Unit 3: Rhythm, Tāla, and Combinatorial Structures

Course Notes

MA375TGR - Mathematics of Music
Semester VII - Institutional Elective

Faculty: Dr. Venugopal K.

Contents

1 Introduction to Rhythm in Music

1.1 Fundamental Concepts

Rhythm is one of the three fundamental elements of music, alongside melody (rāga) and harmony. In Indian classical music, particularly Carnatic music, rhythm achieves a level of sophistication and mathematical complexity unparalleled in other musical traditions.

Definition 1.1 (Rhythm)**.** Rhythm refers to the temporal organization of musical sounds and silences, encompassing duration, onset, and the pattern of strong and weak beats within a musical framework.

Definition 1.2 (Tāla)**.** Tāla (Sanskrit:) is the rhythmic framework or metrical cycle in Indian classical music that provides the temporal structure for composition, improvisation, and accompaniment. It is a fixed-length time cycle that is cyclic in nature.

1.2 Distinction Between Western and Indian Rhythm

Unlike Western music which primarily uses simple and compound meters (2/4, 3/4, 4/4, 6/8, etc.), Indian tāla systems offer:

1. **Cyclic Nature**: Each tāla cycle repeats (āvartana), creating a temporal loop
2. **Additive Structure**: Beats are grouped in sections (vibhāga/aṅga) with potentially unequal lengths
3. **Complex Cycles**: Tālas can range from 3 to 29 beats per cycle
4. **Multiple Time Divisions**: Subdivisions (nadai/gati) can be 3, 4, 5, 7, or 9 per beat
5. **Flexible Accent Patterns**: Not restricted to alternating strong-weak beats

1.3 Mathematical Foundations of Time in Music

Time in music can be represented mathematically as a hierarchical structure:

$$\text{Āvartana} = \sum_{i=1}^{n} \text{Akṣara}_i \tag{1}$$
$$\text{Akṣara} = k \times \text{Mātrā} \tag{2}$$
$$\text{Duration}_{\text{cycle}} = N_{\text{akṣara}} \times k \times \tau_{\text{mātrā}} \tag{3}$$

where:

- $N_{\text{akṣara}}$ = number of akṣaras (beats) in one cycle
- k = nadai (subdivision factor)
- $\tau_{\text{mātrā}}$ = duration of one mātrā (sub-beat unit)

1.4 Questions and Answers

Question 1 (Short Answer): Define tāla and explain its significance in Indian classical music.

Answer: Tāla is the rhythmic framework or metrical cycle in Indian classical music. It is a cyclical pattern of fixed length that provides the temporal structure for musical composition and improvisation. Its significance lies in providing:

- A reference framework for performers and listeners
- A basis for rhythmic improvisation
- Mathematical structure for complex rhythmic variations
- Temporal coordination between different performers

Question 2 (Short Answer): What is the difference between mātrā and akṣara?

Answer: **Mātrā** is the smallest sub-unit of time, while **akṣara** (or kriyā) is the fundamental beat unit that consists of one or more mātras. The relationship is: Akṣara $= k\times$ Mātrā, where k is the nadai (subdivision factor) that can be 3, 4, 5, 7, or 9.

2 The Tāla System in Carnatic Music

2.1 Fundamental Terminology

Definition 2.1 (Āvartana/Āvartam)**.** An āvartana is one complete cycle of a tāla. The term means ”repeating” or ”recurring” in Sanskrit.

Definition 2.2 (Akṣara/Kriyā)**.** An akṣara (literally ”letter” in Sanskrit) is one unit of a tāla cycle. A certain number of akṣaras constitute one āvarta. Each akṣara can be represented by a physical hand gesture (kriyā).

Definition 2.3 (Mātrā)**.** A mātrā is a sub-unit of an akṣara. One or more mātras constitute an akṣara. The number of mātras per akṣara is determined by the nadai/gati.

Definition 2.4 (Vibhāga/Aṅga)**.** The vibhāga or aṅga refers to the sections or divisions within a tāla cycle. Each tāla is divided into (possibly unequal) sections that define its rhythmic character.

Definition 2.5 (Sama)**.** Sama is the first beat of any tāla cycle. It serves as the most important reference point and is where compositions typically begin or cadence.

2.2 The Three Basic Aṅgas

All Carnatic tālas are constructed using three fundamental building blocks:

1. **Laghu (I)**: A pattern with a variable number of beats (3, 4, 5, 7, or 9), depending on the jāti. Notated as ‘I’.

   Kriyā: One clap followed by counting fingers

2. **Dhrutam (O)**: A fixed pattern of 2 beats. Notated as ‘O’.

   Kriyā: One clap (beat) followed by a wave (turn of palm)

3. **Anudhrutam (U)**: A single beat. Notated as 'U'.

   Kriyā: One clap only

Mathematical representation:

$$\text{Anudhrutam} = 1 \text{ beat} \tag{4}$$
$$\text{Dhrutam} = 2 \text{ beats} \tag{5}$$
$$\text{Laghu}(j) = j \text{ beats, where } j \in \{3, 4, 5, 7, 9\} \tag{6}$$

2.3 Jāti and Nadai

Definition 2.6 (Jāti)**.** Jāti refers to the number of akṣaras in a laghu. There are five jātis:

Jāti	Akṣaras in Laghu	Name
Tisra	3	Three-fold
Chatusra	4	Four-fold
Khanda	5	Five-fold
Miśra	7	Seven-fold
Saṅkīrṇa	9	Nine-fold

Table 1: Five Jātis in Carnatic Music

Definition 2.7 (Nadai/Gati)**.** Nadai (also called gati) specifies the number of mātras (subdivisions) per akṣara. There are five nadais corresponding to the five jātis:

Nadai	Mātras per Akṣara
Tisra Nadai	3
Chatusra Nadai	4 (default)
Khanda Nadai	5
Miśra Nadai	7
Saṅkīrṇa Nadai	9

Table 2: Five Nadais in Carnatic Music

The total number of mātras in a cycle is:

$$N_{\text{mātrā}} = N_{\text{akṣara}} \times n \tag{7}$$

where n is the nadai value and $N_{\text{akṣara}}$ is the number of akṣaras in the cycle.

Question 3 (Medium Answer): Explain the concepts of jāti and nadai. How do they relate mathematically?

Answer: Jāti determines the number of akṣaras (beats) in a laghu and can be 3, 4, 5, 7, or 9. **Nadai (gati)** determines the number of mātras (subdivisions) per akṣara and also takes values 3, 4, 5, 7, or 9.

Mathematical relationship:

$$\text{Total Mātras in Cycle} = N_{\text{akṣara}} \times \text{Nadai}$$

For example, Chatusra-jāti means laghu has 4 akṣaras, while Chatusra-nadai means each akṣara has 4 mātras.

