Unit 2: Tuning Systems and Modular Arithmetic in Music

Course Notes

MA375TGR - Mathematics of Music
Semester VII - Institutional Elective

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Contents

1 Introduction to Tuning Systems

The mathematical study of musical tuning systems represents one of the most fascinating intersections between mathematics and music. Throughout history, musicians and mathematicians have grappled with the fundamental challenge of creating scales that are both mathematically elegant and musically pleasing. This unit explores three major tuning systems—Pythagorean tuning, Just Intonation, and Equal Temperament—and examines how modular arithmetic provides a powerful framework for understanding pitch relationships in music.

1.1 Historical Context

The quest for the perfect tuning system dates back to ancient Greece, where Pythagoras discovered that consonant musical intervals correspond to simple ratios of whole numbers. This discovery laid the foundation for Pythagorean tuning. Over centuries, musicians and theorists developed alternative systems to address the limitations of Pythagorean tuning, leading to Just Intonation and eventually Equal Temperament, which dominates Western music today.

1.2 The Fundamental Problem

The central mathematical challenge in tuning theory arises from a profound incompatibility: powers of 2 (representing octaves) and powers of 3 (representing perfect fifths) are *incommensurable*. This means no combination of octaves can exactly equal any combination of perfect fifths, a fact that forces all tuning systems to make compromises.

2 Pythagorean Tuning

2.1 Foundation and Construction

Pythagorean tuning is based on the principle that the most consonant musical intervals correspond to simple frequency ratios. The system uses only two generating intervals: the octave (ratio 2:1) and the perfect fifth (ratio 3:2).

Definition 2.1 (Pythagorean Tuning)**.** Pythagorean tuning is a system of musical intonation in which the frequency ratios of all intervals are derived from the ratio 3:2 (perfect fifth) and its reduction by octaves (ratio 2:1).

2.1.1 Construction Method

To construct a Pythagorean scale starting from C:

1. Begin with the tonic note C (frequency ratio 1:1)
2. Generate notes by ascending in perfect fifths (multiplying by 3/2)
3. Reduce notes to a single octave by dividing by 2 as needed

The sequence of fifths produces:

$$\mathrm{C} = 1 = \frac{2^0}{3^0} \tag{1}$$

$$\mathrm{G} = \frac{3}{2} = \frac{3^1}{2^1} \tag{2}$$

$$\mathrm{D} = \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{9}{8} = \frac{3^2}{2^3} \tag{3}$$

$$\mathrm{A} = \frac{9}{8} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{27}{16} = \frac{3^3}{2^4} \tag{4}$$

$$\mathrm{E} = \frac{27}{16} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{81}{64} = \frac{3^4}{2^6} \tag{5}$$

$$\mathrm{B} = \frac{81}{64} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{243}{128} = \frac{3^5}{2^7} \tag{6}$$

Going down a fifth from C (multiplying by 2/3):

$$\mathrm{F} = 1 \cdot \frac{2}{3} \cdot 2 = \frac{4}{3} = \frac{2^2}{3^1} \tag{7}$$

2.1.2 The Pythagorean Scale

The complete Pythagorean diatonic scale in C is:

Note	C	D	E	F	G	A	B	C'
Ratio	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
Formula	$\frac{2^0}{3^0}$	$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3^1}$	$\frac{3^1}{2^1}$	$\frac{3^3}{2^4}$	$\frac{3^5}{2^7}$	$\frac{2^1}{3^0}$

2.2 Pythagorean Intervals

2.2.1 Whole Tone

Definition 2.2 (Pythagorean Whole Tone)**.** The Pythagorean whole tone is the interval between consecutive scale degrees (e.g., C to D) with frequency ratio:

$$\frac{9}{8} = \frac{3^2}{2^3} \approx 1.125 \text{ (204 cents)} \tag{8}$$

2.2.2 Semitones

In Pythagorean tuning, there are two different sizes of semitones:

Definition 2.3 (Pythagorean Limma (Diatonic Semitone))**.** The limma is the minor semitone, calculated as a perfect fourth minus two whole tones:

$$\frac{256}{243} = \frac{2^8}{3^5} \approx 1.053 \text{ (90 cents)} \tag{9}$$

Definition 2.4 (Pythagorean Apotome (Chromatic Semitone))**.** The apotome is the major semitone, calculated as seven fifths reduced by four octaves:

$$\frac{2187}{2048} = \frac{3^7}{2^{11}} \approx 1.068 \text{ (114 cents)} \tag{10}$$

Note that the limma is *not* exactly half of a whole tone, as $\left(\frac{256}{243}\right)^2 \neq \frac{9}{8}$.

2.3 The Pythagorean Comma

Definition 2.5 (Pythagorean Comma)**.** The Pythagorean comma is the small interval by which twelve perfect fifths exceed seven octaves:

$$\text{Pythagorean Comma} = \frac{(3/2)^{12}}{2^7} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1.0136 \text{ (23.46 cents)} \tag{11}$$

2.3.1 Mathematical Derivation

Starting from C and ascending through twelve perfect fifths:

$$\begin{aligned} \mathrm{C} \to \mathrm{G} \to \mathrm{D} \to \mathrm{A} \to \mathrm{E} \to \mathrm{B} \to \mathrm{F}\sharp \to \mathrm{C}\sharp \to \mathrm{G}\sharp \to \mathrm{D}\sharp \quad (12)\\ \to \mathrm{A}\sharp \to \mathrm{E}\sharp \to \mathrm{B}\sharp \quad (13)\end{aligned}$$

The frequency ratio accumulated is $(3/2)^{12} = 129.746...$

Descending by seven octaves gives: $129.746.../2^7 = 129.746.../128 \approx 1.0136$

This means that B$\sharp$ in Pythagorean tuning is slightly higher than C, differing by one Pythagorean comma.

2.3.2 Implications

The existence of the Pythagorean comma has profound implications:

- The **circle of fifths** does not close—it forms a spiral
- Enharmonic notes (e.g., G$\sharp$ and A$\flat$) are *not* equivalent; they differ by one comma
- Transposition to remote keys introduces increasingly dissonant intervals
- Some intervals become unacceptably out of tune, creating "wolf fifths"

2.4 Advantages and Disadvantages

Advantages:

- Perfect fifths and fourths (except the wolf fifth)
- Strong, clear harmonic structure
- Simple mathematical foundation
- Excellent for monophonic and early polyphonic music

Disadvantages:

- Major thirds are significantly sharp (81/64 vs. ideal 5/4)
- Limited transposition capability
- Presence of wolf intervals in chromatic music
- Unequal semitone sizes complicate chromatic passages

2.5 Practice Questions: Pythagorean Tuning

2.5.1 Short Answer Questions

Q1. What are the two generating intervals in Pythagorean tuning?

Answer: The two generating intervals in Pythagorean tuning are the octave (frequency ratio 2:1) and the perfect fifth (frequency ratio 3:2).

Q2. Calculate the frequency ratio of a Pythagorean major third starting from C.

Answer: A Pythagorean major third (C to E) has the frequency ratio $\frac{81}{64} = \frac{3^4}{2^6} \approx 1.2656$, which is obtained by ascending four perfect fifths and descending two octaves.

Q3. What is the size of the Pythagorean comma in cents?

Answer: The Pythagorean comma measures approximately 23.46 cents, calculated as $1200 \log_2 \left(\frac{3^{12}}{2^{19}}\right) \approx 23.46$ cents.

2.5.2 Medium Answer Questions

Q4. Explain why the Pythagorean tuning system creates two different sizes of semitones. Calculate both semitone sizes.

Answer: Pythagorean tuning creates two different semitone sizes because the system is generated by stacking perfect fifths rather than dividing intervals equally.

The **limma** (diatonic semitone) is calculated as a perfect fourth minus two whole tones:

$$\frac{4/3}{(9/8)^2} = \frac{4/3}{81/64} = \frac{256}{243} \approx 90 \text{ cents}$$

The **apotome** (chromatic semitone) is calculated as seven perfect fifths minus four octaves:

$$\frac{(3/2)^7}{2^4} = \frac{2187}{2048} \approx 114 \text{ cents}$$

The difference between these semitones is the Pythagorean comma: $\frac{2187/2048}{256/243} = \frac{531441}{524288}$. This inequality makes chromatic passages and transposition problematic in Pythagorean tuning.

Q5. Describe the concept of the "circle of fifths" in Pythagorean tuning and explain why it doesn't actually form a closed circle.

Answer: The circle of fifths is a sequence of notes generated by repeatedly ascending (or descending) by perfect fifths. Starting from C and moving clockwise: C $\rightarrow$ G $\rightarrow$ D $\rightarrow$ A $\rightarrow$ E $\rightarrow$ B $\rightarrow$ F$\sharp$ $\rightarrow$ C$\sharp$ $\rightarrow$ G$\sharp$ $\rightarrow$ D$\sharp$ $\rightarrow$ A$\sharp$ $\rightarrow$ E$\sharp$ $\rightarrow$ B$\sharp$.

In theory, after 12 perfect fifths, we should return to the starting note (7 octaves higher). However, mathematically:

$$\left(\frac{3}{2}\right)^{12} = \frac{531441}{262144} \neq 2^7 = \frac{524288}{262144}$$

The ratio $\frac{531441}{524288}$ (the Pythagorean comma) represents the gap that prevents the circle from closing. Instead of forming a circle, the fifths create a spiral that never exactly returns to its starting point. This means B$\sharp$ is not equivalent to C, and enharmonic equivalence does not exist in pure Pythagorean tuning.

2.5.3 Long Answer Questions

Q6. Derive the complete Pythagorean diatonic scale starting from C. Show all mathematical steps and explain the process of octave reduction.

Answer:

Construction Process:

The Pythagorean scale is constructed by generating notes through perfect fifths (ratio 3:2) and reducing them to a single octave when necessary (dividing by 2).

Step 1: Generate notes by ascending fifths from C

Starting with C = 1:

$$\begin{aligned}
\text{C} &= 1 \\
\text{G} &= 1 \times \frac{3}{2} = \frac{3}{2} \quad \text{(already in octave)} \\
\text{D} &= \frac{3}{2} \times \frac{3}{2} = \frac{9}{4} \quad \text{(outside octave, reduce by dividing by 2)} \\
&= \frac{9}{4} \div 2 = \frac{9}{8} \\
\text{A} &= \frac{9}{8} \times \frac{3}{2} = \frac{27}{16} \quad \text{(already in octave)} \\
\text{E} &= \frac{27}{16} \times \frac{3}{2} = \frac{81}{32} \quad \text{(outside octave, reduce)} \\
&= \frac{81}{32} \div 2 = \frac{81}{64} \\
\text{B} &= \frac{81}{64} \times \frac{3}{2} = \frac{243}{128} \quad \text{(already in octave)}
\end{aligned}$$

Step 2: Generate F by descending a fifth from C

To find F, we go down a perfect fifth (multiply by 2/3), then up an octave (multiply by 2):

$$\text{F} = 1 \times \frac{2}{3} \times 2 = \frac{4}{3}$$

Step 3: Arrange in order

The complete Pythagorean diatonic scale:

Note	Frequency Ratio
C	$1 = \frac{1}{1}$
D	$\frac{9}{8} = 1.125$
E	$\frac{81}{64} = 1.2656$
F	$\frac{4}{3} = 1.3333$
G	$\frac{3}{2} = 1.5$
A	$\frac{27}{16} = 1.6875$
B	$\frac{243}{128} = 1.8984$
C'	$2 = \frac{2}{1}$

Interval Analysis:

From this scale, we can identify the interval sizes:

- C–D: $\frac{9/8}{1} = \frac{9}{8}$ (whole tone)

- D–E: $\frac{81/64}{9/8} = \frac{9}{8}$ (whole tone)
- E–F: $\frac{4/3}{81/64} = \frac{256}{243}$ (limma/semitone)
- F–G: $\frac{3/2}{4/3} = \frac{9}{8}$ (whole tone)
- G–A: $\frac{27/16}{3/2} = \frac{9}{8}$ (whole tone)
- A–B: $\frac{243/128}{27/16} = \frac{9}{8}$ (whole tone)
- B–C': $\frac{2}{243/128} = \frac{256}{243}$ (limma/semitone)

The pattern of intervals is: T–T–S–T–T–T–S, where T = tone ($\frac{9}{8}$) and S = semitone ($\frac{256}{243}$).

