ 2N \end{array}
&
\left(\begin{array}{cccc|cccc|cccc}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
\tilde{b}^{11} & \tilde{b}^{12} & \cdots & \tilde{b}^{1N} & b^{11} & b^{12} & \cdots & b^{1R} & b^{1,R+1} & b^{1,R+2} & \cdots & b^{1,N} \\
\tilde{b}^{21} & \tilde{b}^{22} & \cdots & \tilde{b}^{2N} & b^{21} & b^{22} & \cdots & b^{2R} & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & \\
\tilde{b}^{R1} & \tilde{b}^{R2} & \cdots & \tilde{b}^{RN} & b^{R1} & b^{R2} & \cdots & b^{RR} & & & & \\
\hline
\tilde{b}^{R+1,1} & \tilde{b}^{R+1,2} & \cdots & \tilde{b}^{R+1,N} & & & & & & & & \\
\vdots & \vdots & & \vdots & & & & & & & & \\
\tilde{b}^{N1} & \tilde{b}^{N2} & \cdots & \tilde{b}^{NN} & & & & & & & &
\end{array}\right)
\end{array}
\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \\ \hline \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_R \\ \hline \dot{q}_{R+1} \\ \vdots \\ \dot{q}_N \end{pmatrix}
$$

The other $N-R$ velocities cannot be determined. Further, let

$$p^a = g^a(q_i, \dot{q}_b) \quad ; \quad a, b = 1, 2, \cdots, R,$$

and

$$p^\alpha = g^\alpha(q_i, p^a) \quad ; \quad \alpha = R+1, R+2, \cdots, N.$$

The second set of equations specify the $N-R$ constraints among the dynamical variables:

$$\varphi^\alpha \equiv p^\alpha - g^\alpha(q_i, p^a) = 0$$

These constraints reduce the dimensionality of the true phase space of the system. Constraints above are known as primary constraints, and they define a $2N-(N-R) = (N+R)$-dimensional hypersurface Γ_c in the $2N$-dimensional phase space Γ of the system.

Two dynamical variables F and G in Γ are said to be weakly equal to each other $(F \approx G)$ if they are equal on Γ_c, i.e.,

$$(F-G)\Big|_{\Gamma_c} = 0 \Rightarrow F \approx G.$$

We can define the canonical Hamiltonian of the theory as the Legendre transform

$$H_{can.} = \dot{q}_i p^i - L(q_i, \dot{q}_i) = \dot{q}_a p^a + \dot{q}_\alpha g^\alpha - L(q_i, \dot{q}_i).$$

Note that

$$\frac{\partial H_{can.}}{\partial \dot{q}_\alpha} = g^\alpha - \frac{\partial L}{\partial \dot{q}_\alpha} = g^\alpha - p^\alpha \approx 0.$$

So, even though $\dot{q}_\alpha$ cannot be determined, the canonical Hamiltonian on the constraint hypersurface Γ_c does not depend on the velocities:

$$H_{can.} = H_{can.}(q_i, p^a)$$

(presence of constraints)

On the other hand, because of the non-invertibility of the relation $(q_i, \dot{q}_i) \to (q_i, p^i)$, the Hamiltonian of the theory is no longer unique. From the canonical Hamiltonian, we can define a primary Hamiltonian associated with the theory, as:

$$H_p = H_{can.} + \lambda_\alpha \varphi^\alpha,$$

where λ_α are the undetermined Lagrange multipliers, and it follows that

$$H_p \approx H_{can.}$$

(It is called primary Hamiltonian since it incorporates only the primary constraints.)

Also, Hamilton's equations become

$$\dot{q}_i \approx \{q_i, H_p\} = \frac{\partial H_p}{\partial p^i} = \frac{\partial}{\partial p^i}\left(H_{can.} + \lambda_\alpha \varphi^\alpha\right)$$
$$\dot{p}^i \approx \{p^i, H_p\} = -\frac{\partial H_p}{\partial q_i} = -\frac{\partial}{\partial q_i}\left(H_{can.} + \lambda_\alpha \varphi^\alpha\right)$$

Using the above, we can identify $\lambda_\alpha \approx \dot{q}_\alpha$, which remain undetermined.