If Adi Tāla (8 akṣaras) is rendered in Chatusra nadai:

$$\text{Total mātras} = 8 \times 4 = 32 \text{ mātras per cycle}$$

If the same Adi Tāla is rendered in Tisra nadai:

$$\text{Total mātras} = 8 \times 3 = 24 \text{ mātras per cycle}$$

2.4 Sapta Tāla System (The Seven Tālas)

The modern Carnatic system is based on seven basic tālas called Suladi Sapta Tālas:

Tāla	Aṅga Structure	Default Laghu	Total Akṣaras
Dhruva	I O I I	4	14
Matya	I O I	4	10
Rūpaka	O I	4	6
Jhampa	I U O	7	10
Tripuṭa	I O O	4	7
Aṭa	I I O O	5	14
Eka	I	4	4

Table 3: The Seven Suladi Tālas (Default Configuration)

Detailed Structure:
1. Dhruva Tāla: I O I I = Laghu + Dhrutam + Laghu + Laghu
For Chatusra jāti: $4 + 2 + 4 + 4 = 14$ akṣaras
2. Matya Tāla: I O I = Laghu + Dhrutam + Laghu
For Chatusra jāti: $4 + 2 + 4 = 10$ akṣaras
3. Rūpaka Tāla: O I = Dhrutam + Laghu
For Chatusra jāti: $2 + 4 = 6$ akṣaras
4. Jhampa Tāla: I U O = Laghu + Anudhrutam + Dhrutam
For Miśra jāti: $7 + 1 + 2 = 10$ akṣaras
5. Tripuṭa Tāla (Adi Tāla): I O O = Laghu + Dhrutam + Dhrutam
For Tisra jāti: $3 + 2 + 2 = 7$ akṣaras
For Chatusra jāti: $4 + 2 + 2 = 8$ akṣaras (most common - Adi Tāla)
6. Aṭa Tāla: I I O O = Laghu + Laghu + Dhrutam + Dhrutam
For Khanda jāti: $5 + 5 + 2 + 2 = 14$ akṣaras
7. Eka Tāla: I = Laghu only
For Chatusra jāti: 4 akṣaras

2.5 The 35 Tāla System

By varying the jāti of the laghu in each of the 7 tālas, we obtain:

$$\text{Total Tālas} = 7 \text{ (basic tālas)} \times 5 \text{ (jātis)} = 35 \text{ tālas} \tag{8}$$

Range of Cycle Lengths:

- Shortest: Tisra-jāti Eka Tāla = 3 akṣaras
- Longest: Saṅkīrṇa-jāti Dhruva Tāla = $9 + 2 + 9 + 9 = 29$ akṣaras

2.6 The 175 Tāla System

Applying gati-bheda (nadai variations), each of the 35 tālas can be rendered in 5 different nadais:

$$\text{Total Tālas with Nadai} = 35 \times 5 = 175 \text{ tālas} \tag{9}$$

However, in practice, only a few tālas are commonly used.

Example 2.1 (Khanda-jāti Rūpaka Tāla in Chatusra Nadai)**.** Structure: O I (Dhrutam + Laghu)
With Khanda jāti: Laghu = 5 beats
Total akṣaras: $2 + 5 = 7$ akṣaras
With Chatusra nadai: Each akṣara = 4 mātras
Total mātras: $7 \times 4 = 28$ mātras per cycle

Question 4 (Short Answer): How many tālas are theoretically possible in the Carnatic system considering jāti and nadai variations?
Answer: Theoretically, with 7 basic tālas, 5 jātis, and 5 nadais:

$$\text{Total} = 7 \times 5 \times 5 = 175 \text{ tālas}$$

Breaking it down:

- 7 basic tālas (Dhruva, Matya, Rūpaka, Jhampa, Tripuṭa, Aṭa, Eka)
- Each with 5 jāti variations = 35 tālas
- Each of the 35 with 5 nadai variations = 175 tālas

In practice, only a subset of these are commonly used in compositions.

3 Detailed Analysis of Common Tālas

3.1 Adi Tāla (Chatusra-jāti Tripuṭa Tāla)

Adi Tāla is the most widely used tāla in Carnatic music. "Adi" means "primordial" in Sanskrit.
Structure: I O O (Laghu + Dhrutam + Dhrutam)
Akṣaras: With Chatusra jāti: $4 + 2 + 2 = 8$ akṣaras
Kriyā (Hand gestures):

1. Beat 1: Clap (Sama - most important)
2. Beats 2-4: Count on fingers (pinky, ring, middle)
3. Beat 5: Clap
4. Beat 6: Wave (turn palm)
5. Beat 7: Clap
6. Beat 8: Wave

Vibhāga (Sections):

- Laghu: Beats 1-4
- First Dhrutam: Beats 5-6
- Second Dhrutam: Beats 7-8

Mathematical Properties:

$$N_{\text{akṣara}} = 8 \tag{10}$$
$$\text{In Chatusra nadai: } N_{\text{mātrā}} = 8 \times 4 = 32 \tag{11}$$
$$\text{In Tisra nadai: } N_{\text{mātrā}} = 8 \times 3 = 24 \tag{12}$$
$$\text{In Khanda nadai: } N_{\text{mātrā}} = 8 \times 5 = 40 \tag{13}$$

3.2 Rūpaka Tāla

Structure: O I (Dhrutam + Laghu)
Akṣaras: With Chatusra jāti: $2 + 4 = 6$ akṣaras
Kriyā:

1. Beat 1: Clap (note: traditionally it was Beat 3, but modern practice varies)
2. Beat 2: Wave
3. Beat 3: Clap
4. Beats 4-6: Count on fingers

Mathematical Properties:

$$N_{\text{akṣara}} = 6 \tag{14}$$
$$\text{In Chatusra nadai: } N_{\text{mātrā}} = 6 \times 4 = 24 \tag{15}$$
$$\text{In Khanda-jāti: } N_{\text{akṣara}} = 2 + 5 = 7 \tag{16}$$
$$\text{Khanda-jāti in Chatusra nadai: } N_{\text{mātrā}} = 7 \times 4 = 28 \tag{17}$$

3.3 Jhampa Tāla

Structure: I U O (Laghu + Anudhrutam + Dhrutam)
Default jāti: Miśra (7 beats in laghu)
Akṣaras: $7 + 1 + 2 = 10$ akṣaras
Mathematical Properties:

$$N_{\text{akṣara}} = 10 \tag{18}$$
$$\text{In Chatusra nadai: } N_{\text{mātrā}} = 10 \times 4 = 40 \tag{19}$$

3.4 Chapu Tālas

Chapu tālas are special tālas that don't fit neatly into the Suladi Sapta Tāla system. They have unequal divisions within the cycle.

1. Miśra Chapu:

- Total akṣaras: 7
- Grouping: 3 + 2 + 2 (or sometimes 3 + 4)
- Hand gesture pattern: Beat (3 units) + Beat (2 units) + Beat (2 units)

2. Khanda Chapu:

- Total akṣaras: 5
- Grouping: 2 + 3
- Hand gesture pattern: Beat (2 units) + Beat (3 units)

Question 5 (Long Answer): Provide a detailed analysis of Adi Tāla including its structure, mathematical properties, and variations.

Answer: Adi Tāla (Chatusra-jāti Tripuṭa Tāla)

Basic Structure:

- Aṅga composition: I O O (Laghu + Dhrutam + Dhrutam)
- With Chatusra jāti: 4 + 2 + 2 = 8 akṣaras per cycle
- Most commonly used tāla in Carnatic music (approximately 50% of compositions)

Kriyā (Hand Gestures):

1. Akṣara 1: Clap (Sama - the starting point)
2. Akṣaras 2-4: Count fingers (pinky, ring, middle)
3. Akṣara 5: Clap (beginning of first Dhrutam)
4. Akṣara 6: Wave/turn palm
5. Akṣara 7: Clap (beginning of second Dhrutam)
6. Akṣara 8: Wave/turn palm

Mathematical Properties:

In different nadais:

- Chatusra nadai (4 per beat): $8 \times 4 = 32$ mātras
- Tisra nadai (3 per beat): $8 \times 3 = 24$ mātras
- Khanda nadai (5 per beat): $8 \times 5 = 40$ mātras
- Miśra nadai (7 per beat): $8 \times 7 = 56$ mātras
- Saṅkīrṇa nadai (9 per beat): $8 \times 9 = 72$ mātras

