Motivating the Gelfand Transform

One way to make the Gelfand transform easy is to modify the common notation for the eigenvalues of matrices. Let $A, B, \ldots$ be matrices. The common notation for the eigenvalues of A is

$$\lambda_{A,1}, \lambda_{A,2}, \ldots$$

The common notation for the eigenvalues of B is

$$\lambda_{B,1}, \lambda_{B,2}, \ldots$$

Let us modify the notation above as follows. For the eigenvalues of A, let us use

$$\Lambda(A, 1), \Lambda(A, 2), \ldots$$

For the eigenvalues of B, let us use

$$\Lambda(B, 1), \Lambda(B, 2), \ldots$$

The modified notation allows us to see Λ as a mapping with two arguments. The first argument of Λ is a matrix, *e.g.,* $A, B, \ldots$ The second argument of Λ is a positive integer—1 for the first eigenvalue, 2 for the second eigenvalue and so on.

This Λ is nothing but the Gelfand transform!

Let us keep the second argument of Λ fixed at, say, 7 for the seventh eigenvalue. Note that

$$\begin{aligned}
\Lambda(2A, 7) &= 2\Lambda(A, 7) \\
\Lambda(3A, 7) &= 3\Lambda(A, 7) \\
\Lambda(4A, 7) &= 4\Lambda(A, 7) \\
\Lambda(-2A, 7) &= -2\Lambda(A, 7) \\
\Lambda(-3A, 7) &= -3\Lambda(A, 7) \\
\Lambda(-4A, 7) &= -4\Lambda(A, 7)
\end{aligned}$$

And

$$\begin{aligned}
\Lambda(A+B,7) &= \Lambda(A,7)+\Lambda(B,7)\\
\Lambda(A-B,7) &= \Lambda(A,7)-\Lambda(B,7)\\
\Lambda(-911A-22B+828C,7) &= -911\Lambda(A,7)-22\Lambda(B,7)+828\Lambda(C,7)
\end{aligned}$$

These properties make $\Lambda(A,7)$ a linear mapping with respect to the first argument A.

Next, note that

$$\begin{aligned}
\Lambda(A^2,7) &= \Lambda(A,7)^2\\
\Lambda(A^3,7) &= \Lambda(A,7)^3\\
\Lambda(A^4,7) &= \Lambda(A,7)^4
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(AB,7) &= \Lambda(A,7)\Lambda(B,7)\\
\Lambda(ABC,7) &= \Lambda(A,7)\Lambda(B,7)\Lambda(C,7)
\end{aligned}$$

These properties make $\Lambda(A,7)$ a multiplicative mapping with respect to the first argument A.

Thus $\Lambda(A,7)$ is both a linear and a multiplicative mapping with respect to the first argument A. Which makes $\Lambda(A,7)$ a *character* with respect to the first argument A.

Similarly

$$\Lambda(A,1),\Lambda(A,2),\ldots$$

are all characters with respect to the first argument A.

In the passage above, we varied the first argument of Λ and kept the second argument constant. Now, we will do the opposite: keep the first argument of Λ constant and vary the second argument. The set

$$\{\,\Lambda(A,1),\Lambda(A,2),\Lambda(A,3),\ldots\}$$

is the spectrum of A—all the eigenvalues of A. The set

$$\{\,\Lambda(B,1),\Lambda(B,2),\Lambda(B,3),\ldots\}$$

is the spectrum of B—all the eigenvalues of B. That is to say, spectrum is the range of the Gelfand transform when we keep the first argument constant and vary the second argument.

The second argument of the Gelfand transform is an integer in many common examples, but not always. The second argument is best understood as an index that uniquely identifies the ideals of a Banach algebra. What is that Banach algebra?

If the first argument is A, the Banach algebra is the set of all convergent power series in A. If the first argument is B, the Banach algebra is the set of all convergent power series in B.

The number of prime ideals of these sets of power series determine the second argument of the Gelfand transform. If the entities $A, B, \ldots$ are matrices, the number of prime ideals is finite, so integers suffice to act as the index set of the prime ideals. If the entities $A, B, \ldots$ are not matrices, but more advanced concepts, the number of prime ideals need not be finite, or even countably infinite. Then some set—not the integers—must be employed to act as the index set of the prime ideals.