Significance:

This construction demonstrates that all intervals in Pythagorean tuning can be expressed as $\frac{3^m}{2^n}$ where m and n are integers. This pure mathematical elegance was highly valued by ancient theorists but creates practical problems for musicians, particularly the overly-sharp major thirds and the inability to transpose freely to all keys.

3 Just Intonation

3.1 Motivation and Foundation

Just Intonation (JI) arose from the observation that Pythagorean major thirds (ratio 81:64) sound considerably sharper than the natural harmonic third found in the overtone series (ratio 5:4). Musicians naturally gravitates toward intervals that align with the harmonic series, as these create more resonant, acoustically pure sonorities.

Definition 3.1 (Just Intonation)**.** Just Intonation is a tuning system in which all intervals are derived from small whole-number frequency ratios, particularly incorporating ratios involving the prime numbers 2, 3, and 5 (and sometimes 7 and higher primes).

3.2 The Harmonic Series Foundation

The harmonic series provides the theoretical foundation for Just Intonation:

Harmonic	1	2	3	4	5	6	7	8	9
Frequency	f	$2f$	$3f$	$4f$	$5f$	$6f$	$7f$	$8f$	$9f$
Reduced Ratio	1:1	2:1	3:2	2:1	5:4	3:2	7:4	2:1	9:8
Interval Name	Unison	Octave	Fifth	Octave	Maj 3rd	Fifth	Min 7th	Octave	Maj 2nd

3.3 Key Ratios in Just Intonation

3.3.1 Perfect Intervals

- **Octave:** 2:1
- **Perfect Fifth:** 3:2 (same as Pythagorean)
- **Perfect Fourth:** 4:3 (same as Pythagorean)

3.3.2 Major Intervals

- **Just Major Third:** 5:4 = 1.25 (386 cents)
- **Just Major Sixth:** 5:3 = 1.667 (884 cents)
- **Just Major Seventh:** 15:8 = 1.875 (1088 cents)

3.3.3 Minor Intervals

- **Just Minor Third:** 6:5 = 1.2 (316 cents)
- **Just Minor Sixth:** 8:5 = 1.6 (814 cents)
- **Just Minor Seventh:** 9:5 = 1.8 (1018 cents)

3.4 The Just Major Triad

Definition 3.2 (Just Major Triad)**.** A just major triad consists of three notes with frequency ratios 4:5:6. For example, a C major triad would have:

- C: 4 (or 1 when normalized)
- E: 5 (or 5/4)
- G: 6 (or 3/2)

This triad is considered maximally consonant because all three notes are present in the harmonic series of the root note.

3.5 Construction of Just Scales

3.5.1 Just Major Scale

The just major scale in C using 5-limit ratios:

Note	C	D	E	F	G	A	B	C'
Ratio	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Decimal	1.000	1.125	1.250	1.333	1.500	1.667	1.875	2.000

3.5.2 Interval Analysis

The just major scale contains two different sizes of whole tones:

- **Major Tone:** $\frac{9}{8}$ = 1.125 (204 cents) — C to D, F to G, A to B
- **Minor Tone:** $\frac{10}{9}$ = 1.111 (182 cents) — D to E, G to A
- **Diatonic Semitone:** $\frac{16}{15}$ = 1.067 (112 cents) — E to F, B to C'

3.6 The Syntonic Comma

Definition 3.3 (Syntonic Comma)**.** The syntonic comma (also called the Didymus comma) is the difference between a Pythagorean major third and a just major third:

$$\text{Syntonic Comma} = \frac{81/64}{5/4} = \frac{81}{80} \approx 1.0125 \text{ (21.51 cents)} \tag{14}$$

The syntonic comma represents the fundamental incompatibility between Pythagorean and just tuning systems.

3.7 Comparison: Pythagorean vs. Just Intonation

Interval	**Pythagorean**	**Just**	**Difference**
Major Third	$\frac{81}{64}$ (408 cents)	$\frac{5}{4}$ (386 cents)	Syntonic comma
Minor Third	$\frac{32}{27}$ (294 cents)	$\frac{6}{5}$ (316 cents)	Syntonic comma
Major Sixth	$\frac{27}{16}$ (906 cents)	$\frac{5}{3}$ (884 cents)	Syntonic comma
Minor Sixth	$\frac{128}{81}$ (792 cents)	$\frac{8}{5}$ (814 cents)	Syntonic comma
Perfect Fifth	$\frac{3}{2}$ (702 cents)	$\frac{3}{2}$ (702 cents)	None

3.8 Advantages and Disadvantages of Just Intonation

Advantages:

- Maximally consonant major and minor triads (4:5:6 and 10:12:15)
- Intervals align with natural harmonic series
- Pure, beatless thirds and sixths
- Rich, resonant harmonic texture
- Preferred by a cappella vocal ensembles and string quartets

Disadvantages:

- Multiple versions of the same scale degree needed for different harmonic contexts
- Even more limited transposition than Pythagorean tuning
- Some fifths become impure (the "wolf fifth")
- Extremely complex to implement on fixed-pitch instruments
- Chromatic music becomes nearly impossible

3.9 Practice Questions: Just Intonation

3.9.1 Short Answer Questions

Q1. What is the frequency ratio of a just major third, and how does it differ from a Pythagorean major third?

Answer: A just major third has a frequency ratio of 5:4 (1.25 or 386 cents), while a Pythagorean major third has a ratio of 81:64 (approximately 1.266 or 408 cents). The just third is flatter by a syntonic comma ($\frac{81}{80}$ or about 22 cents), making it more consonant and aligned with the harmonic series.

Q2. Define the syntonic comma and calculate its value.

Answer: The syntonic comma is the difference between a Pythagorean major third (81:64) and a just major third (5:4):

$$\frac{81/64}{5/4} = \frac{81}{64} \times \frac{4}{5} = \frac{81}{80} \approx 1.0125 \text{ (21.51 cents)}$$

Q3. What are the frequency ratios of a just major triad built on C?

Answer: A just major triad on C has frequency ratios 4:5:6, which normalized to C=1 gives:

- C: 1 (or 4:4)
- E: $5/4 = 1.25$ (or 5:4)
- G: $3/2 = 1.5$ (or 6:4)

3.9.2 Medium Answer Questions

Q4. Explain why just intonation is considered more consonant than Pythagorean tuning for harmonic music. Support your answer with specific frequency ratios.

Answer: Just intonation is more consonant than Pythagorean tuning because its intervals more closely match the natural harmonic series, which governs acoustic resonance.

The key difference lies in the treatment of thirds and sixths:

Major Third:

- Just: 5:4 — the 5th harmonic in the series, creating zero beating
- Pythagorean: 81:64 = (3^4):(2^6) — does not appear simply in the harmonic series, creates audible beating

Minor Third:

- Just: 6:5 — simple ratio derived from harmonics 5 and 6
- Pythagorean: 32:27 — complex ratio creating more interference

When we form a major triad in just intonation (4:5:6), all three notes appear as consecutive harmonics in the overtone series, creating perfect acoustic resonance with no beating. In contrast, a Pythagorean major triad produces audible interference patterns between the third and the root, making it sound noticeably sharper and less restful.

This is why string quartets and vocal ensembles naturally tend toward just intonation—their ability to adjust pitch continuously allows them to find these acoustically pure intervals.

Q5. Construct a just major scale starting from C and identify the two different sizes of whole tones present in the scale.

Answer:

Just Major Scale Construction:

Note	Ratio	Derivation
C	1:1	Tonic
D	9:8	Major tone above C
E	5:4	Just major third above C
F	4:3	Perfect fourth above C
G	3:2	Perfect fifth above C
A	5:3	Just major sixth above C
B	15:8	Just major seventh above C
C'	2:1	Octave

Analysis of Whole Tones:

Calculating the interval ratios between adjacent notes:

Major Tone (9:8 = 1.125 = 204 cents):

- C to D: $\frac{9/8}{1} = \frac{9}{8}$
- F to G: $\frac{3/2}{4/3} = \frac{9}{8}$
- A to B: $\frac{15/8}{5/3} = \frac{9}{8}$

Minor Tone (10:9 = 1.111 = 182 cents):

- D to E: $\frac{5/4}{9/8} = \frac{10}{9}$
- G to A: $\frac{5/3}{3/2} = \frac{10}{9}$

The difference between these two whole tones is the syntonic comma: $\frac{9/8}{10/9} = \frac{81}{80}$.

This variation in whole tone sizes creates practical problems for modulation and transposition, as the pattern of intervals changes depending on which note serves as the tonic.

3.9.3 Long Answer Questions

Q6. Compare and contrast Pythagorean tuning and Just Intonation systems comprehensively. Include: (a) their mathematical foundations, (b) their strengths and weaknesses, (c) historical contexts of use, and (d) why neither system ultimately became the standard for modern keyboard instruments.

Answer:

(a) Mathematical Foundations

Pythagorean Tuning:

- Based exclusively on powers of 2 (octaves) and 3 (fifths)

- All intervals expressible as $\frac{3^m}{2^n}$
- Generated by the cycle of fifths: repeatedly multiplying by 3/2
- Mathematically elegant but limited to ratios involving only 2 and 3

Just Intonation:

- Based on harmonics 2, 3, 5 (and sometimes 7 and higher)
- Intervals derived from the natural harmonic series
- Major triad as 4:5:6 ratio (consecutive harmonics)
- More complex mathematically but acoustically purer

(b) Strengths and Weaknesses

Pythagorean Tuning Strengths:

- Perfect fifths and fourths (except wolf fifth)
- Suitable for melodic music and early polyphony
- Simple mathematical structure
- Works well in limited keys
- Bright, energetic quality

Pythagorean Tuning Weaknesses:

- Major thirds too sharp (81:64 vs. 5:4) by syntonic comma
- Minor thirds too flat
- Wolf interval makes some keys unusable
- Two different semitone sizes (limma and apotome)
- Limited transposition possibilities

Just Intonation Strengths:

- Pure, beatless major and minor thirds
- Perfect consonance of major triads (4:5:6)
- Intervals match natural acoustic resonance
- Rich harmonic texture
- Ideal for chord-based music

Just Intonation Weaknesses:

- Requires multiple versions of scale degrees for different keys
- Some perfect fifths become impure (wolf fifths)

- Even more limited transposition than Pythagorean
- Practically impossible to implement on keyboards
- Three different whole tone sizes in extended scales

(c) Historical Contexts

Pythagorean Tuning (c. 6th century BCE - 15th century CE):

- Dominant in ancient Greece and medieval Europe
- Well-suited to Gregorian chant and early organum
- Used when music was primarily melodic
- Aligned with Pythagorean philosophy of mathematical perfection
- Gradually abandoned as harmony became more complex

Just Intonation (c. 2nd century CE - 17th century):

- Developed by Didymus and Ptolemy
- Rediscovered in Renaissance (16th century)
- Championed by theorists like Zarlino
- Used by vocal ensembles and unfretted strings
- Still used today by barbershop quartets and string players
- Never successfully implemented on keyboards due to complexity

(d) Why Neither Became Standard

Both systems ultimately failed as keyboard standards due to the fundamental mathematical impossibility of creating a system that simultaneously satisfies all musical requirements:

The Transposition Problem:

- Western music increasingly required modulation to distant keys
- Pythagorean: Transposition creates increasingly sharp thirds
- Just Intonation: Each key requires different pitch sets
- Example: To play in all 24 keys with just intonation would require dozens of different versions of each pitch class

The Comma Problems:

- Pythagorean comma: $\frac{3^{12}}{2^{19}} = \frac{531441}{524288}$ (12 fifths $\neq$ 7 octaves)
- Syntonic comma: $\frac{81}{80}$ (Pythagorean third $\neq$ just third)
- These commas accumulate and create wolf intervals
- No system based on pure ratios can avoid these problems

Practical Constraints:

- Keyboards have fixed pitches—cannot adjust for different harmonic contexts
- Just intonation would require split keys or additional keys per octave
- Some historical organs had up to 19 keys per octave attempting to accommodate just intonation
- Such instruments were too complex and expensive for widespread use

The Solution:

The ultimate solution was Equal Temperament, which abandons pure mathematical ratios entirely in favor of a compromise that makes all keys equally usable (and equally impure). By dividing the octave into 12 equal semitones of ratio $2^{1/12}$, Equal Temperament:

- Makes all keys sound identical
- Eliminates wolf intervals
- Allows unlimited transposition and modulation
- Sacrifices some consonance but gains flexibility

This represents a fundamental shift in values: from seeking mathematically perfect ratios (Pythagorean), to acoustically perfect consonances (Just), to pragmatic equal compromise (Equal Temperament). The choice reflects changing musical styles, from monophonic medieval chant through Renaissance polyphony to the complex chromatic harmonies and distant modulations of Baroque and later music.

