

$$\lceil (d_{\min} - 1) / 2 \rceil.$$

Hamming Codes:

Hamming codes are (n, k) linear block codes, those which satisfies the conditions

- ① No of check bits $q \geq 3$.
- ② Block length $n = 2^q - 1$.
- ③ Number of msg bits $(k) = n - q$.
- ④ Min^m distance $d_{\min} = 3$.

i.e., Code rate $(r) = k/n \Rightarrow r = \frac{n-q}{n}$ for H.C $(k) = n-q$

$$r = 1 - \frac{q}{n}.$$

By substituting $n = 2^q - 1 \Rightarrow r = 1 - \frac{q}{2^q - 1}$

It is observed that $r \approx 1$ if $q \gg 1$.

Error Detection & Error Correction Capabilities of Hamming Codes:

- Min^m distance (d_{\min}) of Hamming code is 3.
- It can be used to detect double errors

(or) to correct single error.

For detecting double (2) errors $\Rightarrow d_{\min} \geq 2+1$ is ~~done~~ $d_{\min} \geq 3$

For correcting upto one (1) errors $\Rightarrow d_{\min} \geq 2+1$ is ~~done~~ $d_{\min} \geq 3$.

—x—

Ex:

$$d_{\min} \geq 2t+1$$

Q Parity check matrix of a particular (7, 4) linear

block code is given by $[H] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$

(i) Find Generator Matrix (G)

(ii) List all the code vectors.

(iii) What is the min^m distance b/w code vectors?

(iv) How many errors can be ~~detected~~ & corrected?

Sol $n = (7, 4) = (n, k)$

$$\Rightarrow n = 7, \quad k = 4.$$

$$r = \text{No of check bits } (n - k) = 3. \quad \boxed{r = 3}$$

$$n = 2^r - 1 = 2^3 - 1 = 8 - 1 = 7$$

$$\boxed{n = 7}$$

Determine 'p' Submatrix:—

→ The parity check matrix is of $r \times n$ size & is given by ~~eq~~ ~~62~~ It is written as,

$$r=3, n=7 \text{ \& } k=4.$$

Q.4

Parity check Matrix. $[H]_{3 \times 7} = \begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{41} & \dots & 1 & 0 & 0 \\ P_{12} & P_{22} & P_{32} & P_{42} & \dots & 0 & 1 & 0 \\ P_{13} & P_{23} & P_{33} & P_{43} & \dots & 0 & 0 & 1 \end{bmatrix}$

$$[H] = [P^T : I_3]$$

Parity check matrix. $P^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}_{3 \times 4} \rightarrow \text{Given.}$

P-Submatrix = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{4 \times 3}$

① Determine Generator Matrix (G):

$$G = [I_k : P_{k \times q}]_{k \times n}$$

$$k=4, q=3 \text{ \& } n=7.$$

$$G = [I_4 : P_{4 \times 3}]_{4 \times 7}.$$

$$G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]_{4 \times 7}.$$

$4 \times 4 \qquad 4 \times 3$

② To find all the codewords:

To obtain eqns for check bits.

$$C = MP$$

$$[C_1 \ C_2 \ C_3]_{1 \times 3} = [m_1 \ m_2 \ m_3 \ m_4]_{1 \times 4} [P]_{4 \times 3}$$

$$[C_1 \ C_2 \ C_3] = [m_1 \ m_2 \ m_3 \ m_4] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$C_1 = (1 \times m_1) \oplus (1 \times m_2) \oplus (1 \times m_3) \oplus (0 \times m_4)$$

$$C_2 = (1 \times m_1) \oplus (1 \times m_2) \oplus (0 \times m_3) \oplus (1 \times m_4)$$

$$C_3 = (1 \times m_1) \oplus (0 \times m_2) \oplus (1 \times m_3) \oplus (1 \times m_4)$$

$$C_1 = m_1 \oplus m_2 \oplus m_3$$

$$C_2 = m_1 \oplus m_2 \oplus m_4$$

$$C_3 = m_1 \oplus m_3 \oplus m_4$$

Consider a msg vector of $\begin{pmatrix} 1 & 0 & 1 & 1 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}$

$$C_1 = 1 \oplus 0 \oplus 1 = 0$$

$$C_2 = 1 \oplus 0 \oplus 1 = 0$$

$$C_3 = 1 \oplus 1 \oplus 1 = 1$$

the check bits are $(C_1 C_2 C_3) = (0 0 1)$.

