Unit I: Fundamentals of Sound and Musical Structure

Mathematics of Music (MA375TGR)

Course Notes

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Contents

1 Introduction to Sound and Music

Music, in its essence, is organized sound. To understand music from a mathematical perspective, we must first understand the physical and perceptual foundations of sound itself. This unit explores the nature of sound as a physical phenomenon, the mathematical relationships that govern musical pitch, and the systems developed across cultures to organize and notate musical information.

The mathematical study of music has ancient roots, dating back to Pythagoras who discovered that harmonious musical intervals correspond to simple numerical ratios. This discovery established a profound connection between mathematics and music that continues to inform our understanding today.

2 Nature of Sound

2.1 Physical Properties of Sound

Sound is fundamentally an organized traveling disturbance in a medium, typically air. When an object vibrates, it creates alternating regions of high and low pressure in the surrounding air molecules. These pressure variations propagate through the medium as a wave, transmitting energy without transmitting matter itself.

2.1.1 Frequency

Definition 2.1 (Frequency)**.** Frequency (f or ν) is the rate of periodic pressure change in a sound wave, measured in cycles per second. The unit of measurement is the Hertz (Hz), where 1 Hz = 1 cycle per second.

The frequency of a sound wave determines its perceived pitch. The human auditory system is capable of detecting frequencies approximately in the range:

$$17 \text{ Hz} \leq f \leq 17,000 \text{ Hz} \tag{1}$$

However, this range varies significantly with age, gender, and individual differences. Young people in good health might hear up to 17 kHz, while older adults typically experience reduced high-frequency hearing, with the upper limit potentially declining to around 12 kHz for women and 5 kHz for men.

Musical Relevance: The range of finest pitch perception occurs between 1 kHz and 4 kHz, which coincidentally corresponds to the frequency range containing most speech information. Musical instruments are generally designed to operate within the effective range of human hearing. For instance:

- Piano: Lowest note A0 ≈ 27.5 Hz, Highest note C8 ≈ 4,186 Hz
- Sounds below approximately 30 Hz progressively lose the quality of pitch and begin to be felt as physical impact rather than heard as tones
- Pitch discrimination deteriorates significantly above 5 kHz

2.1.2 Amplitude

Definition 2.2 (Amplitude)**.** Amplitude is the strength or magnitude of pressure fluctuations in a sound wave. It determines the intensity or energy content of the sound.

For a sinusoidal wave, if we represent the sound as:

$$y(t) = A\sin(2\pi ft + \phi) \tag{2}$$

where:

- A is the amplitude (peak value)
- f is the frequency
- t is time
- ϕ is the phase

Sound intensity is measured in decibels (dB), which employs a logarithmic scale to match human perception. The decibel scale is defined as:

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)\ \text{dB} \tag{3}$$

where I is the intensity of the sound and I_0 is the reference intensity (typically the threshold of hearing).

The human auditory range of intensity spans approximately:

- Threshold of hearing: ≈ 0 dB (in very quiet conditions, possibly 40 dB in typical quiet rooms)
- Threshold of pain: ≈ 120 dB

2.1.3 Energy Relationships

The energy of a wave depends on both its amplitude and frequency. If a wave's amplitude doubles while frequency remains constant, or if frequency doubles while amplitude remains constant, the particle velocity doubles, and consequently the energy quadruples. This follows from the kinetic energy relationship:

$$E_k = \frac{1}{2}mv^2 \tag{4}$$

For rotational motion with radius A and frequency f:

$$v = 2\pi Af \tag{5}$$

Therefore, the energy is:

$$E = m(2\pi Af)^2 \propto A^2f^2 \tag{6}$$

This demonstrates that wave energy is proportional to the square of both amplitude and frequency.

2.2 Waveforms

2.2.1 Simple Harmonic Motion

Definition 2.3 (Simple Harmonic Motion)**.** Simple harmonic motion is the periodic oscillatory motion that results when a system is subject to a restoring force proportional to its displacement from equilibrium.

The differential equation governing simple harmonic motion is:

$$\frac{d^2y}{dt^2} = -\kappa y \tag{7}$$

The general solution to this equation is:

$$y(t) = A\cos(\sqrt{\kappa}t) + B\sin(\sqrt{\kappa}t) \tag{8}$$

or equivalently:

$$y(t) = C\sin(\omega t + \phi) \tag{9}$$

where:

- $\omega = \sqrt{\kappa}$ is the angular frequency
- $C = \sqrt{A^2 + B^2}$ is the amplitude
- $\phi = \arctan(A/B)$ is the phase

Musical Significance: Simple harmonic motion produces a pure sinusoidal waveform, which corresponds to the purest, simplest sounds in nature, such as those produced by a tuning fork. Most musical instruments, however, produce complex waveforms that are combinations of multiple sinusoids.

2.2.2 Complex Waveforms

Real musical sounds are rarely pure sine waves. Instead, they consist of multiple frequency components combined together. A complex periodic waveform can be decomposed into its constituent frequencies using Fourier analysis (which will be studied in detail in Unit IV).

The wave shape of a sound is crucial for determining its timbre or sound quality. Our auditory system is highly sensitive to wave shape, using this information to:

- Identify sound sources (e.g., distinguishing a trumpet from an oboe)
- Determine the relative location of sounds
- Extract important environmental information

Common musical waveforms include:

- **Sine wave:** Pure tone with a single frequency component
- **Square wave:** Contains odd harmonics (1st, 3rd, 5th, ...)
- **Sawtooth wave:** Contains all harmonics (1st, 2nd, 3rd, ...)
- **Triangle wave:** Contains odd harmonics with amplitude decreasing as $1/n^2$

2.3 Harmonics and Overtones

Definition 2.4 (Harmonics)**.** Harmonics are frequency components of a sound that are positive integer multiples of a fundamental frequency. If the fundamental frequency is f, then the harmonics occur at frequencies $f, 2f, 3f, 4f, \ldots$

Definition 2.5 (Overtones)**.** Overtones are all partials above the fundamental frequency. For harmonic sounds, the overtones are the harmonics above the first harmonic (which is the fundamental itself).

The harmonic series is linear:

$$f_n = nf_0, \quad n = 1, 2, 3, 4, \ldots \tag{10}$$

where f_0 is the fundamental frequency and f_n is the n-th harmonic.

The harmonic content of a sound is a primary determinant of its timbre. Musical instruments produce characteristic patterns of harmonic amplitudes:

- **Wind and String Instruments:** Produce harmonic overtones (integer multiples of fundamental)
- **Percussion Instruments:** Often produce inharmonic overtones (not integer multiples)

The fundamental frequency typically determines the perceived pitch of the sound, while the pattern of overtone frequencies and their relative amplitudes constitute the sound's *spectrum*, which our ears use as a crucial cue for timbre recognition.

2.4 Spectrum

Definition 2.6 (Spectrum)**.** The spectrum of a sound is a graph or representation indicating the amplitudes of various different frequencies present in the sound.

Spectra can be classified as:

1. Discrete Spectrum: Contains energy at specific, distinct frequencies (typical of pitched musical tones)

2. Continuous Spectrum: Contains energy distributed across a continuous range of frequencies (typical of noise)

Static vs. Dynamic Spectra:

- **Static Spectrum:** Shows the average energy distribution over a relatively long time period (e.g., the duration of an entire note)
- **Dynamic Spectrum:** Shows how the spectral energy distribution evolves through time, capturing characteristics such as attack, decay, and vibrato

Our auditory system is highly attuned to dynamic spectral changes, using this information to identify instruments and assess performance characteristics.

