Lecture - 4

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PHY781 : Quantum Field Theory - II

4. Relativistic Free Particle

- **Relativistic Free Particle:**

Previously we have studied a simple example of point particle dynamics which was constrained. The constraints of this theory were all second class. Let us next study the Hamiltonian description of a free relativistic massive particle which is described by the equation

$$m\frac{d^2x^\mu}{d\tau^2}=0, \qquad \mu=0,1,2,3,$$

where τ, denoting the proper time, is assumed to label the trajectory of the particle and m is the rest mass of the particle. We note that if we define the relativistic four velocity and momentum of the particle as

$$u^\mu=\frac{dx^\mu}{d\tau}, \quad p^\mu=mu^\mu,$$

then the dynamical equation can be written as

$$\frac{dp^\mu}{d\tau}=0.$$

This dynamical equation can also be obtained as the Euler-Lagrange equation following from the action (recall that $c=1$)

$$\tilde{S}=m\int ds=m\int d\lambda\,(\dot{x}^\mu\dot{x}_\mu)^{\frac{1}{2}}=\int d\lambda\, L,$$

where λ is a parameter labelling the trajectory (worldline) and we have identified

$$\dot{x}^\mu=\frac{dx^\mu}{d\lambda}, \quad \dot{x}_\mu=\eta_{\mu\nu}\dot{x}^\nu,$$

with $\eta_{\mu\nu}$ denoting the Minkowski metric. Note that under a transformation

$$\lambda\to\xi=\xi(\lambda), \quad \dot{x}^\mu=\frac{dx^\mu}{d\lambda}=\frac{d\xi}{d\lambda}\frac{dx^\mu}{d\xi},$$

so that

$$\tilde{S}=m\int d\lambda\left(\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}\right)^{1/2}\longrightarrow m\int d\lambda\left(\frac{d\xi}{d\lambda}\frac{dx^\mu}{d\xi}\frac{d\xi}{d\lambda}\frac{dx_\mu}{d\xi}\right)^{1/2}$$

$$=m\int d\lambda\frac{d\xi}{d\lambda}\left(\frac{dx^\mu}{d\xi}\frac{dx_\mu}{d\xi}\right)^{1/2}=m\int d\xi\left(\frac{dx^\mu}{d\xi}\frac{dx_\mu}{d\xi}\right)^{1/2}=\tilde{S}.$$

Namely, the action $\tilde{S}$ is invariant under the diffeomorphism $\lambda \to \xi = \xi(\lambda)$ which is an example of a local gauge transformation much like in the Maxwell field theory. If we choose a gauge

$$\lambda = \tau,$$

where τ denotes the proper time associated with the particle, then the action $\tilde{S}$ will lead to the dynamical equation as the Euler-Lagrange equation. However, let's proceed with the Hamiltonian description without choosing a gauge at this point. We note from the action $\tilde{S}$ that

$$P_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = \frac{m}{\sqrt{\dot{x}^\rho \dot{x}_\rho}}\left(\eta_{\mu\nu} - \frac{\dot{x}_\mu \dot{x}_\nu}{\dot{x}^\sigma \dot{x}_\sigma}\right),$$

which leads to

$$\dot{x}^\mu P_{\mu\nu} = \frac{m}{\sqrt{\dot{x}^\rho \dot{x}_\rho}}\left(\dot{x}_\nu - \frac{\dot{x}^\mu \dot{x}_\mu \dot{x}_\nu}{\dot{x}^\sigma \dot{x}_\sigma}\right) = 0 = P_{\mu\nu}\dot{x}^\nu$$

Therefore, $P_{\mu\nu}$ is a projection operator and

$$\det.(P_{\mu\nu}) = \det. \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = 0$$

as in the case of the Maxwell theory and the naive passage to the Hamiltonian description fails.

Note that the action $\tilde{S}$ can also be written in an alternate form which is more useful. With the current form of $\tilde{S}$, it is not clear how to take the $m = 0$ limit to describe the motion of a massless particle. However, introducing an auxiliary variable g, we can rewrite the action $\tilde{S}$ in the form

$$S = \int d\lambda\, L = \frac{1}{2}\int d\lambda \sqrt{g}\,\left(g^{-1}\dot{x}^\mu \dot{x}_\mu + m^2\right),$$

where, because of the diffeomorphism invariance of $\tilde{S}$, we can think of g as the metric of the one dimensional manifold of the trajectory of the particle. Viewed in this way, the action S is manifestly diffeomorphism invariant and the massless limit can be now taken in the straightforward way. To see that the two actions $\tilde{S}$ and S are equivalent, we consider the equation for g which takes the form