Code word is $mc = 1011 : 001$

$m_1, m_2, m_3, m_4 \Rightarrow 0000$ to 1111 (16) for C_1, C_2, C_3 ⑤

Sr. No	msg vector m_1, m_2, m_3, m_4	check bits C_1, C_2, C_3	Code word (msg + code vector)	weight of code vector $w(x)$
1	0 0 0 0	0 0 0	$m_1, m_2, m_3, m_4, C_1, C_2, C_3$ 0 0 0 0 0 0 0	0
2	0 0 0 1	0 1 1	0 0 0 1 0 1 1	3
...
16	1 1 1 1	1 1 1	1 1 1 1 1 1 1	7

③ Minimum distance b/n Code Vectors:

In table, $2^R = 2^4 = 16$ Code vectors, along with their weights.

→ The smallest weight of any non-zero Code vector is 3. $d_{min} = [w(x)]_{min}; x \neq (00...0)$

④ Error detection + Correction Capabilities:

Since $d_{min} = 3$

$$d_{min} \geq s + 1$$

$$3 \geq s + 1$$

$$\boxed{s \leq 2}$$

Thus two errors will be detected.

$$d_{min} \geq 2t + 1$$

$$3 \geq 2t + 1$$

$$\boxed{t \leq 1}$$

Thus one error will be corrected.

4, 6, 9, 8, 3
27

The hamming Code. ($d_{min} = 3$) always. two errors a 6
 can be detected and single error can be
 corrected by its property.

Hamming Encoder:

- Encoder is implemented for generator matrix, G
- lower register contains check bits. $C_1, C_2 \& C_3$.

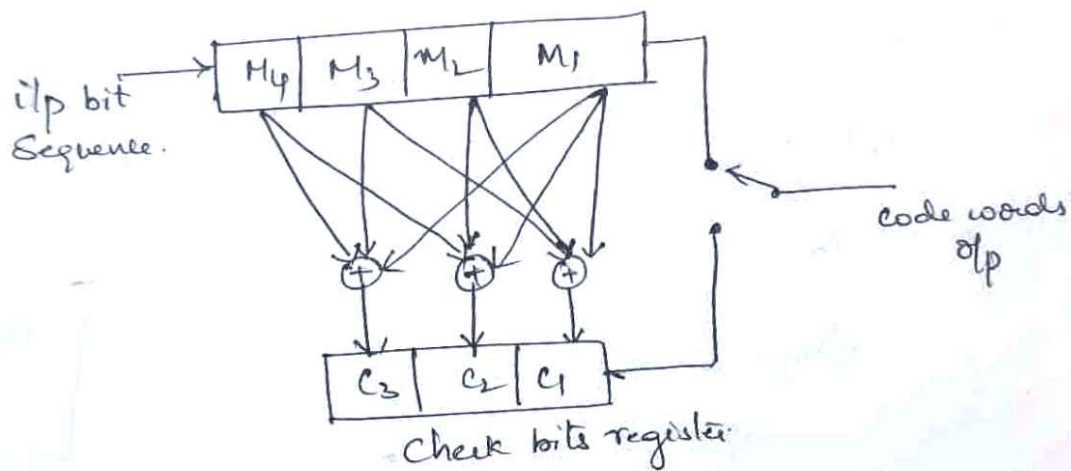


Fig: Hamming Encoder

- These bits are obtained from the msg bits by mod-2 additions. This mod-2 addition operation is nothing but exclusive OR operation.

Syndrome Technique:

Syndrome Decoding

96

The method to correct errors in linear block coding.

→ Let Tx's Code vector be 'X' and corresponding received Code vector be represented by 'Y'.

$X = Y \Rightarrow$ no tx's error.

