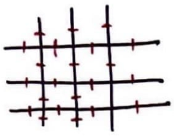
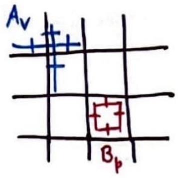
Spontaneous Symmetry Breaking in the Toric Code





v = vertex
p = plaquette
i, j = edges

Put spin $\frac{1}{2}$ d.o.f. on the edges i of a square lattice, and define

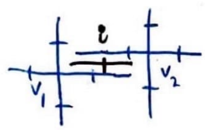
$$H_{TC} = -\sum_v \prod_{i\in v} \tau_i^z - \sum_p \prod_{i\in p} \tau_i^x$$

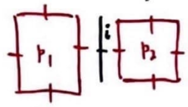
Note $[A_v, B_p] = 0$

as the toric code Hamiltonian.

The excitations are $\underbrace{\tau_i^x |\text{gnd.}\rangle}_{\text{create } e \text{ particles at } v_1, v_2}$, $\underbrace{\tau_i^z |\text{gnd.}\rangle}_{\text{creates } m \text{ particles at } p_1, p_2}$

$(A_{v_1} = -1, A_{v_2} = -1)$



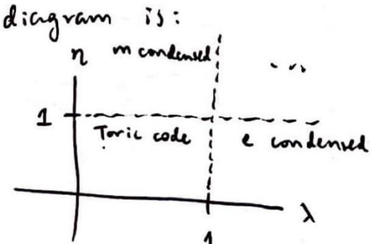


$(B_{p_1} = -1, B_{p_2} = -1)$

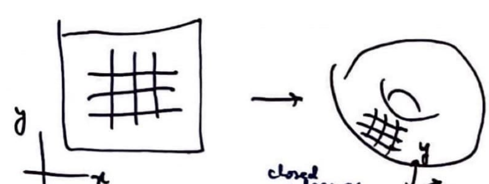
Now, let's modify H:

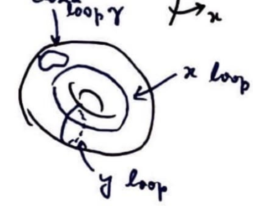
$$H = H_{TC} - \lambda \sum_i \tau_i^x - \eta \sum_i \tau_i^z$$

The phase diagram is:



Put H on a torus





Let $W_e^x = \prod_{i\in x \text{ loop}} \tau_i^x$, $W_m^x = \prod_{i\in y \text{ loop}} \tau_i^z$, similarly for W_e^y, W_m^y.

Toric code phase

Here, λ is small (we'll ignore η), and there are 4 ground states if $\lambda = 0$. These are labelled by the eigenvalues of $W_e^{x,y}$, and $W_m^{x,y}$ act as operators that have nonzero matrix elements between ground states.

$|\pm1, \pm1\rangle$ has $W_m^x |1, 1\rangle = |1, -1\rangle$ since $W_e^y(W_m^x|1,1\rangle) = -W_m^x|1,1\rangle$

W_e^x eigenvalue, W_e^y eigenvalue

$\Rightarrow$ 4 ground states

and similarly for the others.

Therefore, $\langle W_e^{x,y}\rangle = \pm1$ in the ground state (in an appropriately chosen basis), and $\langle W_m^{x,y}\rangle = \pm1$ (in a different appropriately chosen basis).

If we treat the Wilson loop as a (generalized) order parameter, we see that the toric code corresponds to spontaneously broken 1-form electric and magnetic symmetries.

$$\langle W_e^i W_e^j \rangle = \langle W_e^i \rangle \langle W_e^j \rangle (= \pm 1)$$

$\boxed{\text{order param.} \neq 0 \text{ phase}}$

and $\langle W_m^i W_m^j \rangle = \langle W_m^i \rangle \langle W_m^j \rangle (= \pm 1)$

Consider $\langle W_m^\gamma \rangle$ for a closed (contractible) loop γ. For small λ, we can calculate it perturbatively in powers of λ.

$$|\psi\rangle = |\psi^{(0)}\rangle + \frac{1}{E^{(0)} - H^{(0)}} \Big(1 - \sum_{\substack{n \text{ degen.} \\ \text{with } \psi^{(0)}}} |n^{(0)}\rangle\langle n^{(0)}|\Big) H^{(1)} |\psi^{(0)}\rangle + \cdots$$

$|\psi^{(0)}\rangle$ = $\bullet$

$H^{(0)} = H_{TC}$

$H^{(1)} = -\lambda \sum_i \tau_i^x$

$$\rightarrow \; = \frac{1}{E^{(0)} - H^{(0)}} \Big(1 - \sum |n^{(0)}\rangle\langle n^{(0)}|\Big)$$

$——$ $= Q$ (some operator)

$$\langle Q \rangle = \underbrace{\bullet — \bullet}_{\langle Q\rangle_0} + (\bullet\!\rightarrow — \bullet + \bullet — \rightarrow\!\bullet) + \cdots$$