$$\frac{\partial L}{\partial g} = \frac{1}{4\sqrt{g}}\left(-g^{-1}\dot{x}^\mu \dot{x}_\mu + m^2\right) = 0,$$

or, $$g = \frac{\dot{x}^\mu \dot{x}_\mu}{m^2}$$

- **Eliminate g in S to get back $\tilde{S}$.**

$$S = \frac{1}{2}\int d\lambda \sqrt{g}\left(g^{-1}\dot{x}^\mu \dot{x}_\mu + m^2\right), \quad g = \frac{\dot{x}^\mu \dot{x}_\mu}{m^2}$$

$$\Rightarrow \quad S = \frac{1}{2}\int d\lambda \sqrt{g}\; 2m^2 = \int d\lambda \sqrt{g}\, m^2 = m\int d\lambda\; m\sqrt{g}$$

$$= m\int d\lambda \sqrt{\dot{x}^\mu \dot{x}_\mu} = \tilde{S}.$$

However, it is more convenient to work with S. The two dynamical variables of the theory are x^μ and g. The momenta conjugate to these variables are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{\sqrt{g}}\dot{x}_\mu, \quad p_g = \frac{\partial L}{\partial \dot{g}} = 0,$$

so that the primary constraint of the theory is

$$\varphi^1 = p_g \approx 0.$$

The canonical Hamiltonian is

$$H_{can.} = p_\mu \dot{x}^\mu + p_g \dot{g} - L = \frac{1}{\sqrt{g}}\dot{x}_\mu \dot{x}^\mu - L = \frac{1}{2\sqrt{g}}\left(\dot{x}^\mu \dot{x}_\mu - gm^2\right)$$

$$= \frac{\sqrt{g}}{2}\left(p_\mu p^\mu - m^2\right).$$

Adding the primary constraint, we get the primary Hamiltonian of the theory

$$H_p = H_{can.} + \lambda_1 \varphi^1 = \frac{\sqrt{g}}{2}\left(p_\mu p^\mu - m^2\right) + \lambda_1 p_g.$$

The (equal time) canonical Poisson brackets of the theory are

$$\{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = \{g, g\} = \{p_g, p_g\} = 0,$$

$$\{x^\mu, g\} = \{x^\mu, p_g\} = \{p_\mu, g\} = \{p_\mu, p_g\} = 0,$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{g, p_g\} = 1.$$

With these we can now determine the evolution of the primary constraint and requiring the constraint to be time independent, we obtain

$$\dot{\varphi}^1 \approx \{\varphi^1, H_p\} = \left\{p_g, \frac{\sqrt{g}}{2}\left(p_\mu p^\mu - m^2\right) + \lambda_1 p_g\right\}$$
$$= -\frac{1}{4\sqrt{g}}\left(p_\mu p^\mu - m^2\right) \approx 0,$$

which leads to the secondary constraint

$$\varphi^2 = \frac{1}{2}\left(p_\mu p^\mu - m^2\right) \approx 0.$$

We recognize that the secondary constraint is the familiar Einstein relation for a massive relativistic particle (the multiplicative factor is for later calculational simplicity). The time evolution of the secondary constraint leads to

$$\dot{\varphi}^2 \approx \{\varphi^2, H_p\} = \left\{\frac{1}{2}\left(p_\mu p^\mu - m^2\right), \frac{\sqrt{g}}{2}\left(p_\mu p^\mu - m^2\right) + \lambda_1 p_g\right\} = 0,$$

so that it is time independent and the chain of constraints ends.

- **Check the class of the constraints of the theory.**

$$\{\varphi^1, \varphi^2\} = \left\{p_g, \frac{1}{2}\left(p_\mu p^\mu - m^2\right)\right\} = 0 = \{\varphi^1, \varphi^1\} = \{\varphi^2, \varphi^2\},$$

so the constraints are both first class. This is associated with the fact that the theory has a local gauge invariance and is also reflected in the fact that the Lagrange multiplier λ_1 remain arbitrary (the primary constraint is first class). Let us choose the gauge fixing conditions (one for every first class constraint)

$$\chi_1 = g - \frac{1}{m^2} \approx 0,$$
$$\chi_2 = \lambda - x^0 \approx 0,$$

which render all the constraints to be second class. The non vanishing Poisson brackets between the constraints are given by

$$\{\varphi^1, \chi^1\} = \left\{p_g, g - \frac{1}{m^2}\right\} = -1 = -\{\chi^1, \varphi^1\}$$
$$\{\varphi^2, \chi^2\} = \left\{\frac{1}{2}\left(p_\mu p^\mu - m^2\right), \lambda - x^0\right\} = -\frac{1}{2}\{p_\mu p^\mu, x^0\}$$
$$= p^0 = -\{\chi^2, \varphi^2\}.$$

