CL652 Computational Techniques in Control Engineering

End-Term Test, Apr 19, 2017

Time: 90 minutes **Answer all the questions** **Max. Marks: 30**

1. (a) [**2 marks**] In error analysis it is convenient to define the bound on relative error as $\tilde{E}(\hat{x}) = \dfrac{|x-\hat{x}|}{|\hat{x}|}$ instead of $E(x) = \dfrac{|x-\hat{x}|}{|x|}$. Obtain inequalities between $E(\hat{x})$ and $\tilde{E}(x)$, and explain why the new definition is *convenient*.

   (b) [**4 marks**] Show how to rewrite the following expressions to avoid cancellation for the indicated arguments:

   (i) $\sqrt{x+1} - 1$, $x \approx 0$ (ii) $\sin(x) - \sin(y)$, $x \approx y$

   (iii) $x^2 - y^2$, $x \approx y$ (iv) $(1 - \cos(x))/\sin(x)$, $x \approx 0$

2. [**6 marks**] Compute e^A using naive similarity transformation, where
$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \quad (\Lambda = \{-1 \pm j1\})$$

3. [**8 marks**] For the autonomous system $x(k+1) = Ax(k)$ with
$$A = \begin{bmatrix} 2 & 0.5 \\ 0 & 0.8 \end{bmatrix}$$
check for its stability, by transforming the corresponding Lyapunov equation. Justify your answer.

4. [**6 marks**] Given the pair of matrices
$$A = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$
check if this pair makes a controllable system; if yes, compute two different state-feedback controllers K_1 and K_2 such that the eigenvalues of $A + BK_i$, $i = 1, 2$ are $-2 \pm j2$. Check your solutions.

5. [**4 marks**] Write a short note on the eigenvalues and eigenvectors of a Householder matrix $\in \Re^{n \times n}$.

* * * * *

CL 652 Computational Techniques in Control Engg
K E Y for End-Term Test

1. (a) **[2 marks]** Since

$$\tilde{E} = E \times \frac{|x|}{|\hat{x}|} \qquad \text{and} \qquad \hat{x} = x(1 \pm E)$$

we have

$$\frac{E}{1+E} \leq \tilde{E} \leq \frac{E}{1-E}$$

Hence, if $E < 0.01$, say, then there is no difference between E and $\tilde{E}$ for practical purposes.

(b) **[4 marks]**

(i) $\dfrac{x}{\sqrt{x+1} + 1}$ (ii) $2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$

(iii) $(x-y)(x+y)$ (iv) $\dfrac{\sin(x)}{1+\cos(x)}$

In (iii), cancellation has not been avoided, but it is now harmless if x and y are known exactly; in fact, if $y/2 \leq x \leq y$ then $x - y$ is computed exactly (why? prove it as a homework).

This kind of computation arises when squaring a complex number.

2. The matrix is in Controllable Canonical Form, and the eigenvalues are given; hence using

$$T = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda^* \end{bmatrix}$$

you can readily diagonalize A, compute e^{A_d}, and then de-diagonalize it using the T matrix.

3 marks for T matrix and diagonalization, 2 marks for e^{A_d}, and 1 mark for de-diagonalization.

3. The system is unstable; apparently A has no eigenvalues at $+1$; hence we go ahead with the transformations:

$$A_1 = (I - A)^{-1}(I + A), \qquad \text{and} \qquad Q_1 = \left(I + A_1^T\right)(I + A_1)$$

with the assumption that $Q = -2I$. The resulting Lyapunov-like equation is

$$A_1^T P + P A_1 = Q_1$$

Note that Q_1 need not be negative definite; still the system is stable if and only if P is at least positive semi-definite. For the given matrix A, the results are

$$A_1 = \begin{bmatrix} -3 & -5 \\ 0 & 9 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4 & 10 \\ 10 & 125 \end{bmatrix} > 0$$

and

$$P = \frac{1}{324}\begin{bmatrix} -216 & 360 \\ 360 & 2450 \end{bmatrix} = \begin{bmatrix} -0.667 & 1.111 \\ 1.111 & 7.561 \end{bmatrix} \quad \text{is indefinite}$$

3 marks for the transformations; 1 mark to check the sign-definiteness of Q_1; 2 marks for setting up the Kronecker's linear systems of equations; 2 marks for the solution P and its sign-definiteness, and hence the result that A is unstable.

4. The system is controllable, since itself B has full rank; 2 marks for this test. Two possible matrices are

$$K_1 = \begin{bmatrix} 4 & 7 \\ 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 3 & \frac{23}{3} \\ 0 & \frac{2}{3} \end{bmatrix}$$

2 marks for the correct K matrices; there might be many more choices; however, verifying the eigenvalues of $A+BK$ is important, and 2 marks for this.

5. One of the eigenvalue is -1, and the rest $n-1$ are all 1's. All the eigenvectors are orthogonal. You get 6 marks only if you present an algebraic argument; no marks for any numerical illustrations.

* * * * *