2.5 Practice Questions: Nature of Sound

2.5.1 Short Answer Questions (2 marks each)

Question 1: Define frequency and amplitude of a sound wave.

Answer: Frequency is the number of oscillations per second (measured in Hertz, Hz) determining the pitch of a sound. Amplitude is the maximum displacement of particles from their equilibrium position, determining the loudness or intensity of the sound.

Question 2: What is simple harmonic motion and how does it relate to sound?

Answer: Simple harmonic motion is periodic oscillatory motion resulting from a restoring force proportional to displacement from equilibrium. It produces pure sinusoidal sound waves, such as those from a tuning fork. The motion is described by: $y(t) = C\sin(\omega t + \phi)$ where ω is angular frequency, C is amplitude, and ϕ is phase.

2.5.2 Medium Answer Questions (5 marks each)

Question 3: Explain the harmonic series and its relationship to musical timbre.

Answer: The harmonic series consists of frequencies that are integer multiples of a fundamental frequency:

$$f_n = nf_0, \quad n = 1, 2, 3, 4, \ldots \tag{11}$$

Most musical instruments produce complex sounds containing multiple harmonics. The relative amplitudes of these harmonics determine the timbre (tone color) of the instrument. For example:

- Flutes emphasize the fundamental with few harmonics (pure tone)
- Brass instruments have strong odd harmonics (bright tone)
- String instruments have rich harmonic content (warm tone)

Question 4: Describe the common musical waveforms and their harmonic characteristics.

Answer:

1. Sine Wave: Contains only a single frequency component (the fundamental). Produces the purest tone with no overtones.

2. Square Wave: Contains only odd harmonics (1st, 3rd, 5th, ...) with amplitudes decreasing as $1/n$. Produces a hollow, clarinet-like tone.

3. Triangle Wave: Contains odd harmonics with amplitudes decreasing as $1/n^2$. The faster amplitude decay produces a softer, mellow tone.

4. Sawtooth Wave: Contains all harmonics (both odd and even) with amplitudes decreasing as $1/n$. Produces a bright, string-like tone similar to bowed instruments.

The different harmonic content creates distinct timbres that our ears recognize as different instrumental sounds.

2.5.3 Long Answer Questions (10 marks)

Question 5: For a sound wave $p(t) = 0.1\sin(2\pi \cdot 440t)$, find:

(i) Period

(ii) Wavelength (given $v = 343$ m/s)

(iii) Time for 100 oscillations

(iv) The traveling wave form

Solution:
Given: $f = 440$ Hz, $v = 343$ m/s, $A = 0.1$
(i) Period:

$$T = \frac{1}{f} = \frac{1}{440} = 2.27 \times 10^{-3} \text{ s} = 2.27 \text{ ms} \tag{12}$$

(ii) Wavelength: Using the relationship $\lambda = v/f$:

$$\lambda = \frac{343}{440} = 0.78 \text{ m} \tag{13}$$

(iii) Time for 100 oscillations:

$$t_{100} = 100T = 100 \times 2.27 \times 10^{-3} = 0.227 \text{ s} \tag{14}$$

(iv) Traveling wave form: For a wave traveling in the positive x direction, the general form is:

$$p(x,t) = A \sin\left(2\pi f \left(t - \frac{x}{v}\right)\right) \tag{15}$$

Substituting our values:

$$p(x,t) = 0.1 \sin\left(2\pi \cdot 440 \left(t - \frac{x}{343}\right)\right) \tag{16}$$

Alternatively, using the wave number $k = 2\pi/\lambda = 2\pi \cdot 440/343$:

$$p(x,t) = 0.1 \sin(2\pi \cdot 440 \cdot t - kx) \tag{17}$$

This represents the note A4 (concert pitch) propagating through air.
Question 6: A sound wave has an intensity of $I = 10^{-6}$ W/m^2. Find its sound level in decibels and explain the significance of the logarithmic scale.
Solution:
The sound intensity level in decibels is given by:

$$L = 10 \log_{10}\left(\frac{I}{I_0}\right) \text{ dB} \tag{18}$$

where $I_0 = 10^{-12}$ W/m^2 is the reference intensity (threshold of hearing).
Substituting $I = 10^{-6}$ W/m^2:

$$L = 10 \log_{10}\left(\frac{10^{-6}}{10^{-12}}\right) = 10 \log_{10}(10^6) = 10 \times 6 = 60 \text{ dB} \tag{19}$$

Answer: The sound level is **60 dB**, corresponding approximately to normal conversation level.
Significance of the Logarithmic Scale:

1. **Physiological Basis:** Human perception of loudness follows approximately a logarithmic relationship with intensity (Weber-Fechner law). The decibel scale matches our perceptual experience.

2. **Dynamic Range:** Sound intensities vary over an enormous range—from the threshold of hearing to the threshold of pain spans about 10^{12} in intensity. The logarithmic scale compresses this to a manageable 0-120 dB range.

3. **Multiplicative Effects:** Every increase of 10 dB corresponds to a tenfold increase in intensity. This makes it easy to compare sounds that differ by orders of magnitude.

4. **Practical Examples:**
   - 0 dB: Threshold of hearing
   - 20 dB: Whisper
   - 60 dB: Normal conversation
   - 100 dB: Rock concert
   - 120 dB: Threshold of pain

The logarithmic formulation makes the decibel scale both mathematically elegant and perceptually meaningful.

3 Musical Pitch and Logarithmic Perception

3.1 Pitch as a Perceptual Phenomenon

Definition 3.1 (Pitch)**.** Pitch is the subjective psychological attribute of sound corresponding most closely to the physical property of frequency. It is defined as "that auditory attribute of sound according to which sounds can be ordered on a scale from low to high" (ANSI 1999).

While pitch and frequency are closely related, they are not identical:

- Pitch is subjective and perceptual; frequency is objective and physical
- Pitch is influenced by frequency, loudness, duration, and the presence of other frequencies
- Pitch is limited to the range of hearing; frequency is unlimited
- A sound must have a certain minimum duration for its pitch to be perceived; otherwise it is heard as a click

3.2 Octave Equivalence and Logarithmic Perception

One of the most fundamental aspects of pitch perception is *octave equivalence*:

Theorem 3.1 (Octave Equivalence)**.** Pitches related by a frequency ratio of 2:1 are perceived as highly similar, possessing the same pitch quality despite differing in height. This relationship is called the octave.

This phenomenon reveals that pitch perception is not simply one-dimensional. As two tones diverge in frequency, the perceived difference in pitch height increases, but when the frequency ratio reaches 2:1, the tones begin to sound alike again. This pattern repeats at each successive octave.

Remarkably, the octave as a physical frequency ratio of 2:1 always corresponds exactly to the subjective pitch difference of an octave, making the octave a rare instance where objective and subjective measurements align precisely.

