Lecture - 3

17 - 01 - 2025

Arjun Bagchi

PHY781 : Quantum Field Theory - II

3. Classification of Constraints and Dirac Brackets

• Types of Constraints:

Let us denote all the constraints of the theory (primary, secondary, tertiary,...) collectively as

$$\varphi^{\bar{\alpha}} \approx 0, \quad \bar{\alpha} = 1, 2, \ldots, n < 2N,$$

which can be divided into two distinct classes:

1. **First class** constraints, which have weakly vanishing Poisson brackets with all the constraints (including themselves), and are denoted by
$$\psi^{\tilde{\alpha}} \approx 0, \quad \tilde{\alpha} = 1, 2, \ldots, n_1.$$
2. **Second class** constraints, which have at least one non vanishing Poisson bracket with the constraints, and are denoted as
$$\phi^{\hat{\alpha}} \approx 0, \quad \hat{\alpha} = 1, 2, \ldots, 2n_2.$$

Here we have used the observation due to Dirac that **there are an even number of second class constraints in a theory** and we note that $n_1 + 2n_2 = n < 2N$.

It is also worth noting that in this process of determining all the constraints that are time independent (evolution free), all of the Lagrange multipliers may be completely determined or some of them may remain undetermined. In general, if there are primary constraints in the theory which are first class, then there remain undetermined Lagrange multipliers in the primary Hamiltonian after all the constraints have been determined (The number of undetermined Lagrange multipliers equals the number of first class constraints among the primary constraints).

The first class constraints have a special significance in that they are associated with gauge in variances (local invariances) in the theory. As we've seen in the case of Maxwell field theory, a consistent Hamiltonian description, in this case, requires gauge fixing conditions as there are first class constraints. The gauge fixing conditions are generally denoted as

$$\chi^{\tilde{\alpha}} \approx 0, \qquad \tilde{\alpha} = 1, 2, \ldots, n_1,$$

and are chosen such that they convert the first class constraints into second class constraints. Thus, after gauge fixing, we denote all the constraints of the theory (including the gauge fixing conditions), which are all second class, collectively as

$$\phi^A \approx 0, \quad A = 1, 2, \ldots, 2(n_1 + n_2) < 2N,$$

so that the true phase space of the theory is $2(N - n_1 - n_2)$ dimensional.

- **Dirac Brackets:**

Although we have identified all the constraints of the theory, we cannot yet impose them directly in the theory since the canonical Poisson brackets are not compatible with them. Namely, even though

$$\phi^A \approx 0,$$

the Poisson bracket of the constraints with any dynamical variable F does not, in general, vanish:

$$\{F, \phi^A\} \not\approx 0.$$

This issue of incompatibility of the Poisson brackets can be addressed through the use of the Dirac brackets which are constructed as follows. First, note that we can construct the matrix of Poisson brackets of all the constraints of the form $\phi^A \approx 0$ as

$$\{\phi^A, \phi^B\} \approx C^{AB}.$$

This is an even-dimensional anti-symmetric matrix and Dirac had shown that it is nonsingular so that its inverse C^{-1}_{AB} exists, namely,

$$C^{AD} C^{-1}_{DB} = \delta^A_B = C^{-1}_{BD} C^{DA}.$$

In field theories (where the variables depend on space coordinates as well), the C^{AB} matrix of Poisson brackets will be coordinate dependent and the product $C^{AD} C^{-1}_{DB} = \delta^A_B = C^{-1}_{BD} C^{DA}$ will involve integration over the intermediate space coordinate. With this, we can now define the Dirac bracket between any two dynamical variables as

$$\{F, G\}_D = \{F, G\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\},$$

which can be shown to have all the properties of a Poisson bracket (anti-symmetry, Jacobi identity) and in addition satisfies

$$\begin{aligned}\{F, \phi^A\}_D &= \{F, \phi^A\} - \{F, \phi^B\} C^{-1}_{BD} \{\phi^D, \phi^A\} \\ &\approx \{F, \phi^A\} - \{F, \phi^B\} C^{-1}_{BD} C^{DA} \\ &= \{F, \phi^A\} - \{F, \phi^A\} = 0.\end{aligned}$$

- **Exercise**: Establish the anti-symmetry and Jacobi identity for Dirac brackets.