4 Equal Temperament

4.1 Motivation and Development

Equal Temperament (ET) emerged as a pragmatic solution to the fundamental problems inherent in both Pythagorean tuning and Just Intonation. As Western music evolved to include more complex harmonies and frequent modulations to distant keys, the need for a tuning system that treated all keys equally became paramount.

Definition 4.1 (Equal Temperament)**.** Equal Temperament is a tuning system in which the octave is divided into twelve equal semitones, each with a frequency ratio of $2^{1/12} \approx 1.05946$.

4.2 Mathematical Foundation

4.2.1 The Equal Semitone

In 12-tone Equal Temperament (12-TET), the octave (ratio 2:1) is divided into 12 equal parts. If we denote the semitone ratio as r, then:

$$r^{12} = 2 \tag{15}$$

Solving for r:

$$r = 2^{1/12} = \sqrt[12]{2} \approx 1.059463094359 \tag{16}$$

4.2.2 The Equal Tempered Scale

The equal tempered chromatic scale starting from C (frequency f):

Note	Frequency	Decimal Value
C	$f \cdot 2^{0/12}$	$1.0000f$
C♯/D♭	$f \cdot 2^{1/12}$	$1.0595f$
D	$f \cdot 2^{2/12}$	$1.1225f$
D♯/E♭	$f \cdot 2^{3/12}$	$1.1892f$
E	$f \cdot 2^{4/12}$	$1.2599f$
F	$f \cdot 2^{5/12}$	$1.3348f$
F♯/G♭	$f \cdot 2^{6/12}$	$1.4142f$
G	$f \cdot 2^{7/12}$	$1.4983f$
G♯/A♭	$f \cdot 2^{8/12}$	$1.5874f$
A	$f \cdot 2^{9/12}$	$1.6818f$
A♯/B♭	$f \cdot 2^{10/12}$	$1.7818f$
B	$f \cdot 2^{11/12}$	$1.8877f$
C'	$f \cdot 2^{12/12}$	$2.0000f$

4.3 Key Intervals in Equal Temperament

Interval	Semitones	Frequency Ratio
Unison	0	$2^{0/12} = 1.0000$
Minor Second	1	$2^{1/12} = 1.0595$
Major Second	2	$2^{2/12} = 1.1225$
Minor Third	3	$2^{3/12} = 1.1892$
Major Third	4	$2^{4/12} = 1.2599$
Perfect Fourth	5	$2^{5/12} = 1.3348$
Tritone	6	$2^{6/12} = \sqrt{2} = 1.4142$
Perfect Fifth	7	$2^{7/12} = 1.4983$
Minor Sixth	8	$2^{8/12} = 1.5874$
Major Sixth	9	$2^{9/12} = 1.6818$
Minor Seventh	10	$2^{10/12} = 1.7818$
Major Seventh	11	$2^{11/12} = 1.8877$
Octave	12	$2^{12/12} = 2.0000$

4.4 Comparison with Just Intervals

Interval	ET Ratio	Just Ratio	Cents Diff.
Perfect Fifth	$2^{7/12} = 1.4983$	$3/2 = 1.5000$	-1.96
Perfect Fourth	$2^{5/12} = 1.3348$	$4/3 = 1.3333$	-1.96
Major Third	$2^{4/12} = 1.2599$	$5/4 = 1.2500$	$+13.69$
Minor Third	$2^{3/12} = 1.1892$	$6/5 = 1.2000$	-15.64
Major Sixth	$2^{9/12} = 1.6818$	$5/3 = 1.6667$	$+15.64$
Minor Sixth	$2^{8/12} = 1.5874$	$8/5 = 1.6000$	-13.69

The equal tempered fifth is flat by about 2 cents compared to the just fifth, while the major third is sharp by about 14 cents compared to the just major third.

4.5 The Cent System

Definition 4.2 (Cent)**.** A cent is a logarithmic unit of musical interval measure, with 100 cents defined as equal to one equal-tempered semitone. The cent value of an interval with frequency ratio r is:

$$\text{Cents} = 1200\log_2(r) = \frac{1200\ln(r)}{\ln(2)} \tag{17}$$

Examples:

- Just major third: $1200\log_2(5/4) = 386.31$ cents
- ET major third: $1200\log_2(2^{4/12}) = 400$ cents
- Pythagorean major third: $1200\log_2(81/64) = 407.82$ cents
- Syntonic comma: $1200\log_2(81/80) = 21.51$ cents

4.6 Historical Development

4.6.1 Early Proposals

The concept of equal temperament was proposed as early as the 16th century:

- **Vincenzo Galilei (1581)**: Proposed using the ratio 18:17 as an approximation for the semitone (98.96 cents), close to $2^{1/12}$ (100 cents)
- **Simon Stevin (1585)**: First to propose dividing the octave into 12 equal parts mathematically
- **Marin Mersenne (1636)**: Provided detailed calculations for equal temperament

4.6.2 Adoption Timeline

- **Late 16th century**: English virginal composers (John Bull) used forms of equal temperament
- **1702**: Johann Fischer's *Ariadne musica* explored 19 of 24 keys
- **1722**: J.S. Bach's *Well-Tempered Clavier* demonstrated feasibility (though probably not in pure ET)
- **19th century**: Gradual widespread adoption
- **1851**: British organs at Great Exhibition still not equally tempered
- **20th century**: Universal standard for Western music

4.7 Advantages and Disadvantages

Advantages:

- All keys sound identical (enharmonic equivalence)
- Unlimited transposition and modulation
- No wolf intervals
- Practical for keyboard instruments
- Enables complex chromatic and atonal music
- Fixed tuning simplifies instrument construction

Disadvantages:

- No intervals are perfectly in tune (except octave and unison)
- Major thirds are noticeably sharp compared to just intonation
- Loss of key color and character
- Triads have slight beating
- Less acoustically pure than just intonation
- Irrational ratios (cannot be expressed as simple fractions)

4.8 Alternative Equal Temperaments

While 12-TET is standard, other divisions of the octave have been explored:

- **19-TET**: Divides octave into 19 equal parts; provides better approximation of just major third
- **31-TET**: Excellent approximation of quarter-comma meantone
- **53-TET**: Very close approximation of both Pythagorean and just intervals
- **Microtonal systems**: Various composers have explored 24-TET, 72-TET, and other divisions

4.9 Practice Questions: Equal Temperament

4.9.1 Short Answer Questions

Q1. Calculate the frequency ratio of an equal-tempered perfect fifth.
Answer: An equal-tempered perfect fifth spans 7 semitones, so its frequency ratio is:

$$2^{7/12} = \sqrt[12]{2^7} = \sqrt[12]{128} \approx 1.498307077$$

This is approximately 701.955 cents.

Q2. By how many cents does an equal-tempered major third differ from a just major third?
Answer:

- Equal-tempered major third: 400 cents
- Just major third: $1200 \log_2(5/4) = 386.31$ cents
- Difference: $400 - 386.31 = 13.69$ cents (ET is sharper)

Q3. What is the frequency ratio of the tritone in equal temperament?
Answer: The tritone spans 6 semitones (half an octave), giving:

$$2^{6/12} = 2^{1/2} = \sqrt{2} \approx 1.414214$$

4.9.2 Medium Answer Questions

Q4. Explain why equal temperament allows unlimited transposition while Pythagorean tuning and Just Intonation do not.

Answer: Equal temperament allows unlimited transposition because all semitones are exactly identical in size ($2^{1/12}$), making every interval pattern reproducible at any starting pitch. This creates true enharmonic equivalence (G$\sharp$ = A$\flat$) and makes all major scales, minor scales, and chord progressions sound identical regardless of key.

In contrast:

Pythagorean Tuning:

- Generated by perfect fifths, which don't close the circle of fifths
- After 12 fifths, we're off by a Pythagorean comma ($\frac{3^{12}}{2^{19}}$)
- This creates a wolf fifth somewhere in the circle
- Transposing to remote keys increasingly involves this wolf interval
- Enharmonic notes differ by a comma (G$\sharp \neq$ A$\flat$)

Just Intonation:

- Uses pure ratios (3:2, 5:4, etc.) which don't close mathematically
- Contains multiple sizes of whole tones (9:8 and 10:9)
- Each key requires different specific pitch frequencies
- Example: The note D might need to be 9/8 above C in C major, but 10/9 above C in F major
- Would require multiple keys per pitch class to accommodate all keys

Equal temperament sacrifices acoustic purity for complete flexibility, distributing the various commas equally across all intervals rather than accumulating them in specific problematic intervals.

Q5. Using the cent formula, verify that twelve equal-tempered semitones equal exactly one octave.

Answer:

The cent formula for an interval with frequency ratio r is:

$$\text{Cents} = 1200 \log_2(r)$$

For one equal-tempered semitone, $r = 2^{1/12}$:

$$\text{Cents}_{\text{semitone}} = 1200 \log_2(2^{1/12}) = 1200 \cdot \frac{1}{12} = 100 \text{ cents}$$

For twelve semitones:

$$\text{Total cents} = 12 \times 100 = 1200 \text{ cents}$$

Verifying with the octave ratio ($r = 2$):

$$\text{Cents}_{\text{octave}} = 1200 \log_2(2) = 1200 \cdot 1 = 1200 \text{ cents}$$

Therefore, twelve equal-tempered semitones equal exactly one octave, both measuring 1200 cents. This is by definition—the cent system was designed specifically so that 1200 cents = 1 octave, and equal temperament divides this into 12 equal parts of 100 cents each.

Alternative verification using frequency ratios:

$$\begin{aligned} \text{Twelve semitones} &= (2^{1/12})^{12} = 2^{12/12} = 2^1 = 2 \\ \text{One octave} &= 2 \end{aligned}$$

Thus twelve equal-tempered semitones produce exactly the same frequency ratio (2:1) as one octave.

4.9.3 Long Answer Questions

Q6. Create a comprehensive comparison table showing the frequency ratios (in both fraction and decimal form) and cent values for all intervals in Pythagorean tuning, Just Intonation, and Equal Temperament. Then analyze which system provides the best approximation for each interval type.

