

The toric code is a wonderful model!

1. It is exactly solvable (we can exactly diagonalize the model)!

2. It's really easy to work with!

3. It's a quantum error-correcting code!

4. It's a quantum gauge theory!

5. It's a topological phase!

It's very amazing how much you can learn from such a simple model! Let's dive in!

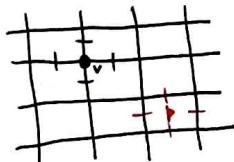
1. The model

We define the toric code Hamiltonian

$$H_{TC} = - \sum_v A_v - \sum_p B_p, \text{ where } A_v = \prod_{i \in v} \tau_i^z,$$

on a square lattice in 2 spatial dimensions:

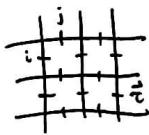
$$B_p = \prod_{i \in p} \tau_i^x$$



$$\begin{array}{c} \tau_2^x \\ \square \\ \tau_1^x \quad op \quad \tau_3^x \\ \tau_4^x \end{array} \quad B_p = \tau_1^x \tau_2^x \tau_3^x \tau_4^x \quad \text{"plaquette" operator}$$

$$\begin{array}{c} \tau_1^z \\ + \\ \tau_2^z \\ \tau_3^z \\ \tau_4^z \end{array} \quad A_v = \tau_1^z \tau_2^z \tau_3^z \tau_4^z \quad \text{"vertex" operator}$$

Here, we've put spin $\frac{1}{2}$ d.o.f. on the edges of the square lattice, labelled by $\vec{\tau}_i$.



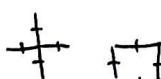
$$[\tau_i^a, \tau_j^b] = i \delta_{ij} \epsilon^{abc} \tau_i^c$$

$i, j \in$ edges of lattice

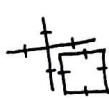
$a, b, c \in x, y, z$

Notice that all terms in H commute: $[A_v, A_{v'}] = [B_p, B_p'] = [A_v, B_p] = 0 \quad \forall v, v', p, p'$.

Why is this? $\{\tau_i^a, \tau_j^b\} = 2 \delta_{ij} \delta^{ab}$, so:



Well-separated A_v 's, B_p 's commute, since there's no overlap.



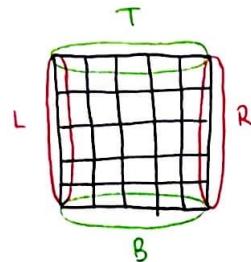
A_v & B_p will overlap at exactly 2 sides if they overlap at all, so you get 2 anticommutations = 2 minus signs = $(-1)^2 = +1$ when going from $A_v B_p$ to $B_p A_v$.

This model is exactly solvable, i.e. all terms in H mutually commute, and we can use their eigenvalues to therefore label states in \mathcal{H} .

(2)

One question is: do we have enough operators to specify all states in the Hilbert space \mathcal{H} ? i.e. By specifying all eigenvalues of A_v and B_p , as $\{|a_v\rangle, |b_p\rangle\}$, does this uniquely determine the state? There could presumably be other quantum numbers attached to the state. We could also have too many A_v & B_p operators, which will turn out to be the case.

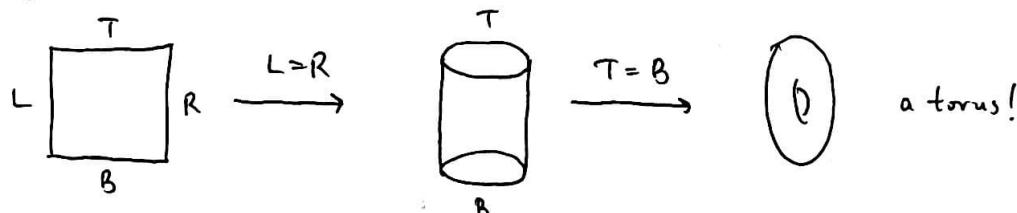
To make progress, let's choose boundary conditions to be periodic:



Identify left and right edges with each other
 $L = R$

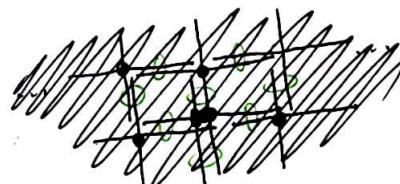
Identify top and bottom edges with each other
 $T = B$

Basically, this is like folding the square lattice into:



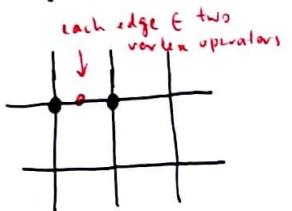
This is why it's called the toric code.

