Weekly Presentation
DeltaGrad: Rapid retraining of machine learning models

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Overview

Motivation

Retaining Problem

Regular Machine Learning Pipeline:

1. Train a ML model from data using a learning algorithm
2. Small change in training data occurs (deletions or additions)
3. Retrain ML model from scratch

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Research Question

Can we retrain models in an efficient manner?

Potential Applications

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- **Continuous Model Updating**: Handle additions, deletions and changes of training samples
- **Data Valuation**: *Leave One Out* tests to find important training samples
- **Bias Reduction**: Speeds up jackknife resampling that requires retrained model parameters

Related Work

Prior Work

- Prior work for specialized problems and ML models, usually for deletion
    - Provenane Based deletions for linear and logistic regression [WTD20]
    - Newton step and noise for *certified data removal* [GGHv20]
    - K-means clustering [GGVZ19]

DeltaGrad

Gradient Descent

- Objective function,

$$F(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{w}),$$

where $F_i(\mathbf{w})$ is loss for i-th sample.

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- Full-batch gradient descent (GD) is on entire data

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \frac{\eta_t}{n}\sum_{i=1}^{n} \nabla F_i\left(\mathbf{w}_t\right)$$

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- Instead rewrite (1) as the "*leave-r-out*" formula

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- $\sum_{i \in R} \nabla F_i \left(\mathbf{w}^U{}_t \right)$ is cheaper to compute

Etymology

- After a small change to the data we need to redo the SGD computations
- We can achieve this by understanding the *delta* of the Gradient Descent

$$n\nabla F(\mathbf{w}) = \sum_{i=1}^{n} \nabla F_i(\mathbf{w}_t) \quad \& \quad n\nabla F(\mathbf{w}^U) = \sum_{i=1}^{n} \nabla F_i\left(\mathbf{w}^U{}_t\right)$$

- Hence, the approach is called *DeltaGrad*

Aprroximating $\nabla F(\mathbf{w}^U)$

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- By Cauchy mean-value theorem[1]

$$\nabla F(\mathbf{w}^U{}_t) - \nabla F(\mathbf{w}_t) = \mathbf{H}_t \cdot (\mathbf{w}^U{}_t - \mathbf{w}_t)$$

Where $\mathbf{H}_t = \int_0^1 \mathbf{H}(\mathbf{w}_t + x(\mathbf{w}^U{}_t - \mathbf{w}_t))dx$ is the integrated hessian

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- This requires a hessian $\mathbf{H}_t$ at each step, which is expensive to maintain and evaluate
- Leverage classical L-BFGS algorithm to approximate $\mathbf{H}_t$

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Review of L-BFGS

- Traditional L-BFGS updates gradients using

$$\nabla F(\mathbf{w}_{t+1}) - \nabla F(\mathbf{w}_t) = \mathbf{B}_t \cdot (\mathbf{w}_{t+1} - \mathbf{w}_t)$$

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Traditional L-BFGS

$$\begin{aligned}
&\nabla F(\mathbf{w}_{t+1}) - \nabla F(\mathbf{w}_t) \approx \mathrm{B}_t (\mathbf{w}_{t+1} - \mathbf{w}_t) \\
&\mathrm{B}_t \approx \mathrm{H}_t \\
&= \int_0^1 \mathrm{H}(\mathbf{w}_t + x(\mathbf{w}_{t+1} - \mathbf{w}_t))\, dx \\
&\mathbf{s}_t = \mathbf{w}_{t+1} - \mathbf{w}_t \\
&\mathbf{y}_t = \nabla F(\mathbf{w}_{t+1}) - \nabla F(\mathbf{w}_t)
\end{aligned}$$

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$$\nabla F(\mathbf{w}_{t+1}) - \nabla F(\mathbf{w}_t) \approx \mathrm{B}_t (\mathbf{w}_{t+1} - \mathbf{w}_t)$$
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$$\mathbf{s}_t = \mathbf{w}_{t+1} - \mathbf{w}_t$$
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L-BFGS for approximating $\nabla F(\mathbf{w}^U)$

$$\nabla F\left({\mathbf{w}^U}_t\right) - \nabla F(\mathbf{w}_t) \approx \mathrm{B}_t \left({\mathbf{w}^U}_t - \mathbf{w}_t\right)$$
$$\mathrm{B}_t \approx \mathrm{H}_t$$
$$= \int_0^1 \mathrm{H}\left(\mathbf{w}_t + x\left({\mathbf{w}^U}_t - \mathbf{w}_t\right)\right) dx$$
$$\mathbf{s}_t = {\mathbf{w}^U}_t - \mathbf{w}_t$$
$$\mathbf{y}_t = \nabla F\left({\mathbf{w}^U}_t\right) - \nabla F(\mathbf{w}_t)$$