3.3 The Logarithmic Nature of Pitch Perception

Human perception of musical intervals corresponds to frequency *ratios* rather than frequency *differences*. This fundamental principle means that:

$$\text{Perceived interval} \propto \log(\text{frequency ratio}) \tag{20}$$

For example:

- The interval from 100 Hz to 200 Hz (ratio 2:1) sounds the same as the interval from 1000 Hz to 2000 Hz (also ratio 2:1)
- Both represent one octave, despite the absolute frequency difference being 100 Hz in the first case and 1000 Hz in the second

This logarithmic relationship is why musical scales are based on frequency ratios. If we want to divide an octave into equal perceptual steps, we must use a geometric (multiplicative) sequence of frequencies, not an arithmetic (additive) sequence.

3.4 The Cent Scale

To provide a practical logarithmic measure of musical intervals, Alexander Ellis introduced the *cent* scale around 1875.

Definition 3.2 (Cent)**.** A cent is a logarithmic unit of measurement for musical intervals. There are 1200 cents in an octave, making each semitone in the equal-tempered scale equal to 100 cents.

The conversion formulas are:

From frequency ratio r to cents:

$$c = 1200 \log_2(r) = \frac{1200 \ln(r)}{\ln(2)} \tag{21}$$

From cents c to frequency ratio:

$$r = 2^{c/1200} \tag{22}$$

Example 3.1. Calculate the interval in cents between frequencies with ratio 3:2 (a perfect fifth):

$$c = 1200 \log_2(3/2) = \frac{1200 \ln(1.5)}{\ln(2)} \approx 701.96 \text{ cents} \tag{23}$$

3.5 Just Noticeable Difference (JND)

Definition 3.3 (Just Noticeable Difference for Pitch)**.** The just noticeable difference (JND) or difference limen (DL) for pitch is the minimum threshold of frequency difference required for two pitches to be perceived as distinct.

Key characteristics of pitch JND:

- The JND is not constant across the frequency range
- The JND for high frequencies covers a larger span than for low frequencies
- Typical JND values range from approximately 0.5% to 3% of the fundamental frequency
- The JND depends on:
  - Frequency range
  - Sound intensity
  - Duration of tone
  - Rate of frequency change
  - Individual differences and musical training

The JND provides a measure of the *precision* or *resolution* of our pitch perception, analogous to the finest markings on a ruler.

3.6 Practice Questions: Pitch and Logarithmic Perception

3.6.1 Short Answer Questions (2 marks each)

Question 1: Write the formula for sound intensity level in decibels.
Answer:

$$L = 10 \log_{10} \left(\frac{I}{I_0} \right) \text{ dB} \tag{24}$$

where I is the sound intensity and $I_0 = 10^{-12}$ W/m² is the reference intensity (threshold of hearing at 1000 Hz).

Question 2: What is octave equivalence? Give its mathematical representation.

Answer: Octave equivalence is the perceptual phenomenon where pitches related by a frequency ratio of 2:1 are perceived as highly similar, possessing the same pitch quality despite differing in height. Mathematically:

$$f_{\text{octave}} = 2 \times f_{\text{reference}} \tag{25}$$

This 2:1 ratio is universal across all musical cultures and forms the basis of the octave as a fundamental musical interval.

3.6.2 Medium Answer Questions (5 marks each)

Question 3: Explain the logarithmic nature of pitch perception and the cent scale.

Answer:

Logarithmic Perception:

Human pitch perception follows a logarithmic relationship. The perceived pitch P is proportional to $\log_2(f)$, not to f itself. This means:

$$P = k \log_2 \left(\frac{f}{f_0} \right) \tag{26}$$

Equal frequency *ratios* produce equal perceived intervals, regardless of absolute frequencies.

Evidence:

- Octave equivalence: Doubling frequency always produces one octave, whether from 100 Hz to 200 Hz or from 1000 Hz to 2000 Hz
- Musical scales are based on frequency ratios, not differences
- Transposition maintains musical relationships because it multiplies all frequencies by a constant ratio

The Cent Scale:

To provide a practical logarithmic measure, the cent is defined as 1/1200 of an octave:

$$\text{cents} = 1200 \log_2 \left(\frac{f_2}{f_1} \right) \tag{27}$$

This divides the octave into 1200 equal perceptual units.

Examples:

- Octave (2:1): $1200 \log_2(2) = 1200$ cents
- Perfect fifth (3:2): $1200 \log_2(1.5) \approx 702$ cents
- Major third (5:4): $1200 \log_2(1.25) \approx 386$ cents

The cent scale provides a uniform, logarithmic measure that matches human perception.

Question 4: Explain the Just Noticeable Difference (JND) for pitch and its implications.

Answer:

The Just Noticeable Difference (JND) is the smallest change in a stimulus that can be reliably detected. For pitch perception:

Magnitude of JND:

- At middle frequencies (around 1000 Hz): approximately 1% or about 5-6 cents
- Varies with frequency, being larger at very low and very high frequencies
- Trained musicians may achieve JND as small as 1-3 cents

Implications:

1. Resolution of Pitch Space: Within one octave (1200 cents), there are theoretically:

$$\frac{1200 \text{ cents}}{5 \text{ cents/JND}} = 240 \text{ distinguishable pitches} \tag{28}$$

2. Musical Scale Design:

- Scale steps must be significantly larger than JND to be clearly perceived
- Semitones (100 cents) are about 20 times the JND, providing clear differentiation
- Microtonal systems use intervals closer to JND for subtle expression

3. Intonation Tolerance: Small deviations (¡ 10 cents) from ideal pitch are generally acceptable in performance, as they approach the perceptual threshold.

4. Context Dependence: JND depends on:

- Duration of the tone (shorter tones have larger JND)
- Presence of other sounds (masking effects)
- Musical training and attention

3.6.3 Long Answer Questions (10 marks)

Question 5: Derive the Weber-Fechner law and explain its application to musical perception.

Answer:

Weber-Fechner Law:

The Weber-Fechner law states that perceived sensation is proportional to the logarithm of stimulus intensity:

$$S = k \ln\left(\frac{I}{I_0}\right) \tag{29}$$

where:

- S is the perceived sensation (e.g., perceived pitch or loudness)
- I is the physical intensity (e.g., frequency or sound pressure)
- I_0 is the threshold intensity
- k is a constant

Derivation from Weberś Law:

Weberś Law states that the JND is proportional to the stimulus magnitude:

$$\frac{\Delta I}{I} = c \tag{30}$$

where c is a constant.

Integrating this relationship:

$$\Delta S \propto \frac{\Delta I}{I} \tag{31}$$

$$dS = k\frac{dI}{I} \tag{32}$$

$$S = k\ln(I) + C \tag{33}$$

Setting $S = 0$ at threshold $I = I_0$:

$$S = k\ln\left(\frac{I}{I_0}\right) \tag{34}$$

Applications to Musical Perception:
1. Pitch Perception: For frequency f, perceived pitch follows:

$$P = k\log_2\left(\frac{f}{f_0}\right) \tag{35}$$

- Equal pitch intervals correspond to equal frequency *ratios*
- Doubling frequency always produces one octave
- Musical scales use multiplicative (geometric) progressions

2. Loudness Perception: The decibel scale directly implements Weber-Fechner:

$$L = 10\log_{10}\left(\frac{I}{I_0}\right) \tag{36}$$

- Each 10 dB increase represents a tenfold intensity increase
- But perceived loudness increases by a constant amount
- Matches human perception of loudness changes

3. Musical Implications:

- Explains why octaves (2:1) are the fundamental organizing principle
- Justifies the use of cents (logarithmic scale) for measuring intervals
- Explains transposition: multiplying all frequencies by constant ratio preserves musical relationships
- Informs the design of musical instruments with logarithmically-spaced controls

Conclusion: The Weber-Fechner law provides the mathematical foundation for understanding why music is fundamentally organized around ratios rather than differences, and why logarithmic scales like cents and decibels are natural choices for musical measurement.

