IC 021 Computational Techniques in Control Engineering

Cycle Test I, Feb 16, 2017

Time: 1 hour **Answer all the questions** **Max. Marks: 20**

1. (a) [**3 marks**] Let $\mathcal{V}^3$ be a 3-dimensional vector space with b_1, b_2, b_3 as a basis. Prove that

$$\{b_1,\, b_1 + b_2,\, b_1 + b_2 + b_3\}$$

is also a basis for the vector space $\mathcal{V}^3$.

(b) [**3 marks**] Prove that

$$\{b_1 + b_2,\, b_2 + b_3,\, b_1 - b_3\}$$

is not a basis for the vector space $\mathcal{V}^3$.

(c) [**2 marks**] Let $\mathcal{V}^n$ be a vector space with b_1, b_2, $\cdots$, b_n as a basis. Assume that the vectors b_3, b_4, $\cdots$, b_n span the space $\mathcal{W} = w_1, w_2, \cdots, .$ If, for some non-zero scalars α_1 and α_2, the linear combination $w_j = \alpha_1 b_1 + \alpha_2 b_2$ belongs to $\mathcal{W}$, then show that $w_j = \bar{0}$, the zero vector. What is the dimension of $\mathcal{W}$?

2. (a) [**2 marks**] Explain why, in a similarity transformation of a square matrix, the eigenvalues remain invariant. How about the eigenvectors?

(b) [**2 marks**] Two similar matrices have the same set of eigenvalues, but is the converse true? Why or why not?

3. (a) [**3 marks**] Give the algorithm for QR decomposition, based on the Gram-Schmidt process, of a square matrix $A \in \Re^{n \times n}$.

(b) [**2 marks**] To improve the numerical properties, how do you modify this algorithm?

(c) [**3 marks**] Illustrate the modified algorithm in decomposing

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

Justify that $\mathcal{O}\left(n^3\right)$ floating-point operations are involved.

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021 Computational Techniques in Control Engg
K E Y for Cycle Test I

1. (a) If you call the three vectors as q_1, q_2, q_3, then the linear combinations q_1, $q_2 - q_1$, $q_3 - q_2$ reduce to the original basis.

   (b) It is easy to see that the third vector $b_1 - b_3$ is a linear combination of the first two, and such a linearly dependent set cannot form a basis.

   (c) The dimension of $\mathcal{W}$ must be $n-2$. The origin of this sub-space must be the linear combination:

   $$0 \times b_3 \; + \; 0 \times b_4 \; + \; \cdots \; 0 \times b_n$$

   Any other arbitrary linear combination generates $\mathcal{W}$. Since b_1 and b_2 are not the basis vectors of $\mathcal{W}$, the only possibility for the linear combination $w_j = \alpha_1 b_1 + \alpha_2 b_2$ to belong to $\mathcal{W}$ must be $w_j = \bar{0}$. In fact, while defining the span of a sub-space, all linear combinations of the left-out vectors are considered as origin. There are some more technical issues here, which are left to you for further exploration.

2. Being similar means the two square matrices A and A' represent the same "Linear Transformation", but with respect to different bases.

   (a) Any similarity transformation, primarily out of change of basis, keeps the characteristic polynomial intact, and hence the eigenvalues remain invariant. However, owing to the change of basis, the eigenvectors would appear different, but they are related to the non-singular transformation matrix T, i.e., if $\bar{e}$ is an eigenvector of A then $T^{-1}\bar{e}$ is an eigenvector of A'.

   (b) If a matrix is *similar* to another one, the characteristic polynomial is invariant, and hence the eigenvalues; however, two arbitrary matrices having an identical set of eigenvalues need not be similar. A quick counterexample is

   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is } \textit{not} \text{ similar to} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

   Since both represent the same linear transformation, not only the eigenvalues of A and $T^{-1}AT$ are invariant, their algebraic and geometric multiplicities are also invariant; this fact may not be found explicitly in many texts, but must be understood by the reader in the spirit of change of basis.

3. (a) The basic algorithm is

$$\begin{aligned}
R(1,1) &= \|A(:,1)\|_2 \\
Q(:,1) &= A(:,1)/R(1,1)
\end{aligned}$$

for $k = 2 : n$ {

$$\begin{aligned}
R(1:k-1,k) &= Q(1:n,1:k-1)^T A(1:n,k) \\
z &= A(1:n,k) - Q(1:n,1:k-1)R(1:k-1,k) \\
R(k,k) &= \|z\|_2 \\
Q(1:n,k) &= z/R(k,k)
\end{aligned}$$

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Here, we first compute an orthogonal vector and then normalize it make it a unit vector. However, if the matrix A itself is normalized and then the orthonormal vectors are built, using the same G-S process, we'll have better accuracy in the results. This modification is shown next.

(b)

for $k = 1 : n$ {

$$\begin{aligned}
R(k,k) &= \|A(1:n,k)\|_2 \\
Q(1:n,k) &= A(1:n,k)/R(k,k)
\end{aligned}$$

for $j = k+1 : n$ {

$$\begin{aligned}
R(k,j) &= Q(1:n,k)^T A(1:n,j) \\
A(1:n,j) &= A(1:n,j) - Q(1:n,k)R(k,j)
\end{aligned}$$

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(c) The QR decomposition is

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

The outer loop runs n times with $\mathcal{O}(n)$ multiplications plus $n-1$ times $\mathcal{O}(n)$ multiplications within the inner loop, thus have $\mathcal{O}(n^3)$ multiplications. For the given example, where $n = 3$, we have $2n^3 = 54$ operations including the square-root computation for the norms. The original A matrix will be replaced by the Q matrix, and R is generated in terms of the upper triangular elements, R_{11} etc., and we need to pack them as the matrix R.

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