Using L-BFGS

- Maintain m historical observations of $\mathbf{Y} = (\mathbf{y}_t, \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-m})$ and $\mathbf{S} = (\mathbf{s}_t, \mathbf{s}_{t-1}, \dots, \mathbf{s}_{t-m})$
- Let g be a function defined by L-BFGS, then we can approximate $\mathbf{B}_t \cdot \mathbf{v}$ using

$$g(\mathbf{Y}, \mathbf{S}, \mathbf{v})$$

  Where, $\mathbf{v}$ is an arbitrary vector.
- Therefore,

$$\mathbf{B}_t \cdot (\mathbf{w}^U{}_t - \mathbf{w}_t) = g(\mathbf{Y}, \mathbf{S}, \mathbf{w}^U{}_t - \mathbf{w}_t)$$

- Hence we obtain the approximation as

$$\nabla F(\mathbf{w}^U{}_t) \approx \nabla F(\mathbf{w}_t) + \mathbf{B}_t \cdot (\mathbf{w}^U{}_t - \mathbf{w}_t)$$

Rewriting

- Denoting $\mathbf{w}^I$ as the approximate $\mathbf{w}^U$ we have

$$\nabla F({\mathbf{w}^I}_t) \approx \nabla F(\mathbf{w}_t) + \mathbf{B}_t \cdot ({\mathbf{w}^I}_t - \mathbf{w}_t).$$

- replacing in (2)

$$\begin{aligned} {\mathbf{w}^I}_{t+1} &= {\mathbf{w}^I}_t - \frac{\eta_t}{n-r}\left[\sum_{i=1}^{n} \nabla F_i\left({\mathbf{w}^I}_t\right) - \sum_{i\in R} \nabla F_i\left({\mathbf{w}^I}_t\right)\right] \\ &= {\mathbf{w}^I}_t - \frac{\eta_t}{n-r}\left\{n[\mathbf{B}_t({\mathbf{w}^I}_t - \mathbf{w}_t) + \nabla F(\mathbf{w}_t)] - \sum_{i\in R} \nabla F({\mathbf{w}^I}_t)\right\} \end{aligned}$$

Problem with Error Bound

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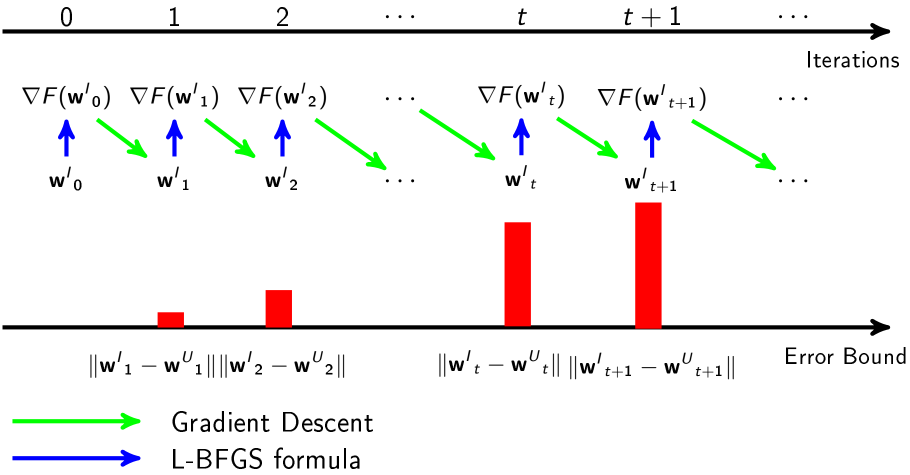
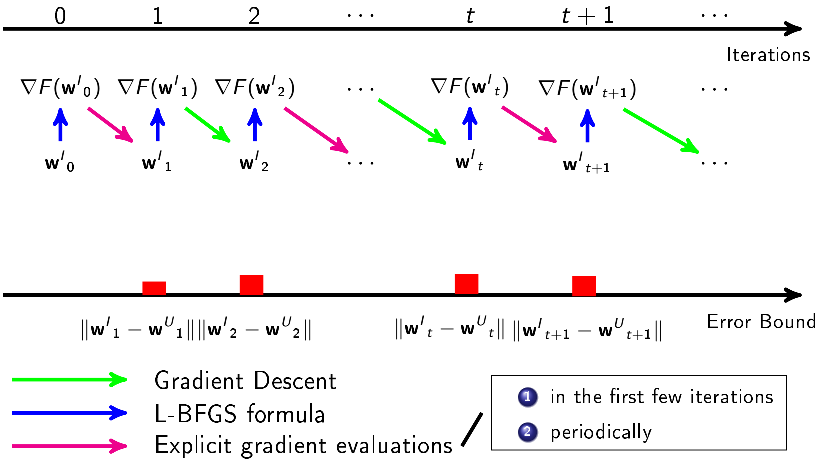