4 Indian Musical Concepts: Shruti, Swara, and Saptak

4.1 Overview of Indian Music Theory

Indian classical music, both Carnatic (South Indian) and Hindustani (North Indian), has developed sophisticated mathematical systems for organizing pitch. Unlike Western music's emphasis on harmony and modulation, Indian classical music emphasizes melodic elaboration and microtonal nuances.

4.2 Swara

Definition 4.1 (Swara)**.** A swara is a musical note or tone in Indian music, analogous to Western notes but with more refined categorizations. The seven basic swaras are: Sa, Ri (or Re), Ga, Ma, Pa, Dha, and Ni.

In the Carnatic system, these seven main swaras can be elaborated:

Swara	**Position**	**Name**
Sa	1	Shadjamam (fixed)
Ri	2-4	Rishabham (3 variants: R1, R2, R3)
Ga	5-7	Gandharam (3 variants: G1, G2, G3)
Ma	8-9	Madhyamam (2 variants: M1, M2)
Pa	10	Panchamam (fixed)
Dha	11-13	Dhaivatam (3 variants: D1, D2, D3)
Ni	14-16	Nishadam (3 variants: N1, N2, N3)
Sa	1	Shadjamam (upper octave)

Some variants are enharmonically equivalent:

- R2 = G1
- R3 = G2
- D2 = N1
- D3 = N2

This yields 7 main swaras plus 5 semitone variants, giving 12 distinct pitch positions, plus the upper Sa for 13 total within an octave (8 unique positions when considering the octave structure).

In the Hindustani system, these seven main swaras are elaborated as follows:

Swara	**Position**	**Name**
Sa	1	Shadja (fixed)
Ri	2-3	Rishabh (2 variants: r, R)
Ga	4-5	Gandhar (2 variants: g, G)
Ma	6-7	Madhyam (2 variants: m, M)
Pa	8	Pancham (fixed)
Dha	9-10	Dhaivat (2 variants d, D)
Ni	11-12	Nishad (2 variants n, N)
Sa	1	Shadja (upper octave)

4.3 Shruti

Definition 4.2 (Shruti)**.** A shruti is the smallest interval of pitch that can be perceived and used musically in Indian classical music. The term derives from the root "śru" meaning "to hear."

The classical Hindustani 22-shruti scale is one of the most refined pitch systems in world music. The 22 shrutis per octave provide:

5 Complete 22-Shruti Frequency Table

Table 1: 22 Shrutis with Frequencies and Ratios (Sa = 100 Hz)

No.	**Shruti**	**Description**	**Frequency**	**Ratio**	**×100**
1	S	Shadja	100	1/1	X 100
2	r1	Atikomal Rishabh (Lower)	105.3497942	256/243	X 100
3	r2	Komal Rishabh (Higher)	106.666666	16/15	X 100
4	R1	Shuddha Rishabh (Lower)	111.111111	10/9	X 100
5	R2	Teevra Rishabh (Higher)	112.5	9/8	X 100
6	g1	Atikomal Gandhaar (Lower)	118.51851	32/27	X 100
7	g2	Komal Gandhaar (Higher)	120	6/5	X 100
8	G1	Shuddha Gandhaar (Lower)	125	5/4	X 100
9	G2	Teevra Gandhaar (Higher)	126.5625	81/64	X 100
10	M1	Shuddha Madhyam (Lower)	133.33333	4/3	X 100
11	M2	Ekashruti Madhyam (Higher)	135	27/20	X 100
12	m1	Teevra Madhyam (Lower)	140.625	45/32	X 100
13	m2	Teevratama Madhyam (Higher)	142.3828125	729/512	X 100
14	P	Pancham	150	3/2	X 100
15	d1	Atikomal Dhaivat (Lower)	158.0246913	128/81	X 100
16	d2	Komal Dhaivat (Higher)	160	8/5	X 100
17	D1	Shuddha Dhaivat (Lower)	166.666666	5/3	X 100
18	D2	Teevra Dhaivat (Higher)	168.75	27/16	X 100

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Table 1 – *Continued from previous page*

No.	Shruti	Description	Frequency	Ratio	×100
19	n1	Atikomal Niashad (Lower)	177.777777	16/9	X 100
20	n2	Komal Nishad (Higher)	180	9/5	X 100
21	N1	Shuddha Nishad (Lower)	187.5	15/8	X 100
22	N2	Teevra Nishad (Higher)	189.84375	243/128	X 100

Note: All frequencies are calculated assuming Sa (Shadja) = 100 Hz. The "X 100" column indicates that these values are multiplied by 100 when Sa is taken as 100 Hz.

5.1 Naming Convention Used in This Table

- **Lowercase letters (r, g, d, n)**: Komal (flat) and Atikomal (extremely flat) variants
- **Uppercase letters (R, G, M, D, N)**: Shuddha (natural) and Teevra (sharp) variants
- **S and P**: Always uppercase as they are Achal (immovable) swaras
- **Numbers (1, 2)**: Indicate lower and higher variants within the same category
- **Atikomal**: Extremely flat version
- **Komal**: Flat version
- **Shuddha**: Natural/pure version
- **Teevra**: Sharp version (note: "Teevra" is typically used only for Madhyam, but this table extends it to other swaras)
- **Ekashruti**: Special name for one particular Madhyam shruti (M2)
- **Teevratama**: Special name for the highest Madhyam shruti (m2)

5.2 Saptak

Definition 5.1 (Saptak)**.** Saptak (from sapta meaning "seven") refers to an octave in Indian music, literally meaning "a group of seven." It encompasses the seven basic swaras within the range of an octave.

Indian music theory recognizes three saptaks:

1. **Mandra Saptak:** Lower octave
2. **Madhya Saptak:** Middle octave (reference octave)
3. **Tara Saptak:** Upper octave

Just as in Western music, the saptak represents the fundamental repeating unit of pitch organization, based on the frequency ratio of 2:1. The universality of octave-based organization across diverse musical cultures provides evidence for fundamental psychoacoustic principles in human music perception.

5.3 Practice Questions: Indian Classical Music

5.3.1 Short Answer Questions (2 marks each)

Question 1: State two differences between Shruti and Swara.

Answer:

Shruti	Swara
22 microtonal divisions per octave (theoretical framework)	7 main notes used in practice (Sa, Ri, Ga, Ma, Pa, Dha, Ni)
Smallest perceivable pitch intervals	Actual notes sung or played in melodies
Not all directly used in a single raga	Selected from shrutis to form melodic structures

Question 2: What is a Saptak? Name its three registers.

Answer: A Saptak (from Sanskrit *sapta* meaning "seven") is an octave consisting of seven Swaras (Sa, Ri, Ga, Ma, Pa, Dha, Ni) spanning a 2:1 frequency ratio.