Different kalai (speed multiplicities):

- 1st kalai: 32 mātras (in Chatusra nadai)
- 2nd kalai: 64 mātras
- 3rd kalai: 128 mātras

Vibhāga Structure: The cycle divides into three unequal sections:

1. Laghu section: 4 akṣaras (50% of cycle)
2. First Dhrutam: 2 akṣaras (25% of cycle)
3. Second Dhrutam: 2 akṣaras (25% of cycle)

This asymmetric division (4:2:2) creates the characteristic feel of Adi Tāla.
Compositional Usage:

- Many kritis (compositions) are set to Adi Tāla
- Allows for complex rhythmic variations while maintaining clarity
- Provides good balance between simplicity and sophistication

Question 6 (Medium Answer): Calculate the total number of mātras in one cycle of Khanda-jāti Rūpaka Tāla rendered in Miśra nadai.
Answer: Given:

- Tāla: Rūpaka
- Structure: O I (Dhrutam + Laghu)
- Jāti: Khanda (Laghu = 5 beats)
- Nadai: Miśra (7 mātras per akṣara)

Calculation:
Number of akṣaras:

$$N_{\text{akṣara}} = \text{Dhrutam} + \text{Laghu} = 2 + 5 = 7 \text{ akṣaras}$$

Total mātras with Miśra nadai:

$$N_{\text{mātrā}} = N_{\text{akṣara}} \times \text{Nadai} = 7 \times 7 = 49 \text{ mātras}$$

Answer: One cycle contains **49 mātras**.

4 Combinatorial Structures in Tāla

4.1 Eduppu (Starting Point)

Definition 4.1 (Eduppu)**.** Eduppu refers to the starting point of a composition relative to the sama (first beat) of the tāla cycle. It can be measured in akṣaras or mātras and represents an offset from the sama.

Mathematically, if a composition starts at position p (where $p = 0$ represents sama):

$$\text{Eduppu offset} = p \mod N_{\text{akṣara}} \tag{20}$$

Common Eduppu patterns:

- Samam (on sama): $p = 0$
- Anu-samam (just after sama): $p = 0^+$
- Atīta (before sama, from previous cycle): $p < 0$
- After specific beat: $p = k$, where $k \in \{1, 2, ..., N_{\text{akṣara}} - 1\}$

4.2 Combinatorial Mathematics in Tāla Variations

The Carnatic tāla system exhibits rich combinatorial structure:

1. **Choice of Tāla:** 7 basic tālas
2. **Choice of Jāti:** 5 options
3. **Choice of Nadai:** 5 options
4. **Choice of Kalai (Speed):** Typically 3-4 levels
5. **Total Theoretical Combinations:**

$$C_{\text{total}} = 7 \times 5 \times 5 \times 4 = 700 \text{ basic combinations} \tag{21}$$

When we add eduppu variations (8 possible starting points in Adi Tāla):

$$C_{\text{with eduppu}} = 700 \times 8 = 5,600 \text{ combinations} \tag{22}$$

4.3 Graha-bheda (Change of Reference Point)

Graha-bheda refers to the technique of changing the reference point or sama within a composition, creating polyrhythmic effects.

If a pattern of length L mātras is repeated with an offset of k mātras relative to the tāla cycle of length N mātras, the pattern will realign with sama after:

$$n_{\text{cycles}} = \frac{\text{lcm}(L, N)}{N} \tag{23}$$

where lcm denotes the least common multiple.

Example 4.1 (Graha-bheda Pattern)**.** Consider a 5-mātrā pattern repeated in Adi Tāla (32 mātras in Chatusra nadai):

$$n_{\text{cycles}} = \frac{\text{lcm}(5, 32)}{32} = \frac{160}{32} = 5 \text{ cycles}$$

The pattern will realign with sama after 5 cycles of Adi Tāla.

Question 7 (Short Answer): Define eduppu and explain its mathematical significance.

Answer: Eduppu is the starting point of a composition relative to the sama (first beat) of the tāla cycle. It represents an offset measured in akṣaras or mātras.

Mathematical Significance:

- Creates phase shift in rhythmic patterns
- If eduppu $= k$ akṣaras, the composition begins at beat $(k+1)$ of the cycle
- Eduppu $\equiv p \pmod{N_{\text{akṣara}}}$, where $N_{\text{akṣara}}$ is the cycle length
- Different eduppus create different relationships between melodic phrases and tāla structure
- In Adi Tāla (8 beats), there are 8 possible eduppus: 0, 1, 2, ..., 7

5 Korvais: Structured Rhythmic Patterns

5.1 Definition and Structure

Definition 5.1 (Korvai)**.** A korvai (Tamil: , meaning "joining" or "beading") is a structured rhythmic composition consisting of two distinct parts that "join" together to create a cadential pattern. It is typically repeated three times and concludes at the sama.

Two Parts of a Korvai:

1. **Purvāṅga (First Part):** The introductory or developmental section
2. **Uttarāṅga (Last Part):** The concluding or cadential section

Mathematical Structure:
If a korvai is to end at sama and is repeated 3 times:

$$3 \times (\text{Purvāṅga} + \text{Uttarāṅga}) = k \times N_{\text{cycle}} \tag{24}$$

where k is an integer and N_{cycle} is the number of mātras in one tāla cycle.

5.2 Mathematical Properties of Korvais

1. Length Constraint:
For a korvai starting at sama and ending at sama:

$$P + U \equiv 0 \pmod{N_{\text{cycle}}/3} \tag{25}$$

where P = length of purvāṅga in mātras, U = length of uttarāṅga in mātras.
2. Balanced Ratio:
The ratio of purvāṅga to uttarāṅga typically falls in the range:

$$\frac{P}{U} \in [0.3, 2.33] \quad \text{or} \quad \frac{P}{P+U} \in [0.23, 0.70] \tag{26}$$

Common ratios: 30:70, 40:60, 50:50, 60:40, 70:30
3. Repetition Factor:
Korvais are traditionally repeated 3 times, but can be repeated r times where:

$$r \times (P + U) = k \times N_{\text{cycle}}, \quad k \in \mathbb{Z}^+ \tag{27}$$

5.3 Types of Korvais

1. Simple Korvai: Fixed purvāṅga and uttarāṅga, repeated identically three times.
Structure: $ABC\ ABC\ ABC$, where A = purvāṅga, BC = uttarāṅga

2. Progressive Korvai: Purvāṅga progressively increases or decreases.
Example structure: $(n)(3x)\ (n+1)(3x)\ (n+2)(3x)$

3. Yati Korvai: Follows specific patterns like fibonacci, geometric, or arithmetic progressions.

4. Porutham: A special korvai that matches the melodic pattern of the eduppu.

5.4 Arithmetic in Korvai Construction

Example 5.1 (Simple Korvai in Adi Tāla)**.** Construct a korvai spanning 2 cycles of Adi Tāla (Chatusra nadai).

Total mātras needed: $2 \times 32 = 64$ mātras

For 3 repetitions: Each statement $= 64 \div 3 = 21.33$ mātras (doesn't divide evenly)

Adjust to span exactly 2 cycles with unequal repetitions, or choose: - 3 statements of 21 mātras each = 63 mātras (end 1 mātrā before sama) - First two statements: 21 mātras each - Third statement: 22 mātras (to complete the cycle)

Example 5.2 (Progressive Korvai)**.** Using arithmetic progression in purvāṅga:

Pur vāṅga lengths: 2, 3, 4, 5, 6, 7 mātras

Sum: $\sum_{i=2}^{7} i = 27$ mātras

Uttarāṅga: $3 \times 7 = 21$ mātras (repeated 3 times)

Total per repetition: $27 + 7 = 34$ mātras

For 3 repetitions: Doesn't fit standard Adi Tāla cycle precisely, needs adjustment.