With this, let's imagine that we've specified all eigenvalues of A_v and B_p , as a_v and b_p . First, notice that multiplying all A_v 's together gives 1 , since every edge appears in two A_v 's:

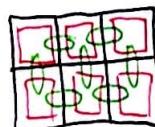


every edge appears twice!

$$\prod_v A_v = 1$$



The same exact thing holds for the plaquette operators!



$$\prod_p B_p = 1$$

This means that the eigenvalues must also satisfy this condition, $\prod_v a_v = \prod_p b_p = 1$.

So not all of the operators are independent, i.e. by knowing all but 1 of the A_v 's, you know the last A_v , and similarly for the B_p 's.

३

Let's do some counting. On an $L \times L$ lattice, there are L^2 vertices and plaquettes, and $2L^2$ edges.

$$\text{Number of d.o.f.} = \text{number of edges} = 2L^2$$

$$\Rightarrow \dim \mathcal{H} = 2^{\# \text{ of rows}} = 2^{2L^2}$$

$$\text{Number of distinct values of } \{a_v, b_p\} \text{ (subject to } \prod_v a_v = \prod_p b_p = 1) = 2^{(L^2-1)+(L^2-1)}$$

So given $\{a_p, b_p\}$, there's a Hilbert space of dimension $\frac{2^{2^2}}{2^{2(2^2-1)}} = 4$ left over. This means that knowing $\{a_p, b_p\}$ is not enough to nail down the state.

Consider the ground state(s): $H = -\sum_v A_v - \sum_p B_p$ is minimized when $A_v = +1 \geq b_p \forall v, p$.

This 4-fold degeneracy means that there are 4 ground states!

2. Constructing the ground states

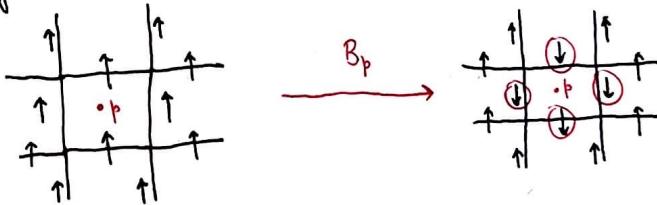
To construct the ground states, let's start with a reference configuration:

$$= \langle \text{all } \uparrow \rangle$$

Here, all A_{ij} 's are +1.

But this is not an eigenstate of the B_p 's.

What does B_p do?



$$\text{because } \tau_i^x |\uparrow_i\rangle = |\downarrow_i\rangle$$

$$\tau_i^x |\downarrow_i\rangle = |\uparrow_i\rangle$$

so B_p flips the spins around a plaquette.

Say $H = -\sum A_p - B_p$ for just one of the B_p 's. A ground state would be an equal weight superposition of our two configurations:

$$\frac{1}{\sqrt{2}} \left(| \# \# \# \# \rangle + | \# \# \# \# \# \rangle \right)$$

$$\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} = | \uparrow \rangle \\ = | \downarrow \rangle \end{array} \right)$$

For two Bp's, it'll look like this:

$$\frac{1}{2} \left(| \# \# \# \rangle + | \# \# \# \# \rangle + | \# \# \# \# \# \rangle + | \# \# \# \# \# \# \rangle \right)$$

(4)

and so the pattern becomes

$$|g.s.\rangle = |\begin{array}{|c|c|} \hline \# & \# \\ \hline \# & \# \\ \hline \end{array}\rangle + |\begin{array}{|c|c|} \hline \# & \# \\ \hline \# & \square \\ \hline \end{array}\rangle + |\begin{array}{|c|c|} \hline \# & \# \\ \hline \square & \# \\ \hline \end{array}\rangle + |\begin{array}{|c|c|} \hline \# & \# \\ \hline \# & \# \\ \hline \end{array}\rangle + \dots$$

= all possible loop configurations superposed

We can formally write this as $|g.s.\rangle = \prod_p \frac{1+B_p}{2} |all\uparrow\rangle$, since $\frac{1+B_p}{2}$ is a projector onto the $B_p=+1$ subspace. And that's exactly this loop superposition if you expand it out!