Three registers:

1. **Mandra Saptak:** Lower octave (low register)
2. **Madhya Saptak:** Middle octave (reference/main register)
3. **Tara Saptak:** Upper octave (high register)

Each successive saptak doubles the frequency: $f_{\text{Tara}} = 2 \times f_{\text{Madhya}} = 4 \times f_{\text{Mandra}}$.

5.3.2 Medium Answer Questions (5 marks each)

Question 3: Describe Shruti, Swara, and Saptak with their mathematical basis.

Answer:

Shruti:

Shruti (Sanskrit: "what is heard") represents the smallest interval of pitch discernible by the human ear in Indian classical music. Traditional theory divides the octave into 22 shrutis of unequal size.

Mathematical Model:

$$f_k = f_0 \cdot 2^{c_k/1200} \tag{37}$$

where f_k is the frequency of the k-th shruti, f_0 is the tonic (Sa), and c_k is the cumulative cent position (non-uniformly spaced).

The 22-shruti system is based on just-intonation ratios from the harmonic series, with three basic interval sizes:

- 256/243 ($\approx$ 90 cents)
- 25/24 ($\approx$ 71 cents)
- 81/80 ($\approx$ 22 cents) — called *murchana*

Swara:

Swara refers to the 7 main notes selected from the 22 shrutis for melodic use: Sa, Ri, Ga, Ma, Pa, Dha, Ni.

Each swara (except Sa and Pa, which are fixed) has variants:

- **Komal:** Flat (lowered)
- **Shuddha:** Natural
- **Tivra:** Sharp (raised, primarily for Ma)

Just Intonation Ratios:

Swara	Ratio	Cents
Sa	1:1	0
Ri	9:8	204
Ga	5:4	386
Ma	4:3	498
Pa	3:2	702
Dha	5:3	884
Ni	15:8	1088
Sa'	2:1	1200

Saptak:
Saptak (from *sapta* = seven) is an octave formed by 7 swaras with frequency doubling.
Mathematical Relationship:

$$f_{\text{upper}} = 2 \times f_{\text{lower}} \tag{38}$$

Three registers (Mandra, Madhya, Tara) extend the pitch range while maintaining octave equivalence through the 2:1 ratio.

Unified Mathematical Framework:
All three concepts rely on logarithmic pitch perception:

$$P = k \log_2(f/f_0) \tag{39}$$

This ensures that:

- Saptaks maintain perceptual equivalence across registers
- Swaras retain identity under octave transposition
- Shrutis provide microtonal nuance within this framework

5.3.3 Long Answer Questions (10 marks)

Question 5: If Sa = 240 Hz, calculate frequencies of all Swaras using just-intonation ratios. Verify that Sa' = 2 × Sa.

Solution:
Given: Sa (tonic) = 240 Hz
Using just-intonation ratios for the seven main swaras:

Swara	Ratio	Calculation	Frequency (Hz)
Sa	1:1	240×1	240
Ri	9:8	$240 \times 9/8$	270
Ga	5:4	$240 \times 5/4$	300
Ma	4:3	$240 \times 4/3$	320
Pa	3:2	$240 \times 3/2$	360
Dha	5:3	$240 \times 5/3$	400
Ni	15:8	$240 \times 15/8$	450
Sa'	2:1	240×2	480

Detailed Calculations:

$$f_{\text{Sa}} = 240 \times \frac{1}{1} = 240 \text{ Hz} \tag{40}$$
$$f_{\text{Ri}} = 240 \times \frac{9}{8} = \frac{2160}{8} = 270 \text{ Hz} \tag{41}$$
$$f_{\text{Ga}} = 240 \times \frac{5}{4} = \frac{1200}{4} = 300 \text{ Hz} \tag{42}$$
$$f_{\text{Ma}} = 240 \times \frac{4}{3} = \frac{960}{3} = 320 \text{ Hz} \tag{43}$$
$$f_{\text{Pa}} = 240 \times \frac{3}{2} = \frac{720}{2} = 360 \text{ Hz} \tag{44}$$
$$f_{\text{Dha}} = 240 \times \frac{5}{3} = \frac{1200}{3} = 400 \text{ Hz} \tag{45}$$
$$f_{\text{Ni}} = 240 \times \frac{15}{8} = \frac{3600}{8} = 450 \text{ Hz} \tag{46}$$
$$f_{\text{Sa'}} = 240 \times \frac{2}{1} = 480 \text{ Hz} \tag{47}$$

Verification of Octave Relationship:

$$f_{\text{Sa'}} = 2 \times f_{\text{Sa}} \quad \Rightarrow \quad 480 = 2 \times 240 \quad \checkmark \tag{48}$$

Additional Verifications:
1. Perfect Fifth (Sa to Pa):

$$\frac{f_{\text{Pa}}}{f_{\text{Sa}}} = \frac{360}{240} = \frac{3}{2} \quad \checkmark \tag{49}$$

2. Perfect Fourth (Sa to Ma):

$$\frac{f_{\text{Ma}}}{f_{\text{Sa}}} = \frac{320}{240} = \frac{4}{3} \quad \checkmark \tag{50}$$

3. Major Third (Sa to Ga):

$$\frac{f_{\text{Ga}}}{f_{\text{Sa}}} = \frac{300}{240} = \frac{5}{4} \quad \checkmark \tag{51}$$

Interval Sizes in Cents:

For reference, the cent values of these intervals:

$$\text{Sa}\to\text{Ri:} \quad 1200\log_2(9/8) \approx 204 \text{ cents} \tag{52}$$
$$\text{Sa}\to\text{Ga:} \quad 1200\log_2(5/4) \approx 386 \text{ cents} \tag{53}$$
$$\text{Sa}\to\text{Ma:} \quad 1200\log_2(4/3) \approx 498 \text{ cents} \tag{54}$$
$$\text{Sa}\to\text{Pa:} \quad 1200\log_2(3/2) \approx 702 \text{ cents} \tag{55}$$
$$\text{Sa}\to\text{Dha:} \quad 1200\log_2(5/3) \approx 884 \text{ cents} \tag{56}$$
$$\text{Sa}\to\text{Ni:} \quad 1200\log_2(15/8) \approx 1088 \text{ cents} \tag{57}$$
$$\text{Sa}\to\text{Sa':} \quad 1200\log_2(2/1) = 1200 \text{ cents} \tag{58}$$

Summary:
The frequencies range from 240 Hz to 480 Hz, following just-intonation ratios derived from the harmonic series. These ratios ensure:

- Pure consonances with minimal beating
- Natural alignment with overtone structure
- Maximum harmonic clarity

The octave relationship (2:1) is perfectly satisfied, demonstrating the fundamental principle of saptak organization in Indian classical music.

6 Musical Notation Systems

6.1 The Need for Musical Notation

Musical notation systems serve several crucial functions:

- **Communication:** Allow musicians to communicate musical ideas across time and space
- **Preservation:** Enable music to be recorded and transmitted to future generations
- **Analysis:** Provide a framework for studying and understanding musical structure
- **Pedagogy:** Facilitate the teaching and learning of music
- **Composition:** Allow composers to explore and develop complex musical ideas

6.2 Western Common Music Notation (CMN)

6.2.1 Historical Development

Western musical notation evolved over many centuries:

- **9th-10th centuries:** Early neumes (symbols showing relative pitch direction)
- **11th-12th centuries:** Development of staff notation
- **15th century:** Standardization of five-line staff
- **16th-17th centuries:** Development of modern notation conventions

6.2.2 The Staff System

The staff consists of five horizontal lines and four spaces, providing a visual representation of pitch space.