Controlling the Errors

- Do explicit evaluations for j_0 "burn-in" iterations and then periodically every T_0 iterations



Benefit of DeltaGrad

- DeltaGrad can be extended to when r samples are added rather than deleted
- Change the $+$ to minus in the update formula to get

$$\mathbf{w^I}_{t+1} = \mathbf{w^I}_t - \frac{\eta_t}{n+r}\left\{n[\mathbf{B}_t(\mathbf{w^I}_t - \mathbf{w}_t) + \nabla F(\mathbf{w}_t)] + \sum_{i \in R} \nabla F(\mathbf{w^I}_t)\right\}$$

- Here $\sum_{i \in R} \nabla F(\mathbf{w^I}_t)$ is the gradient of the added r samples

Algorithm 1: DeltaGrad

Input : The full training set $(\mathbf{X}, \mathbf{Y})$, model parameters cached during the training phase over the full training samples $\{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_t\}$ and corresponding gradients $\{\nabla F(\mathbf{w}_0), \nabla F(\mathbf{w}_1), \dots, \nabla F(\mathbf{w}_t)\}$, the indices of the removed training samples R, period T_0, total iteration number T, history size m, "burn-in" iteration number j_0, learning rate η_t

Output: Updated model parameter $\mathbf{w}^I{}_t$

Initialize $\mathbf{w}^I{}_0 \leftarrow \mathbf{w}_0$

Initialize an array $\Delta G = []$

Initialize an array $\Delta W = []$

for $t = 0; t < T; t++$ **do**

  if $[((t - j_0) \mod T_0) == 0]$ *or* $t \leq j_0$ **then**

    compute $\nabla F(\mathbf{w}^I{}_t)$ exactly

    compute $\nabla F(\mathbf{w}^I{}_t) - \nabla F(\mathbf{w}_t)$ based on the cached gradient $\nabla F(\mathbf{w}_t)$

    set $\Delta G[k] = \nabla F(\mathbf{w}^I{}_t) - \nabla F(\mathbf{w}_t)$

    set $\Delta W[k] = \mathbf{w}^I{}_t - \mathbf{w}_t$, based on the cached parameters $\mathbf{w}_t$

    $k \leftarrow k + 1$

    compute $\mathbf{w}^I{}_{t+1}$ by using exact GD update (equation (1))

  else

    Pass $\Delta W[-m:]$, $\Delta G[-m:]$, the last m elements in ΔW and ΔG, which are from the $j_1^{th}, j_2^{th}, \dots, j_m^{th}$ iterations where $j_1 < j_2 < \dots < j_m$ depend on t, $\mathbf{v} = \mathbf{w}^I{}_t - \mathbf{w}_t$, and the history size m, to the L-BFGFS Algorithm to get the approximation of $\mathbf{H}(\mathbf{w}_t)\mathbf{v}$, i.e., $\mathbf{B}_{j_m}\mathbf{v}$

    Approximate $\nabla F(\mathbf{w}^I{}_t) = \nabla F(\mathbf{w}_t) + \mathbf{B}_{j_m}(\mathbf{w}^I{}_t - \mathbf{w}_t)$

    Compute $\mathbf{w}^I{}_{t+1}$ by using the "leave-r-out" gradient formula, based on the approximated $\nabla F(\mathbf{w}^I{}_t)$

  end

end

return $\mathbf{w}^I{}_t$

Theoretical Results

Results for GD

Theorem (Bound between true and incrementally updated iterates)