Answer:

Comprehensive Interval Comparison:

Interval	**Pythagorean**		**Just**		**Equal Temp.**	
	Ratio	Cents	Ratio	Cents	Ratio	Cents
Unison	1:1	0.00	1:1	0.00	$2^{0/12}$	0.00
Minor 2nd	256:243	90.22	16:15	111.73	$2^{1/12}$	100.00
Major 2nd	9:8	203.91	9:8	203.91	$2^{2/12}$	200.00
Minor 3rd	32:27	294.13	6:5	315.64	$2^{3/12}$	300.00
Major 3rd	81:64	407.82	5:4	386.31	$2^{4/12}$	400.00
Perfect 4th	4:3	498.04	4:3	498.04	$2^{5/12}$	500.00
Tritone	729:512	611.73	45:32	590.22	$2^{6/12}$	600.00
			64:45	609.78		
Perfect 5th	3:2	701.96	3:2	701.96	$2^{7/12}$	700.00
Minor 6th	128:81	792.18	8:5	813.69	$2^{8/12}$	800.00
Major 6th	27:16	905.87	5:3	884.36	$2^{9/12}$	900.00
Minor 7th	16:9	996.09	9:5	1017.60	$2^{10/12}$	1000.00
Major 7th	243:128	1109.78	15:8	1088.27	$2^{11/12}$	1100.00
Octave	2:1	1200.00	2:1	1200.00	$2^{12/12}$	1200.00

Analysis by Interval Type:
1. Perfect Intervals (Octave, Fifth, Fourth):

- **Best:** Pythagorean and Just (identical at 3:2 for fifth, 4:3 for fourth)
- **ET deviation:** Fifth is flat by 1.96 cents, fourth by 1.96 cents
- **Conclusion:** The perfect fifth is most important structurally; ET's 2-cent compromise is barely perceptible

2. Major Third:

- **Best:** Just (5:4 = 386.31 cents) — pure harmonic
- **ET:** 400 cents — 13.69 cents sharp
- **Pythagorean:** 407.82 cents — 21.51 cents sharp (worse)
- **Conclusion:** Just intonation wins decisively; ET is closer to just than Pythagorean

3. Minor Third:

- **Best:** Just (6:5 = 315.64 cents)
- **ET:** 300 cents — 15.64 cents flat
- **Pythagorean:** 294.13 cents — 21.51 cents flat (worse)
- **Conclusion:** Just intonation best; ET closer than Pythagorean

4. Major Sixth:

- **Best:** Just (5:3 = 884.36 cents)
- **ET:** 900 cents — 15.64 cents sharp
- **Pythagorean:** 905.87 cents — 21.51 cents sharp (worse)
- **Conclusion:** Pattern mirrors minor third (intervals are inversions)

5. Minor Sixth:

- **Best:** Just (8:5 = 813.69 cents)
- **ET:** 800 cents — 13.69 cents flat
- **Pythagorean:** 792.18 cents — 21.51 cents flat (worse)
- **Conclusion:** Pattern mirrors major third (intervals are inversions)

Overall Assessment:

1. **For acoustic purity:** Just Intonation is superior for thirds and sixths (the most harmonically important intervals in Western tonal music). Perfect fifths and fourths are equally pure in Pythagorean and Just.

2. **For practical flexibility:** Equal Temperament wins despite being acoustically imperfect. Its compromises are distributed evenly, allowing:
   - Free modulation to all keys
   - Simplified instrument construction
   - No wolf intervals
   - Enharmonic equivalence
3. **Pythagorean's role:** Historically important and excellent for melody-based music with perfect fifths. Less suitable for harmony-based music due to sharp thirds.
4. **Perceptibility:** Most trained musicians can detect differences of about 5-10 cents. The ET deviations from just intonation (14-16 cents for thirds/sixths) are clearly audible to sensitive listeners, which is why string players and vocalists naturally adjust toward just intonation when playing in ensemble.

Modern Practice:
Today's Western music uses a hybrid approach:

- Fixed-pitch instruments (piano, guitar, etc.): Equal Temperament
- Flexible-pitch instruments (strings, voice): Tendency toward just intonation for sustained harmonies, with adjustments as needed
- Electronic music: Can use any tuning system precisely

5 The Circle of Fifths and Irrationality of $\log(3/2)$

5.1 The Circle of Fifths Concept

Definition 5.1 (Circle of Fifths)**.** The circle of fifths is a geometric representation of the relationships among the twelve pitch classes of the chromatic scale, arranged such that each pitch class is a perfect fifth above the previous one.

In equal temperament, the circle of fifths truly forms a closed circle because enharmonic equivalence holds (e.g., G$\sharp$ = A$\flat$). However, in Pythagorean tuning and just intonation, the circle doesn't close—it forms a spiral.

5.2 The Mathematical Question

The fundamental question is: *Can any number of perfect fifths equal any number of octaves?*

Mathematically, this asks: Do there exist integers m and n such that:

$$\left(\frac{3}{2}\right)^m = 2^n \tag{18}$$

This is equivalent to asking whether $\log_2(3/2)$ is rational.

5.3 Proof of Irrationality

Theorem 5.1. The number $\log_2(3/2)$ is irrational. Equivalently, $\log_2(3)$ is irrational.

Proof. We prove that $\log_2(3)$ is irrational (noting that $\log_2(3/2) = \log_2(3) - 1$, so if $\log_2(3)$ is irrational, so is $\log_2(3/2)$).

Suppose, for the sake of contradiction, that $\log_2(3)$ is rational. Then there exist positive integers m and n such that:

$$\log_2(3) = \frac{m}{n}$$

This means:

$$3 = 2^{m/n}$$

Raising both sides to the n-th power:

$$3^n = 2^m$$

Now we analyze this equation:

- The left side, 3^n, is odd for any positive integer n (since 3 is odd)
- The right side, 2^m, is even for any positive integer $m > 0$ (since it's a power of 2)

An odd number cannot equal an even number, so we have reached a contradiction.

Therefore, our assumption must be false, and $\log_2(3)$ is irrational. Consequently, $\log_2(3/2) = \log_2(3) - 1$ is also irrational. □

5.4 Musical Implications

5.4.1 The Spiral of Fifths

Since $\log_2(3/2)$ is irrational, no sequence of perfect fifths can ever exactly equal any sequence of octaves. This has profound implications:

- The "circle" of fifths in Pythagorean tuning is actually a spiral that never closes
- Enharmonic notes (like G♯ and A♭) are *not* equivalent in Pythagorean tuning
- Each time we traverse the spiral, we're off by one Pythagorean comma

5.4.2 Approximations

Although perfect fifths never exactly match octaves, we can find good rational approximations to $\log_2(3/2)$ using continued fractions.

5.5 Continued Fraction Expansion

The continued fraction expansion of $\log_2(3/2)$ begins:

$$\log_2(3/2) = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}}}} \tag{19}$$

Written in compact notation: $[0; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, \ldots]$

5.6 Convergents and Musical Scales

The convergents of this continued fraction give us good rational approximations:

Convergent	**Approximation**	**Fifths : Octaves**	**Musical Significance**
$\frac{1}{2}$	0.5000	1:2	Poor
$\frac{3}{5}$	0.6000	3:5	5-tone scale
$\frac{7}{12}$	0.5833	7:12	**12-tone equal temperament**
$\frac{24}{41}$	0.5854	24:41	41-tone scale
$\frac{31}{53}$	0.5849	31:53	**53-tone scale** (Mercator-Holder)
$\frac{179}{306}$	0.5850	179:306	306-tone scale
$\frac{389}{665}$	0.5850	389:665	**665-tone scale** (very accurate)

The actual value is $\log_2(3/2) \approx 0.584962500721\ldots$

5.6.1 The 12-Tone System

The convergent $\frac{7}{12}$ tells us that 7 fifths approximately equal 12 semitones (one octave in 12-TET):

$$\begin{aligned}
\text{Seven ET fifths} &= (2^{7/12})^7 = 2^{49/12} \approx 14.917 \\
\text{Four octaves} &= 2^4 = 16 \\
\text{Ratio} &= \frac{16}{14.917} = 1.0726
\end{aligned}$$

The error is about 7.26

5.6.2 The 53-Tone System

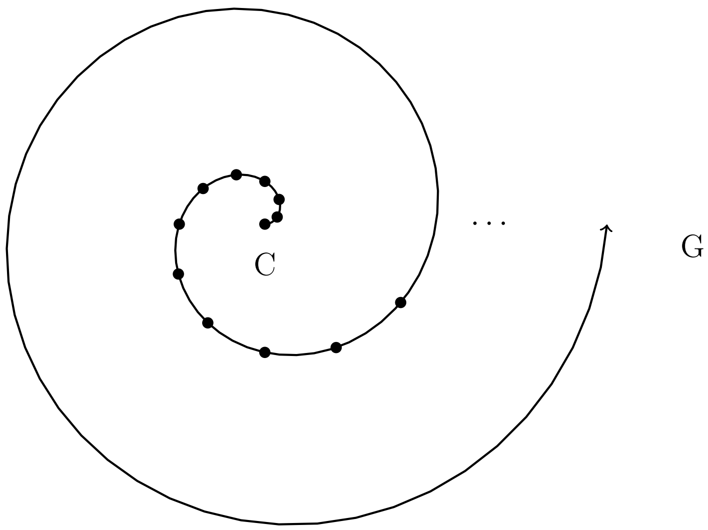
The convergent $\frac{31}{53}$ provides an even better approximation:

- 31 perfect fifths in 53-tone equal temperament almost perfectly equal 18 octaves

- Error is only about 0.07 cents—essentially imperceptible
- 53-TET was advocated by Mercator (1679) and Holder (1694)
- Provides excellent approximations to both Pythagorean and just intonation intervals

5.7 Geometric Visualization

Consider plotting the circle of fifths in Pythagorean tuning on a spiral:



Each complete rotation represents traversing 12 fifths, but we never return exactly to the starting pitch—we're always one Pythagorean comma away.

5.8 Practice Questions: Circle of Fifths and Irrationality

5.8.1 Short Answer Questions

Q1. State whether $\log_2(3/2)$ is rational or irrational, and explain the musical significance of your answer.

Answer: $\log_2(3/2)$ is **irrational**. This means that no number of perfect fifths (ratio 3:2) can ever exactly equal any number of octaves (ratio 2:1). Musically, this explains why the "circle of fifths" in pure intonation doesn't actually close into a circle—it forms a spiral. This fundamental mathematical fact forces all practical tuning systems to make compromises.

Q2. Which convergent of the continued fraction expansion of $\log_2(3/2)$ corresponds to 12-tone equal temperament?

Answer: The convergent $\frac{7}{12}$ corresponds to 12-tone equal temperament. This means 7 semitones (one equal-tempered fifth) approximates $\log_2(3/2)$ when the octave is divided into 12 equal parts. In other words, raising $2^{7/12}$ to the 12th power gets us close to raising 3/2 to the 7th power (both being approximately 128).

Q3. Calculate the numerical value of $\log_2(3/2)$ to five decimal places.

Answer:

$$\log_2(3/2) = \log_2(3) - \log_2(2) = \log_2(3) - 1 = \frac{\ln(3)}{\ln(2)} - 1 \approx 1.58496 - 1 = 0.58496$$

5.8.2 Medium Answer Questions

Q4. Explain how the irrationality of $\log_2(3/2)$ leads to the existence of the Pythagorean comma. Include the relevant mathematical calculations.

Answer:

The irrationality of $\log_2(3/2)$ means that there exist no integers m and n such that $(3/2)^m = 2^n$. However, we can get arbitrarily close approximations.

The 12-Fifth Approximation:

Consider stacking 12 perfect fifths:

$$(3/2)^{12} = \frac{3^{12}}{2^{12}} = \frac{531441}{4096} = 129.746...$$

This should approximately equal 7 octaves:

$$2^7 = 128$$

The ratio between these two values is:

$$\frac{(3/2)^{12}}{2^7} = \frac{531441/4096}{128} = \frac{531441}{524288} \approx 1.0136$$

This ratio is the **Pythagorean comma** (approximately 23.46 cents).

Why the Comma Exists:

Because $\log_2(3/2)$ is irrational, we can write:

$$12 \cdot \log_2(3/2) \neq 7$$

The *difference* between $12 \cdot \log_2(3/2)$ and 7 is:

$$12 \cdot \log_2(3/2) - 7 = \log_2\left(\frac{(3/2)^{12}}{2^7}\right) = \log_2\left(\frac{531441}{524288}\right) \approx 0.0195$$

This difference, when converted to cents:

$$1200 \times 0.0195 \approx 23.46 \text{ cents}$$

This is precisely the Pythagorean comma. Its existence is a direct consequence of the irrationality of $\log_2(3/2)$—if this logarithm were rational, the circle of fifths would close perfectly, and there would be no comma.

Q5. What is a continued fraction? Calculate the first three convergents of $\log_2(3/2)$ and verify which one gives the 12-tone equal temperament approximation.