Clefs: Symbols that establish the pitch reference for the staff

- **Treble Clef (G clef):** The spiral encircles the second line from bottom, designating it as G above middle C
- **Bass Clef (F clef):** The two dots bracket the second line from top, designating it as F below middle C
- **Alto/Tenor Clefs (C clefs):** Center on middle C

Note Symbols: Indicate both pitch (vertical position) and duration:

- Whole note: Open oval (4 beats in 4/4 time)
- Half note: Open oval with stem (2 beats)
- Quarter note: Filled oval with stem (1 beat)
- Eighth note: Filled oval with stem and flag (1/2 beat)
- Sixteenth note: Filled oval with stem and two flags (1/4 beat)

Accidentals:

- Sharp ($\sharp$): Raises pitch by one semitone
- Flat ($\flat$): Lowers pitch by one semitone
- Natural ($\natural$): Cancels previous sharp or flat
- Double sharp ($\times$): Raises pitch by two semitones
- Double flat ($\flat\flat$): Lowers pitch by two semitones

6.2.3 Additional Notational Elements

Key Signature: A collection of sharps or flats at the beginning of the staff indicating the tonal center

Time Signature: Two numbers indicating:

- Top number: Number of beats per measure
- Bottom number: Note value receiving one beat

Dynamic Markings: Indicate loudness levels

- *ppp* (pianississimo): As soft as possible
- *pp* (pianissimo): Very soft
- *p* (piano): Soft

- *mp* (mezzo piano): Moderately soft
- *mf* (mezzo forte): Moderately loud
- *f* (forte): Loud
- *ff* (fortissimo): Very loud
- *fff* (fortississimo): As loud as possible

Tempo Markings: Indicate speed of performance, either through:

- Metronome markings (e.g., = 120 beats per minute)
- Italian terms (e.g., Allegro, Andante, Adagio)

6.3 Indian Music Notation

Indian classical music has traditionally been an oral tradition, with knowledge passed from guru to student. However, several notation systems have been developed:

6.3.1 Bhatkhande Notation (Hindustani)

Developed by Vishnu Narayan Bhatkhande in the early 20th century:

- Uses letters or numbers (Sa Re Ga Ma Pa Dha Ni or S R G M P D N or 1 2 3 4 5 6 7)
- Dots above notes indicate higher octave (tara saptak)
- Dots below notes indicate lower octave (mandra saptak)
- Lines indicate duration
- Relatively simple but effective for basic melodic representation

6.3.2 Carnatic Notation

- Uses similar swara notation (Sa Ri Ga Ma Pa Da Ni)
- Employs various symbols to indicate gamakas (ornamentations)
- Includes notation for complex rhythmic patterns (konnakol)
- Different styles exist for different purposes (concert, pedagogy, analysis)

6.4 Comparative Analysis of Notation Systems

Aspect	Western CMN	Indian Systems
Pitch representation	Absolute (staff position determines exact pitch)	Relative (swaras defined relative to Sa)
Microtones	Limited (quarter-tones rarely used)	Extensive (shrutis, gamakas)
Rhythm notation	Precise durational values	More flexible, with tala framework
Ornamentation	Indicated by specific symbols	Integral to notation, extensive gamaka symbols
Harmony	Facilitates polyphonic writing	Primarily melodic, harmony secondary
Improvisation	Generally fully specified	Provides framework for improvisation

Table 2: Comparison of Western and Indian notation systems

6.5 Mathematical Representation of Pitch

Beyond traditional notation, mathematical representations of pitch include:

6.5.1 Frequency Formula

For a given reference frequency f_{ref} and number of semitones n above or below that reference:

$$f = f_{\text{ref}} \cdot 2^{n/12} \tag{59}$$

6.5.2 Pitch Class Sets

Modern music theory uses pitch class sets for analysis:

- Pitch classes are represented modulo 12: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
- C = 0, C♯ = 1, D = 2, etc.
- This representation facilitates mathematical analysis of musical structures

6.6 Practice Questions: Musical Notation Systems

6.6.1 Short Answer Questions (2 marks each)

Question 1: What is the purpose of musical notation systems?

Answer: Musical notation systems serve to preserve, transmit, and communicate musical ideas across time and space. They provide a standardized way to represent pitch, duration, dynamics, and other musical parameters, enabling musicians to reproduce compositions accurately and allowing composers to document their creative work for future generations.

Question 2: Explain the staff notation system and how pitch is represented.

Answer: The staff (or stave) consists of five horizontal lines and four spaces. Pitch is represented by the vertical position of note symbols on the staff:

- Higher positions represent higher pitches
- A clef at the beginning specifies which pitches the lines represent
- The treble clef (G clef) positions middle C below the staff
- The bass clef (F clef) positions middle C above the staff
- Ledger lines extend the range above and below the staff

6.6.2 Medium Answer Questions (5 marks each)

Question 3: Compare Western staff notation with Indian swara notation systems.
Answer:
Western Staff Notation:

- Uses a five-line staff with absolute pitch representation
- Fixed clefs determine the pitch of each line and space
- Includes detailed rhythmic notation with note values and time signatures
- Specifies dynamics, articulation, and expression marks
- Best suited for ensemble music and polyphonic textures

Indian Swara Notation:

- Uses letter abbreviations (Sa, Ri, Ga, Ma, Pa, Dha, Ni)
- Relative pitch system—Sa can be any pitch chosen by performer
- Dots above/below indicate octave (tara/mandra saptak)
- Rhythmic notation less detailed, relies on oral tradition
- Suited for melodic music and improvisation
- Komal (flat) and Tivra (sharp) indicated by underlining or apostrophes

Key Differences:

1. Absolute vs. relative pitch representation
2. Visual/spatial vs. symbolic representation
3. Detailed rhythm vs. emphasis on melodic contour
4. Suited for different musical traditions and practices

Question 4: Explain pitch class sets and their mathematical representation.
Answer:
Pitch class sets are a modern mathematical approach to representing and analyzing musical structures, particularly in atonal and contemporary music.
Mathematical Representation:
Pitch classes are represented using modulo 12 arithmetic:

- The octave is divided into 12 equal semitones
- Each pitch class is assigned a number: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
- Typically: C = 0, C = 1, D = 2, ..., B = 11
- All octave transpositions of a pitch belong to the same pitch class

Operations on Pitch Classes:
1. Transposition: Adding a constant (mod 12)

$$T_n(x) = (x + n) \mod 12 \tag{60}$$

2. Inversion: Reflection about an axis

$$I_n(x) = (n - x) \mod 12 \tag{61}$$

Applications:

- Analyzing atonal music and twelve-tone compositions
- Identifying transformational relationships between musical patterns
- Computer-assisted composition and analysis
- Mathematical music theory and group theory applications

Example: A C major triad {C, E, G} is represented as {0, 4, 7}. Transposing up a perfect fifth (7 semitones) gives:

$$T_7(\{0, 4, 7\}) = \{7, 11, 2\} = \{G, B, D\} \tag{62}$$

6.6.3 Long Answer Questions (10 marks)

Question 5: Discuss the evolution and mathematical basis of musical notation systems across different cultures.