Assuming $F(\mathbf{w})$ is strongly convex, for large enough iterations t, the result ${\mathbf{w}^I}_t$ of *DeltaGrad* approximates the correct iteration values ${\mathbf{w}^U}_t$ at the rate of

$$\|{\mathbf{w}^U}_t - {\mathbf{w}^I}_t\| = o\left(\frac{r}{n}\right)$$

So $\|{\mathbf{w}^U}_t - {\mathbf{w}^I}_t\|$ is of a lower order than r/n. r/n is the "baseline error rate" of the original weights $\mathbf{w}_t$, i.e., $\|\mathbf{w}_t - {\mathbf{w}^I}_t\| = o(\frac{r}{n})$

architecture of proof

Results for SGD

Theorem(Bound between true and incrementally updated iterates in SGD)

Assuming $F(\mathbf{w})$ is strongly convex, for large enough iterations t and mini-batch size B, the result ${\mathbf{w}^I}_t$ of *DeltaGrad* approximates the correct iteration values ${\mathbf{w}^U}_t$ at the rate of

$$\|{\mathbf{w}^U}_t - {\mathbf{w}^I}_t\| = o\left(\frac{r}{n} + \frac{1}{B^{\frac{1}{4}}}\right)$$

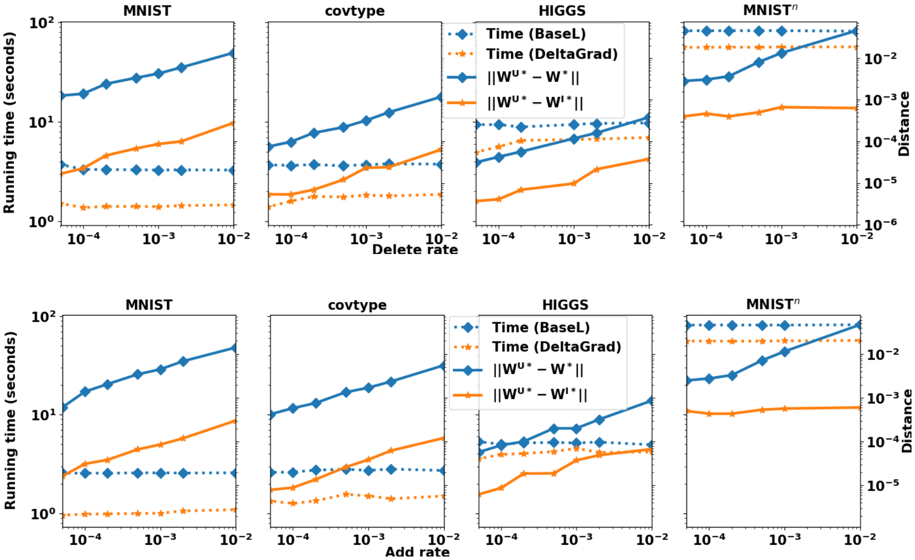
with high probability

Experimental Results

Setup

- **Datasets**: MNIST, RCV1, HIGGS
- **Model**: Logistic regression with L2 regularization
- **Baseline**: Naive retraining (BaseL)
- **Hyperparameters**: $j_0 = \{10, 10, 300\}$ and $T_0 = \{5, 10, 3\}$

Results



▸ Batch Performance ▸ Online Performance ▸ Large Deletions ▸ ResNet512 Results

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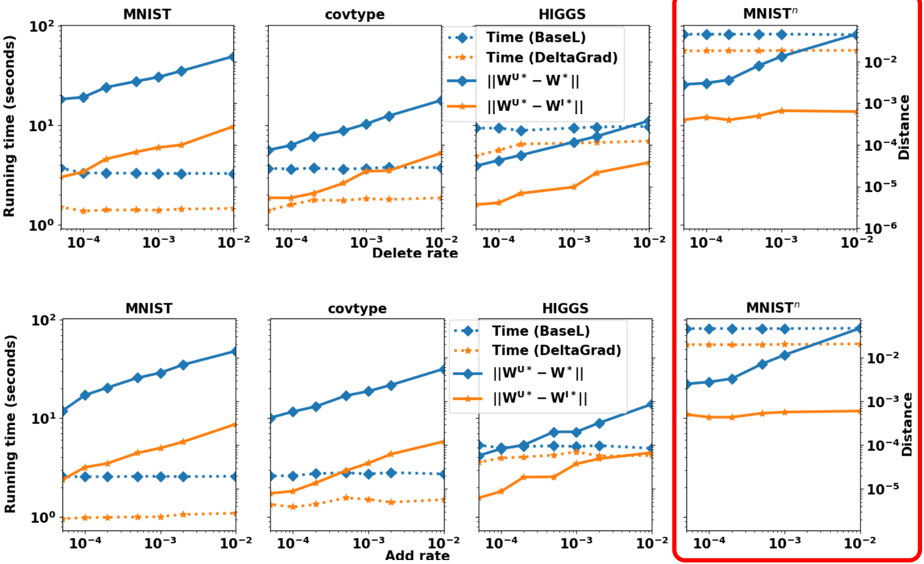