Answer:

Definition:

A continued fraction is an expression of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Written compactly as $[a_0; a_1, a_2, a_3, \ldots]$, where the a_i are called the partial quotients.
For $\log_2(3/2)$:
The continued fraction expansion begins: $[0; 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]$
Calculating Convergents:
Convergents are successive finite truncations of the continued fraction. We use the recurrence relation:

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}$$

Starting with $p_{-1} = 1, p_0 = a_0 = 0$ and $q_{-1} = 0, q_0 = 1$:
First convergent ($[0; 1] = 0 + \frac{1}{1}$):

$$\begin{aligned} p_1 &= 1 \cdot 0 + 1 = 1 \\ q_1 &= 1 \cdot 1 + 0 = 1 \\ \text{Convergent} &= \frac{1}{1} = 1.0000 \end{aligned}$$

Second convergent ($[0; 1, 1] = 0 + \cfrac{1}{1 + \frac{1}{1}}$):

$$\begin{aligned} p_2 &= 1 \cdot 1 + 0 = 1 \\ q_2 &= 1 \cdot 1 + 1 = 2 \\ \text{Convergent} &= \frac{1}{2} = 0.5000 \end{aligned}$$

Third convergent ($[0; 1, 1, 2] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}$):

$$\begin{aligned} p_3 &= 2 \cdot 1 + 1 = 3 \\ q_3 &= 2 \cdot 2 + 1 = 5 \\ \text{Convergent} &= \frac{3}{5} = 0.6000 \end{aligned}$$

Fourth convergent ($[0; 1, 1, 2, 2]$):

$$\begin{aligned} p_4 &= 2 \cdot 3 + 1 = 7 \\ q_4 &= 2 \cdot 5 + 2 = 12 \\ \text{Convergent} &= \frac{7}{12} = 0.58\overline{3} \end{aligned}$$

Verification:
The fourth convergent, $\frac{7}{12}$, corresponds to 12-tone equal temperament:

- Numerator 7: number of semitones in a perfect fifth
- Denominator 12: number of semitones in an octave
- Ratio 7:12 means 7 fifths approximately close after 12 semitones

Comparing to the actual value:

$$\log_2(3/2) \approx 0.584962..., \quad \frac{7}{12} \approx 0.583333...$$

The error is only about 0.0016, or less than 2 cents when converted to musical intervals—remarkably close and well within perceptual tolerance.

5.8.3 Long Answer Questions

Q6. Provide a complete mathematical proof that $\log_2(3)$ is irrational. Then explain in detail the musical consequences of this irrationality for (a) the circle of fifths, (b) enharmonic equivalence, and (c) the development of tuning systems.

Answer:

PART I: Mathematical Proof

Theorem: $\log_2(3)$ is irrational.

Proof by Contradiction:

Step 1: Assumption

Assume, for the sake of contradiction, that $\log_2(3)$ is rational. Then there exist positive integers m and n (with no common factors, i.e., in lowest terms) such that:

$$\log_2(3) = \frac{m}{n}$$

Step 2: Algebraic Manipulation

By the definition of logarithm:

$$3 = 2^{m/n}$$

Raising both sides to the n-th power:

$$3^n = (2^{m/n})^n = 2^m$$

Therefore:

$$3^n = 2^m$$

Step 3: Parity Analysis

Now we analyze the parity (odd/even nature) of both sides:

Left side analysis: 3^n

- The number 3 is odd
- Any power of an odd number is odd
- Therefore, 3^n is odd for all positive integers n

Right side analysis: 2^m

- For $m \geq 1$, 2^m is divisible by 2

- Therefore, 2^m is even for all positive integers $m \geq 1$

Step 4: Contradiction
We have shown that:

- 3^n is odd
- 2^m is even (for $m \geq 1$)
- But we stated that $3^n = 2^m$

This is impossible: an odd number cannot equal an even number.
Step 5: Conclusion
Our initial assumption must be false. Therefore, $\log_2(3)$ is irrational.
Corollary: Since $\log_2(3/2) = \log_2(3) - \log_2(2) = \log_2(3) - 1$, and subtracting 1 from an irrational number yields an irrational number, we conclude that $\log_2(3/2)$ is also irrational.
PART II: Musical Consequences
(a) The Circle of Fifths
The irrationality of $\log_2(3/2)$ has profound implications for the circle of fifths:
1. The Spiral Structure:
In Pythagorean tuning, if we continuously ascend by perfect fifths:

$$\mathrm{C} \to \mathrm{G} \to \mathrm{D} \to \mathrm{A} \to \mathrm{E} \to \mathrm{B} \to \mathrm{F}\sharp \to \mathrm{C}\sharp \to \mathrm{G}\sharp \to \mathrm{D}\sharp \to \mathrm{A}\sharp \to \mathrm{E}\sharp \to \mathrm{B}\sharp$$

After 12 fifths, we should theoretically return to C (seven octaves higher):

$$(3/2)^{12} \stackrel{?}{=} 2^7$$

But since $\log_2(3/2)$ is irrational:

$$12 \cdot \log_2(3/2) \neq 7$$

The actual ratio is:

$$\frac{(3/2)^{12}}{2^7} = \frac{531441}{524288} \approx 1.0136 \text{ (Pythagorean comma)}$$

This means B♯ is NOT equal to C—it's higher by one Pythagorean comma.
2. The Infinite Spiral:
The circle never closes. If we continue:

- After 12 fifths: B♯ (one comma above C)
- After 24 fifths: B♯♯ (two commas above C)
- After 36 fifths: three commas above C
- And so on, infinitely

Mathematically, since $\log_2(3/2)$ is irrational, the sequence $\{n \cdot \log_2(3/2) \pmod 1\}$ for $n = 1, 2, 3, \ldots$ is dense in the interval $[0, 1)$. This means we can get arbitrarily close to any pitch, but never exactly return to our starting pitch.

(b) Enharmonic Equivalence

The irrationality destroys enharmonic equivalence in pure tuning systems:

Definition: Enharmonic equivalence is the principle that notes with different names but supposedly the same pitch (like G♯ and A♭) are actually the same frequency.

In Pythagorean Tuning:

Consider G♯ vs. A♭:

G♯: Eight fifths up from C, reduced by octaves

$$\begin{aligned} \text{G}\sharp &= (3/2)^8/2^{something} \\ &= \frac{6561}{4096} \text{ (after octave reduction)} \end{aligned}$$

A♭: Four fifths down from C, adjusted by octaves

$$\begin{aligned} \text{A}\flat &= (2/3)^4 \times 2^{something} \\ &= \frac{128}{81} \text{ (after octave reduction)} \end{aligned}$$

The ratio between them:

$$\frac{6561/4096}{128/81} = \frac{6561 \times 81}{4096 \times 128} = \frac{531441}{524288} = \text{Pythagorean comma}$$

They differ by exactly one Pythagorean comma! This pattern holds for ALL enharmonic pairs: C♯ vs. D♭, D♯ vs. E♭, etc.

Practical Consequences:

- Keyboard instruments require different keys for enharmonic notes
- Some historical organs had split black keys (e.g., G♯/A♭)
- Notation becomes ambiguous: does the composer want G♯ or A♭?
- Transposition becomes problematic

In Equal Temperament:

Equal temperament *imposes* enharmonic equivalence by definition:

$$\text{G}\sharp = \text{A}\flat = 2^{8/12} \cdot f_C$$

This is an artificial equivalence that abandons pure ratios in favor of practical convenience.

(c) Development of Tuning Systems

The irrationality of $\log_2(3/2)$ drove the entire historical development of tuning theory:

1. Ancient Period (Pythagorean Dominance):

- Music was primarily melodic, not harmonic
- Perfect fifths more important than thirds

- Pythagorean tuning worked well for its purpose
- The problem wasn't yet recognized as critical

2. Medieval to Renaissance (Crisis):

- Development of polyphony highlighted sharp Pythagorean thirds
- Musicians naturally sang thirds that sounded better (just intonation)
- Growing gap between theory and practice
- Recognition that no pure system could satisfy all requirements

3. Renaissance to Baroque (Compromise Solutions):
Quarter-Comma Meantone (1523):

- Deliberately mistune fifths by 1/4 syntonic comma
- Makes major thirds pure (5:4)
- Sacrifices fifths slightly
- Works in limited number of keys
- Wolf intervals appear in remote keys

Well Temperament (1691):

- Irregular temperament (Werckmeister, Kirnberger, Vallotti)
- Distributes comma unequally
- Some keys better than others
- All keys usable but each has different character
- Possibly what Bach used for *Well-Tempered Clavier*

4. 18th-20th Century (Equal Temperament Victory):
As Western music developed:

- Increased chromaticism (Wagner, Liszt, etc.)
- Atonal and twelve-tone music (Schoenberg)
- Jazz with its free modulation
- Need for all keys to be truly equal

Equal temperament became necessary because:

- It's the ONLY system where all twelve notes are mathematically equivalent
- It distributes the irrationality equally: every interval is equally "wrong"
- The errors are small enough to be acceptable (2 cents for fifth, 14 cents for third)

- It enables the full exploitation of the chromatic gamut

5. Modern Era (Pluralism):

- Equal temperament standard for fixed-pitch instruments
- Flexible instruments (strings, voice) use context-dependent intonation
- Electronic music can use any tuning precisely
- Renewed interest in historical temperaments
- Exploration of microtonal systems (31-TET, 53-TET, etc.)

Philosophical Conclusion:
The irrationality of $\log_2(3/2)$ represents a fundamental incompatibility in nature: the mathematical elegance of pure ratios conflicts with the practical needs of musical flexibility. This conflict drove centuries of theoretical and practical development, ultimately leading to a pragmatic compromise that sacrifices acoustic purity for compositional freedom. It's a beautiful example of how mathematical truth constrains and shapes artistic practice.

6 Modular Arithmetic and Pitch Class Sets

6.1 Introduction to Modular Arithmetic

Modular arithmetic provides a powerful mathematical framework for analyzing pitch relationships in music, particularly in twelve-tone and atonal compositions. By treating pitches as equivalence classes rather than absolute frequencies, we gain insights into the deep structural properties of musical scales and compositions.

Definition 6.1 (Modular Arithmetic)**.** Modular arithmetic (also called clock arithmetic) is a system of arithmetic for integers where numbers "wrap around" after reaching a certain value called the modulus. We write $a \equiv b \pmod{n}$ to mean that a and b have the same remainder when divided by n.

Example: In arithmetic modulo 12:

- $13 \equiv 1 \pmod{12}$ (because $13 = 12 \cdot 1 + 1$)
- $25 \equiv 1 \pmod{12}$ (because $25 = 12 \cdot 2 + 1$)
- $-1 \equiv 11 \pmod{12}$ (because $-1 = 12 \cdot (-1) + 11$)

6.2 Pitch Classes

Definition 6.2 (Pitch Class)**.** A pitch class is the set of all pitches that are octave equivalents. In twelve-tone equal temperament, there are exactly 12 pitch classes, typically labeled 0 through 11 or C through B.

Standard Labeling:

Pitch Class	0	1	2	3	4	5	6	7	8	9	10	11
Note Name	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B

All C's (C_0, C_1, C_2, ..., C_8) belong to pitch class 0. All C♯'s belong to pitch class 1, and so on.

6.3 Operations on Pitch Classes

6.3.1 Transposition

Definition 6.3 (Transposition)**.** Transposition by n semitones maps pitch class p to pitch class $(p+n) \bmod 12$. We denote this operation as T_n.

Example: Transpose C major triad (pitch classes 0, 4, 7) up by 5 semitones:

$$\begin{aligned} T_5(0) &= (0+5) \bmod 12 = 5 \text{ (F)} \\ T_5(4) &= (4+5) \bmod 12 = 9 \text{ (A)} \\ T_5(7) &= (7+5) \bmod 12 = 12 \bmod 12 = 0 \text{ (C)} \end{aligned}$$

Result: F major triad (pitch classes 5, 9, 0) = F, A, C

6.3.2 Inversion

Definition 6.4 (Inversion)**.** Inversion around axis a maps pitch class p to pitch class $(a-p) \bmod 12$ or equivalently $(2a-p) \bmod 12$ when considering the axis of symmetry. We denote this operation as I_a.