Answer:
Historical Development:
1. Ancient Systems:

- Early notations were primarily mnemonic devices
- Greek letter notation for pitch
- Neumatic notation in medieval Europe (9th century)
- Oral tradition dominant in many cultures

2. Western Development:

- Staff notation emerged in 11th century (Guido dÁrezzo)
- Mathematical organization: five lines, logarithmic pitch spacing
- Mensural notation for rhythm (13th century)

- Modern notation standardized by 17th-18th centuries

3. Indian Systems:

- Ancient texts described swaras conceptually
- Written notation developed relatively late (19th-20th centuries)
- Emphasis on oral transmission (guru-shishya parampara)
- Multiple notation systems coexist (Bhatkhande, Paluskar)

Mathematical Principles:
1. Pitch Representation:

- Western: Spatial/visual mapping (higher = higher pitch)
- Reflects logarithmic perception through equally-spaced lines
- Indian: Symbolic representation with relative pitch
- Modern: Numerical (MIDI, frequency, pitch class)

2. Rhythmic Representation:

- Western: Proportional duration symbols (whole, half, quarter notes)
- Binary division: 2^n subdivision of beats
- Indian: Matras and tala cycles, often asymmetric

3. Information Encoding:
Different notation systems encode different aspects:

System	Pitch Detail	Rhythm Detail	Expression
Western Staff	High	High	High
Indian Swara	Medium	Low	Low
Tablature	Fingering	Medium	Low
Modern Symbolic	High	High	Variable

Contemporary Developments:

- Digital notation: MIDI, MusicXML
- Mathematical representations: pitch class sets, transformational theory
- Spectrographic notation for electronic music
- Graphic scores for experimental music

Conclusion: Musical notation systems reflect both the mathematical organization of music and the cultural priorities of musical traditions. Western notation emphasizes precise pitch and rhythm for ensemble performance, while Indian notation prioritizes melodic contour and allows for improvisation. Modern mathematical approaches like pitch class sets provide tools for analysis that transcend traditional notation systems.

7 Perception of Sound

7.1 Physical vs. Perceptual Attributes

The relationship between physical properties of sound and their perception is complex:

Physical Property	**Perceptual Correlate**	**Relationship**
Frequency	Pitch	Logarithmic
Intensity/Amplitude	Loudness	Logarithmic (complex)
Wave shape	Timbre	Complex, multidimensional
Onset time	Attack/Rhythm	Direct
Duration	Length/Rhythm	Direct

Table 3: Physical properties and their perceptual correlates

7.2 Psychoacoustic Phenomena

Several important psychoacoustic phenomena affect musical perception:

1. Masking: A loud sound at one frequency can prevent perception of a quieter sound at a nearby frequency. Sounds at lower frequencies mask higher frequencies more effectively than vice versa.

2. Critical Bands: The auditory system analyzes frequency in terms of critical bandwidths. Sounds within the same critical band interact strongly, affecting perception of consonance and dissonance.

3. The Missing Fundamental: When presented with harmonics at $2f$, $3f$, $4f$, etc., we often perceive a pitch at the (missing) fundamental frequency f, even when that frequency is not physically present.

4. Octave Equivalence: As discussed earlier, pitches separated by octaves (2:1 frequency ratio) sound remarkably similar.

7.3 Weber-Fechner Law

The Weber-Fechner law describes the relationship between physical stimulus intensity and perceived sensation:

$$S = k \ln(I/I_0) \tag{63}$$

where:

- S is the perceived sensation
- I is the physical intensity
- I_0 is the threshold intensity
- k is a constant

This logarithmic relationship explains why:

- Musical intervals correspond to frequency ratios (not differences)
- The decibel scale is logarithmic
- Equal perceptual steps require multiplicative (not additive) physical changes

7.4 Practice Questions: Perception of Sound

7.4.1 Short Answer Questions (2 marks each)

Question 1: Distinguish between physical and perceptual attributes of sound.

Answer: Physical attributes of sound are objective, measurable properties like frequency (Hz), intensity (W/m²), and wave shape. Perceptual attributes are subjective experiences like pitch, loudness, and timbre. The relationship is complex and often non-linear:

- Frequency → Pitch (logarithmic relationship)
- Intensity → Loudness (logarithmic relationship)
- Wave shape → Timbre (complex, multidimensional)

Question 2: What is the missing fundamental phenomenon?

Answer: The missing fundamental is a psychoacoustic phenomenon where the human auditory system perceives a pitch at a fundamental frequency even when that frequency is not physically present in the sound. When harmonics at 2f, 3f, 4f, etc., are present, the brain reconstructs and perceives the pitch at the missing fundamental frequency f. This demonstrates that pitch perception is a cognitive process, not just a passive response to frequency.

7.4.2 Medium Answer Questions (5 marks each)

Question 3: Explain the concept of critical bands and their role in musical perception.

Answer:

Critical Bands:

Critical bands are frequency ranges within which the auditory system processes sound as a unified entity. They represent the frequency resolution of the ear.

Characteristics:

- Width: Approximately 100 Hz at low frequencies, increasing to about 3500 Hz at high frequencies
- About 24 critical bands span the audible range
- Roughly constant on a logarithmic frequency scale

Role in Musical Perception:

1. Consonance and Dissonance:

- When two frequencies fall within the same critical band, they interact strongly
- Close intervals (within critical band) sound rough or dissonant
- Wider intervals (outside critical band) sound smoother or consonant
- This explains why minor seconds (semitone) sound dissonant while octaves sound consonant

2. Masking Effects:

- A loud sound masks quieter sounds within the same critical band
- Important for mixing and orchestration
- Instruments in different frequency ranges are easier to distinguish

3. Timbre Perception:

- Harmonics within a critical band blend together
- Distribution of energy across critical bands creates timbral character
- Explains why instruments with different harmonic distributions sound distinct

4. Musical Implications:

- Voice leading: Wide spacing in low registers, closer in high registers
- Chord voicing: Critical bands wider at low frequencies
- Acoustic design: Consider critical band interactions

Question 4: Discuss masking effects and their applications in music production.
Answer:
Masking:

Masking occurs when a louder sound (the masker) makes a softer sound (the maskee) inaudible or less perceptible. This is particularly pronounced when sounds are close in frequency.

Types of Masking:
1. Simultaneous Masking:

- Both sounds occur at the same time
- Most effective within the same critical band
- Low frequencies mask high frequencies more effectively than vice versa

2. Temporal Masking:

- Forward masking: Loud sound masks subsequent quiet sounds
- Backward masking: Loud sound masks preceding quiet sounds
- Duration: Up to 200ms for forward, 20ms for backward

Applications in Music Production:
1. Mixing and Balance:

- Separate instruments in different frequency ranges to reduce masking
- Use equalization to carve out frequency space for each instrument
- Example: Cut low frequencies from guitars to prevent bass masking

2. Audio Compression:

- MP3 and other lossy formats exploit masking

- Remove masked components that are inaudible
- Significantly reduces file size with minimal perceptual loss

3. Orchestration:

- Distribute instruments across frequency spectrum
- Avoid simultaneous loud passages in similar frequency ranges
- Traditional orchestra seating considers masking effects

4. Sound Design:

- Layer sounds knowing quieter elements may be masked
- Use sidechain compression to temporarily reduce masking
- Strategic use of silence and space

Conclusion: Understanding masking is crucial for audio engineering and music production. It informs decisions about frequency balance, dynamic range, and spatial arrangement, ultimately contributing to clarity and intelligibility in musical mixes.