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Future Work

Our Research Directions

- **What can we forget?**
  Selectively cache $\mathbf{w}_t$ and $\nabla F(\mathbf{w}_t)$ during original training, and still uphold the update approximation guarantee
- **How to perform consecutive updates?**
  Are there issues with cumulative approximations? Compare online machine learning with deletions to DeltaGrad.
- **When should one retrain?** After how many additions/deletions does $\mathbf{w}_t$ and $\mathbf{w}^U{}_t$ diverge beyond approximation guarantees? Can a complete retraining benefit from prior updates performed?

References I

Chuan Guo, Tom Goldstein, Awni Hannun, and Laurens van der Maaten.
Certified Data Removal from Machine Learning Models.
arXiv:1911.03030 [cs, stat], August 2020.

Antonio Ginart, Melody Guan, Gregory Valiant, and James Y Zou.
Making AI Forget You: Data Deletion in Machine Learning.
In H. Wallach, H. Larochelle, A. Beygelzimer, F. d\textquotesingle Alché-Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems 32*, pages 3518–3531. Curran Associates, Inc., 2019.

References II

Yinjun Wu, Val Tannen, and Susan B. Davidson.
PrIU: A Provenance-Based Approach for Incrementally Updating Regression Models.
In *Proceedings of the 2020 ACM SIGMOD International Conference on Management of Data*, pages 447–462, Portland OR USA, June 2020. ACM.

Additional Results

Batch Performance

Table 1. Prediction accuracy of BaseL and DeltaGrad with batch addition/deletion. MNIST^n refers to MNIST with a neural net.

Dataset		BaseL(%)	DeltaGrad(%)
Add (0.005%)	MNIST	87.530 ± 0.0025	87.530 ± 0.0025
	MNIST^n	92.340 ± 0.002	92.340 ± 0.002
	covtype	62.991 ± 0.0027	62.991 ± 0.0027
	HIGGS	55.372 ± 0.0002	55.372 ± 0.0002
	RCV1	92.222 ± 0.00004	92.222 ± 0.00004
Add (1%)	MNIST	87.540 ± 0.0011	87.542 ± 0.0011
	MNIST^n	92.397 ± 0.001	92.397 ± 0.001
	covtype	63.022 ± 0.0008	63.022 ± 0.0008
	HIGGS	55.381 ± 0.0007	55.380 ± 0.0007
	RCV1	92.233 ± 0.00010	92.233 ± 0.00010
Delete (0.005%)	MNIST	86.272 ± 0.0035	86.272 ± 0.0035
	MNIST^n	92.203 ± 0.004	92.203 ± 0.004
	covtype	62.966 ± 0.0017	62.966 ± 0.0017
	HIGGS	52.950 ± 0.0001	52.950 ± 0.0001
	RCV1	92.241 ± 0.00004	92.241 ± 0.00004
Delete (1%)	MNIST	86.082 ± 0.0046	86.074 ± 0.0048
	MNIST^n	92.373 ± 0.003	92.370 ± 0.003
	covtype	62.943 ± 0.0007	62.943 ± 0.0007
	HIGGS	52.975 ± 0.0002	52.975 ± 0.0002
	RCV1	92.203 ± 0.00007	92.203 ± 0.00007

Online Performance

Table 2. Distance and prediction performance of BaseL and DeltaGrad in online deletion/addition

