Wednesday, November 6, 2019 7:57 AM

Solutions - 2018 Exam #1

1) $k = 3, 4$ $f(x,y) = \sqrt{4 + x^2 + 4y^2} = k$ level curves

$k^2 = 4 + x^2 + 4y^2$

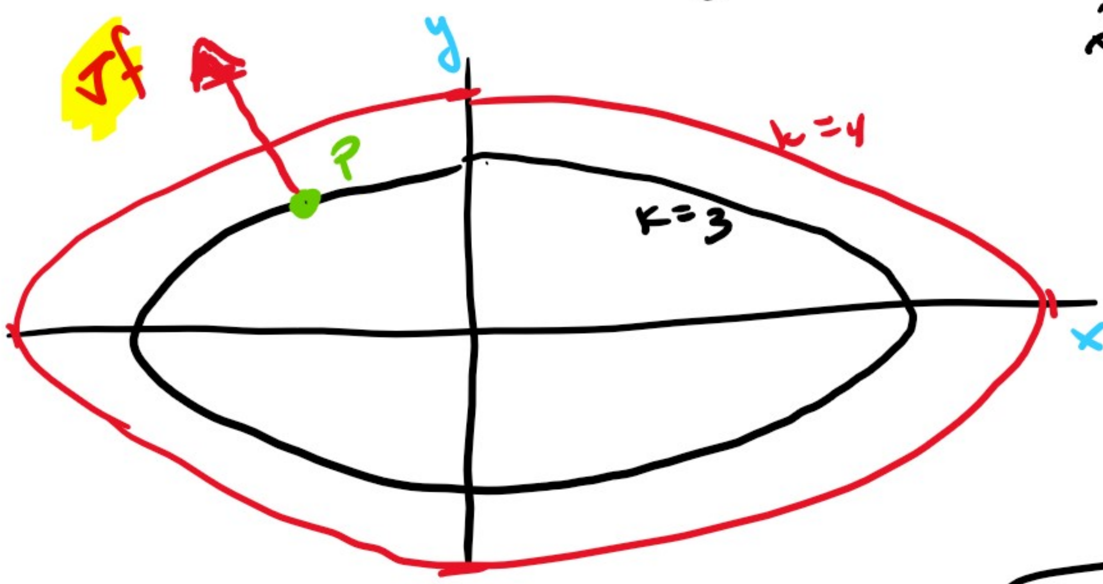
$x^2 + 4y^2 = k^2 - 4$ ellipses concentric

$x^2 + 4y^2 = 5$ $k = 3$

$x^2 + 4y^2 = 12$ $k = 4$

$x = 0$ $y = \pm\sqrt{3}$

$y = 0$ $x = \pm 2\sqrt{3}$



$y = 0$ $x = \pm\sqrt{5}$

$x = 0$ $y = \pm\frac{\sqrt{5}}{2}$

$\nabla f(-1,1)$? $f(-1,1) = \sqrt{4 + 1 + 4} = 3$

P

∇f is $\perp$ to the $k = 3$ level curve, pointing outward (increasing function)

You don't need to compute it, but if you did:

$$\nabla f = \left\langle \frac{x}{\sqrt{4+x^2+4y^2}}, \frac{4y}{\sqrt{4+x^2+4y^2}} \right\rangle$$