7.4.3 Long Answer Questions (10 marks)

Question 5: Analyze the relationship between physical sound properties and human perception, including relevant psychoacoustic phenomena.

Answer:

Physical Properties and Perceptual Correlates:

The relationship between physical sound properties and their perception is complex and multifaceted:

1. Frequency and Pitch:
Physical: Frequency (Hz) — objective, linear scale
Perceptual: Pitch — subjective, logarithmic scale
Relationship:

$$P = k \log_2 \left(\frac{f}{f_0} \right) \tag{64}$$

Phenomena:

- Octave equivalence: 2:1 frequency ratio always perceived as one octave
- Missing fundamental: Pitch perceived even when fundamental absent
- Pitch depends on loudness, duration, and context
- JND: Approximately 1% or 5-6 cents at middle frequencies

2. Intensity and Loudness:
Physical: Intensity (W/m²) — objective, enormous dynamic range
Perceptual: Loudness (sones or phons) — subjective
Relationship:

$$L = 10 \log_{10} \left(\frac{I}{I_0} \right) \text{ dB} \tag{65}$$

Phenomena:

- Fletcher-Munson curves: Frequency-dependent loudness perception
- Ear most sensitive around 3-4 kHz
- Low frequencies require higher intensity to sound equally loud
- Dynamic range: 0-120 dB (factor of 10^{12} in intensity)

3. Wave Shape and Timbre:
Physical: Spectral content, harmonic structure
Perceptual: Timbre or tone color
Relationship: Complex and multidimensional
Phenomena:

- Determined by relative amplitudes of harmonics
- Strongly influenced by attack transients
- Critical band interactions affect timbral perception
- Even without attack, instruments distinguishable by spectral envelope

Key Psychoacoustic Phenomena:
1. Critical Bands:

- Frequency resolution of auditory system
- About 24 bands spanning audible range
- Sounds within same band interact strongly
- Basis for consonance/dissonance perception

2. Masking:

- Loud sounds mask nearby quiet sounds
- Most effective within critical bands
- Low frequencies mask high more than reverse
- Temporal masking: Forward and backward

3. Missing Fundamental:

- Brain reconstructs fundamental from harmonics
- Demonstrates pitch is cognitive, not purely physical
- Important for speech intelligibility and music perception

4. Octave Equivalence:

- Universal across cultures
- 2:1 frequency ratio perceived as ”same” pitch quality

- Basis for pitch class concept
- Fundamental to all musical scale systems

Mathematical Framework:
The Weber-Fechner law provides the unifying principle:

$$S = k \ln \left(\frac{I}{I_0} \right) \tag{66}$$

This logarithmic relationship explains why:

- Musical scales use frequency ratios, not differences
- Decibel scale is logarithmic
- Cent scale divides octave into 1200 logarithmic units
- Equal perceptual steps require multiplicative physical changes

Implications for Music:
1. Scale Construction:

- Based on ratios, not absolute frequency differences
- Octave (2:1) universal across all systems
- Transposition preserves musical relationships

2. Instrumentation and Orchestration:

- Consider critical band spacing
- Avoid masking between instruments
- Use frequency separation for clarity

3. Audio Engineering:

- Logarithmic scales for controls (volume, frequency)
- Audio compression exploits masking
- Mixing considers critical bands and masking

Conclusion:
The relationship between physical sound properties and perception is fundamentally logarithmic and nonlinear. Understanding psychoacoustic phenomena like critical bands, masking, and the missing fundamental is essential for music theory, composition, performance, and audio engineering. These principles bridge the gap between the physics of sound waves and the psychology of musical experience, providing the mathematical foundation for the art of music.

8 Conclusion and Summary

This unit has established the foundational understanding necessary for the mathematical study of music. We have explored:

1. **Physical Nature of Sound:**
   - Sound as pressure waves characterized by frequency, amplitude, and waveform
   - Simple harmonic motion as the basis for pure tones
   - Harmonics and overtones as building blocks of musical timbre
   - The spectrum as a representation of frequency content
2. **Logarithmic Pitch Perception:**
   - The distinction between physical frequency and perceptual pitch
   - Octave equivalence and the fundamental role of 2:1 frequency ratios
   - Logarithmic relationship between frequency ratios and perceived intervals
   - The cent scale as a practical logarithmic measure
   - Just noticeable difference as a measure of perceptual resolution
3. **Indian Musical Concepts:**
   - Swaras as the basic pitch units with multiple variants
   - The 22-shruti system as a refined microtonal framework
   - Saptak as the octave organization principle
   - Mathematical structure of Indian scale systems
4. **Musical Notation:**
   - Western staff notation and its conventions
   - Indian swara notation systems
   - Mathematical representations of pitch
   - Comparative analysis of different notational approaches
5. **Psychoacoustic Foundations:**
   - Relationship between physical and perceptual attributes
   - Weber-Fechner law and logarithmic perception
   - Phenomena like masking and the missing fundamental

These fundamentals provide the necessary framework for understanding more advanced topics in the mathematics of music, including tuning systems (Unit II), rhythm and tala (Unit III), spectral analysis (Unit IV), and algorithmic composition (Unit V).

9 Key Formulas and Relationships

9.1 Wave Properties

$$\text{Sine wave: } y(t) = A\sin(2\pi ft + \phi) \tag{67}$$
$$\text{Angular frequency: } \omega = 2\pi f \tag{68}$$
$$\text{Wave energy: } E \propto A^2 f^2 \tag{69}$$

9.2 Pitch and Frequency

$$\text{Frequency from semitones: } f = f_{\text{ref}} \cdot 2^{n/12} \tag{70}$$
$$\text{Octave relationship: } f_{\text{octave}} = 2f \tag{71}$$
$$\text{Harmonic series: } f_n = nf_0, \quad n = 1, 2, 3, \ldots \tag{72}$$

9.3 Logarithmic Scales

$$\text{Cents: } c = 1200\log_2(r) = \frac{1200\ln(r)}{\ln(2)} \tag{73}$$
$$\text{Decibels: } L = 10\log_{10}(I/I_0) \tag{74}$$
$$\text{Weber-Fechner: } S = k\ln(I/I_0) \tag{75}$$

10 Practice Problems

1. Calculate the frequency of the note A one octave above A440.
2. How many cents are in a perfect fifth (frequency ratio 3:2)?
3. If a sound has a frequency of 256 Hz and another has a frequency of 512 Hz, what is the interval between them in cents?
4. Given that the JND for pitch is approximately 1% at middle frequencies, how many perceptually distinct pitches can exist within one octave?
5. In the 22-shruti system, calculate the frequency ratio represented by the interval from shruti 1 to shruti 5.
6. If two sounds differ in intensity by 20 dB, what is the ratio of their actual power intensities?
7. A string vibrating at 440 Hz produces harmonics. List the frequencies of the first 8 harmonics.
8. Convert the following frequency ratio to cents: 5:4 (major third in just intonation).

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