Dataset	Distance		Prediction accuracy (%)	
	$\|\mathbf{w}^{U*} - \mathbf{w}^*\|$	$\|\mathbf{w}^{I*} - \mathbf{w}^{U*}\|$	BaseL	DeltaGrad
MNIST (Addition)	5.7×10^{-3}	2×10^{-4}	87.548 ± 0.0002	87.548 ± 0.0002
MNIST (Deletion)	5.0×10^{-3}	1.4×10^{-4}	87.465 ± 0.002	87.465 ± 0.002
covtype (Addition)	8.0×10^{-3}	2.0×10^{-5}	63.054 ± 0.0007	63.054 ± 0.0007
covtype (Deletion)	7.0×10^{-3}	2.0×10^{-5}	62.836 ± 0.0002	62.836 ± 0.0002
HIGGS (Addition)	2.1×10^{-5}	1.4×10^{-6}	55.303 ± 0.0003	55.303 ± 0.0003
HIGGS (Deletion)	2.5×10^{-5}	1.7×10^{-6}	55.333 ± 0.0008	55.333 ± 0.0008
RCV1 (Addition)	0.0122	3.6×10^{-6}	92.255 ± 0.0003	92.255 ± 0.0003
RCV1 (Deletion)	0.0119	3.5×10^{-6}	92.229 ± 0.0006	92.229 ± 0.0006

MNIST Deletions upto 20%

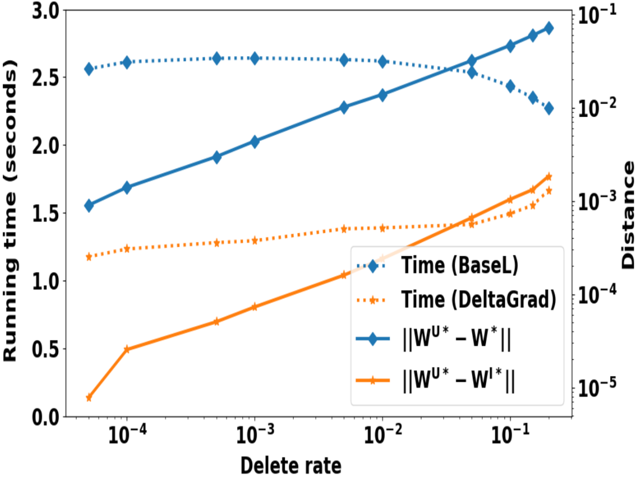


Figure S1. Running time and distance with varied deletion rate up to 20%

ResNet512 Results

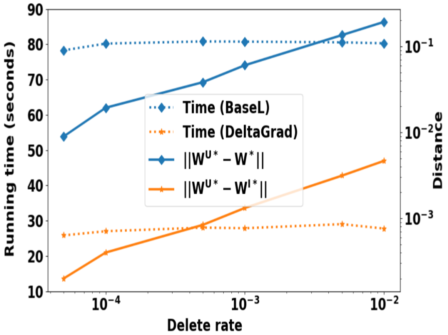
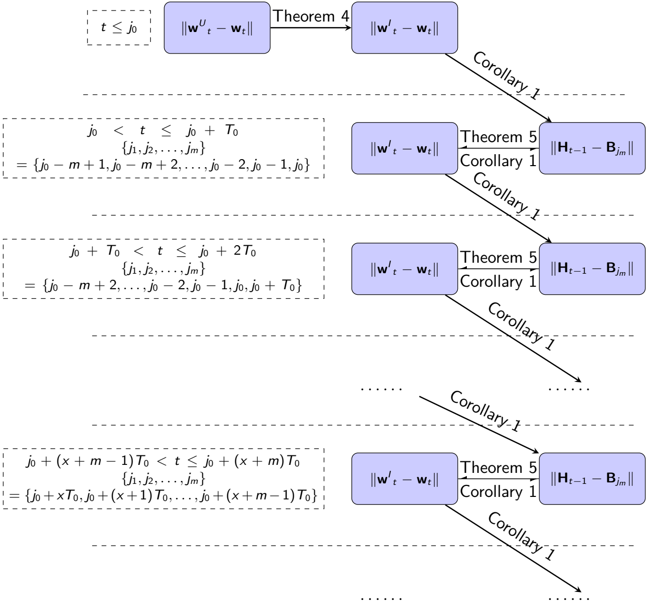


Figure S5. Comparison of DeltaGrad and BaseL on the CIFAR-10 dataset with pre-trained ResNet152 network

Proof Architecture

Reursive Architecture of Proof



◂ Theoretical Results