We have

$$\{F, G\}_D = \{F, G\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\},$$

$$\begin{aligned}\Rightarrow \{F, G\}_D + \{G, F\}_D &= \{F, G\} + \{G, F\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\} - \{G, \phi^A\} C^{-1}_{AB} \{\phi^B, F\} \\ &= -\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\} - \{G, \phi^A\} C^{-1}_{AB} \{\phi^B, F\} \\ &= -\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\} - \{G, \phi^B\} C^{-1}_{BA} \{\phi^A, F\} \\ &= -\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\} - \{F, \phi^A\} C^{-1}_{BA} \{\phi^B, G\} \\ &= -\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\} + \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\} \\ &= 0.\end{aligned}$$

Thus, the Dirac bracket is antisymmetric. Also,

$$\begin{aligned}\{F, \{G, K\}_D\}_D &= \{F, \{G, K\}_D\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, K\}_D\} \\ &= \{F, \{G, K\} - \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} \\ &\quad - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, K\} - \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} \\ &= \{F, \{G, K\}\} - \{F, \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} \\ &\quad - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, K\}\} + \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\}\end{aligned}$$

Now, we make use of the facts that the Poisson brackets satisfy the Jacobi identity and for second class constraints ϕ^A, $\{F, \phi^B\}_D \approx 0$:

$$\begin{aligned}&\{F, \{G, K\}_D\}_D + \{G, \{K, F\}_D\}_D + \{K, \{F, G\}_D\}_D \\ =\ &\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} \\ &- \{F, \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} - \{G, \{K, \phi^P\} C^{-1}_{PQ} \{\phi^Q, F\}\} - \{K, \{F, \phi^P\} C^{-1}_{PQ} \{\phi^Q, G\}\} \\ &- \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, K\}\} - \{G, \phi^A\} C^{-1}_{AB} \{\phi^B, \{K, F\}\} - \{K, \phi^A\} C^{-1}_{AB} \{\phi^B, \{F, G\}\} \\ &+ \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} + \{G, \phi^A\} C^{-1}_{AB} \{\phi^B, \{K, \phi^P\} C^{-1}_{PQ} \{\phi^Q, F\}\} \\ &+ \{K, \phi^A\} C^{-1}_{AB} \{\phi^B, \{F, \phi^P\} C^{-1}_{PQ} \{\phi^Q, G\}\}\end{aligned}$$

$$
\begin{aligned}
= \quad & -\{F, \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} - \{G, \{K, \phi^P\} C^{-1}_{PQ} \{\phi^Q, F\}\} - \{K, \{F, \phi^P\} C^{-1}_{PQ} \{\phi^Q, G\}\} \\
& - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, K\}\} - \{G, \phi^A\} C^{-1}_{AB} \{\phi^B, \{K, F\}\} - \{K, \phi^A\} C^{-1}_{AB} \{\phi^B, \{F, G\}\} \\
& + \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, \{G, \phi^P\} C^{-1}_{PQ} \{\phi^Q, K\}\} + \{G, \phi^A\} C^{-1}_{AB} \{\phi^B, \{K, \phi^P\} C^{-1}_{PQ} \{\phi^Q, F\}\} \\
& + \{K, \phi^A\} C^{-1}_{AB} \{\phi^B, \{F, \phi^P\} C^{-1}_{PQ} \{\phi^Q, G\}\} \\
\approx 0. &
\end{aligned}
$$

In contrast to the canonical Poisson brackets, the Dirac bracket is compatible with the constraints in the sense that any dynamical variable of the theory has a vanishing Dirac bracket with a constraint. In fact, through the use of Lagrange brackets, we can show that the Dirac brackets are indeed the correct Poisson brackets of the theory subject to the constraints $\phi^A \approx 0$. As a result, we can work with the Dirac brackets and set the constraints to zero in the theory (namely, in the definitions of the Hamiltonian, energy-momentum tensor and other observables of the theory since their Dirac bracket with any variable vanishes) and it is the Dirac brackets that can be quantized (promoted to commutators or anti-commutators) in a quantum theory. The Dirac bracket has the interesting property that it can be constructed in stages. Namely, when the number of constraints is large, rather than dealing with a large matrix of Poisson brackets and its inverse, we can equivalently choose a smaller set of an even number of constraints and define an intermediate Dirac bracket and then construct the final Dirac bracket of the theory building on this structure. It is also worth emphasizing here that in dealing with field theories where the matrix C^{AB} is coordinate dependent, the inverse needs to be defined carefully with the appropriate boundary condition relevant for the problem.

Let us go back to our example of a point particle on a sphere.