$X \neq Y \Rightarrow$ errors are created during Tx's.

→ Decoder. detects (or) corrects those errors in 'Y', using the stored bit pattern in the decoder about the code.

→ For larger block lengths more & more bits are required to be stored in the decoder.

→ This is the m/r requirements and adds to the complexity and cost of the system. To avoid these problems,

Syndrome decoding is used in linear block codes.

→ The parity check matrix (H) is $H = [P^T | I]_{q \times n}$.

→ The transpose of H is given as:-

$$H^T = \begin{bmatrix} P \\ I_q \end{bmatrix}_{n \times q} \quad P \rightarrow \text{Submatrix of size } k \times q$$

$I_q \rightarrow$ identity matrix of $q \times q$

→ The transpose of parity check matrix (H^T) exhibits an important property.

$$x H^T = (0, 0, 0, \dots)$$

ie, the product of any code vector x and the transpose of the parity check matrix will be always '0'.

⇒ ~~the~~ At receiver, the property for the detection of errors in the received code as

$$y H^T = (0, 0, 0, \dots, 0) \text{ if no error.}$$

$$y H^T \neq (0, 0, \dots, 0) \text{ if error exists in Rx'd codeword.}$$

Syndrome And its Use for Error detection:

The syndrome is defined as the non-zero o/p of the product $y H^T$. Thus, the non-zero syndrome represents the presence of errors in the received Codeword.

The syndrome is represented by 's' and is ^{a ⊕} expressed as $S = YH^T$

$$[S]_{1 \times q} = [Y]_{1 \times n} [H^T]_{n \times q}$$

When $S = 0$ — $\begin{cases} X = Y & \text{no error.} \\ X \neq Y & Y \text{ is some other valid Codeword (other than } X \text{).} \end{cases}$

⇒ Error Vector:

The non zero elements of 's' represent error in ^{of} p.

Consider. ~~an~~ n-bit error vector E. Let this vector be position of tx'n errors in Y.

$$\begin{aligned} \rightarrow X &= (1 \ 0 \ 1 \ 1 \ 0) \rightarrow \text{Tx'd vector} \\ &\quad \quad \quad \uparrow \quad \quad \uparrow \\ Y &= (1 \ 0 \ 0 \ 1 \ 1) \rightarrow \text{Rx'd vector.} \end{aligned}$$

$$E = (0 \ 0 \ 1 \ 0 \ 1) \rightarrow \text{Error vector.}$$

Mod-2 addition X-OR addition.

eg.

$$Y = X \oplus E$$

$$= (1 \ 0 \ 1 \ 1 \ 0) \oplus (0 \ 0 \ 1 \ 0 \ 1)$$

$$= (1 \oplus 0 \ 0 \oplus 0 \ 1 \oplus 1 \ 0 \oplus 0 \ 0 \oplus 1)$$

$$Y = 1 \quad 0 \quad 0 \quad 1 \quad 1$$

$$\text{If } X = Y \oplus E \Rightarrow (1 \ 0 \ 0 \ 1 \ 1) \oplus (0 \ 0 \ 1 \ 0 \ 1) \Rightarrow (1 \ 0 \ 1 \ 1 \ 0)$$

Relation b/n syndrome vector S + error vector E .

From In Syndrome eqn, $S = YH^T$

If we substitute $Y = X \oplus E$, then we get

$$S = (X \oplus E)H^T$$

$$= XH^T \oplus EH^T$$

From the property we stated earlier $XH^T = 0$

$$\boxed{S = EH^T}$$

Syndrome 'S' depends on error pattern.

Example problem:

Q The parity check matrix of a (7,4).

Hamming code is given as.

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & : & 0 & 0 & 1 \end{bmatrix}_{3 \times 7}$$

Calculate syndrome vector for single bit error

Soln 7,4 linear block code.

$$n = 7, \quad k = 4, \quad r = 3.$$

- ① Determine error pattern for single bit error.
- ② Calculation of syndromes.