What about the other three ground states?

Note that B_p 's acting on a state only produce closed loops. So if we start with an $|all\uparrow\rangle$ configuration, the winding number of \vec{J} 's around any noncontractible handle of the torus will be preserved.

If we start with something like this:

$$|\textcircled{6}\rangle$$

then doing $\prod_p \frac{1+B_p}{2}$ on it gives this:

$$|\textcircled{6}\rangle + |\textcircled{5}\rangle + |\textcircled{4}\rangle + \dots$$

which is a distinct ground state.

So the 4 ground states are:

$$\prod_p \frac{1+B_p}{2} |\textcircled{6}\rangle = |\{a_v=1, b_p=1\}, n_x=0, n_y=0\rangle \text{ which are } \underline{\text{topologically distinct!}}$$

$$\prod_p \frac{1+B_p}{2} |\textcircled{5}\rangle = |\{a_v=1, b_p=1\}, n_x=1, n_y=0\rangle$$

$$\prod_p \frac{1+B_p}{2} |\textcircled{4}\rangle = |\{a_v=1, b_p=1\}, n_x=0, n_y=1\rangle$$

$$\prod_p \frac{1+B_p}{2} |\textcircled{3}\rangle = |\{a_v=1, b_p=1\}, n_x=1, n_y=1\rangle$$

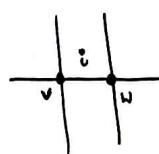
3. A Topological Phase of Matter

What is a particle? Anything that carries a quantum number.

What's a quantum number? Something you can measure.

We'll now discuss the excited states of the toric code, and their interpretation as charges and fluxes.

Consider doing $\tau_i^x |\circlearrowleft\rangle \rightarrow \cancel{\text{what}}$

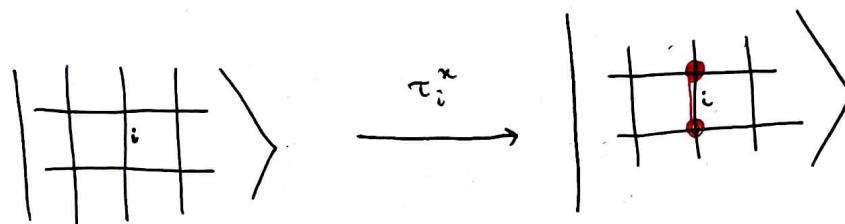


$$\{\tau_i^x, A_v\} = \{\tau_i^x, A_w\} = 0$$

so $A_v |\circlearrowleft\rangle = 1$ means that

$$A_v \tau_i^x |\circlearrowleft\rangle = -\tau_i^x A_v |\circlearrowleft\rangle = -\tau_i^x |\circlearrowleft\rangle$$

so ~~what~~ $\tau_i^x |\circlearrowleft\rangle$ has $a_v = a_w = -1$.

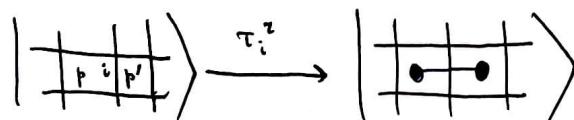


All a_v 's = +1

All b_p 's = +1

Two of the a_v 's now = -1
(indicated by 0's)
All b_p 's = +1

The same thing happens for $\tau_i^z |\circlearrowleft\rangle$, except that x and z are flipped (and A_v, B_p are flipped):



All b_p 's = +1

All a_v 's = +1

Two of the b_p 's now = -1

(indicated by 0's)

All a_v 's = +1

changes
magnetic fluxes

We'll refer to τ_i^x excitations (vertex excitations) as e , and τ_i^z excitations (plaquette $\frac{1}{2}$ excitations) as m .