5.5 Backward Counting in Korvai Construction

Professional musicians often construct korvais by counting backward from sama:

Process:

1. Determine the target sama position
2. Decide uttarāṅga length U
3. Calculate: Number of mātras before sama in previous cycles
4. Distribute remaining mātras to purvāṅga with mathematical elegance

Example 5.3 (Backward Counting Example)**.** Target: End at sama in Adi Tāla (32 mātras/cycle)

Spanning 2 cycles = 64 mātras total

Choose uttarāṅga: $3 \times 7 = 21$ mātras (7 mātras repeated 3 times)

Remaining for purvāṅga: $64 - 21 = 43$ mātras

But we need this to be repeated 3 times, so: Purvāṅga per statement: Must fit the remaining space

Alternative: Fix uttarāṅga as 21, purvāṅga as 33 Total per statement: $33 + 7 = 40$ mātras Three statements: $3 \times 40 = 120$ mātras = 3.75 cycles

Adjust to exact cycle fit.

Question 8 (Long Answer): Explain the structure of a korvai and demonstrate how to construct a korvai that spans exactly 2 cycles of Adi Tāla in Chatusra nadai.

Answer: Structure of a Korvai:

A korvai consists of two parts:

1. **Purvāṅga (P)**: Introductory/developmental section
2. **Uttarāṅga (U)**: Concluding/cadential section

The pattern is typically repeated 3 times and ends at sama.

Construction for 2 Cycles of Adi Tāla:

Given:

- Adi Tāla in Chatusra nadai
- Mātras per cycle: $8 \times 4 = 32$
- Target: 2 cycles = 64 mātras
- 3 repetitions

Method:

Since $64 \div 3 = 21.33$ (not evenly divisible), we need to be strategic.

Solution 1: Unequal Statements

Let the first two statements be 21 mātras each, and the third be 22 mātras:

- Statement 1: $P_1 + U_1 = 21$ mātras
- Statement 2: $P_2 + U_2 = 21$ mātras
- Statement 3: $P_3 + U_3 = 22$ mātras
- Total: $21 + 21 + 22 = 64$ mātras

Choose: $P = 14$ mātras, $U = 7$ mātras (for first two statements)

For third statement: $P = 15$ mātras, $U = 7$ mātras

Pattern: $(14, 7)$ $(14, 7)$ $(15, 7)$

Solution 2: Three Cycles

Alternatively, span exactly 3 cycles (96 mātras):

- Per statement: $96 \div 3 = 32$ mātras
- Choose: $P = 25$ mātras, $U = 7$ mātras
- Pattern: $(25, 7)$ repeated 3 times

Verification:

$$3 \times (25 + 7) = 3 \times 32 = 96 = 3 \times 32 \text{ mātras} = 3 \text{ cycles}$$

Purvāṅga-Uttarāṅga Ratio:

$$\frac{P}{U} = \frac{25}{7} \approx 3.57$$
$$\frac{P}{P+U} = \frac{25}{32} \approx 0.78$$

This is acceptable but on the higher side; more balanced would be closer to 50:50 or 60:40.

Question 9 (Medium Answer): A korvai has a purvāṅga of 18 mātras and an uttarāṅga of 12 mātras. If it is repeated 3 times, how many cycles of Adi Tāla (Chatusra nadai) does it span?

Answer: Given:

- Purvāṅga (P) = 18 mātras
- Uttarāṅga (U) = 12 mātras
- Repetitions (r) = 3
- Adi Tāla in Chatusra nadai: 32 mātras/cycle

Calculation:

Total length per statement:

$$L = P + U = 18 + 12 = 30 \text{ mātras}$$

Total length for 3 repetitions:

$$L_{\text{total}} = r \times L = 3 \times 30 = 90 \text{ mātras}$$

Number of Adi Tāla cycles:

$$n = \frac{L_{\text{total}}}{32} = \frac{90}{32} = 2.8125 \text{ cycles}$$

Answer: The korvai spans **2.8125 cycles** or **2 complete cycles plus 26 mātras into the 3rd cycle**.

Note: This korvai does not end precisely at sama. To end at sama, adjustments would be needed.

6 Tihais: Repetitive Cadential Patterns

6.1 Definition and Structure

Definition 6.1 (Tihai)**.** A tihai (also spelled t ihāi) is a rhythmic pattern repeated exactly three times, with the third repetition ending precisely at the sama. It is a common feature in both Hindustani and Carnatic music.

Basic Structure:

$$\text{Pattern} \quad \text{Gap} \quad \text{Pattern} \quad \text{Gap} \quad \text{Pattern} \rightarrow \text{SAMA}$$

6.2 Mathematical Formula for Tihai

Let:

- P = length of the pattern (in mātras)
- G = length of the gap between repetitions (in mātras)

- T = total length to fill (in mātras)
- r = number of repetitions (typically 3)

The tihai must satisfy:

$$r \times P + (r-1) \times G = T \tag{28}$$

For a standard tihai with $r = 3$:

$$3P + 2G = T \tag{29}$$

Solving for G:

$$G = \frac{T - 3P}{2} \tag{30}$$

Constraint: $G \geq 0$ and G must be an integer (or at least a rational number of mātras).

6.3 Types of Tihais

1. Damdar Tihai: Gap length $G > 0$. The pattern is separated by silences or sustained notes.

2. Bedam Tihai: Gap length $G = 0$. The pattern is repeated immediately without pause.

$$3P = T \quad \Rightarrow \quad P = \frac{T}{3} \tag{31}$$

3. Chakkardar Tihai: The entire tihai structure is itself repeated, creating a "tihai of tihais."

Inner tihai: $3P_1 + 2G_1 = L_1$
Outer tihai: $3L_1 + 2G_2 = T$
Substituting:

$$3(3P_1 + 2G_1) + 2G_2 = T \tag{32}$$

$$9P_1 + 6G_1 + 2G_2 = T \tag{33}$$

6.4 Examples of Tihai Constructions

Example 6.1 (Simple Bedam Tihai)**.** Fill 24 mātras with a bedam tihai:

$$3P = 24 \quad \Rightarrow \quad P = 8 \text{ mātras}$$

Structure: [8 mātras] [8 mātras] [8 mātras] → SAMA
Total: 24 mātras

Example 6.2 (Damdar Tihai)**.** Fill 35 mātras with pattern of 7 mātras:

$$3 \times 7 + 2G = 35$$
$$21 + 2G = 35$$
$$2G = 14$$
$$G = 7 \text{ mātras}$$

Structure: [7] gap[7] [7] gap[7] [7] → SAMA
Total: $7 + 7 + 7 + 7 + 7 = 35$ mātras

Example 6.3 (Tihai in Adi Tāla). Create a tihai to fill one cycle of Adi Tāla (32 mātras) with a 5-mātrā pattern:

$$3 \times 5 + 2G = 32$$
$$15 + 2G = 32$$
$$2G = 17$$
$$G = 8.5 \text{ mātras}$$

Structure: [5] gap[8.5] [5] gap[8.5] [5] → SAMA
This works if we can divide mātras into half-units.

Question 10 (Medium Answer): Derive the formula for calculating the gap length in a tihai given the pattern length and total duration.