Example: Invert C major triad (0, 4, 7) around axis 0:

$$\begin{aligned} I_0(0) &= (0-0) \bmod 12 = 0 \text{ (C)} \\ I_0(4) &= (0-4) \bmod 12 = -4 \bmod 12 = 8 \text{ (G}\sharp/\text{A}\flat) \\ I_0(7) &= (0-7) \bmod 12 = -7 \bmod 12 = 5 \text{ (F)} \end{aligned}$$

So we get 0, 8, 5 = C, A♭, F, which rearranged in ascending order is F, A♭, C = F minor triad!

6.4 Pitch Class Sets

Definition 6.5 (Pitch Class Set)**.** A pitch class set is an unordered collection of pitch classes, typically written in set notation or as a sequence in ascending order.

Examples:

- Major triad: {0, 4, 7}
- Minor triad: {0, 3, 7}
- Diminished triad: {0, 3, 6}
- Whole-tone scale: {0, 2, 4, 6, 8, 10}
- Chromatic scale: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

6.5 Set Class and Prime Form

Definition 6.6 (Set Class)**.** A set class is the collection of all pitch class sets related by transposition and/or inversion. Sets in the same set class share the same intervallic structure.

Definition 6.7 (Prime Form)**.** The prime form of a pitch class set is its most compact representation, beginning with 0, and arranged to be as "left-packed" as possible.

Algorithm for finding prime form:

1. Transpose the set so it begins with 0
2. Arrange in ascending order
3. Check the inversion (apply I and transpose to 0)
4. Choose the most left-packed version
5. If tied, choose the one with smallest second number, then third, etc.

6.6 Interval Vectors

Definition 6.8 (Interval Vector)**.** The interval vector of a pitch class set is a six-element array $[i_1, i_2, i_3, i_4, i_5, i_6]$ where i_k represents the number of times interval class k appears between pairs of pitch classes in the set.

Interval classes:

- 1: minor second / major seventh (1 or 11 semitones)
- 2: major second / minor seventh (2 or 10 semitones)
- 3: minor third / major sixth (3 or 9 semitones)
- 4: major third / minor sixth (4 or 8 semitones)
- 5: perfect fourth / perfect fifth (5 or 7 semitones)
- 6: tritone (6 semitones)

Example: Major triad {0, 4, 7}
Intervals between pairs:

- 0 to 4: 4 semitones $\rightarrow$ interval class 4
- 0 to 7: 7 semitones $\rightarrow$ interval class 5
- 4 to 7: 3 semitones $\rightarrow$ interval class 3

Interval vector: [0, 0, 1, 1, 1, 0]

6.7 Twelve-Tone Composition

In twelve-tone (dodecaphonic) composition developed by Arnold Schoenberg, a tone row is a specific ordering of all 12 pitch classes. Modular arithmetic is used to generate transformations of the row:

- **Prime (P):** Original row
- **Retrograde (R):** Row in reverse order
- **Inversion (I):** Each pitch class p becomes $-p \pmod{12}$
- **Retrograde-Inversion (RI):** Inversion of the retrograde

Each of these can be transposed to any of the 12 pitch classes, giving 48 possible row forms (12×4) from a single tone row.

6.8 Practice Questions: Modular Arithmetic and Pitch Class Sets

6.8.1 Short Answer Questions

Q1. Calculate: $(7 + 8) \bmod 12$ and explain its musical significance.
Answer:

$$(7 + 8) \bmod 12 = 15 \bmod 12 = 3$$

Musically, this represents transposing a perfect fifth (7 semitones, pitch class G when starting from C) up by a minor sixth (8 semitones). The result is pitch class 3 (D♯/E♭).

Q2. What is the pitch class set representation of a C major triad?
Answer: The C major triad consists of the notes C, E, and G, which correspond to pitch classes 0, 4, and 7. The pitch class set is written as {0, 4, 7}.

Q3. Apply the transposition operator T_5 to the pitch class set {2, 6, 9}.
Answer:

$$\begin{aligned}
T_5(2) &= (2 + 5) \bmod 12 = 7 \\
T_5(6) &= (6 + 5) \bmod 12 = 11 \\
T_5(9) &= (9 + 5) \bmod 12 = 14 \bmod 12 = 2
\end{aligned}$$

Result: {7, 11, 2} or {2, 7, 11} in ascending order.

6.8.2 Medium Answer Questions

Q4. Calculate the interval vector for a minor triad {0, 3, 7}. Show all steps.
Answer:
An interval vector $[i_1, i_2, i_3, i_4, i_5, i_6]$ counts occurrences of each interval class.
Step 1: Identify all intervals
For the set {0, 3, 7}, we calculate intervals between all pairs:
Pair 0-3:

- Forward: $3 - 0 = 3$ semitones

- Backward: $12 - 3 = 9$ semitones
- Interval class: $\min(3, 9) = 3$

Pair 0-7:

- Forward: $7 - 0 = 7$ semitones
- Backward: $12 - 7 = 5$ semitones
- Interval class: $\min(5, 7) = 5$

Pair 3-7:

- Forward: $7 - 3 = 4$ semitones
- Backward: $12 - 4 = 8$ semitones
- Interval class: $\min(4, 8) = 4$

Step 2: Count occurrences

- Interval class 1: 0 occurrences
- Interval class 2: 0 occurrences
- Interval class 3: 1 occurrence (from pair 0-3)
- Interval class 4: 1 occurrence (from pair 3-7)
- Interval class 5: 1 occurrence (from pair 0-7)
- Interval class 6: 0 occurrences

Result: The interval vector is [0, 0, 1, 1, 1, 0].
Note: This is identical to the major triad's interval vector, which makes sense since major and minor triads are inversions of each other and thus belong to the same set class.

Q5. Explain how modular arithmetic allows us to treat all octaves of a note as equivalent. Why is this useful for analyzing modern music?
Answer:
Mathematical Explanation:
In twelve-tone equal temperament, pitches separated by an octave have a frequency ratio of 2:1. If we assign integers to pitches chromatically (C=0, C♯=1, D=2, etc.), then moving up one octave adds 12 to the pitch number:

$$\mathrm{C}_4 = 60, \quad \mathrm{C}_5 = 72, \quad \mathrm{C}_6 = 84, \text{ etc.}$$

Modular arithmetic modulo 12 maps all these to the same pitch class:

$$60 \bmod 12 = 0$$
$$72 \bmod 12 = 0$$
$$84 \bmod 12 = 0$$

All C's (regardless of octave) belong to pitch class 0.
Why This Is Useful:
1. Simplification:

- Reduces infinite pitch space to 12 pitch classes
- Focuses on pitch *quality* rather than register
- Makes patterns and relationships clearer

2. Structural Analysis:

- Modern music often emphasizes pitch class relationships over specific registers
- Twelve-tone music is explicitly organized around pitch class sets
- Allows identification of motivic relationships across octaves

3. Transposition and Symmetry:

- Transposition becomes simple addition modulo 12
- Inversion becomes subtraction modulo 12
- Set class theory identifies equivalent structures
- Enables systematic exploration of all possible pitch combinations

4. Atonal Music Analysis:

- Without traditional tonal hierarchy, pitch class becomes primary
- Interval vectors reveal sonority relationships
- Composers like Schoenberg, Webern, Berg used pitch class systematically
- Modern set theory (Allen Forte, Milton Babbitt) relies on modular arithmetic

Example Application:
Consider analyzing a Webern piece. A motive appears in different octaves:

- Bar 5: C_4-E_4-$G\sharp_5$ (pitches 60-64-68)
- Bar 12: C_6-E_5-$G\sharp_4$ (pitches 84-76-56)

In pitch class space:

- Bar 5: $\{0, 4, 8\}$
- Bar 12: $\{0, 4, 8\}$

Modular arithmetic reveals they're the same pitch class set (an augmented triad), helping us understand the underlying structure despite the different registers and ordering.

6.8.3 Long Answer Questions

Q6. Derive a complete twelve-tone matrix for the row P = [0, 11, 3, 1, 7, 9, 5, 6, 2, 4, 10, 8]. Show: (a) all 12 transpositions of the prime form (P_0 through P_{11}), (b) all 12 transpositions of the inversion (I_0 through I_{11}), (c) explain how to read retrograde (R) and retrograde-inversion (RI) forms from the matrix, and (d) discuss how this matrix embodies the mathematical structure of twelve-tone composition.

Answer:

(a) Constructing the Twelve-Tone Matrix

A twelve-tone matrix is a 12×12 array showing all 48 forms of a tone row. The given prime row is:

$$P_0 = [0, 11, 3, 1, 7, 9, 5, 6, 2, 4, 10, 8]$$

Step 1: Write P_0 in the first row

Step 2: Compute I_0 (inversion around 0)

To invert, we calculate $(-p)$ mod 12 for each element:

$$I_0 = [0, 1, 9, 11, 5, 3, 7, 6, 10, 8, 2, 4]$$

Verification: - -0 mod 12 = 0 ✓- -11 mod 12 = 1 ✓- -3 mod 12 = 9 ✓- etc.

Step 3: Write I_0 in the first column

Step 4: Complete the matrix

Each row P_n is P_0 transposed by n semitones (add n to each element mod 12). Each column I_n is I_0 transposed by n semitones.

Alternatively, the element at position (i, j) is: $(P_0[j] + I_0[i])$ mod 12

Complete Matrix:

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
I_0	0	11	3	1	7	9	5	6	2	4	10	8
I_1	1	0	4	2	8	10	6	7	3	5	11	9
I_9	9	8	0	10	4	6	2	3	11	1	7	5
I_{11}	11	10	2	0	6	8	4	5	1	3	9	7
I_5	5	4	8	6	0	2	10	11	7	9	3	1
I_3	3	2	6	4	10	0	8	9	5	7	1	11
I_7	7	6	10	8	2	4	0	1	9	11	5	3
I_6	6	5	9	7	1	3	11	0	8	10	4	2
I_{10}	10	9	1	11	5	7	3	4	0	2	8	6
I_8	8	7	11	9	3	5	1	2	10	0	6	4
I_2	2	1	5	3	9	11	7	8	4	6	0	10
I_4	4	3	7	5	11	1	9	10	6	8	2	0

Row Labels: - Rows reading left to right: $P_0, P_1, P_9, P_{11}, P_5, P_3, P_7, P_6, P_{10}, P_8, P_2, P_4$ - Columns reading top to bottom: $I_0, I_1, I_3, I_1, I_7, I_9, I_5, I_6, I_2, I_4, I_{10}, I_8$

(b) All Transpositions

Prime Forms (P): Read rows left to right - $P_0 = [0, 11, 3, 1, 7, 9, 5, 6, 2, 4, 10, 8]$ (first row) - $P_1 = [1, 0, 4, 2, 8, 10, 6, 7, 3, 5, 11, 9]$ (second row) - ... and so on for all 12 rows

Inversion Forms (I): Read columns top to bottom - $I_0 = [0, 1, 9, 11, 5, 3, 7, 6, 10, 8, 2, 4]$ (first column) - $I_{11} = [11, 0, 8, 10, 4, 2, 6, 5, 9, 7, 1, 3]$ (second column) - ... and so on for all 12 columns

(c) Retrograde Forms

Retrograde (R): Read rows right to left - $R_0 = [8, 10, 4, 2, 6, 5, 9, 7, 1, 3, 11, 0]$ (first row backwards) - This is also equal to reading the bottom-right to top-left diagonal

Retrograde-Inversion (RI): Read columns bottom to top - $RI_0 = [4, 2, 8, 10, 6, 7, 3, 5, 11, 9, 1, 0]$ (first column upwards)

Summary of Reading Directions:

- Prime (P): Rows, left $\rightarrow$ right
- Inversion (I): Columns, top $\rightarrow$ bottom
- Retrograde (R): Rows, right $\rightarrow$ left
- Retrograde-Inversion (RI): Columns, bottom $\rightarrow$ top

(d) Mathematical Structure

The twelve-tone matrix embodies several deep mathematical principles:

1. Group Theory:

The set of transposition operators $\{T_0, T_1, \ldots, T_{11}\}$ forms a group under composition:

- **Closure:** $T_m \circ T_n = T_{(m+n) \bmod 12}$
- **Identity:** T_0 is the identity element.
- **Inverses:** $T_n^{-1} = T_{12-n}$
- **Associativity:** $(T_m \circ T_n) \circ T_p = T_m \circ (T_n \circ T_p)$

Similarly, inversions and transpositions together form the dihedral group D_{12}.