As a result, if we combine all the constraints into $\phi^A = \{\varphi^1, \varphi^2, \chi^1, \chi^2\}$, then it follows from above that the matrix of Poisson brackets of constraints is

$$C^{AB} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & p^0 \\ 1 & 0 & 0 & 0 \\ 0 & -p^0 & 0 & 0 \end{pmatrix},$$

whose inverse is given by

$$C^{-1}_{AB} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{p^0} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{p^0} & 0 & 0 \end{pmatrix},$$

The equal time Dirac bracket between any two dynamical variables is given by

$$\{F, G\}_D = \{F, G\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\}.$$

- **Compute the equal time Dirac brackets between the fundamental dynamical variables.**

$$\varphi^1 = p_g, \quad \varphi^2 = \tfrac{1}{2}(p_\mu p^\mu - m^2), \quad \chi^1 = g - \frac{1}{m^2}, \quad \chi^2 = \lambda - x^0$$

$$\{x^\mu, x^\nu\}_D = \{x^\mu, x^\nu\} - \{x^\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, x^\nu\} = 0,$$

$$\{p_\mu, p_\nu\}_D = \{p_\mu, p_\nu\} - \{p_\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, p_\nu\} = -\{p_\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, p_\nu\} = 0,$$

$$\{g, g\}_D = \{g, g\} - \{g, \phi^A\} C^{-1}_{AB} \{\phi^B, g\} = 0,$$

$$\{p_g, p_g\}_D = \{p_g, p_g\} - \{p_g, \phi^A\} C^{-1}_{AB} \{\phi^B, p_g\} = 0,$$

$$\{x^\mu, g\}_D = \{x^\mu, g\} - \{x^\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, g\} = 0,$$

$$\{x^\mu, p_g\}_D = \{x^\mu, p_g\} - \{x^\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, p_g\} = 0,$$

$$\{p_\mu, g\}_D = \{p_\mu, g\} - \{p_\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, g\} = 0,$$

$$\{p_\mu, p_g\}_D = \{p_\mu, p_g\} - \{p_\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, p_g\} = 0,$$

$$\{g, p_g\}_D = \{g, p_g\} - \{g, \phi^A\} C^{-1}_{AB} \{\phi^B, p_g\} = 1 - \{g, p_g\} C^{-1}_{13} \{g - \tfrac{1}{m^2}, p_g\} = 0,$$

$$\begin{aligned}\{x^\mu, p_\nu\}_D &= \{x^\mu, p_\nu\} - \{x^\mu, \phi^A\} C^{-1}_{AB} \{\phi^B, p_\nu\} = \delta^\mu_\nu - \{x^\mu, \phi^2\} C^{-1}_{24} \{\phi^4, p_\nu\} \\ &= \delta^\mu_\nu - \{x^\mu, \tfrac{1}{2}(p_\alpha p^\alpha - m^2)\}\left(-\tfrac{1}{p^0}\right)\{\lambda - x^0, p_\nu\} \\ &= \delta^\mu_\nu - \frac{1}{2p^0} 2 p^\alpha \delta^\mu_\alpha \delta^0_\nu = \delta^\mu_\nu - \frac{p^\mu}{p^0}\delta^0_\nu.\end{aligned}$$

With the Dirac brackets we can set the constraints to zero in which case we see that the primary Hamiltonian $H_p = \frac{\sqrt{g}}{2}(p_\mu p^\mu - m^2) + \lambda_1 p_g$ vanishes. On the other hand, if we set the constraints strongly to zero, $\varphi^2 = \frac{1}{2}(p^\mu p_\mu - m^2)$ in particular leads to $H = p^0 = \sqrt{\mathbf{p}^2 + m^2}$ which can be taken as the Hamiltonian. Note that the first gauge fixing condition implies

$$\chi^1 = g - \frac{1}{m^2} = 0 \Rightarrow \frac{\dot{x}^\mu \dot{x}_\mu}{m^2} - \frac{1}{m^2} = 0 \Rightarrow \frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} = 1.$$

On the other hand, from the definition of the infinitesimal invariant length in the Minkowski space, we have ($c = 1$)

$$d\tau^2 = dx^\mu dx_\mu,$$

or,

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = 1.$$

It follows, therefore, that the first gauge fixing condition corresponds to choosing $\lambda = \tau$.