This is because A_v is a dot-like gauss' law

If $\tau_i^z \sim e^{i\vec{E}}$, then $A_v \sim e^{i\nabla \cdot \vec{E}}$, so $A_v = a_v \Rightarrow e^{i(\nabla \cdot \vec{E})/2} = e^{ip}$.

So $a_v = +1 \Rightarrow \nabla \cdot \vec{E} = 0$ (no charge)

And $a_v = -1 \Rightarrow \nabla \cdot \vec{E} = \pi$ (π charge)

⑥

And B_p is like flux:

$$\text{If } \tau_i^x \sim e^{i\hat{\phi}}, \text{ then } B_p \begin{array}{c} \text{square} \\ \text{with} \\ \text{+ sign} \end{array} \sim e^{i\oint \hat{\phi} \cdot d\vec{a}} = e^{i \oint \vec{B} \cdot d\vec{a}}, \text{ so } B_p = b_p$$

$$\rightarrow e^{i \oint \vec{B} \cdot d\vec{a}} = e^{i \Phi_B}.$$

$$\text{So } \cancel{b_p = +1} \Rightarrow \cancel{\Phi_B = 0} \quad (\text{no flux})$$

$$\text{And } b_p = -1 \Rightarrow \cancel{\Phi_B = \pi} \quad (\pi \text{ flux})$$

There's a really good reason for thinking about the model as a \mathbb{Z}_2 version of electromagnetism, which we'll talk about later.

These e and m particles have important properties:

$$1. \text{ They're both bosons (Why? } [\tau_i^x, \tau_i^x] = [\tau_i^z, \tau_i^z] = 0, \text{ so } \frac{e}{e} \left| \begin{array}{c} e \\ e \end{array} \right\rangle = \left| \begin{array}{c} e \\ e \end{array} \right\rangle$$

and similarly for m). ~~(Why?~~

$$\begin{array}{c} \diagup \quad \diagdown \\ a \quad b \end{array} = \theta_{ab} \begin{array}{c} \diagup \quad \diagdown \\ a \quad b \end{array}$$

$$2. \text{ They have a phase of } -1 \text{ when braiding } e \text{ around } m.$$

$$(\text{Why? } \{ \tau^x, \tau^z \} = 0, \text{ so } \frac{e}{m} \left| \begin{array}{c} m \\ m \end{array} \right\rangle = (-1) e \left| \begin{array}{c} m \\ m \end{array} \right\rangle)$$

$$\begin{array}{c} \diagup \quad \diagdown \\ e \quad m \end{array} = \theta_{em} \begin{array}{c} \diagup \quad \diagdown \\ e \quad m \end{array}. \text{ This is the Aharonov-Bohm phase } (\mathbb{Z}_2).$$

$$\begin{array}{c} \diagup \quad \diagdown \\ a \quad a \end{array} = \theta_{aa} \begin{array}{c} \diagup \quad \diagdown \\ a \quad a \end{array}$$

$$3. 4 = e \times m \text{ (think of a bound state of } e \text{ and } m \text{ that move together) is a fermion!}$$

$$\begin{array}{c} \diagup \quad \diagdown \\ e \quad m \end{array} = \theta_{ee} \begin{array}{c} \diagup \quad \diagdown \\ e \quad m \end{array}$$

$$= \theta_{ee} \theta_{mm} \begin{array}{c} \diagup \quad \diagdown \\ e \quad m \end{array}$$

$$= \theta_{ee} \theta_{mm} |\theta_{em}|^2 \begin{array}{c} \diagup \quad \diagdown \\ e \quad m \end{array}$$

$$= 1 \cdot 1 \cdot (-1) = -1$$

So exchanging two ψ 's gives a -1 phase.

The e, m, ψ are called "anyons." In general, exchanging 2 particles doesn't need to give ± 1 in 2+2 spatial dimensions. This has to do with the fact that you can unbind worldlines in 3+1d but not in 2+1d.

(in 3+1d, you have an extra dimension to unwrap the worldlines with).

Representations of the braid group are related to possible anyons.

braids can't be unwound in 2+1d.



