Background Review

Presented By

Haoying(Jack) Zhou [TA]

Background Review

- Differential Equation

- Dynamics

Differential equations

- It includes functions and derivatives

e.g. : $\frac{dy}{dx} = y' = f(x)$

- ODE (Ordinary Differential Equation)

Consist of 1 independent variable ($\frac{dy}{dx}$)

- PDE (Partial Differential Equation)

Consist of multiple independent variables ($\frac{\partial y}{\partial x}$)

Differential equation's order

- The order of the differential equation is the highest order **derivative** presented in the differential equation.

e.g.

$\sin x \frac{dy}{dx} + e^{-ky} = 1$ is a first-order differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = F(x)$ is a second-order differential equation

Linear Differential Equation

- It has no product of functions and its derivatives
- The power of the functions and derivatives is 1.

- It can be written as:

$$a_0 y(t) + a_1 y'(t) + a_2 y''(t) + \cdots + a_n y^{(n)}(t) = F(t)$$

- If F(t) = 0, it's a homogenous linear equation.

Initial Conditions

- A set of conditions at the initial state of the equation
- With initial conditions, we can solve a differential equation at specific points.
- Given the initial conditions, the problem is called IVP (initial value problem). If the conditions are bounded, the problem is called BIVP (bounded initial value problem)

Procedures of solving differential equation

- Rewrite differential equation, separate variables $\int dy$
- Integrate both sides $y' = 1$ $\frac{dy}{dx} = 1$ $dy = dx$ $y = x \, (+C)$
- Use initial conditions to find C (constant term of the integration)
- Solve the equation (the solution should be a function)

Examples

- (a) $y' + xy = 0$ $(y' = \frac{dy}{dx})$
- (b) $3y'' + y' + y = 0$

(a) $y' + xy = 0$

$$\frac{dy}{dx} = -xy$$

$$\frac{1}{y}\,dy = -x\,dx$$

$$\ln y = -\frac{1}{2}x^2 + C$$

$$y = e^{-\frac{1}{2}x^2} + C$$

(b) $3y'' + y' - y = 0$

$e^{ix} = \cos x + i\sin x$ (if complex)

assume / let $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$; $y'' = \lambda^2 \cdot e^{\lambda x}$

$(3\lambda^2 + \lambda - 1) \cdot \underbrace{e^{\lambda x}}_{\neq 0} = 0$

$\underline{y = C_1 \cdot e^{\lambda_1 x} + C_2 e^{\lambda_2 x}}$

$3\lambda^2 + \lambda - 1 = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{13}}{6}$

$$a \cdot b = 0$$

$$b \neq 0$$

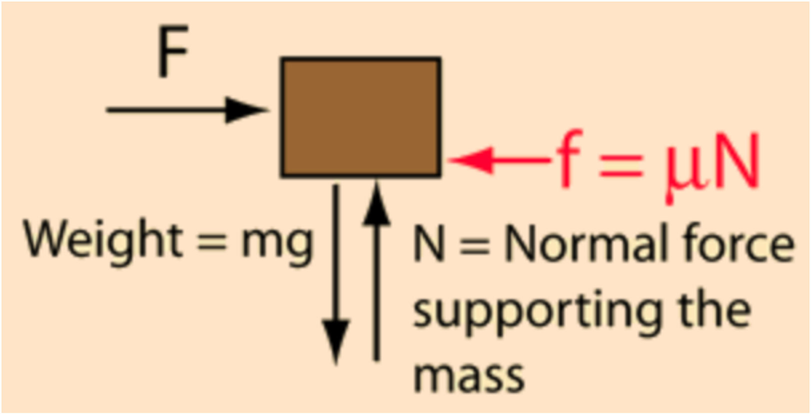
$$a = 0$$

Dynamics

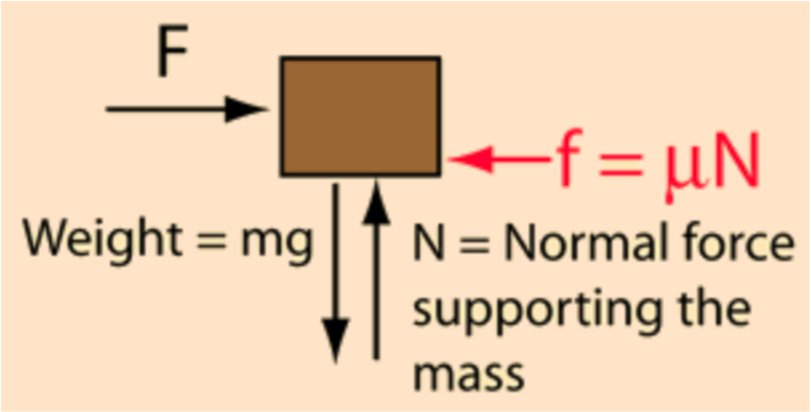
- Describe rigid bodies in motion
- Kinematics → without F
- Kinetics