[Check this!]

$$H_p = H_{can.} + \lambda_\alpha \varphi^\alpha$$
$$\Rightarrow \quad \dot{q}_\alpha \approx \{q_\alpha, H_p\} = \frac{\partial H_p}{\partial p^\alpha} = \frac{\partial}{\partial p^\alpha}\left(H_{can.} + \lambda_\beta \varphi^\beta\right)$$
$$= \frac{\partial H_{can.}}{\partial p^\alpha} + \lambda_\beta \frac{\partial \varphi^\beta}{\partial p^\alpha} = \lambda_\alpha$$
$$\left[\because \ \varphi^\beta = p^\beta - g^\beta(q_i, p^a) \qquad \frac{\partial \varphi^\beta}{\partial p^\alpha} = \delta^\beta_\alpha\right]$$

Also, for any dynamical variable F, we have

$$\dot{F} \approx \{F, H_p\} = \{F, H_{can.} + \lambda_\alpha \varphi^\alpha\}$$

Note that the constraints of the system should be invariant under time evolution:

$$\dot{\varphi}^\alpha \approx \{\varphi^\alpha, H_{can.} + \lambda_\beta \varphi^\beta\}$$
$$= \{\varphi^\alpha, H_{can.}\} + \lambda_\beta\{\varphi^\alpha, \varphi^\beta\} + \{\varphi^\alpha, \lambda_\beta\}\varphi^\beta \qquad [\because \varphi^\beta \approx 0]$$
$$\approx \{\varphi^\alpha, H_{can.}\} + \lambda_\beta\{\varphi^\alpha, \varphi^\beta\} \approx 0$$

The above makes it clear that the condition $\dot{\varphi}^\alpha \approx 0$ could determine some of the Lagrange multipliers or may lead to new constraints, known as secondary constraints.

We continue this process of invariance under time evolution again on the secondary constraints which again may determine some other Lagrange multipliers or may generate newer (tertiary) constraints. We continue this process until all constraints are evolution - free.

- **Example** – A non-relativistic point particle on a sphere:

As a simple example of constrained systems, let us consider the motion of a point particle constrained to move on the surface of an n-dimensional unit sphere (with

Euclidean metric). The coordinates q_i, $i = 1, 2, \ldots, n$ of the particle satisfy

$$q_i q_i = 1$$

where summation over repeated indices is assumed. This simple system is an interesting example in the study of constrained quantization and is a prototype of the field theoretic model known as the nonlinear sigma model. The dynamics of the system is described by the Lagrangian

$$L = \frac{1}{2}\left[\dot{q}_i \dot{q}_i - F(q_i q_i - 1)\right],$$

where F is a Lagrange multiplier field (an auxiliary field without independent dynamics) whose Euler-Lagrange equation

$$\frac{\partial L}{\partial F} = -\frac{1}{2}(q_i q_i - 1) = 0,$$

gives the constraint on the dynamics. If we combine the dynamical variables q_i, F into $q_a = (q_i, F)$, $a = 1, 2, \ldots, n, n+1$, then the matrix of highest derivatives has the form

$$\frac{\partial^2 L}{\partial \dot{q}_a \partial \dot{q}_b} = \left(\begin{array}{c|c} \underset{N\times N}{\delta_{ij}} & \underset{N\times 1}{0} \\ \hline \underset{1\times N}{0} & \underset{1\times 1}{0} \end{array}\right),$$

which is indeed singular, reflecting the fact that there are constraints in the theory and the naive Hamiltonian description would not hold.

The conjugate momenta of the theory are given by

$$p^i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i\,, \quad p_F = \frac{\partial L}{\partial \dot{F}} = 0.$$

Thus, the theory has a primary constraint given by

$$\varphi^1 = p_F \approx 0$$

The canonical Hamiltonian has the form

$$H_{can.} = p^i \dot{q}_i + p_F \dot{F} - L = \frac{1}{2}\left[p^i p^i + F(q_i q_i - 1)\right]$$

which leads to the primary Hamiltonian

$$H_p = H_{can.} + \lambda_1 \varphi^1 = \frac{1}{2}\left[p^i p^i + F(q_i q_i - 1)\right] + \lambda_1 p_F.$$

