2 (a) To show H is symmetric Prove $H^T = H$

$$H = X(X^TX)^{-1}X^T \qquad X = \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{N \times (d+1)}$$

$$H^T = \left(X(X^TX)^{-1}X^T\right)^T$$
$$= (X^T)^T\left((X^TX)^{-1}\right)^T X^T$$

since $(A^{-1})^T = (A^T)^{-1}$

$$\Rightarrow H^T = X\left((X^TX)^T\right)^{-1}X^T$$
$$H^T = X(X^TX)^{-1}X^T = H$$

Hence $H^T = H \Rightarrow H$ is symmetric

(b) Show $H^k = H$

We can prove it by induction

Base Case ~~let~~ We know $H^1 = H$ $[k=1]$

$H^2 = H$ $[k=2]$

$$H^2 = X \underbrace{(X^TX)^{-1}X^T X}_{I} (X^TX)^{-1}X^T \qquad [\because X^TX \text{ is invertible}]$$

$\Rightarrow H^2 = H$ ——— 1

Inductive Step Let us assume its true for $k = m$ for m

i.e $H^m = H$

We prove that it is also true for $k = m+1$

i.e $H^{m+1} = H$

$$H^{m+1} = H^m H$$
$$= H \cdot H \qquad [\text{Since we assume } H^m = H]$$
$$H^{m+1} = H^2$$

$\Rightarrow \boxed{H^{m+1} = H}$ ⟶ from 1

(c) We will prove it by induction

Base Case $K=1$

$(I-H)^1 = I-H$

Illy for $K=2$

$(I-H)^2 = (I-H)(I-H) = I^2 - HI - HI + H^2$

Since I is identity $\Rightarrow I^2 = I$ & $HI = IH = H$
and $H^2 = H$ from part (b)

$= I - H - \cancel{H} + \cancel{H} = I-H$

$\Rightarrow (I-H)^2 = I-H$ —— 1

Inductive Step :

Assume its true for $K=m$ where $m>2$

i.e $(I-H)^m = I-H$ —— 2

→ We will prove that its true for $k = m+1$

$(I-H)^{m+1} = (I-H)^m (I-H)$

$= (I-H)(I-H) \longrightarrow$ from 2

$= (I-H)^2$

$\boxed{(I-H)^{m+1} = I-H} \longrightarrow$ from 1

(d) Prove trace$(H) = d+1$

We know rank of a matrix X ie rank$(X) = \min(N, d+1)$
$= d+1$

for a matrix $X = N \times (d+1)$

trace$(H) = $ trace $(X(X^TX)^{-1}X^T)$

$= $ trace $(\underbrace{X^TX}\underbrace{(X^TX)^{-1}})$ [Trace is invariant under cyclic permutations

$= $ trace(I_{d+1}) where $I \in (d+1) * (d+1)$

$\Rightarrow \boxed{\text{trace}(H) = d+1}$

③

$$\mathbb{P}\left[\,|E_{in}(g) - E_{out}(g)| > \epsilon\,\right] \leq 2Me^{-2\epsilon N}$$

Replace M by $m_H(N)$

$m_H(N) \longrightarrow$ is polynomial bounded by N^{k-1}

$k > 0$ & $k < \infty$

Find $\lim_{N\to\infty} N^{k-1}e^{-\epsilon N}$

$$\lim_{N\to\infty} \frac{N^{k-1}}{e^{\epsilon N}} \quad \begin{matrix} N^{k-1} \to f(n) \\ e^{\epsilon N} \to g(n) \end{matrix}$$

Both go to ∞ as $N \to \infty$

Hence $\lim_{N\to\infty} f(n) = \infty$ and $\lim_{N\to\infty} g(n) = \infty$

$\Rightarrow$ We can use L' Hôpital's rule to compute this limit

$$\frac{f'(n)}{g'(n)} = \frac{(k-1)N^{(k-2)}}{\epsilon e^{\epsilon N}}$$

This will still be of the form $\frac{\infty}{\infty}$

$\Rightarrow$ We will ~~take~~ ~~repeat~~ ~~do~~ apply L' Hôpitals rule repeatedly

$$= \lim_{N\to\infty} \frac{(k-1)(k-2)N^{(k-3)}}{\epsilon^2 e^{\epsilon N}} = \lim_{N\to\infty} \frac{(k-1)(k-2)(k-3)N^{(k-4)}}{\epsilon^3 e^{\epsilon N}}$$

$$= \lim_{N\to\infty} \frac{[(k-1)(k-2)(k-3)\ldots\ldots 1]\,N^0}{\epsilon^{k-1} e^{\epsilon N}}$$

$$= \lim_{N\to\infty} \frac{(k-1)!}{\epsilon^{k-1} e^{\epsilon N}}$$

$$= \frac{(k-1)!}{\epsilon^{k-1}} \lim_{N\to\infty} \frac{1}{e^{\epsilon N}} \xrightarrow[\text{form}]{\text{is of the}} \frac{\text{Const}}{\infty} = 0$$

$$\Rightarrow \lim_{N\to\infty} N^{k-1}e^{-\epsilon N} = \frac{(k-1)!}{\epsilon^{k-1}} \times \frac{1}{\infty} = 0$$

$$\Rightarrow \boxed{\lim_{N\to\infty} N^{k-1}e^{-\epsilon N} = 0}$$