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4. The ground states are topological

The reason why the toric code is interesting is because information can be protected under weak measurements/perturbations. This really comes from the ground states being topologically distinct.

Consider doing $H_{TC} \rightarrow H_{TC} - \lambda \sum_i \tau_i^x$ for some small λ . Here, we're imagining that the environment adds some local, weak perturbations $-\lambda \tau_i^x$. Let's imagine how this'll affect the ground states. If the ground states (which are orthogonal at $\lambda=0$) get some overlap that's large, then trying to encode information into the ground states of the toric code will be hard. But intuitively, the topology of the ground states should help.

In perturbation theory, roughly,

$$|\circlearrowleft\rangle = |\circlearrowleft^{(0)}\rangle + \lambda |\circlearrowleft^{(1)}\rangle + \lambda^2 \left(|\circlearrowleft^{(2)}\rangle + |\circlearrowleft^{(3)}\rangle \right) + \dots$$

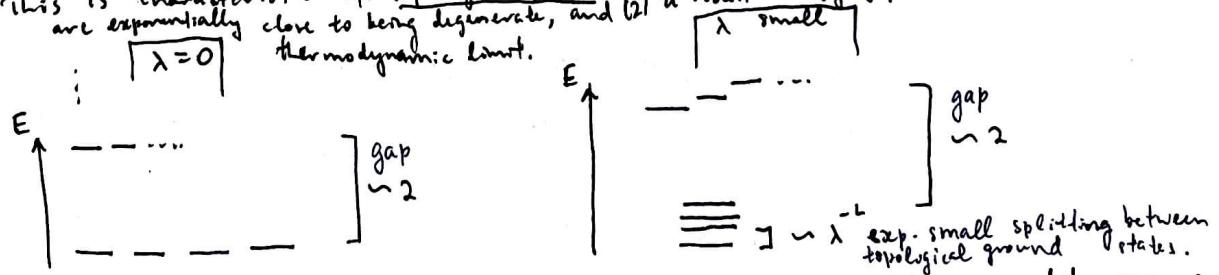
where $|\circlearrowleft^{(0)}\rangle$ is the ground state of H_{TC} , and $|\circlearrowleft\rangle$ is the ground state of H .

The overlap between different (exact) ground states is, e.g.,

$$\langle \circlearrowleft | \circlearrowleft \rangle = \left(\langle \circlearrowleft^{(0)} | + \lambda \langle \circlearrowleft^{(1)} | + \dots + \lambda^{L_x} \langle \circlearrowleft^{(L_x)} | + \dots \right) \times \left(|\circlearrowleft^{(0)}\rangle + \lambda |\circlearrowleft^{(1)}\rangle + \dots + \lambda^{L_x} |\circlearrowleft^{(L_x)}\rangle + \dots \right)$$

$$\approx \lambda^{L_x} = e^{-L_x \ln \lambda}$$

So the ground states split by an amount exponentially small from system size. This is characteristic of topological phases, which have (1) topological ground states that are exponentially close to being degenerate, and (2) a robust energy gap to excited states in the thermodynamic limit.

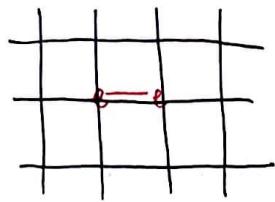


Indeed, information stored as the ground states can be robustly encoded, even in the presence of arbitrary local perturbations!

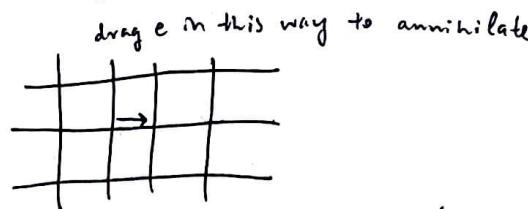
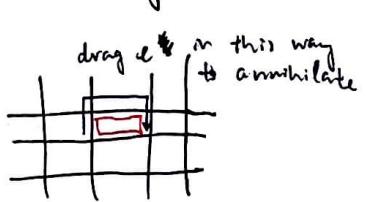
However, you might argue that going to any excited state will also destroy the information. You'd be correct, but those errors are easy to correct.