Answer: Derivation:

Given:

- Pattern length: P mātras
- Gap length: G mātras
- Number of repetitions: $r = 3$
- Total duration to fill: T mātras

Structure:

$$\underbrace{P}_{1st} + \underbrace{G}_{\text{gap}} + \underbrace{P}_{2nd} + \underbrace{G}_{\text{gap}} + \underbrace{P}_{3rd} = T$$

Equation:

$$3P + 2G = T$$

Solving for G:

$$2G = T - 3P$$
$$G = \frac{T - 3P}{2}$$

Constraints:

1. $G \geq 0$ (gap cannot be negative)
2. Therefore: $T \geq 3P$
3. G must be expressible in the mātrā system (integer or simple fraction)

Special Case - Bedam Tihai: When $G = 0$:

$$3P = T \quad \Rightarrow \quad P = \frac{T}{3}$$

This requires T to be divisible by 3.

Question 11 (Short Answer): What is the difference between a damdar tihai and a bedam tihai?

Answer: Damdar Tihai:

- Has gaps ($G > 0$) between repetitions of the pattern
- Formula: $3P + 2G = T$
- Creates dramatic effect with silences
- Example: [Pattern]-GAP-[Pattern]-GAP-[Pattern]

Bedam Tihai:

- No gaps ($G = 0$) between repetitions
- Formula: $3P = T$
- Continuous repetition without pause
- Example: [Pattern][Pattern][Pattern]
- Requires total duration T to be exactly divisible by 3

The choice depends on the total duration available and the musical effect desired.

7 Algorithmic Composition of Rhythmic Phrases

7.1 Mathematical Patterns in Korvai Construction

1. Arithmetic Progression:
Purvāṅga uses consecutive integers:

$$P = a + (a+1) + (a+2) + ... + (a+n-1) = na + \frac{n(n-1)}{2}$$

Example 7.1. $a = 2, n = 6$:

$$P = 2 + 3 + 4 + 5 + 6 + 7 = 27 \text{ mātras}$$

2. Fibonacci Sequence:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1$$

Sequence: 1, 1, 2, 3, 5, 8, 13, 21, ...
Example korvai: Purvāṅga with phrases of lengths 3, 4, 7, 11, 18 mātras (similar pattern)

3. Geometric Progression:
Pattern lengths: $a, ar, ar^2, ar^3, ...$
Sum: $P = a\frac{r^n-1}{r-1}$

Example 7.2. $a = 2, r = 2, n = 5$:

$$P = 2 + 4 + 8 + 16 + 32 = 62 \text{ mātras}$$

4. Squares and Cubes:
Purvāṅga: n^2 for $n = 1, 2, 3, ...$

$$P = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 7.3. $n = 5$:

$$P = 1 + 4 + 9 + 16 + 25 = 55 \text{ mātras}$$

7.2 Polite Number Representation

A **polite number** is a positive integer that can be written as the sum of two or more consecutive positive integers.

Formula for representation:

$$n = \frac{d(2q + d - 1)}{2}$$

where d is an odd divisor of n and $q = n/d$.

Example 7.4 (Polite Number Korvai)**.** Represent 35 mātras:

$35 = 5 \times 7$, choose $d = 5$, $q = 7$

Consecutive integers centered at 7:

$$35 = 5 + 6 + 7 + 8 + 9$$

Use this as purvāṅga structure in a korvai.

7.3 Modular Arithmetic in Rhythm

Cycle Completion:

For a pattern of length L repeating in a cycle of length N:

Pattern aligns with sama when:

$$k \times L \equiv 0 \pmod{N}$$

Smallest k:

$$k = \frac{N}{\gcd(L, N)}$$

Number of cycles to realign:

$$n_{\text{cycles}} = \frac{\operatorname{lcm}(L, N)}{N} = \frac{L}{\gcd(L, N)}$$

Example 7.5 (Pattern Alignment)**.** Pattern: 7 mātras

Cycle: Adi Tāla = 32 mātras

$$\gcd(7, 32) = 1$$
$$k = \frac{32}{1} = 32 \text{ repetitions}$$
$$n_{\text{cycles}} = \frac{7 \times 32}{32} = 7 \text{ cycles}$$

The 7-mātrā pattern repeats 32 times before realigning with sama, spanning 7 complete cycles.

7.4 Algorithmic Generation of Korvais

Algorithm 1: Target-Based Construction

```
Input: Target duration T, desired purvanga ratio r
Output: Korvai structure (P, U)

1. Calculate total per statement: S = T / 3
2. Set P = floor(r * S)
3. Set U = S - P
4. Verify: 3 * (P + U) = T
5. If not exact, adjust P or U
6. Return (P, U)
```

Algorithm 2: Progressive Pattern

```
Input: Starting value a, increment d, number of terms n
Output: Purvanga structure

1. Generate arithmetic sequence: a, a+d, a+2d, ..., a+(n-1)d
2. Calculate sum: P = n*a + d*n*(n-1)/2
3. Choose uttaranga U to fit desired total
4. Return sequence and (P, U)
```

Algorithm 3: Backward Construction

```
Input: Target sama position, uttaranga length U
Output: Complete korvai

1. Determine total mātras to fill: T
2. Reserve for uttaranga: 3 * U
3. Remaining for purvanga: R = T - 3*U
4. Distribute R using chosen pattern (arithmetic, etc.)
5. Verify total = T
6. Return complete korvai structure
```

7.5 Recursive Structures

Nested Korvais:

A korvai can contain smaller korvais within its purvāṅga:

$$\text{Outer Korvai} = 3 \times [\underbrace{(3 \times \text{Inner})}_{\text{Purvāṅga}} + \text{Uttarāṅga}]$$

Depth of recursion can increase complexity exponentially.

Question 12 (Long Answer): Design an algorithmic approach to automatically generate a korvai of specified length using mathematical patterns. Implement it for a 64-mātrā korvai with progressive structure.

Answer: Algorithmic Approach:

Objective: Generate a 64-mātrā korvai with progressive purvāṅga structure.

Design Specifications:

- Total duration: $T = 64$ mātras
- Number of repetitions: 3
- Per statement: $S = 64/3 \approx 21.33$ mātras

Step 1: Choose Pattern Type
Use arithmetic progression for purvāṅga: $2, 3, 4, 5, 6, 7$

$$P = \sum_{i=2}^{7} i = 2 + 3 + 4 + 5 + 6 + 7 = 27 \text{ mātras}$$

Step 2: Calculate Uttarāṅga
For exact fit with adjustment: - First two statements: 21 mātras each - Third statement: 22 mātras

Let $U = 6$ mātras (repeated 3 times in uttarāṅga)

Check: $P + 3U = 27 + 18 = 45$ (for one complete purvāṅga + uttarāṅga section)

This doesn't fit 21, so revise:

Revised Approach:
Use smaller progression: $2, 3, 4, 5 = 14$ mātras for purvāṅga
Uttarāṅga: 7 mātras
Per statement: $14 + 7 = 21$ mātras
Three statements: $3 \times 21 = 63$ mātras
Add 1 mātrā to third uttarāṅga: $7 + 1 = 8$
Final structure:

- Statement 1: $(2, 3, 4, 5) + (7) = 21$ mātras
- Statement 2: $(2, 3, 4, 5) + (7) = 21$ mātras
- Statement 3: $(2, 3, 4, 5) + (8) = 22$ mātras
- Total: $21 + 21 + 22 = 64$ mātras

Algorithm Pseudocode:

```
function generate_progressive_korvai(T, pattern_type):
    S = T / 3  // target per statement

    if pattern_type == "arithmetic":
        // Generate arithmetic sequence
        start = 2
        increment = 1
        terms = []
        sum = 0
        i = 0
        while sum < S * 0.7:  // purvanga target ~70%
            term = start + i * increment
            terms.append(term)
            sum += term
            i += 1
```

```
P = sum
U_per_rep = floor((S - P/3))

// Adjust for exact fit
statement1 = P + U_per_rep
statement2 = P + U_per_rep
statement3 = T - statement1 - statement2

return {
    "purvanga": terms,
    "uttaranga": [U_per_rep, U_per_rep, statement3 - P],
    "total": T
}
```

Verification:

$$2 + 3 + 4 + 5 = 14$$

$$14 + 7 = 21$$

$$14 + 7 = 21$$

$$14 + 8 = 22$$

$$21 + 21 + 22 = 64 \text{ mātras}$$

This algorithm can be adapted for different mathematical patterns (Fibonacci, geometric, etc.) by changing the pattern generation logic.

