**Lecture - 1**

09 - 01 - 2025

Arjun Bagchi

**PHY781 : Quantum Field Theory - II**

Evaluation: 4 Assignments + 1 Take Home Exam (24 hrs),

+ Mid-semester Project : 30 min. of talk → 50% marks.

TUE : 17:15 - ...

WED : 12:00 - 13:30 } L9

THU : 17:15 - ...

# 1. Constrained Systems

Dynamical phase space variables may not be all independent. Some of those variables may satisfy constraints following the structure of the theory. This is what happens e.g. in electrodynamics. Such systems are known as constrained systems and the transition from the Lagrangian to the Hamiltonian description in the naive way fails. In these cases, following a procedure by Dirac, one can get to the Hamiltonian formulation from the Lagrangian. This is what we will do in the first part of the course.

---

Consider a classical system of point particles described by the Lagrangian

$$L = L(q_i, \dot{q}_i) \; ; \quad i = 1, 2, \ldots, N.$$

Here $q_i$, $\dot{q}_i$ are all independent → the configuration space is 2N dimensional. We have the canonically conjugate momenta:

$$p^i = \frac{\partial L}{\partial \dot{q}_i} \qquad \text{———— (1)}$$

If this relation is invertible, velocities can be expressed in terms of coordinates and momenta and one can switch to the phase space from the configuration space:

$$(q_i, \dot{q}_i) \longrightarrow (q_i, p^i)$$

The Hamiltonian is defined by the Legendre transformation:

$$H(q_i, p^i) = p^i \dot{q}_i - L(q_i, \dot{q}_i).$$

The phase space is spanned by the independent coordinates and momenta and is 2N dimensional. It is equipped with an equal time Poisson bracket structure:

$$\{q_i, q_j\} = 0 = \{p^i, p^j\} \; ; \quad \{q_i, p^j\} = \delta_i^{\,j} = -\{p^j, q_i\}.$$

For any two dynamical variables $A_1(q_i, p^i)$ and $A_2(q_i, p^i)$,

$$\{A_1, A_2\} = \frac{\partial A_1}{\partial q_i}\frac{\partial A_2}{\partial p^i} - \frac{\partial A_1}{\partial p^i}\frac{\partial A_2}{\partial q_i}$$

There are 2N first-order dynamical equations,

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p^i} \quad ; \quad \dot{p}^i = \{p^i, H\} = -\frac{\partial H}{\partial q_i}$$

These are equivalent to the Euler-Lagrange equations of motion.

For any dynamical variable A,

$$\dot{A}(q_i, p^i) = \{A(q_i, p^i), H\}.$$

Difficulty in transitioning from Lagrangian to Hamiltonian formulation arises when Eq (1) is non-invertible. The transformation to phase space can in general be written as

$$\begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{A}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \tilde{\mathbf{b}} & \mathbf{b} \end{pmatrix}\begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

[Here, $\mathbf{A}$ is a $2N \times 2N$ matrix; $\mathbf{1}$, $\mathbf{O}$, $\tilde{\mathbf{b}}$, $\mathbf{b}$ are $N \times N$ matrices each; $q$, $\dot{q}$, $p$ are $N \times 1$ column matrices each.]

Thus, $\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ -\mathbf{b}^{-1}\tilde{\mathbf{b}} & \mathbf{b}^{-1} \end{pmatrix}$.

[ Let $\mathbf{A}^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, so $\begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \tilde{\mathbf{b}} & \mathbf{b} \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{pmatrix}$,

$$\Rightarrow \quad X = \mathbf{1}, \quad Y = \mathbf{O}, \quad \tilde{\mathbf{b}} + \mathbf{b}Z = \mathbf{O} \Rightarrow Z = -\mathbf{b}^{-1}\tilde{\mathbf{b}}, \quad \mathbf{b}W = \mathbf{1} \Rightarrow W = \mathbf{b}^{-1}.$$ ]

$\Rightarrow$ $\mathbf{A}^{-1}$ exists only when $\mathbf{b}$ is invertible.

[Note that L is quadratic in $\dot{\mathbf{q}}$. The relation $\frac{\partial L}{\partial \dot{\mathbf{q}}} = \mathbf{p} = \tilde{\mathbf{b}}\mathbf{q} + \mathbf{b}\dot{\mathbf{q}}$ indicates that $\mathbf{b} = \mathbf{b}(\mathbf{q})$ and $\tilde{\mathbf{b}} = \tilde{\mathbf{b}}(\mathbf{q})$.]

$$\Rightarrow \quad p^i = \tilde{b}^{ij}(\mathbf{q})\, q_j + b^{ij}(\mathbf{q})\,\dot{q}_j \quad \Rightarrow b^{ij}(q) = \frac{\partial p^i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}.$$

So, the transformation $\mathbf{A}$ exists if $\det.\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0$. Naïve transition to Hamiltonian formulation is possible only in this case.