CL652 Computational Techniques in Control Engineering

Mid-Term Test, Mar 07, 2017

Time: 90 minutes **Answer all the questions** **Max. Marks: 30**

1. [**3 marks**] Does the set of all three-dimensional vectors inside a spere of finite radius constitute a linear vector space? Present your argument in terms of the axioms of the vector space.

2. [**4 marks**] A mirror lies in the plane defined by $-2\hat{\imath} + 3\hat{\jmath} + \hat{k} = 0$. Find the reflected image of $y = [4 \quad -2 \quad 3]^T$. Also, find its orthogonal projection on the plane of the mirror.

3. [**5 marks**] Compute the eigenvalues and all corresponding eigenvectors of the following matrix.
$$\begin{bmatrix} 2 & 5 & -2 \\ 5 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

4. [**6 marks**] Assume that a given matrix $A \in \Re^{n\times n}$ has a set of linearly independent eigenvectors $\{v_1, \cdots, v_n\}$. Further, suppose that there is a vector x such that $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$. Show that
$$A^q x = c_1 \lambda_1^q v_1 + c_2 \lambda_2^q v_2 + \cdots + c_n \lambda_n^q v_n$$
where $A^q = \underbrace{A \times A \cdots \times A}_{q \text{ times}}$, and λ_i, $i = 1, \cdots, n$ is an eigenvalue of A.

5. [**6 marks**] If a matrix $A \in \Re^{n\times n}$ has the following properties:

   (a) all the entries are non-negative, and

   (b) the sum of all entries in each row is one

   Show that $\lambda = 1$ is an eigenvalue of A.

6. [**6 marks**] Given:
$$A = \begin{bmatrix} 2 & 5 & -2 \\ 5 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -0.3714 & -0.9285 & 0 \\ -0.9285 & 0.3714 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5.3852 & -3.7139 & -0.1857 \\ 0 & -3.8996 & 2.2283 \\ 0 & 0 & -3.0000 \end{bmatrix}$$
Compute e^{At} using the algorithm based on Real Schur Decomposition. Verify your results up to 6 significant digits.

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1. No, it is not a vector space - consider two vectors on the outer surface of the sphere and add them - the first axiom itself is violated.

2. The relation between a vector y and its reflection y_r at a plane surface defined by a unit normal $\hat{n}$ is

$$y_r = y - 2 < \hat{n}, y > \hat{n}$$

i.e., y and y_r are equal except for a sign change in the component along $\hat{n}$.

This can be re-written as

$$\begin{aligned} y_r &= y - 2\hat{n} >< \hat{n}\, y \\ &= [I - 2\hat{n} >< \hat{n}]\, y \\ &= \underbrace{\left[I - 2\frac{nn^T}{n^T n}\right]}_{\triangleq \text{ Householder reflection}} y \qquad (1) \end{aligned}$$

where $u >< v$ is called as the **outer product** of the vectors, which gives us the square matrix

$$\frac{uv^T}{u^T v} \in \Re^{n \times n}$$

Note that this outer product does not always commute.

A linear transformation which maps y into its orthogonal projection on a hyperplane with a unit normal vector $\hat{n}$ is

$$\begin{aligned} \mathcal{A}_p &= [I - \hat{n} >< \hat{n}] \\ &= \left[I - \frac{nn^T}{n^T n}\right] \qquad (2) \end{aligned}$$

Observe that

$$\mathcal{A}_p^2 = \mathcal{A}_p \qquad (3)$$

Derive the above 3 results as a homework; it is easy if you carefully understand the case in 2 dimensions.

For the given problem,

$$y_r = \frac{1}{7}\begin{bmatrix} 6 \\ 19 \\ 32 \end{bmatrix} \quad \text{and} \quad y_p = \frac{1}{14}\begin{bmatrix} 34 \\ 5 \\ 53 \end{bmatrix}$$

3. You need to expand the determinant, $|\lambda I - A|$, and this is straight if you make use of the last row:

$$\chi = (\lambda + 3)\left[(\lambda - 2)^2 - 25\right]$$

from which, the eigenvalues are -3, -3, & 7. (You don't have to solve a cubic polynomial).

The corresponding eigenvectors are:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The eigenvalue -3 has a geometric multiplicity of only 1, and hence the first two eigenvectors are linearly dependent. You can also find an independent eigenvector for $\lambda = -3$ as

$$[\lambda I - A]\, e_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

from which you may get, for instance,

$$\begin{bmatrix} 1/15 \\ 0 \\ 2/3 \end{bmatrix}$$

4. Actually, this question is fit for only 1 mark! Just plug in, realize that eigenvectors v_i are simply scaled by the corresponding eigenvalues, and that $A^q v_i = \lambda^q v_i$.

5. This question is not even worth 1 mark! Since the sum of all entries in each row of the matrix is given to be 1, simply choose a vector

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \Re^{n \times 1}$$

Then, you will immediately see that

$$Ax = x$$

which means that $\lambda = 1$ is an eigenvalue.

Never attempt to prove something using an example

6. An upper triangular matrix is given, but you should have been careful to see that it is only the QR decomposition of A. Therefore, using the given Q matrix, you need to iterate, around 4 times, to get a *similar* matrix to A. This is the Schur decomposition.

$$A \sim \begin{bmatrix} 7 & 0 & -1/\sqrt{2} \\ 0 & -3 & 2.1213 \\ 0 & 0 & -3 \end{bmatrix}$$

From question 3 above, the eigenvalues - -3, -3, 7 - are available for you to cross check.

Then you use the algorithm to compute e^{Rt}, in the upper triangular form.

e^{At} may be obtained by the inverse similarity transformation:

$$e^{A} = \begin{bmatrix} 548.3415 & 548.2917 & -54.9038 \\ 548.2917 & 548.3415 & -54.7545 \\ 0 & 0 & 0.0498 \end{bmatrix}$$

I strongly suggest that you submit the answer to this question as a quick homework before the next class.

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- Derivation for Q2:

  Let y be the vector, and $\hat{n}$ be the unit normal vector upon which we seek the reflection of y. In 2 dimensions, refer to the figure below.

  It readily follows that

  $$r = \left(y^T \hat{n}\right) \hat{n} \text{ is the projection of } y \text{ on } \hat{n}$$

  If x is the *mirror image* of y, then $x + y$ lies on $\hat{n}$ by parallelogram law of addition, and hence

  $$\begin{aligned} x + y &= 2\|r\| \, \hat{n} \\ &= \left(2y^T \hat{n}\right) \hat{n} \end{aligned}$$

  Therefore,

  $$x = \left(2y^T \hat{n}\right) \hat{n} - y$$

  If y_r is the *reflection* of y, then $y_r = -x$, i.e.,

  $$\begin{aligned} y_r &= y - \left(2y^T \hat{n}\right) \hat{n} \\ &= \left[I - 2nn^T\right] y \end{aligned}$$

  Since r is the projection of y on $\hat{n}$, $y - r$ is the perpendicular distance from r to y. Hence the orthogonal projection is

  $$y - r = \left[I - nn^T\right] y$$

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