- **What class does each of the constraints belong to?**

 We have the constraints $\varphi^1 = p_F$, $\varphi^2 = \frac{1}{2}(q_i q_i - 1)$, $\varphi^3 = q_i p^i$, and $\varphi^4 = p^i p^i - F$. Now,

$$
\begin{aligned}
\{\varphi^1, \varphi^4\} &= \{p_F, p^i p^i - F\} = 1 = -\{\varphi^4, \varphi^1\}, \\
\{\varphi^2, \varphi^3\} &= \{\tfrac{1}{2}(q_i q_i - 1), q_j p^j\} = q_i q_j \{q_i, p^j\} = q_i q_i \approx 1 = -\{\varphi^3, \varphi^2\}, \\
\{\varphi^3, \varphi^4\} &= \{q_i p^i, p^j p^j - F\} = 2\{q_i, p^j\} p^i p^j - q_i \{p^i, F\} = 2 p^i p^i = 2\mathbf{p}^2
\end{aligned}
$$

$$= -\{\varphi^4, \varphi^3\}.$$

where we've identified $p^i p^i = \mathbf{p}^2$. Hence, all the constraints are second class.

Collecting all the constraints into $\phi^A = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$, $A = 1, 2, 3, 4$, we can calculate the matrix of the (equal time) Poisson brackets of constraints:

$\{\phi^A, \phi^B\} = C^{AB}$, and its inverse, C^{-1}_{AB}:

$$C^{AB} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2\mathbf{p}^2 \\ -1 & 0 & -2\mathbf{p}^2 & 0 \end{pmatrix}, \quad C^{-1}_{AB} = \begin{pmatrix} 0 & -2\mathbf{p}^2 & 0 & -1 \\ 2\mathbf{p}^2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- **Exercise**: Compute the Dirac brackets for the non-relativistic particle constrained to move on a sphere.

The Dirac brackets between any two dynamical variables are defined as

$$\{F, G\}_D = \{F, G\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, G\}$$

The fundamental dynamical variables are F, p_F, q_i, p^i. Let's now compute their Dirac brackets:

$$\{F, F\}_D = \{F, F\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, F\} = -\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, F\}$$
$$= -\{F, \phi^B\} C^{-1}_{BA} \{\phi^A, F\} = -\{\phi^B, F\} C^{-1}_{BA} \{F, \phi^A\}$$
$$= -\{F, \phi^A\} C^{-1}_{BA} \{\phi^B, F\} = \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, F\}$$

Thus, we see that $-\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, F\} = \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, F\}$, so that

$$\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, F\} = 0.$$

Consequently, $\boxed{\{F, F\}_D = 0}$ In the same way, $\boxed{\{p_F, p_F\}_D = 0.}$

$$\{F, p_F\}_D = \{F, p_F\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, p_F\}$$
$$= 1 - \{F, \phi^1\} C^{-1}_{12} \{\phi^2, p_F\} - \{F, \phi^1\} C^{-1}_{14} \{\phi^4, p_F\} - \{F, \phi^2\} C^{-1}_{21} \{\phi^1, p_F\}$$
$$- \{F, \phi^2\} C^{-1}_{23} \{\phi^3, p_F\} - \{F, \phi^3\} C^{-1}_{32} \{\phi^2, p_F\} - \{F, \phi^4\} C^{-1}_{41} \{\phi^1, p_F\}$$

But, $\{F, \phi^1\} = \{F, p_F\} = 1$, $\{\phi^2, p_F\} = \{\frac{1}{2}(q_i q_i - 1), p_F\} = 0$,

$\{F, \phi^2\} = \{F, \frac{1}{2}(q_i q_i - 1)\} = 0$, $\{\phi^1, p_F\} = \{p_F, p_F\} = 0$, and

$\{\phi^4, p_F\} = \{p^i p^i - F, p_F\} = -1.$

Thus, $\{F, p_F\}_D = 1 - \{F, \phi^1\} C^{-1}_{14} \{\phi^4, p_F\} = 1 - 1.(-1)\cdot(-1) = 0.$

$$\Rightarrow \boxed{\{F, p_F\}_D = 0}$$

$$\{q_i, F\}_D = \{q_i, F\} - \{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, F\} = -\{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, F\}$$

$$= -\{q_i, \phi^1\} C^{-1}_{12} \{\phi^2, F\} - \{q_i, \phi^1\} C^{-1}_{14} \{\phi^4, F\}$$

$$-\{q_i, \phi^2\} C^{-1}_{21} \{\phi^1, F\} - \{q_i, \phi^2\} C^{-1}_{23} \{\phi^3, F\}$$

$$-\{q_i, \phi^3\} C^{-1}_{32} \{\phi^2, F\} - \{q_i, \phi^4\} C^{-1}_{41} \{\phi^1, F\}$$