8 Advanced Topics in Rhythmic Combinatorics

8.1 Nadai Bheda (Gait Change)

Nadai bheda involves changing the subdivision pattern within a tāla cycle.

Cross-Rhythm Effect:

When a pattern in nadai n_1 is played in a tāla with nadai n_2:

Perceived beats: $\frac{n_1}{n_2}$ times the original

Example 8.1. Pattern in Tisra nadai (3) played in Chatusra nadai (4) tāla:

Each akṣara of the pattern occupies $\frac{3}{4}$ of a tāla akṣara.

A 4-akṣara pattern in Tisra $= 4 \times 3 = 12$ mātras

In Chatusra tāla: $12/4 = 3$ akṣaras

Realignment: After $\text{lcm}(3, 4) = 12$ mātras $= 3$ Chatusra akṣaras $= 4$ Tisra akṣaras

8.2 Yati (Phrasing Shapes)

Yati refers to specific patterns of phrase lengths:

Types of Yati:

1. **Sama Yati** (Rectangular): Equal phrase lengths

   Example: 8-8-8-8

2. **Srotovaha Yati** (Increasing): Phrases increase

   Example: 4-6-8-10

3. **Gopuchha Yati** (Decreasing): Phrases decrease

   Example: 10-8-6-4

4. **Mridanga Yati** (Expanding then contracting): Bell curve

   Example: 4-6-8-6-4

5. **Pippilika Yati** (Contracting then expanding): Inverse bell

   Example: 8-6-4-6-8

Mathematical representation:
For Srotovaha Yati with n phrases starting at a with increment d:

$$\text{Total} = na + \frac{dn(n-1)}{2}$$

8.3 Kuraippu (Reduction)

Kuraippu is a technique where phrase lengths are systematically halved:
Starting length L: Subsequent lengths $L/2, L/4, L/8, ...$
Geometric series sum:

$$S = L(1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{n-1}}) = L \cdot \frac{2^n - 1}{2^{n-1}}$$

As $n \to \infty$: $S \to 2L$

Example 8.2 (Kuraippu Pattern)**.** Starting with 16 mātras, reduce 4 times:
$16 + 8 + 4 + 2 + 1 = 31$ mātras

8.4 Permutation and Combination in Rhythm

Number of Unique Rhythmic Patterns:
For n time units with k possible durations:

$$N_{\text{patterns}} = k^n$$

Example 8.3. 4 beats with 3 duration options (short, medium, long):

$$N = 3^4 = 81 \text{ patterns}$$

Patterns Ending at Sama:
If patterns must complete at cycle boundary, constraint:

$$\sum_{i=1}^{m} d_i \equiv 0 \pmod{N_{\text{cycle}}}$$

This significantly reduces the number of valid patterns.

8.5 Graph Theory Representation

Tāla cycles can be represented as directed graphs:

Nodes: Beat positions (0 to $N-1$)

Edges: Possible transitions with pattern lengths

Paths: Valid rhythmic phrases

Hamiltonian Cycle: Patterns that visit each beat exactly once before returning to sama.

Question 13 (Medium Answer): Explain the concept of yati and provide mathematical formulations for at least three different yati patterns.

Answer: Yati refers to specific patterns or shapes formed by the lengths of successive phrases in a musical composition.

Three Main Yati Patterns:

1. Srotovaha Yati (Increasing/Ascending Pattern):

Phrase lengths increase in arithmetic progression:

$$L_i = a + (i-1)d, \quad i = 1, 2, ..., n$$

where a = initial length, d = common difference

Total length:

$$S = \sum_{i=1}^{n} [a + (i-1)d] = na + \frac{dn(n-1)}{2}$$

Example: 4-6-8-10-12 mātras

$$S = 4 + 6 + 8 + 10 + 12 = 40 \text{ mātras}$$

2. Gopuchha Yati (Decreasing/Descending Pattern):

Phrase lengths decrease:

$$L_i = a - (i-1)d, \quad i = 1, 2, ..., n$$

Total length:

$$S = na - \frac{dn(n-1)}{2}$$

Example: 12-10-8-6-4 mātras

$$S = 12 + 10 + 8 + 6 + 4 = 40 \text{ mātras}$$

3. Mridanga Yati (Bell-shaped/Symmetric Pattern):

Phrases increase then decrease symmetrically:

$$L_i = \begin{cases} a + (i-1)d & \text{for } i \leq \lceil n/2 \rceil \\ L_{n-i+1} & \text{for } i > \lceil n/2 \rceil \end{cases}$$

Example: 4-6-8-10-8-6-4 mātras

For symmetric pattern with $2m+1$ phrases:

$$S = 2\sum_{i=1}^{m} L_i + L_{m+1}$$

where L_{m+1} is the central (longest) phrase.

Musical Significance:

- Creates dramatic arc in rhythmic compositions
- Provides structural framework for improvisation
- Enhances listener engagement through predictable yet varied patterns

9 Applications and Examples

9.1 Complete Korvai Analysis

Example 9.1 (Detailed Korvai Construction)**. Objective:** Create a korvai for Adi Tāla spanning 3 cycles

Given:

- Tāla: Adi Tāla (Chatusra jāti, Chatusra nadai)
- Cycles: 3
- Total mātras: $3 \times 32 = 96$ mātras
- Pattern: Progressive (using polite numbers)

Design:
Per statement: $96/3 = 32$ mātras
Purvāṅga using polite number representation of 33:

$$33 = 3 + 4 + 5 + 6 + 7 + 8$$

Uttarāṅga: Three repetitions of 7 mātras = 21 mātras
But $33 + 21 = 54 \neq 32$, so we need to adjust.
Revised:
Use 25 for purvāṅga (polite): $25 = 3 + 4 + 5 + 6 + 7$
Uttarāṅga: $32 - 25 = 7$ mātras
Final Structure:
Statement 1: $(3, 4, 5, 6, 7) + (7) = 32$ mātras
Statement 2: $(3, 4, 5, 6, 7) + (7) = 32$ mātras
Statement 3: $(3, 4, 5, 6, 7) + (7) = 32$ mātras
Total: $3 \times 32 = 96$ mātras = 3 cycles
Purvāṅga-Uttarāṅga Ratio:

$$\frac{P}{U} = \frac{25}{7} \approx 3.57$$

$$\frac{P}{P+U} = \frac{25}{32} = 0.78125 \approx 78\%$$

This is acceptable though slightly purvāṅga-heavy.