The equal time canonical Poisson brackets of the theory are given by

$$\{q_i, q_j\} = \{p^i, p^j\} = \{F, F\} = \{p_F, p_F\} = 0$$

$$\{q_i, F\} = \{q_i, p_F\} = \{p^i, F\} = \{p^i, p_F\} = 0$$

$$\{q_i, p^j\} = \delta_i^j, \quad \{F, p_F\} = 1$$

Using these, we can now determine the time evolution of any dynamical variable. In particular, requiring the primary constraint to be independent of time leads to

$$\begin{aligned}
\dot{\varphi}^1 &\approx \{\varphi^1, H_p\} \\
&= \{p_F, \tfrac{1}{2}\left[p^i p^i + F(q_i q_i - 1)\right] + \lambda_1 p_F\} \\
&= -\tfrac{1}{2}(q_i q_i - 1) \approx 0.
\end{aligned}$$

Let us denote this secondary constraint as

$$\varphi^2 = \tfrac{1}{2}(q_i q_i - 1) \approx 0.$$

Requiring φ^2 to be time-independent, we obtain

$$\begin{aligned}
\dot{\varphi}^2 &\approx \{\varphi^2, H_p\} \\
&= \{\tfrac{1}{2}(q_i q_i - 1), \tfrac{1}{2}\left[p^j p^j + F(q_j q_j - 1)\right] + \lambda_1 p_F\} \\
&= \tfrac{1}{4}\{q_i q_i, p^j p^j\} \\
&= q_i p^i \approx 0,
\end{aligned}$$

which generates a new constraint

$$\varphi^3 = q_i p^i.$$

Again, requiring its time-independence, we have

$$\begin{aligned}
\dot{\varphi}^3 &\approx \{\varphi^3, H_p\} \\
&= \{q_i p^i, \tfrac{1}{2}\left[p^j p^j + F(q_j q_j - 1)\right] + \lambda_1 p_F\} \\
&= \tfrac{1}{2}\left[\{q_i p^i, p^j p^j\} + \{q_i p^i, F(q_j q_j - 1)\}\right] \\
&= \tfrac{1}{2}\left[\{q_i, p^j p^j\} p^i + q_i\{p^i, F(q_j q_j - 1)\} + \{q_i, F(q_j q_j - 1)\} p^i\right] \\
&= \tfrac{1}{2}\left[2 p^i p^j \{q_i, p^j\} + q_i F\{p^i, q_j q_j\}\right] \\
&= p^i p^j \{q_i, p^j\} + q_i F q_j \{p^i, q_j\} \\
&= p^i p^i - F q_i q_i \\
&\approx p^i p^i - F \approx 0
\end{aligned}$$

so that we can identify

$$\varphi^4 = p^i p^i - F \approx 0$$

Requiring its time - indepence as well leads to

$$\begin{aligned}
\dot{\varphi}^4 &\approx \{\varphi^4, H_p\} \\
&= \left\{p^i p^i - F, \tfrac{1}{2}\left[p^j p^j + F(q_j q_j - 1)\right] + \lambda_1 p_F\right\} \\
&= \tfrac{1}{2} F \{p^i p^i, q_j q_j\} - \lambda_1 \{F, p_F\} \\
&= 2 p^i F q_j \{p^i, q_j\} - \lambda_1 \{F, p_F\} \\
&= -2F q_i p^i - \lambda_1 \\
&\approx -\lambda_1 \approx 0,
\end{aligned}$$

which determines the Lagrange multiplier λ_1 and the chain of constraints terminates.

- **Appendix:**

Let $A_{n\times n}$ be an $n \times n$ matrix. We know,

$$\det(A^T) = \det(A)$$

If A is antisymmetric, then $A^T = -A$. Hence,

$$\det(A^T) = \det(-A) = (-1)^n \det(A)$$

$\Rightarrow$ $\det(A) = 0$ if A is antisymmetric.

$\Rightarrow$ An antisymmetric matrix of odd dimensions is not invertible.

- **Number of second class constraints is always even:**

Let ϕ^A be a second class constraint, and let's consider the matrix $C_{AB} = \{\phi^A, \phi^B\}$ of Poisson brackets of all the second class constraints. By definition, C_{AB} is antisymmetric. And Dirac had shown that C_{AB} is always invertible. Hence, C_{AB} must be even dimensional $\longrightarrow$ hence the theorem.