① We know that syndrome vector is given by a(8)

$$S = EH^T$$

$$S_{1 \times 3} = E_{1 \times 7} [H^T]_{7 \times 3}$$

Various errors can be taken as $2^3 = 8$.

$$n = 2^3 - 1 \Rightarrow 2^3 - 1 = 7; \boxed{n=7}$$

Error Vector 'E' is a n bit vector representing error pattern.

SNo	Error Vectors	Bit in error.
1.	<u>1</u> 0 0 0 0 0 0	First
2.	0 <u>1</u> 0 0 0 0 0	Second
3.	0 0 <u>1</u> 0 0 0 0	Third
4.	0 0 0 <u>1</u> 0 0 0	Fourth
5.	0 0 0 0 <u>1</u> 0 0	Fifth
6.	0 0 0 0 0 <u>1</u> 0	Sixth
7.	0 0 0 0 0 0 <u>1</u>	Seventh.

② Calculation of syndromes:

$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Syndrome for first bit error $\Rightarrow S = EH^T = [1000000]$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} (1.1 \oplus 0.1 \oplus 0.1 \oplus 0.0 \oplus 0.1 \oplus 0.0 \oplus 0.0), \\ (0.1 \oplus 0.1 \oplus 0.1 \oplus 0.1 \oplus 0.0 \oplus 0.1 \oplus 0.0), \\ (1.1 \oplus 0.1 \oplus 0.0 \oplus 0.1 \oplus 0.0 \oplus 0.0 \oplus 0.1) \\ 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0, \\ 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0, \\ 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0. \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Synonyms for Second but in error.
(1817)

$$S = EH^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0, & 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0, \\ & 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \end{bmatrix}$$

ly after calculating other bit errors.

SNo	Error Vector 'E' showing Single bit error.	Syndrome 'S'	
1.	0 0 0 0 0 0 0	0 0 0	
2.	1 0 0 0 0 0 0	1 0 1	1st row(H^T)
3.	0 1 0 0 0 0 0	1 1 1	2nd row(H^T)
4.	0 0 1 0 0 0 0	1 1 0	3rd row of H^T
5.	0 0 0 1 0 0 0	0 1 1	4th row of H^T
6.	0 0 0 0 1 0 0	1 0 0	5th row of H^T
7.	0 0 0 0 0 1 0	0 1 0	6th row of H^T
8.	0 0 0 0 0 0 1	0 0 1	7th row of H^T

Q. Parity check matrix of a (7,4) Hamming code

is: $H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{3 \times 7}$

Error Correction Using Syndrome Vector.

Procedure to use Syndrome Vector for the error correction:

① For transmitted code vector $x = (0, 1, 0, 0, 1, 1, 0)$
the received code vector $y = (0, 1, \underline{0}, 0, 1, 1, 0)$.
Assume error to be in 3rd position.

② Calculate the corresponding syndrome vector.
 $S = yH^T$

③ ~~The error vectors are $S = EH^T$ if $S = E$.~~

④ Check row of H^T which is same as 'S'.

⑤ From Syndrome vector, we obtain error vector.

⑥ From error vector, ~~the~~ $x = y \oplus E$.

Example

$y = [1, 0, 1, 1, 1, 0, 0]$

$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

① Obtain Syndrome vector

② Determine row of H^T is same as 'S'. ③ To determine 'E'. ④ Obtain correct value.

② Syndrome Vector 'S'.

$$S = YH^T = [1011110]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow [1 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 0, 0 \oplus 0 \oplus 1 \oplus 1 \oplus 0 \oplus 0 \oplus 0, 1 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0]$$

$$= [110]$$

$$S = YH^T = EH^T \Rightarrow 110.$$

③ row of H^T . In table, if we observe, $S = 110$ is the 3rd row of H^T .

④ The error pattern corresponding to this syndrome $E = (0010000)$

This shows that there is an error in the third bit of 'Y'.

⑤ To obtain correct vector.

$$\text{from eq: } x = Y \oplus E$$

$$= [1011110] \oplus [0010000]$$

$$x = [1001110]$$

which is same as the code vector.

Thus a single bit error can be corrected using syndrome decoding.