9.2 Tihai in Different Tālas

Example 9.2 (Tihai in Rūpaka Tāla)**. Given:**

- Tāla: Rūpaka (Chatusra jāti, Chatusra nadai)
- Mātras per cycle: $6 \times 4 = 24$
- Fill: 2 cycles = 48 mātras
- Pattern: 9 mātras

Calculate Gap:

$$3 \times 9 + 2G = 48$$
$$27 + 2G = 48$$
$$2G = 21$$
$$G = 10.5 \text{ mātras}$$

Structure: [9 mātras] gap[10.5] [9 mātras] gap[10.5] [9 mātras] → SAMA
Verification:

$$9 + 10.5 + 9 + 10.5 + 9 = 48 \text{ mātras}$$

Example 9.3 (Nested Tihai)**. Objective:** Create a chakkardar tihai (tihai of tihais)
Inner Tihai:

- Pattern: 3 mātras
- Gap: 2 mātras
- Structure: [3]-[2]-[3]-[2]-[3]
- Total: $3 \times 3 + 2 \times 2 = 13$ mātras

Outer Tihai:

- Pattern: Inner tihai (13 mātras)
- Target: 48 mātras (2 cycles of Adi Tāla in Tisra nadai)
- Gap: ?

$$3 \times 13 + 2G = 48$$
$$39 + 2G = 48$$
$$G = 4.5 \text{ mātras}$$

Complete Structure:
[Inner tihai: 13] gap[4.5] [Inner tihai: 13] gap[4.5] [Inner tihai: 13] → SAMA
Complexity:
This creates 9 repetitions of the basic 3-mātrā pattern nested within a 3-repetition outer structure.

Feature	**Korvai**	**Tihai**
Definition	Structured pattern with two parts (purvāṅga + uttarāṅga)	Pattern repeated exactly 3 times
Structure	Typically repeated 3 times, but not mandatory	Always 3 repetitions
Parts	Two distinct parts that "join"	Single pattern with optional gaps
Flexibility	More flexible; parts can vary in each repetition	Fixed pattern in all repetitions
Mathematical Focus	Sum and ratio of two parts	Pattern length and gap calculation
Ending	Must end at sama	Must end at sama
Complexity	Can be more elaborate with progressive structures	Simpler, more direct
Origin	Tamil word ("joining")	Hindi/Sanskrit word

Question 14 (Long Answer): Compare and contrast korvais and tihais. Design one example of each for Adi Tāla spanning 2 cycles, and explain the mathematical principles behind your designs.

Answer: Comparison of Korvais and Tihais:
Example Designs for Adi Tāla (2 Cycles = 64 mātras):
KORVAI DESIGN:
Specifications:

- Per statement: Not necessarily equal
- Use progressive purvāṅga
- Total: 64 mātras across 3 statements

Structure:
Purvāṅga pattern: $2, 3, 4, 5 = 14$ mātras
Uttarāṅga: 7 mātras
Statement 1: $14 + 7 = 21$ mātras
Statement 2: $14 + 7 = 21$ mātras
Statement 3: $14 + 8 = 22$ mātras (extra mātrā in uttarāṅga)
Total: $21 + 21 + 22 = 64$ mātras
Mathematical Principles:

- Arithmetic progression in purvāṅga
- Purvāṅga:Uttarāṅga ratio = $14 : 7 = 2 : 1$ (67%:33%)
- Adjustment in final statement to achieve exact cycle fit
- Division: $64 \div 3 = 21.33...$, so use 21, 21, 22

TIHAI DESIGN:
Specifications:

- Pattern length: 10 mātras
- Total: 64 mātras
- Type: Damdar tihai (with gaps)

Calculation:

$$3P + 2G = 64$$
$$3(10) + 2G = 64$$
$$30 + 2G = 64$$
$$G = 17 \text{ mātras}$$

Structure:
[10 mātras] gap[17] [10 mātras] gap[17] [10 mātras] → SAMA
Mathematical Principles:

- Linear equation solving: $3P + 2G = T$
- Large gaps (17 mātras) create dramatic pauses
- Exact division: $10 + 17 + 10 + 17 + 10 = 64$
- Gap is valid: $G = 17 > 0$ and G is integer

Key Differences in Design:

1. **Structural complexity**: Korvai has two-part structure allowing more variation; tihai has uniform repetition
2. **Mathematical approach**: Korvai uses sum decomposition ($P+U$); tihai uses gap calculation from total
3. **Flexibility**: Korvai allows progressive patterns in purvāṅga; tihai requires identical pattern repetitions
4. **Aesthetic**: Korvai builds up gradually; tihai provides rhythmic insistence through repetition

Both demonstrate the sophisticated mathematical thinking embedded in Carnatic rhythmic composition.

10 Summary and Key Formulas

10.1 Essential Definitions Recap

- **Tāla**: Cyclical rhythmic framework
- **Āvartana**: One complete cycle
- **Akṣara**: Fundamental beat unit
- **Mātrā**: Sub-beat unit

- **Sama**: First beat of cycle
- **Vibhāga/Aṅga**: Sections within cycle
- **Jāti**: Number of akṣaras in laghu (3,4,5,7,9)
- **Nadai/Gati**: Number of mātras per akṣara (3,4,5,7,9)
- **Korvai**: Two-part rhythmic pattern (purvāṅga + uttarāṅga)
- **Tihai**: Pattern repeated three times ending at sama

10.2 Key Mathematical Formulas

Cycle Length:

$$N_{\text{mātrā}} = N_{\text{akṣara}} \times \text{Nadai} \tag{34}$$

Aṅga Values:

$$\text{Anudhrutam (U)} = 1 \tag{35}$$
$$\text{Dhrutam (O)} = 2 \tag{36}$$
$$\text{Laghu (I)} = j \in \{3, 4, 5, 7, 9\} \tag{37}$$

35 Tāla System:

$$N_{\text{tāla}} = 7 \times 5 = 35 \tag{38}$$

175 Tāla System:

$$N_{\text{tāla+nadai}} = 7 \times 5 \times 5 = 175 \tag{39}$$

Korvai Structure:

$$r \times (P + U) = k \times N_{\text{cycle}}, \quad r = 3 \text{ typically} \tag{40}$$

Tihai Formula:

$$3P + 2G = T \tag{41}$$
$$G = \frac{T - 3P}{2} \tag{42}$$

Pattern Alignment:

$$n_{\text{cycles}} = \frac{\text{lcm}(L, N)}{N} \tag{43}$$

Arithmetic Progression Sum:

$$S = na + \frac{dn(n-1)}{2} \tag{44}$$

10.3 Common Tāla Structures

1. **Adi Tāla**: I O O, Chatusra jāti = 8 akṣaras
2. **Rūpaka Tāla**: O I, Chatusra jāti = 6 akṣaras
3. **Jhampa Tāla**: I U O, Miśra jāti = 10 akṣaras
4. **Miśra Chapu**: 7 akṣaras (3+2+2 grouping)
5. **Khanda Chapu**: 5 akṣaras (2+3 grouping)

11 Practice Problems

Question 15 (Short Answer): Calculate the number of mātras in one cycle of Dhruva Tāla with Khanda jāti in Miśra nadai.

Answer: Given:

- Tāla: Dhruva (I O I I)
- Jāti: Khanda (Laghu = 5 akṣaras)
- Nadai: Miśra (7 mātras per akṣara)

Calculation:

Number of akṣaras:

$$N_{\text{akṣara}} = 5 + 2 + 5 + 5 = 17 \text{ akṣaras}$$

Total mātras:

$$N_{\text{mātrā}} = 17 \times 7 = 119 \text{ mātras}$$

Answer: 119 mātras per cycle.

Question 16 (Medium Answer): A musician wants to create a tihai using an 8-mātrā pattern to fill exactly 2 cycles of Rūpaka Tāla (Chatusra jāti, Chatusra nadai). Calculate the required gap length.