⑧

Consider, for example, creating some e 's at lowest order in perturbation theory.



All we know, really, is the position of the two e 's, since the ground state has fluctuating strings/loops. So we can close the loop however we want, so long as we don't accidentally wind the e 's all the way around.



This can be done without creating an error up to $O(\lambda^{\frac{L}{2}})$.

5. The Toric Code as a Gauge Theory

Whenever you hear the words "gauge symmetry", replace them with "local symmetry" (or "local redundancy") — that's really what it means to be a gauge theory.

i.e. gauge = local.

Consider the models $\begin{cases} H_0 = -\sum_v A_v = -\sum_v \prod_{i \in v} \tau_i^z \\ H = -\sum_v A_v - \lambda \sum_i \tau_i^x \end{cases}$. This has a local symmetry, that is $[H, B_p] = 0 \quad \forall p$ ($B_p = \prod_{i \in p} \tau_i^x$). Such local symmetries are really redundancies in our description, since my choice of B_p shouldn't affect ~~which~~ your measurement if they are truly symmetries. To remove this massive redundancy (there are 2^{LxL} equivalent states, equivalent by all possible B_p transformations),

we gauge fix. We can do this by, e.g., choosing a gauge where $B_p = 1 \forall p$.

This amounts to choosing $\begin{cases} H_0 = -\sum_v A_v - \infty \sum_p B_p \\ H = -\sum_v A_v - \infty \sum_p B_p - \lambda \sum_i \tau_i^x \end{cases}$, which is -- almost the toric code --

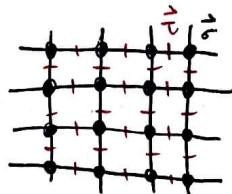
In this language, having the coefficient of $\sum_p B_p$ not be ∞ doesn't really make sense — the toric code as a model really moves beyond this restriction.

⑨

5.1 changing the transverse field Ising model

Consider $H = -h \sum_v \sigma_v^z - J \sum_{\langle vw \rangle} \sigma_v^z \sigma_w^z$, the transverse field Ising model, on a square lattice in 2d. This has a global symmetry, $U = \prod_v \sigma_v^z$, i.e. $[U, H] = 0$.

We gauge H by adding degrees of freedom ~~and~~ and modifying our symmetry so that the symmetry becomes local.



Add d.o.f. τ_i^z on the links

Modify $U = \prod_v \sigma_v^z$ to $U(v) = \sigma_v^z \times \text{some } \tau_i^z$ near v

If we only do $U(v) = \sigma_v^z$, then $[U(v), H] \neq 0$. This makes sense — the Ising model is not a gauge theory.

The idea is that the problematic term is $\sigma_v^z \sigma_w^z$, since σ_i^z anticommutes with it.

What if we add something on the link $\langle vw \rangle$ that anticommutes with $\sigma_v^z \sigma_w^z$? Well, as it is, that'd have to be adding σ_v^z or σ_w^z , but that'd be modifying the σ_i^z part of H . This is why we introduce d.o.f. on the links.

If we do $\sigma_v^z \sigma_w^z \rightarrow \sigma_v^z \tau_{vw}^z \sigma_w^z$, that adds a d.o.f. on the link that can

$$\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & & \\ \hline & & & \\ \hline \end{array} \right) \rightarrow \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & & \\ \hline & & \times & \\ \hline \end{array} \right)$$

now potentially introduce our extra sign^z provided that we modify $U(v)$.

To modify $U(v)$, we want $U(v)$ to commute with $\sigma_v^z \tau_{vw}^z \sigma_w^z + \langle vw \rangle$, so we

take the simplest choice:

$$U(v) = \sigma_v^z \rightarrow U(v) = \sigma_v^z \prod_{i \in v} \tau_i^z$$

$$\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & & \\ \hline & & & \\ \hline \end{array} \right) \rightarrow \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & & \\ \hline & & \times & \\ \hline \end{array} \right)$$