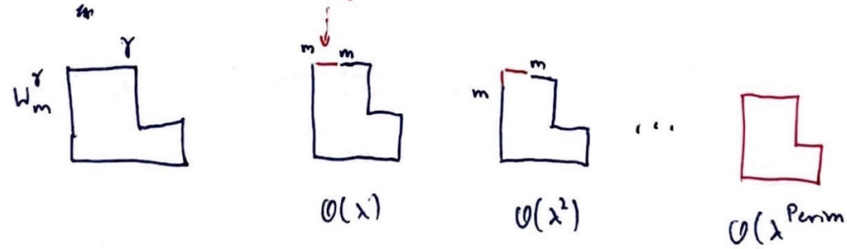
At n^{th} order, $\langle Q \rangle \sim \bullet (\rightarrow)^n — \bullet$

Then, $\langle W_m^\gamma \rangle = \underbrace{\overset{\langle W_m^\gamma\rangle_0}{\bullet — \bullet}}_{=1} + \underbrace{(\bullet\!\leftarrow — \bullet + \bullet — \rightarrow\!\bullet)}_{\substack{=0 \text{ because } H^{(1)} = -\lambda\sum_i \tau_i^x \\ \text{takes you to an excited state}}} + \cdots + \lambda^{\text{Perim}(\gamma)} + \cdots$

$\mathcal{O}(1)$ (because it shuffles around stringnet configs.)

$\mathcal{O}(\lambda)$

$\underbrace{\lambda^{\text{Perim}(\gamma)}}$ smallest nonzero contribution because you need τ^x operators to trace out the loop and have no excitations



$\mathcal{O}(\lambda)$ $\mathcal{O}(\lambda^2)$ $\mathcal{O}(\lambda^{\text{Perim.}(\gamma)})$

The particles in this phase are deconfined. This phase corresponds to weak coupling, because the $\frac{1}{4g^2} \sum_p W_p + W_p^\dagger$ in the lattice Yang-Mills action is strong at weak coupling, corresponding to $\sum A_v + \sum B_p$ being stronger than the other terms in this model.

(λ is also like a gauge coupling, since $\lambda \tau_i^x \sim \lambda \sigma_v^x \tau_{vw}^x \sigma_w^x \sim \lambda \bar{\psi} \cancel{A} \psi$)

The perimeter law is $\boxed{\langle W_m^\gamma \rangle - \langle W_m^\gamma \rangle_0 \sim \lambda^{\text{Perim}(\gamma)}}$.

e condensed phase

Now, let λ be large. The ground state (with $\lambda = \infty$) is $|\text{all}+\rangle$.

Consider inserting an open Wilson loop, $W^m_{\gamma_{\text{open}}} |\text{all}+\rangle$. Since

$$H\,\tau_i^z |\text{all}+\rangle = \left(E_{\text{gnd.}} - \lambda \tau_i^x \tau_i^z\right)|\text{all}+\rangle = \left(E_{\text{gnd.}} + \lambda\right)|\text{all}+\rangle,$$

we have $\langle \text{all}+ | W^{m\,\dagger}_{\gamma_{\text{open}}} H\, W^m_{\gamma_{\text{open}}} |\text{all}+\rangle \propto \text{len}(\gamma_{\text{open}})$,

which corresponds to a linearly confining potential between m particles.

Additionally, the fact that the creation operator for e, τ_0^x, has a finite expectation value, $\langle \tau_i^x \rangle = 1$, is why we say e has condensed in this phase.

$$\text{Now, } \langle W_m^{\gamma_{\text{closed}}} \rangle = \underbrace{\langle \text{all}+ | W_m^\gamma | \text{all}+\rangle}_{O(1)} + \underbrace{\left(\langle +\ddagger- | W_m^\gamma |\text{all}+\rangle + \langle \text{all}+ | W_m^\gamma | +\ddagger- \rangle\right)}_{O(\frac{1}{\lambda})} + \dots$$

$H^{(0)} = -\lambda \sum_i \tau_i^x$

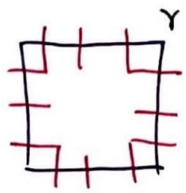
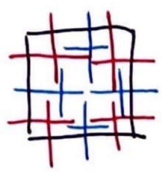
$H^{(1)} = H_{TC}$

i.e. each successive order in perturbation theory flips $|+\rangle$ to $|-\rangle$ (denoted by red links) in $\frac{1}{E^{(0)} - H^{(0)}}\left(1 - \sum |n^{(0)}\rangle\langle n^{(0)}|\right) \underbrace{H^{(1)}}_{\text{Toric code terms}}$.