2. Symmetry:

The matrix is *symmetrical* in a deep sense: - Rows and columns are dual transformations - $P_n[m] = I_m[n]$ (matrix is symmetric about its diagonal) - This reflects the mathematical duality between transposition and inversion - The four operations (P, I, R, RI) relate through symmetries: rotation (R) and reflection (I)

3. Completeness:

The matrix contains all 48 possible forms of the row:

- 12 prime forms (P_0 through P_{11}),
- 12 inversion forms (I_0 through I_{11}),
- 12 retrograde forms (R_0 through R_{11}), and
- 12 retrograde-inversion forms (RI_0 through RI_{11}).

This exhaustive generation ensures the composer has access to maximum variety while maintaining unity (all forms derive from one row).

4. Serialism Philosophy:

The matrix embodies Schoenberg's principle of "developing variation": - Unity: All 48 rows are transformations of one source - Variety: Each row presents the pitch classes in different order - Democracy: No pitch class is more important than others (anti-tonal) - Structure: The transformations are systematic, not arbitrary

5. Modular Arithmetic Foundation:

Every element in the matrix is computed using modulo 12 arithmetic:

$$\text{Element}[i, j] = (P_0[j] + I_0[i]) \bmod 12$$

This makes the twelve-tone system a concrete application of abstract algebra to music composition.

6. Invariance Properties:

Some rows may have special relationships: - **Hexachordal combinatoriality:** Some row pairs can be combined so their first/last hexachords contain all 12 pitch classes - **Self-inversion:** Some rows equal their own inversion at certain transpositions - **Symmetrical rows:** Some rows are palindromic or have other symmetries

These properties can be systematically explored using the matrix.

Compositional Usage:

A composer using this matrix might:

1. Choose rows to maintain or avoid intervallic relationships
2. Create canons by using different forms simultaneously
3. Modulate between forms to control harmonic density
4. Exploit symmetries for structural cohesion
5. Use retrograde and RI forms to create mirror structures

The matrix thus serves as both analytical tool and compositional resource, encoding the complete mathematical structure of the twelve-tone system in a single, elegant diagram.

7 Construction of Scales: Melakartha Rāga and Thaat Systems

7.1 Introduction to Indian Classical Music Scales

Indian classical music has developed sophisticated mathematical frameworks for organizing scales. The two primary systems—the Melakartha system in Carnatic (South Indian) music and the Thaat system in Hindustani (North Indian) music—represent remarkable achievements in systematic musical organization.

7.2 The Melakartha Rāga System

7.2.1 Foundation and Structure

The Melakartha system, formalized by Venkatamakhin in 1650 in his treatise *Chaturdandi Prakasika*, is a complete theoretical framework encompassing all possible seven-note scales (heptatonic) in Carnatic music.

Definition 7.1 (Melakartha Rāga)**.** A Melakartha rāga (also called Janaka or parent rāga) is a seven-note scale that satisfies:

1. Contains all seven swaras (notes): Sa, Ri, Ga, Ma, Pa, Da, Ni

2. Includes the upper Sa (octave), making eight notes total
3. Uses the same notes in both arohanam (ascending) and avarohanam (descending)
4. Notes are in strict ascending frequency order (no zigzag patterns)

Such scales are called *Krama Sampoorna* (complete and ordered) rāgas.

7.2.2 The Seven Swaras

The seven basic swaras with their variations:

Swara	Name	Variants	Pitch Class
Sa	Shadjamam	(Fixed)	0
Ri	R1: Shuddha Rishabham		1
	R2: Chatusruti Rishabham	(= G1)	2
	R3: Shatsruti Rishabham	(= G2)	3
Ga	G1: Shuddha Gandharam		2
	G2: Sadharana Gandharam		3
	G3: Antara Gandharam		4
Ma	M1: Shuddha Madhyamam		5
	M2: Prati Madhyamam		6
Pa	Panchamam	(Fixed)	7
Da	D1: Shuddha Daivatam		8
	D2: Chatusruti Daivatam	(= N1)	9
	D3: Shatsruti Daivatam	(= N2)	10
Ni	N1: Shuddha Nishadam		9
	N2: Kaisika Nishadam		10
	N3: Kakali Nishadam		11

Note: Some variants are enharmonically equivalent (e.g., R2 = G1, R3 = G2).

7.2.3 Mathematical Derivation of 72 Melakarthas

Fixed Elements:

- Sa (position 1): Fixed
- Ma (position 4): 2 choices (M1 or M2)
- Pa (position 5): Fixed
- Upper Sa (position 8): Fixed

Variable Elements (Purvanga - first half):
Choosing Ri and Ga:

Due to the strict ascending requirement: - If Ri = R1, then Ga can be G1, G2, or G3 (3 choices) - If Ri = R2 (= G1), then Ga can be G2 or G3 (2 choices) - If Ri = R3 (= G2), then Ga can only be G3 (1 choice)

Total Ri-Ga combinations: $3 + 2 + 1 = 6$

Variable Elements (Uttaranga - second half):

Choosing Da and Ni:

By the same logic: - If Da = D1, then Ni can be N1, N2, or N3 (3 choices) - If Da = D2 (= N1), then Ni can be N2 or N3 (2 choices) - If Da = D3 (= N2), then Ni can only be N3 (1 choice)

Total Da-Ni combinations: $3 + 2 + 1 = 6$

Total Number of Melakarthas:

Since Ma has 2 choices, Ri-Ga has 6 combinations, and Da-Ni has 6 combinations, and these are independent:

$$\text{Total} = 2 \times 6 \times 6 = 72$$

Therefore, there are exactly **72 possible Melakartha rāgas**.

7.2.4 Organization into Chakras

The 72 Melakarthas are organized into 12 Chakras (groups) of 6 rāgas each:

First 36 (with M1 - Shuddha Madhyamam):

1. **Indu Chakra** (1-6)
2. **Netra Chakra** (7-12)
3. **Agni Chakra** (13-18)
4. **Veda Chakra** (19-24)
5. **Bana Chakra** (25-30)
6. **Rutu Chakra** (31-36)

Next 36 (with M2 - Prati Madhyamam):

7. **Rishi Chakra** (37-42)
8. **Vasu Chakra** (43-48)
9. **Brahma Chakra** (49-54)
10. **Disi Chakra** (55-60)
11. **Rudra Chakra** (61-66)
12. **Aditya Chakra** (67-72)

Within each Chakra, all six rāgas share the same Ri-Ga combination but differ in their Da-Ni combinations.

7.2.5 Algorithm for Determining Scale from Rank Order

Given a Melakartha number n (1 to 72), determine its scale:

Step 1: Determine Madhyamam

- If $n \leq 36$: Use M1 (Shuddha), proceed with $m = n$
- If $n > 36$: Use M2 (Prati), set $m = n - 36$

Step 2: Calculate Quotient and Remainder

- Compute: $(m - 1) \div 6 = q$ remainder r
- Quotient q (0-5): Determines Ri-Ga combination (Purvanga)
- Remainder r (0-5): Determines Da-Ni combination (Uttaranga)

Step 3: Determine Ri-Ga from Quotient

Quotient	Ri-Ga Combination
0	R1, G1
1	R1, G2
2	R1, G3
3	R2, G2
4	R2, G3
5	R3, G3

Step 4: Determine Da-Ni from Remainder

Remainder	Da-Ni Combination
0	D1, N1
1	D1, N2
2	D1, N3
3	D2, N2
4	D2, N3
5	D3, N3

Examples:

Example 1: Melakartha 28 (Harikambhoji)

- $28 \leq 36$, so Ma = M1, and $m = 28$
- $(28 - 1) \div 6 = 27 \div 6 = 4$ remainder 3
- Quotient 4 $\rightarrow$ R2, G3
- Remainder 3 $\rightarrow$ D2, N2
- Scale: S R2 G3 M1 P D2 N2 S

Example 2: Melakartha 56 (Shanmukhapriya)

- $56 > 36$, so Ma = M2, and $m = 56 - 36 = 20$

- $(20 - 1) \div 6 = 19 \div 6 = 3$ remainder 1
- Quotient 3 $\rightarrow$ R2, G2
- Remainder 1 $\rightarrow$ D1, N2
- Scale: S R2 G2 M2 P D1 N2 S

7.2.6 Katapayadi Scheme

The Melakartha rāgas are named using the ancient Katapayadi numerical encoding system:

1	2	3	4	5	6	7	8	9	0
k, ṭ	kh, ṭh	g, ḍ	gh, ḍh	ṅ, ṇ	c, t	ch, th	j, d	jh, dh	ñ, n
					p	ph	b	bh	m
					y	r	l	v	ś, ṣ, s, h

The rāga name encodes its number. For example: - **Shanmukhapriya**: Sha=5, mu=5 $\rightarrow$ 55, reverse to get $55 + 1 = 56$ ✓- **Harikambhoji**: ha=0, ri=2 $\rightarrow$ 02, reverse to 20, add offset for M1 $= 20 + 8 = 28$ ✓

7.3 The Thaat System (Hindustani Music)

7.3.1 Foundation

The Thaat system was formalized by Vishnu Narayan Bhatkhande (1860-1936) for Hindustani classical music. Unlike the comprehensive Melakartha system, the Thaat system identifies only 10 parent scales.

Definition 7.2 (Thaat)**.** A Thaat is a parent scale framework used to classify rāgas in Hindustani music. It serves as a theoretical reference rather than being performed directly.

7.3.2 The Ten Thaats

Thaat	**Notes**	**Similar Melakartha**
Bilawal	S R G m P D N S	Dheerasankarabharanam (29)
Khamaj	S R G m P D n S	Harikambhoji (28)
Kafi	S R g m P D n S	Kharaharapriya (22)
Asavari	S R g m P d n S	Natabhairavi (20)
Bhairavi	S r g m P d n S	Hanumattodi (8)
Kalyan	S R G M P D N S	Mechakalyani (65)
Marwa	S r G M P D N S	Gamanashrama (53)
Poorvi	S r G M P d N S	Kamavardhani (51)
Todi	S r g M P d N S	Shubhapantuvarali (45)
Bhairav	S r G m P d N S	Mayamalavagowla (15)

Note: Lowercase letters indicate komal (flat) swaras, uppercase indicate shuddha (natural) or tivra (sharp for Ma).

7.3.3 Comparison: Melakartha vs. Thaat

Feature	Melakartha	Thaat
Number of parent scales	72	10
System coverage	Complete (all possibilities)	Selected (common patterns)
Strict ascending/descending	Yes	No
Direct performance	Rarely	Never
Geographical origin	South India	North India
Theoretical vs. Practical	More theoretical	More practical
Derivation	Mathematical	Empirical

7.4 Mathematical Significance

Both systems demonstrate:

1. **Combinatorial completeness** (Melakartha): Systematic enumeration of all possibilities
2. **Algorithmic construction**: Clear rules for generating scales
3. **Hierarchical organization**: Grouping by shared properties
4. **Encoding schemes**: Using linguistic/numerical systems to represent structure

7.5 Practice Questions: Melakartha and Thaat Systems

7.5.1 Short Answer Questions

Q1. Why are there exactly 72 Melakartha rāgas? Show the calculation.
Answer: The number 72 arises from the combinatorial possibilities:

- Sa and Pa are fixed
- Ma has 2 variants (M1, M2)
- Ri-Ga combinations (maintaining ascending order): 6 possibilities
- Da-Ni combinations (maintaining ascending order): 6 possibilities

Total: $2 \times 6 \times 6 = 72$ Melakartha rāgas.