To be systematic, we do this:

$$\text{First, } \left[\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & \times & \\ \hline & & & \\ \hline \end{array} \right] \neq 0$$

$$\text{but } \left[\begin{array}{|c|c|c|c|} \hline & & z & \\ \hline & \bullet & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & x & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right] = 0 \Rightarrow \text{modify } \frac{z}{\bullet} \text{ to } \frac{z}{\circ}$$

$$\text{Next, } \left[\begin{array}{|c|c|c|c|} \hline & & z & \\ \hline & \bullet & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & x & \\ \hline & & & \\ \hline & & \times & \\ \hline \end{array} \right] \neq 0$$

$$\text{but } \left[\begin{array}{|c|c|c|c|} \hline & & z & \\ \hline & \bullet & & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & x & \\ \hline & & & \\ \hline & & \times & \\ \hline \end{array} \right] = 0 \Rightarrow \text{modify } \frac{\circ}{\bullet} \text{ to } \frac{\circ}{\circ}$$

and so on.

Doing this gives:

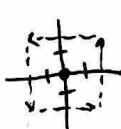
$$U(v) = \frac{\sigma_v^2 + \tau_v^2}{\tau_v^2 + \tau_v^2} = \sigma_v^2 \prod_{i \in v} \tau_i^2 = \sigma_v^2 A_v$$

This is a symmetry of our Hamiltonian that, right now, looks like

$$H = -h \sum_v \sigma_v^2 - J \sum_{(vw)} \sigma_v^x \tau_{vw}^x \sigma_w^x$$

Let's think about what A_v means. Earlier, we said that if we think of τ_i^2 as e^{iE} , then

$$A_v = \prod_i \tau_i^2 = e^{i \oint \vec{E} \cdot d\vec{a}} = e^{i \int \nabla \cdot \vec{E}}$$



Now, if we think of σ_v^2 as $\tau_v^2 = e^{-ip}$, then $U(v) = e^{i \int \nabla \cdot \vec{E} - p}$
the exponential of clamp' law.

Having clamp' law in a gauge theory is non-negotiable — it is a physical equation of the system that relates matter charges (the σ 's) to gauge field configurations (the τ 's). So if we want to interpret our model as a gauge theory, we need gauge' law. We impose this energetically in H by adding it with a large negative coefficient.

$$H = -h \sum_v \sigma_v^2 - J \sum_{(vw)} \sigma_v^x \tau_{vw}^x \sigma_w^x - \Delta_v \sum_v \underbrace{\sigma_v^2 \prod_{i \in v} \tau_i^2}_{\text{large}}$$

And, since this has a gauge symmetry, we gauge fix to obtain:

$$H = -h \sum_v \sigma_v^2 - J \sum_{(vw)} \sigma_v^x \sigma_w^x \tau_{vw}^x - \Delta_v \sum_v \underbrace{\sigma_v^2 \prod_{i \in v} \tau_i^2}_{\text{large}} - \Delta_p \sum_p \prod_{i \in p} \tau_i^x$$

To get the toric code, we go away from thinking about this rigidly as a gauge theory, and we instead allow ourselves to tune the coefficients arbitrarily.

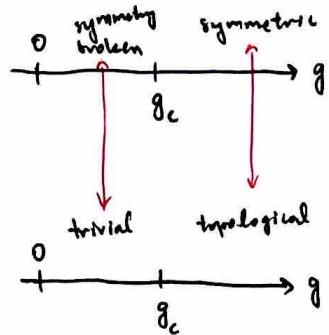
$b_f \rightarrow \infty$: Basically, H becomes const. $-H_{TC}$ in this limit, and is the toric code.

$\frac{h}{J} \rightarrow 0$: Here, $H = -J \sum_{(vw)} \sigma_v^x \sigma_w^x \tau_{vw}^x - \Delta_v \sum_v \sigma_v^2 A_v - \Delta_p \sum_p B_p$. With some inspiration, it's easy to see that there are no constraints between the operators, so this limit has 1 ground state, and is not a topological phase. We call this a "trivial" phase.

The other limits aren't as important.

(11)

Let $g = \frac{h}{J}$. We have the phase diagrams:



(Ungauged) transverse field Ising model.

(Gauged) Toric code/trivial.