And the only way to get a nonzero value for this is to tile out the Wilson loop:

$$\boxed{\begin{gathered}\langle W_m^i W_m^j \rangle \propto \delta_{ij} \\ \langle W_m^i \rangle \langle W_m^j \rangle = 0\end{gathered}}$$

($i, j = x, y$ noncontractible loops)

$W_m^\gamma =$  $=$ 

$\prod_{i \in \gamma} \tau_i^x$ $\qquad$ $\prod_{v \text{ inside } \gamma} A_v$

(blue and red terms both)

Therefore, $\boxed{\langle W_m^\gamma \rangle \sim \left(\frac{1}{\lambda}\right)^{\text{Area}(\gamma)}}$, which is the area law of confinement.

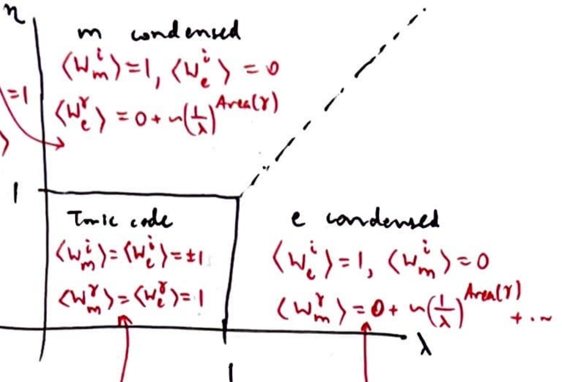
And, $\langle W_m^x \rangle = 0 = \langle W_m^y \rangle$ to lowest order, which is like $\boxed{\text{order param.} = 0}$ phase. This is what we mean by the magnetic 1-form symmetry being unbroken.

Since $\langle W_e^{x,y} \rangle = 1$ still, the electric symmetry is still broken.

$\langle W_m^i \rangle \langle W_m^j \rangle = \langle W_m^i W_m^j \rangle = 1$

$\langle W_e^i \rangle \langle W_e^j \rangle \neq \langle W_e^i W_e^j \rangle$

linearly confining potential between m's



$\langle W_e^i W_e^j \rangle = \langle W_e^i \rangle \langle W_e^j \rangle = \pm 1$

$\langle W_m^i W_m^j \rangle = \langle W_m^i \rangle \langle W_m^j \rangle = \pm 1$

(multiple equivalent ground states, just like usual SSB)

$\langle W_e^i W_e^j \rangle = \langle W_e^i \rangle \langle W_e^j \rangle = 1$

$\langle W_m^i W_m^j \rangle \propto \delta_{ij} \neq \langle W_m^i \rangle \langle W_m^j \rangle = 0$

linearly confining potential between e's

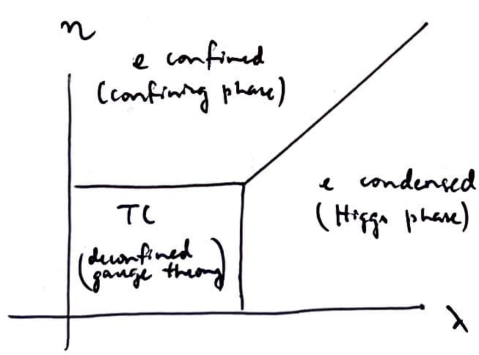
Wilson loop $\sim$ order param.

SSB : $\langle W_i W_j \rangle = \langle W_i \rangle \langle W_j \rangle$

$\langle W_i \rangle \neq 0$

Note that $\langle W_m^i \rangle = \begin{cases} 0 \text{ to all orders}, & \text{e condensed} \\ \pm 1 + O(\lambda^{L_x}), & \text{TC} \end{cases}$ (because TC operators won't change winding # of a state, they'll ~~[illegible]~~ give an orthogonal state always in $W_m^i |\text{all}+\rangle$) (by pert. theory, $\lambda \tau^x$ will only

unbroken : $\langle W_i \rangle = 0, \quad \langle W_i W_j \rangle \neq \langle W_i \rangle \langle W_j \rangle$



Phases of gauge theories (this holds more generally)

Higgs phase : $\langle W_e \rangle \sim e^{-\text{Perim}(\gamma)}$

$\langle W_m \rangle \sim e^{-\text{Area}(\gamma)}$

Confined phase : $\langle W_e \rangle \sim e^{-\text{Area}(\gamma)}$

$\langle W_m \rangle \sim e^{-\text{Perim}(\gamma)}$

Deconfined phase : $\langle W_e \rangle \sim e^{-\text{Perim}(\gamma)}$

$\langle W_m \rangle \sim e^{-\text{Perim}(\gamma)}$

$A_\mu J^\mu$

Higgs phase : e condensed $\Rightarrow$ in $|D_\mu \phi|^2$, $\phi \to \langle \phi \rangle$ $\Rightarrow$ A_μ gets a mass

Confined phase : $\langle W_e \rangle \sim e^{-\text{Area}(\gamma)} \sim e^{-R \cdot T} \sim e^{-E_{\text{pair}} \cdot T}$ $\Rightarrow E_{\text{pair}} \sim R$ $\Rightarrow$ A_μ gets confined

(glueballs are gapped)

Deconfined phase : usual gauge theory at weak coupling