Q2. What is the Katapayadi scheme and why is it useful in the Melakartha system?
Answer: The Katapayadi scheme is an ancient Indian alphanumeric encoding system that assigns numerical values to consonants in Sanskrit. In the Melakartha system, rāga names are constructed using this scheme so that the name itself encodes the rāga's numerical position (1-72). This allows musicians to determine a rāga's scale structure directly from its name.

Q3. State the scale of Melakartha rāga number 1 (Kanakangi).
Answer: For Melakartha 1:

- $1 \leq 36$, so Ma = M1, $m = 1$
- $(1 - 1) \div 6 = 0$ remainder 0
- Quotient 0 $\rightarrow$ R1, G1
- Remainder 0 $\rightarrow$ D1, N1

Scale: S R1 G1 M1 P D1 N1 S

7.5.2 Medium Answer Questions

Q4. Determine the scale of Melakartha rāga number 65 (Mechakalyani). Show all steps of the algorithm.

Answer:

Step 1: Determine Madhyamam

$$n = 65 > 36 \Rightarrow \text{Use M2 (Prati Madhyamam)}$$

$$m = 65 - 36 = 29$$

Step 2: Calculate Quotient and Remainder

$$(m - 1) = 29 - 1 = 28$$

$$28 \div 6 = 4 \text{ remainder } 4$$

So $q = 4$ and $r = 4$.

Step 3: Determine Ri-Ga (Purvanga)

From quotient $q = 4$, we get:

$$\text{Ri-Ga combination: R2, G3}$$

Step 4: Determine Da-Ni (Uttaranga)

From remainder $r = 4$, we get:

$$\text{Da-Ni combination: D2, N3}$$

Step 5: Construct Complete Scale

$$\text{Melakartha 65 (Mechakalyani): S R2 G3 M2 P D2 N3 S}$$

In Western notation (if S = C):

$$\text{C D E F}\sharp\text{ G A B C}$$

This corresponds to the Western major scale with raised fourth (Lydian mode), confirming that Mechakalyani is equivalent to the Western Lydian scale.

Q5. Explain the relationship between the Ri-Ga and Da-Ni combinations in the Melakartha system. Why do both have exactly 6 possibilities?

Answer:

The Ri-Ga and Da-Ni combinations are structurally parallel due to the Melakartha requirement of strict ascending order.

Analysis of Ri-Ga Combinations:

The three Ri variants are:

- R1 (position 1)
- R2 = G1 (position 2)
- R3 = G2 (position 3)

The three Ga variants are:

- G1 (position 2)
- G2 (position 3)
- G3 (position 4)

For strict ascending order, Ga must be higher than or equal to Ri:
If Ri = R1 (position 1):

- Ga can be G1 (position 2) ✓
- Ga can be G2 (position 3) ✓
- Ga can be G3 (position 4) ✓
- Total: 3 combinations

If Ri = R2 = G1 (position 2):

- Ga cannot be G1 (same position, not strictly ascending)
- Ga can be G2 (position 3) ✓
- Ga can be G3 (position 4) ✓
- Total: 2 combinations

If Ri = R3 = G2 (position 3):

- Ga cannot be G1 (lower)
- Ga cannot be G2 (same position)
- Ga can be G3 (position 4) ✓
- Total: 1 combination

Total Ri-Ga combinations: $3 + 2 + 1 = 6$
Analysis of Da-Ni Combinations:
By identical logic, since Da and Ni have the same structure (three variants each with overlapping positions), they also produce exactly 6 combinations:
If Da = D1: 3 choices for Ni (N1, N2, N3)
If Da = D2: 2 choices for Ni (N2, N3)
If Da = D3: 1 choice for Ni (N3)
Total Da-Ni combinations: $3 + 2 + 1 = 6$
Significance:
This symmetry between Purvanga (first half) and Uttaranga (second half) reflects the octave's natural division at the perfect fifth (Pa). The parallel structure allows for a

systematic organization of all 72 Melakarthas into 12 Chakras, where each Chakra shares the same Purvanga but varies the Uttaranga.

This elegant mathematical structure demonstrates that the Melakartha system is not arbitrary but arises naturally from:

1. The chromatic division of the octave
2. The requirement of strict ascending order
3. The enharmonic equivalences built into the swara system

7.5.3 Long Answer Questions

Q6. Compare and contrast the Melakartha (Carnatic) and Thaat (Hindustani) systems in detail. Discuss: (a) their mathematical structures, (b) their completeness and coverage of possible scales, (c) their practical applications, (d) why the systems differ, and (e) which system is more mathematically comprehensive.

Answer:
(a) Mathematical Structures
Melakartha System (Carnatic):
Foundation:

- Systematic enumeration based on combinatorics
- Fixed elements: Sa (1st), Pa (5th), upper Sa (8th)
- Variable elements: Ri, Ga, Ma, Da, Ni
- Ma has 2 variants; Ri, Ga, Da, Ni each have 3 variants
- Constraint: Strict ascending order (Krama Sampoorna)

Calculation:

$$\text{Total rāgas} = (\text{Ma variants}) \times (\text{Ri-Ga combinations}) \times (\text{Da-Ni combinations})$$

$$= 2 \times 6 \times 6 = 72$$

Organization:

- Hierarchical: 12 Chakras $\times$ 6 rāgas
- Algorithmic: Scale determinable from rank number
- Encoded: Katapayadi scheme embeds number in name

Thaat System (Hindustani):
Foundation:

- Empirical selection of common scale patterns
- Based on practical usage in Hindustani rāgas
- No strict ascending/descending requirement

- Focuses on melodic behavior rather than mathematical completeness

Structure:

- 10 parent Thaats selected from practice
- Each Thaat represents a family of related rāgas
- No algorithmic derivation—historically determined
- No encoding scheme

(b) Completeness and Coverage
Melakartha: Mathematically Complete
The Melakartha system covers ALL possible seven-note scales with:

- Fixed Sa and Pa
- Strict ascending order
- Same notes in arohanam and avarohanam

This mathematical exhaustiveness means:

- Every possible combination is accounted for
- No "gaps" in the system
- Completely predictable and systematic
- Can generate any heptatonic scale structure

Thaat: Practically Selective
The Thaat system covers only 10 scales:

- Represents commonly used melodic patterns
- Omits many theoretical possibilities
- Focused on practical musical tradition
- Incomplete from a combinatorial perspective

Example of Gap:
The scale S R2 G3 m P D1 N3 S is theoretically possible but doesn't appear in the 10 Thaats, yet it exists in the Melakartha system (as rāga #27, Sarasangi). This demonstrates the Melakartha's greater coverage.

(c) Practical Applications
Melakartha System:
Theoretical Framework:

- Classification: Every Carnatic rāga can be traced to a Melakartha parent
- Janya rāgas: Derived rāgas are created by omitting notes from Melakarthas
- Pedagogy: Students learn systematic scale construction

- Composition: Composers can explore all possible tonal spaces

Performance Practice:

- Melakarthas themselves are rarely performed
- Serve as reference scales for deriving performed rāgas
- Example: Rāga Mohanam (S R2 G3 P D2 S) is derived from Melakartha 28 (Harikambhoji)

Innovation:

- Allows systematic exploration of new rāgas
- Modern composers have used previously rare Melakarthas
- Expands creative possibilities within traditional framework

Thaat System:
Classification Tool:

- Organizes thousands of Hindustani rāgas into 10 families
- Simplifies learning by grouping similar rāgas
- More practical than exhaustive

Pedagogical Simplicity:

- Easier to remember 10 Thaats than 72 Melakarthas
- Focuses on commonly performed rāgas
- Less theoretical abstraction

Flexibility:

- Allows for asymmetric scales (different ascending/descending)
- Accommodates curved melodic patterns (vakra phrasing)
- More aligned with actual performance practice

(d) Why the Systems Differ
Historical Context:
Melakartha (Formalized 1650):

- Carnatic music emphasized theoretical systematization
- Influence of Sanskrit grammatical and mathematical traditions
- Court patronage supported scholarly music theory
- Venkatamakhin sought complete mathematical framework
- Southern philosophical emphasis on order and classification

Thaat (Formalized 1900):

- Greater emphasis on improvisatory performance
- Oral tradition more prominent
- Bhatkhande's system aimed for practical pedagogy
- Focus on actual performed rāgas rather than theoretical completeness

Musical Philosophy:
Carnatic Approach:

- Composition-centric: Emphasis on pre-composed kritis
- Structured improvisation within compositional framework
- Theoretical rigor valued
- Mathematical elegance appreciated

Hindustani Approach:

- Improvisation-centric: Emphasis on alap, jor, jhala
- Greater freedom in melodic development
- Mood (rasa) and time (samay) associations critical
- Practical performance more important than mathematical completeness

Cultural Factors:

- North-South cultural differences in India
- Different patronage systems (Hindu temples vs. Mughal courts)
- Varying degrees of Persian/Islamic musical influence
- Different linguistic and philosophical traditions

(e) Mathematical Comprehensiveness
Winner: Melakartha System
The Melakartha system is unambiguously more mathematically comprehensive:
1. Completeness:

- Melakartha: 72/72 possible scales (100%)
- Thaat: 10/72 possible scales ($\approx$ 14%)
- Melakartha covers all heptatonic possibilities within its constraints

2. Systematic Derivation:

- Melakartha: Pure combinatorial mathematics
- Thaat: Empirical selection

- Melakartha can be derived from first principles

3. Predictability:

- Melakartha: Algorithm determines scale from number
- Thaat: Must memorize each Thaat individually
- Melakartha enables systematic exploration

4. Information Encoding:

- Melakartha: Katapayadi scheme encodes structure in name
- Thaat: Names don't encode structure
- Melakartha demonstrates mathematical elegance

5. Organizational Structure:

- Melakartha: Hierarchical (12 Chakras $\times$ 6)
- Thaat: Flat (10 independent Thaats)
- Melakartha reveals deeper mathematical relationships

However, Practical Usefulness is Context-Dependent:
While Melakartha wins on mathematical grounds, the Thaat system has practical advantages:

- Easier to learn and remember
- More directly applicable to performance
- Focuses on commonly used patterns
- Less theoretical abstraction

Conclusion:
The contrast between Melakartha and Thaat systems illustrates a fundamental tension in music theory:
Melakartha represents:

- Theoretical completeness
- Mathematical elegance
- Systematic exploration
- Comprehensive coverage

Thaat represents:

- Practical usability
- Performance relevance

- Pedagogical simplicity
- Historical accuracy

From a pure mathematics perspective, the Melakartha system is a remarkable achievement—a complete enumeration of musical possibilities organized with systematic precision. It demonstrates that Indian music theory achieved a level of mathematical sophistication comparable to Western equal temperament, but with different goals (complete coverage of heptatonic scales rather than enharmonic equivalence).

The Thaat system, while less comprehensive, serves its purpose effectively by focusing on the scales that matter most in actual Hindustani musical practice. It sacrifices mathematical completeness for practical relevance.

Both systems are valuable: Melakartha for theoretical understanding and systematic exploration, Thaat for practical classification and pedagogy. The choice between them depends on whether one values mathematical elegance or practical applicability—a choice that reflects broader philosophical differences between Carnatic and Hindustani musical traditions.

8 Conclusion

Unit 2 has explored the deep mathematical foundations underlying tuning systems and scale construction. From the ancient Pythagorean system based on perfect fifths to the modern equal temperament that enables unlimited modulation, we have seen how mathematical constraints and compromises shape musical practice.

Key insights include:

- The fundamental irrationality of $\log_2(3/2)$ makes perfect tuning systems impossible
- Different tuning systems represent different compromises between acoustic purity and practical flexibility
- Modular arithmetic provides a powerful framework for analyzing pitch relationships
- Indian classical music systems demonstrate sophisticated mathematical organization of scales

The mathematical study of tuning systems reveals music as a domain where abstract mathematics meets human perception, where theoretical elegance confronts practical necessity, and where different cultures have developed distinct yet equally sophisticated solutions to universal problems.

References and Further Reading

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