$$= -\{q_i, \phi^4\} C^{-1}_{41} \{\phi^1, F\} = -(2p^i) \cdot 1 \cdot (-1) = 2p^i$$

$$\Rightarrow \boxed{\{q_i, F\}_D = 2p^i}$$

$$\{q_i, p_F\}_D = \{q_i, p_F\} - \{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, p_F\} = -\{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, p_F\} = 0$$

$$\Rightarrow \boxed{\{q_i, p_F\}_D = 0}$$

$$\{F, p^i\}_D = \{F, p^i\} - \{F, \phi^A\} C^{-1}_{AB} \{\phi^B, p^i\} = -\{F, \phi^A\} C^{-1}_{AB} \{\phi^B, p^i\}$$

$$= -\{F, \phi^1\} C^{-1}_{12} \{\phi^2, p^i\} = -\{F, p^F\}(-2\mathbf{p}^2)\{\tfrac{1}{2}(q_j q_j - 1), p^i\}$$

$$= 2\mathbf{p}^2 q_j \delta_j^i = 2 q_i \mathbf{p}^2$$

$$\Rightarrow \boxed{\{F, p^i\}_D = 2 q_i \mathbf{p}^2}$$

$$\{p^i, p_F\}_D = \{p^i, p_F\} - \{p^i, \phi^A\} C^{-1}_{AB} \{\phi^B, p_F\} = -\{p^i, \phi^A\} C^{-1}_{AB} \{\phi^B, p_F\} = 0$$

$$\Rightarrow \boxed{\{p^i, p_F\}_D = 0}$$

$$\{q_i, q_j\}_D = \{q_i, q_j\} - \{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, q_j\} = -\{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, q_j\} = 0$$

$$\Rightarrow \boxed{\{q_i, q_j\}_D = 0}$$

$$\{p^i, p^j\}_D = \{p^i, p^j\} - \{p^i, \phi^A\} C^{-1}_{AB} \{\phi^B, p^j\} = -\{p^i, \phi^A\} C^{-1}_{AB} \{\phi^B, p^j\}$$

$$= -\{p^i, \phi^2\} C^{-1}_{23} \{\phi^3, p^j\} - \{p^i, \phi^3\} C^{-1}_{32} \{\phi^2, p^j\}$$

$$= -\{p^i, \tfrac{1}{2}(q_k q_k - 1)\}(-1)\{q_l p^l, p^j\} - \{p^i, q_k p^k\} \cdot 1 \cdot \{\tfrac{1}{2}(q_l q_l - 1), p^j\}$$

$$= -\tfrac{1}{2}\, 2 q_k (-\delta^i_k)(-1) p^l \delta^j_l - p^k(-\delta^i_k) \cdot 1 \cdot \tfrac{1}{2}\, 2 q_l \delta^j_l$$

$$= -q_i p^j + p^i q_j$$

$$\Rightarrow \boxed{\{p^i, p^j\}_D = -q_i p^j + p^i q_j}$$

$$\{q_i, p^j\}_D = \{q_i, p^j\} - \{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, p^j\} = \delta_i^j - \{q_i, \phi^A\} C^{-1}_{AB} \{\phi^B, p^j\}$$
$$= \delta_i^j - \{q_i, \phi^3\} C^{-1}_{32} \{\phi^2, p^j\}$$
$$= \delta_i^j - \{q_i, q_k p^k\} \cdot 1 \cdot \{\tfrac{1}{2}(q_l q_l - 1), p^j\}$$
$$= \delta_i^j - q_k \delta_i^k \tfrac{1}{2} \cdot 2 q_l \delta_l^j = \delta_i^j - q_i q_j$$
$$\Rightarrow \boxed{\{q_i, p^j\}_D = \delta_i^j - q_i q_j}$$

With the use of the Dirac brackets, we can set the constraints to zero so that the true independent dynamical variables are (q_i, p^i) (we note that $F = p^i p^i$ because of the constraint $\varphi^4 = p^i p^i - F \approx 0$) and the Hamiltonian for the theory is given by

$$H_p = \frac{1}{2} p^i p^i,$$

where we have used $\dot{\varphi}^4 \approx 0 \Rightarrow -\lambda_1 \approx 0$. This is the starting point for the quantization of this theory.

To quantize such a system, we use Dirac brackets instead of using Poisson brackets:

Previously, we did : $\{\cdot, \cdot\}_{PB} \longrightarrow \frac{i}{\hbar}[\cdot, \cdot]$

Now, we do : $\{\cdot, \cdot\}_D \longrightarrow \frac{i}{\hbar}[\cdot, \cdot]$