$$\nabla f(-1,1) = \left\langle -\frac{1}{3}, \frac{4}{3} \right\rangle$$

so in the $\langle -1, 4 \rangle$ direction

2) $f(x,y) = 100 - 4x^2 - 25y^2 = z$

a.) in y-z plane, so $x = 0$ $z = 100 - 25y^2$

parabola

vertex $(0,0,100)$

intercepts $(0, \pm 2, 0)$

b.) x-z plane, so $y = 0$ $z = 100 - 4x^2$

intercepts $(\pm 5, 0, 0)$

c) horizontal cross sections - level curves

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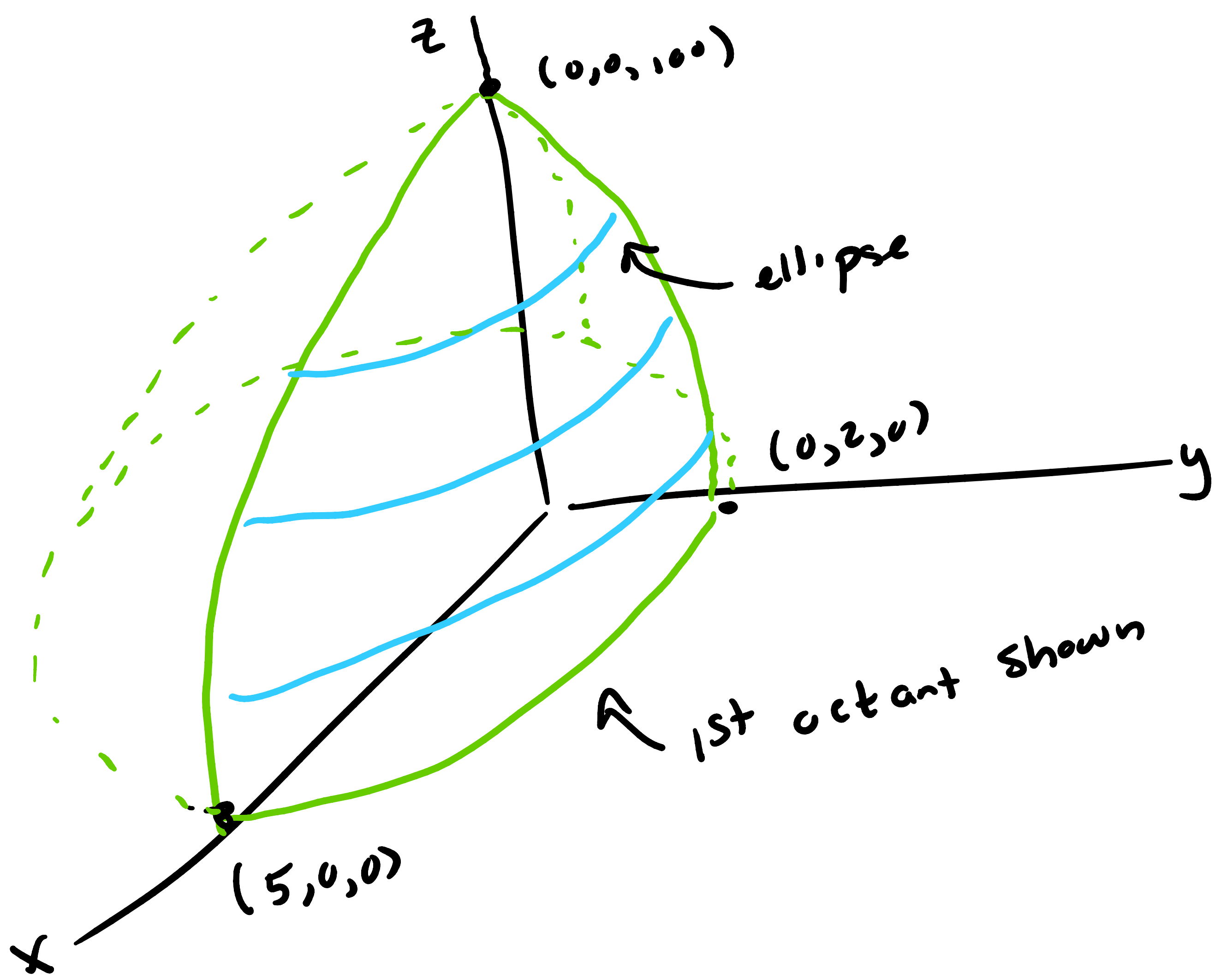
$z = k$ $\qquad k = 25 - 4x^2 - 25y^2$

$4x^2 + 25y^2 = 25 - k \qquad k \leq 25$

ellipses

3D image: elliptic paraboloid opening down



3) $f(x,y) = \frac{x^3y^2}{3} - xy + y$ at $(1,3)$ $\langle -1,4 \rangle$

$\nabla f = \langle x^2y^2 - y, \frac{2}{3}x^3y - x + 1 \rangle$

a) $\nabla f(1,3) = \langle 6, 2 \rangle$

$\hat{u} = \frac{\langle -1, 4 \rangle}{\sqrt{17}}$

$D_{\hat{u}} f(1,3) = \frac{\langle -1,4 \rangle}{\sqrt{17}} \cdot \langle 6,2 \rangle = +\frac{2}{\sqrt{17}}$

b) go in $-\nabla f$ direction so $\langle -6, -2 \rangle$

c) $-\sqrt{40}$ decrease in $-\nabla f$ direction

d) need a normal vector at $(1,3,3)$

$$z = \frac{x^3y^2}{3} - xy + y$$

level surface for F

$$F(x,y,z) = \frac{x^3y^2}{3} - xy + y - z = 0 \text{ on the graph}$$

So ∇F is normal*:

$$\nabla F = \langle x^2y^2 - y,\ \tfrac{2}{3}x^3y - x + 1,\ -1 \rangle$$

$$\vec{N} = \nabla F(1,3,3) = \langle 6, 2, -1 \rangle$$

Plane: $\langle 6,2,-1 \rangle \cdot \langle x-1, y-3, z-3 \rangle = 0$

or $6(x-1) + 2(y-3) - (z-3) = 0$

or $6x + 2y - z = 9$

(your choice)

* Could also have gotten 2 tangent vectors from $(x, y, f(x,y))$ via 2 partial derivatives. then do Cross Product to get $\vec{N}$. Discussed this in class. Equivalent result.

4.) $f(x,y) = \ln(x^2+y^2)$ $\quad \dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$$

Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2x^2 - 2y^2}{(x^2+y^2)^2}$

Sum: $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \dfrac{2y^2 - 2x^2}{(x^2+y^2)^2} + \dfrac{(2x^2 - 2y^2)}{(x^2+y^2)^2} = 0$ ✓

Sum: $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \dfrac{2y^2 - 2x^2}{(x^2+y^2)^2} + \dfrac{(2x^2 - 2y^2)}{(x^2+y^2)^2} = 0$ ✓

satisfied

5.) a.) if I am at $(1,5)$ then

i) for every unit in the $+x$ direction, the concentration goes down by 3 ppm

ii) for every unit I go in the $+y$ direction, the concentration goes up 4 ppm

b.) $-\nabla C$, so in the $\langle 3,-4 \rangle$ direction

c) $\perp$ to ∇C so $\langle 4,3 \rangle$ or $\langle -4,-3 \rangle$ or any scalar multiple of that vector (req: dot product $=0$)

d) $\dfrac{\partial C}{\partial x} \Delta x = (-3)(5) = -15$

concentration will go down about 15 ppm, so it will be about 285 ppm