This reflects a general fact: phases of the ungauged theory are in 1-to-1 correspondence with phases of the gauged theory. This can be understood by thinking about the evolution from the ungauged to the gauged theory as adiabatic evolution. Since everything's gapped, this evolution is well-defined and won't introduce any sort of phase transition. That means a finite-depth unitary circuit here, and is why this correspondence is true.

5.2 Dualizing the transverse field Ising model

We'll get the toric code in a different way: rewriting the transverse field Ising model in terms of domain walls.

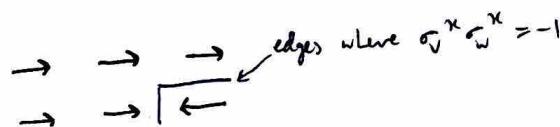
Consider $H = -h \sum_v \sigma_v^z - J \sum_{\langle vw \rangle} \sigma_v^x \sigma_w^x$. If we're in a state



(Spins drawn at an angle for clarity)

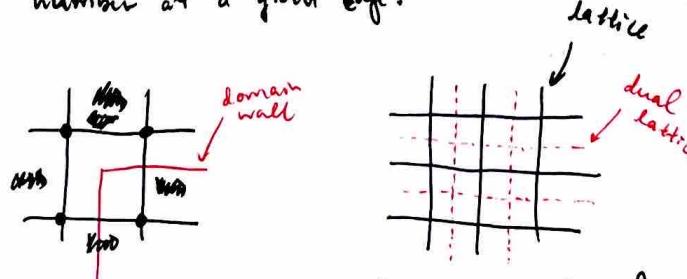
then flipping a spin will create a domain wall, which forms a closed loop.

Basically, a domain wall is just an edge where $\sigma_v^x \sigma_w^x = -1$, and they must form closed loops on the



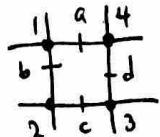
"dual" lattice.

We'll call (suggestively) $\tau_{vw}^x \tau_{vw}^x = \tau_{vw}^z$ as the domain wall occupation number at a given edge.



Note that no matter what domain wall configuration we choose, the product of τ_{vw}^z 's around a plaquette is always 1.

(12) Thus is because $\tau_a^z \tau_b^z \tau_c^z \tau_d^z = \sigma_4^x \sigma_1^x \cdot \sigma_1^x \sigma_2^x \cdot \sigma_2^x \sigma_3^x \cdot \sigma_3^x \sigma_4^x = 1$.



So thinking about domain walls instead of spins gives us the constraint that

$A_p = \prod_{i \in p} \tau_i^z = 1 \vee p$. This is the analogue of gauge law — field lines (domain walls) can only end on charges, and there are no charges here.

Now, how does σ_v^z map to the domain wall picture?

$$\sigma_v^z \left| \begin{array}{c} \text{grid with arrows} \\ \text{all pointing right} \end{array} \right\rangle = \left| \begin{array}{c} \text{grid with arrows} \\ \text{some pointing left, some right} \end{array} \right\rangle$$

It creates domain walls by taking $\tau_{vw}^z \rightarrow -\tau_{vw}^z \forall w \in \text{n.n.}(v)$ (for all vertices w that are nearest neighbors of v). What operator does this? Exactly this:

$$B_v = \prod_{i \in v} \tau_i^z.$$

So in terms of domain walls, $H = -h \sum_v \sigma_v^z - J \sum_{\langle vw \rangle} \sigma_v^x \sigma_w^x$ becomes

$$H = -h \sum_v B_v - J \sum_i \tau_i^z \text{ with } A_p = 1 \vee p.$$

We'll rewrite this on the dual lattice, where plaquettes \leftrightarrow vertices. Then,

$$H = -h \sum_p B_p - J \sum_i \tau_i^z \text{ with } A_v = 1 \vee v.$$

"DUALITY"!

There's a good reason for mapping to dual lattice when doing this procedure — it's more general.

Basically, $H = -h \sum_p B_p - J \sum_i \tau_i^z - \infty \sum_v A_v$. And moving away from the

need for A_v to be 1, we again obtain the toric code!

This is a duality: an equivalent description of the same Hamiltonian.

There's a reason why dualizing and gauging gives the same thing when acting on systems with global symmetries: it's really the only thing you can do. This behavior is indeed more general.