Laplace Transform Tables

Time Function	LaPlace Transform
Unit Impulse, $\delta(t)$	1
Unit step, $u_s(t)$ $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$

Time Function	LaPlace Transform
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)$ for $(\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{-\omega_n^2 e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\theta)$ where $\theta=\cos^{-1}(\zeta)$ and $(\zeta<1)$	$\frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

2nd order dynamic response

Properties of Laplace Transform:

1. Superposition:

$$\mathcal{L}\left[a f_1(t) + b f_2(t) \right] = a F_1(s) + b F_2(s)$$

2. Time Delay:

$$\mathcal{L}\left[f(t-\tau) \right] = e^{-s\tau} F(s)$$

3. multiplication of $f(t)$ by $e^{-\alpha t}$

$$\left[e^{-\alpha t} f(t) \right] = F(s+\alpha)$$

4. Time scaling:

$$\mathcal{L}\left[f(at) \right] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$ → useful to switch btw units.

5. Differentiation:

$$\mathcal{L}[\dot{f}(t)] = (s F(s) - f(0))$$

$$\mathcal{L}[\ddot{f}(t)] = s^2 F(s) - s(f(0)) - \dot{f}(0)$$

$$= s[\mathcal{L}(\dot{f}(t))] - \dot{f}(0)$$

Higher-order Derivatives:

$$\mathcal{L}(f^{(n)}(t)) = s\mathcal{L}[f^{(n-1)}(t)] - (f^{(n-1)}(0))$$

$$\mathcal{L}[f^{(n)}(t)] = \underline{s^n} F(s) - \underbrace{s^{n-1} f(0) - s^{n-2} \dot{f}(0) - \dots - f^{(n-1)}(0)}_{\text{usually all initial values will be zero.}}$$

6. Integration:

$$\mathcal{L}\left[\int_0^\infty f(t)\,dt\right] = \frac{1}{s} F(s)$$

7) Convolution:

$$\mathcal{L}[f(t) * g(t)] = F(s)\,G(s).$$

Inverse Laplace Transform:

→ Converts function from s-domain to t-domain.

→ Workplan:

1) Take dynamic system and apply $\mathcal{L}[\cdot]$
2) Perform algebraic operations → to solve/simplify in s-domain
3) Take $\mathcal{L}^{-1}[\cdot]$ to convert to t-domain.

$$f(t) = \underbrace{\mathcal{L}^{-1}[F(s)]} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

∞: ω

only valid for $(t>0)$

$f(t)\cdot 1(t)$

Problem: $F(s)$ is a general function, unlikely on the Laplace table

↳ sweeping all frequencies for $s = \sigma + j\omega$ (Fourier Transform)

↳ you don't need to take the transform explicitly.

Partial Fraction Expansion:

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

$$\underbrace{\mathcal{L}^{-1}[F(s)]}_{f(t)} = \underbrace{\mathcal{L}^{-1}[F_1(s)]}_{f_1(t)} + \underbrace{\mathcal{L}^{-1}[F_2(s)]}_{f_2(t)} + \cdots + \underbrace{\mathcal{L}^{-1}[F_n(s)]}_{f_n(t)}$$

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$$

→ in a system model (transfer function), $F(s)$ is of the form:

$$F(s) = \frac{B(s)}{A(s)}$$

1) $B(s)$, $A(s)$ polynomials

2) Degree of $B(s)$ < degree of $A(s)$

$$\boxed{m < n}$$

$$F(s) = \frac{b_m s^{m} + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^{n} + a_{n-1} s^{n-1} + \cdots + a_0}$$

→ written in factored form:

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Roots of numerator are called the zeroes of $F(s)$

poles of $F(s)$

→ After partial Fraction Expansion (PFE)

$$\boxed{F(s) = \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}}$$

→ Each term has a simple $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\left[\frac{c_i}{s+p_i}\right] = c_i e^{-p_i t}$$

$$\rightarrow \boxed{f(t) = c_1 e^{-p_1 t} + c_2 e^{-p_2 t} + \cdots + c_n e^{-p_n t}}$$

There are 3 possible cases:

1) Poles are real and distinct:

↳ no repeating roots.

Example: $\boxed{F(s) = \frac{2}{(s+1)(s+2)}} = \underbrace{\frac{K_1}{s+1} + \frac{K_2}{s+2}}$

- PFE w/ unknown coefficients.
- 1 term per pole

Find K_1 by multiplying both sides by $(s+1)$

$$\frac{2}{\cancel{(s+1)}(s+2)}\cancel{(s+1)} = \frac{K_1}{\cancel{s+1}}\cancel{(s+1)} + \frac{K_2}{s+2}(s+1)$$

$$\frac{2}{s+2} = \boxed{K_1 + \frac{K_2(s+1)}{s+2}}$$

Let $s=-1$ $\Rightarrow \frac{2}{-1+2} = K_1 + \cancel{K_2(0)} \Rightarrow \boxed{K_1 = 2}$

$$K_i = \left[(s+p_i)F(s)\right]_{s=-p_i}$$

For K_2, multiply by $(s+2)$, evaluate at $s=-2$

$$K_2 = \left[\frac{2}{(s+1)\cancel{(s+2)}}\cancel{(s+2)}\right]_{s=-2} = \frac{2}{-2+1} \Rightarrow \boxed{K_2 = -2}$$

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2} \Rightarrow \boxed{f(t) = 2e^{-t} - 2e^{-2t}}$$

$$\boxed{t>0}$$