Answer: Given:

- Pattern length: $P = 8$ mātras
- Rūpaka Tāla: 6 akṣaras
- Chatusra nadai: $6 \times 4 = 24$ mātras per cycle
- 2 cycles: $T = 2 \times 24 = 48$ mātras

Calculation:

Using tihai formula:

$$3P + 2G = T$$
$$3(8) + 2G = 48$$
$$24 + 2G = 48$$
$$2G = 24$$
$$G = 12 \text{ mātras}$$

Structure: [8] gap[12] [8] gap[12] [8] $\rightarrow$ SAMA

Verification:

$$8 + 12 + 8 + 12 + 8 = 48 \text{ mātras}$$

Answer: Gap length = **12 mātras**.

Question 17 (Long Answer): Design a complete korvai for Khanda-jāti Aṭa Tāla in Chatusra nadai, spanning 2 complete cycles. Use a Fibonacci-like sequence for the purvāṅga and explain all mathematical steps.

Answer: Step-by-Step Design:

1. Determine Cycle Structure

Tāla: Aṭa = I I O O
Jāti: Khanda (Laghu = 5 akṣaras)
Akṣaras per cycle: $5 + 5 + 2 + 2 = 14$ akṣaras
Nadai: Chatusra (4 mātras per akṣara)
Mātras per cycle: $14 \times 4 = 56$ mātras
Total for 2 cycles: $2 \times 56 = 112$ mātras
2. Design Korvai Structure
For 3 repetitions: $112 \div 3 = 37.33...$ mātras per statement
Use: 37, 37, 38 mātras (or 38, 37, 37)
3. Create Fibonacci-like Purvāṅga
Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21...
Modified for our purpose: Use 2, 3, 5, 8, 13
Sum: $2 + 3 + 5 + 8 + 13 = 31$ mātras
But this is for the entire purvāṅga repeated once. For better structure, use:
3, 5, 8 = 16 mātras per statement
4. Calculate Uttarāṅga
For statements of 37 and 38 mātras:
Statement 1: $P + U = 37$

$$U_1 = 37 - 16 = 21 \text{ mātras}$$

Statement 2: $P + U = 37$

$$U_2 = 37 - 16 = 21 \text{ mātras}$$

Statement 3: $P + U = 38$

$$U_3 = 38 - 16 = 22 \text{ mātras}$$

5. Uttarāṅga Structure
Break uttarāṅga into 3 repetitions: - Statements 1 2: $21 \div 3 = 7$ mātras each - Statement 3: Use $7, 7, 8$ to get 22 mātras total
6. Complete Korvai
Statement 1:

- Purvāṅga: (3, 5, 8) = 16 mātras
- Uttarāṅga: (7, 7, 7) = 21 mātras
- Total: 37 mātras

Statement 2:

- Purvāṅga: (3, 5, 8) = 16 mātras
- Uttarāṅga: (7, 7, 7) = 21 mātras
- Total: 37 mātras

Statement 3:

- Purvāṅga: (3, 5, 8) = 16 mātras

- Uttarāṅga: (7, 7, 8) = 22 mātras
- Total: 38 mātras

7. Verification
Total: $37 + 37 + 38 = 112$ mātras
Number of cycles: $112 \div 56 = 2$ cycles
8. Ratio Analysis
Purvāṅga:Uttarāṅga ratio:
For statements 1 2: $16 : 21 \approx 43 : 57$
For statement 3: $16 : 22 \approx 42 : 58$
Both ratios are within acceptable range (typically 30:70 to 70:30).
9. Mathematical Properties

- Uses Fibonacci-like progression: each term roughly sum of previous two
- Progression: 3, 5, 8 (where $5 \approx 3 + 2$, $8 \approx 5 + 3$)
- Symmetric uttarāṅga with slight variation in third statement
- Exact cycle completion through calculated adjustment

Conclusion:
This korvai demonstrates:

1. Integration of mathematical sequences (Fibonacci)
2. Precise calculation for cycle completion
3. Balance between mathematical structure and musical aesthetics
4. Flexibility in adjusting final statement for exact fit

A Quick Reference Tables

A.1 Sapta Tāla Quick Reference

Tāla	Structure	Tisra	Chatusra	Khanda	Miśra	Saṅkīrṇa	Default
Dhruva	I O I I	11	14	17	23	29	C
Matya	I O I	8	10	12	16	20	C
Rūpaka	O I	5	6	7	9	11	C
Jhampa	I U O	6	7	8	10	12	M
Tripuṭa	I O O	7	8	9	11	13	T (Adi)
Aṭa	I I O O	10	14	18	26	34	K
Eka	I	3	4	5	7	9	C

Table 4: Total Akṣaras for Each Tāla-Jāti Combination

Note: Default column shows default jāti - T=Tisra, C=Chatusra, K=Khanda, M=Miśra

Tāla/Nadai	**Tisra (×3)**	**Chatusra (×4)**	**Khanda (×5)**	**Miśra (×7)**	**Saṅkīrṇ**
Adi (8 akṣ)	24	32	40	56	72
Rūpaka (6 akṣ)	18	24	30	42	54
Jhampa (10 akṣ)	30	40	50	70	90
Miśra Chapu (7 akṣ)	21	28	35	49	63
Khanda Chapu (5 akṣ)	15	20	25	35	45

Table 5: Total Mātras for Common Tāla-Nadai Combinations

A.2 Nadai Multiplier Table

B Conclusion

Unit 3 has explored the sophisticated mathematical structures underlying rhythm in Indian classical music, particularly the Carnatic tāla system. From the fundamental concepts of mātrā and akṣara to the complex combinatorial patterns in korvais and tihais, we have seen how mathematical principles enable infinite rhythmic creativity within structured frameworks.

Key insights include:

- The hierarchical organization of time in Indian music (mātrā → akṣara → āvartana)
- The systematic enumeration of 35 (and theoretically 175) tālas through combinatorial mathematics
- The mathematical elegance of korvai construction with purvāṅga-uttarāṅga balance
- The precise formulation of tihais: $3P + 2G = T$
- Algorithmic approaches to rhythmic composition using number theory
- Applications of modular arithmetic in pattern alignment and graha-bheda

The Carnatic tāla system demonstrates remarkable mathematical sophistication comparable to any Western theoretical framework. Its combination of rigid mathematical structure with creative flexibility provides a powerful model for understanding how abstract mathematics can serve artistic expression. The system's reliance on combinatorics, number theory, and algorithmic thinking reveals rhythm as a domain where mathematical elegance and musical beauty converge.

The study of tāla also illustrates broader principles applicable beyond music: how finite elements can generate infinite combinations, how constraints enable rather than limit creativity, and how mathematical systems can be simultaneously rigorous and flexible.

References and Further Reading

1. Benson, D. J. (2006). *Music: A Mathematical Offering*. Cambridge University Press.
2. Loy, G. (2006). *Musimathics: The Mathematical Foundations of Music* (Vols. 1 & 2). MIT Press.

3. Fauvel, J., Flood, R., & Wilson, R. (Eds.). (2003). *Music and Mathematics: From Pythagoras to Fractals*. Oxford University Press.

4. Nelson, D. P. (2008). *Solkattu Manual: An Introduction to the Rhythmic Language of South Indian Music*. Wesleyan University Press.

5. Balaji, M. (2024). *A Practical Guide to South Indian Rhythm through Konnakkol*.

6. Clayton, M. (2000). *Time in Indian Music: Rhythm, Metre, and Form in North Indian Rāg Performance*. Oxford University Press.

7. Viswanathan, T., & Allen, M. H. (2004). *Music in South India: The Karnatak Concert Tradition and Beyond*. Oxford University Press.