Free Body Diagram

- Vectors for each force on an object



Newton's Law



- Newton's second law:

$$\sum \vec{F} = F - f = ma = m\vec{a}$$

- Newton's third law:

Equal and opposite F

(e.g. weight and normal force in previous FBD: W = N)

Equation of motion

- $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
- $v = \frac{ds}{dt}$
- $\sum F = ma, \sum M = I\alpha$

mass moment of inertia

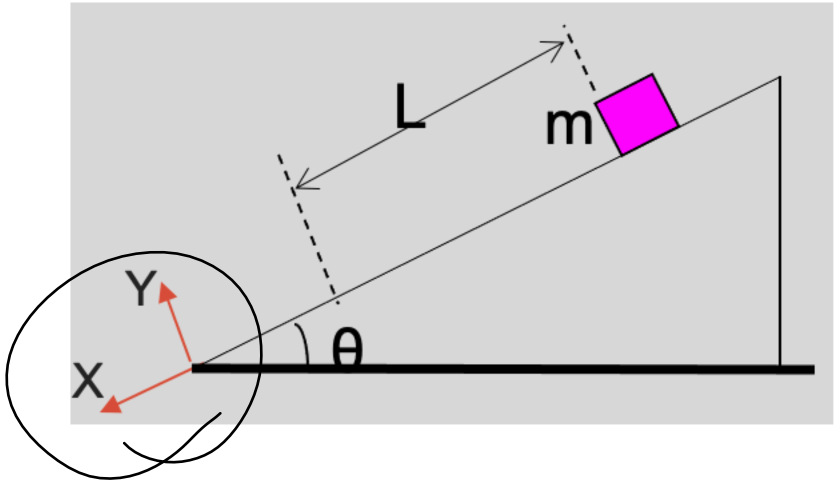
angular acceleration

- Based on above equations, we can obtain:
- $v = v_0 + at$
- $s = s_0 + v_0 t + \frac{1}{2}at^2$
- $v^2 = {v_0}^2 + 2as$

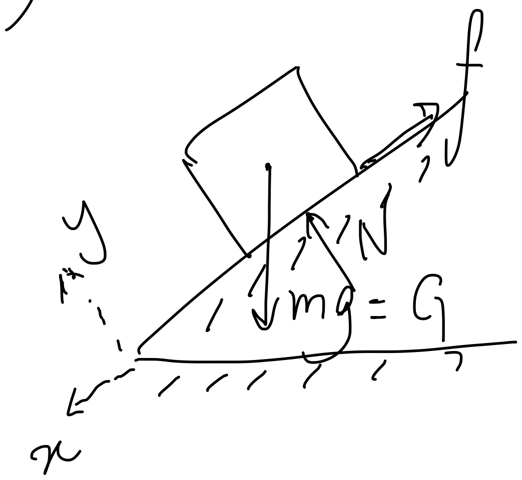
Examples

- (a)

Example: A block of mass m rests on a wedge of angle θ. Assume the surface is rough and the static and kinetic friction coefficients are μs and μk, respectively. Find (a) the critical θ such that the block starts motion downwards and (b) for a given angle (greater than the critical one) find the time it travels down a distance of L



(a)



$$\sum_i F_x = m a_x$$

$$\sum_i F_y = m a_y \qquad a_y = 0$$

$$\begin{cases} \underline{mg\sin\theta - f} = m\boxed{a} \rightarrow a_x \\ N - mg\cos\theta = 0 \end{cases}$$

$$mg\sin\theta - f_s = 0$$

$$f = \mu \cdot N$$

static $\underline{f_s = \mu_s N}$

moving $f_k = \mu_k \cdot N$

$a_x = 0$

$$\begin{cases} f_s = \mu_s \cdot mg\cos\theta & (1) \\ f_s = mg\sin\theta & (2) \end{cases}$$

$\tan\theta \geq \mu_s$

$$\boxed{\theta_c = \tan^{-1}\mu_s}$$

$\theta > \theta_c$

$$\begin{cases} mg\sin\theta - f = ma \\ N - mg\cos\theta = 0 \end{cases}$$

$\Downarrow$

$$\begin{cases} N = mg\cos\theta \\ f = \mu_k N = \mu_k \cdot mg\cos\theta \\ a = \dfrac{mg\sin\theta - f}{m} \end{cases}$$

$= (\sin\theta - \mu_s\cos\theta)\cdot g$

$$a = (\sin\theta - \mu_k \cos\theta) \cdot g$$

$$v_0 = 0$$

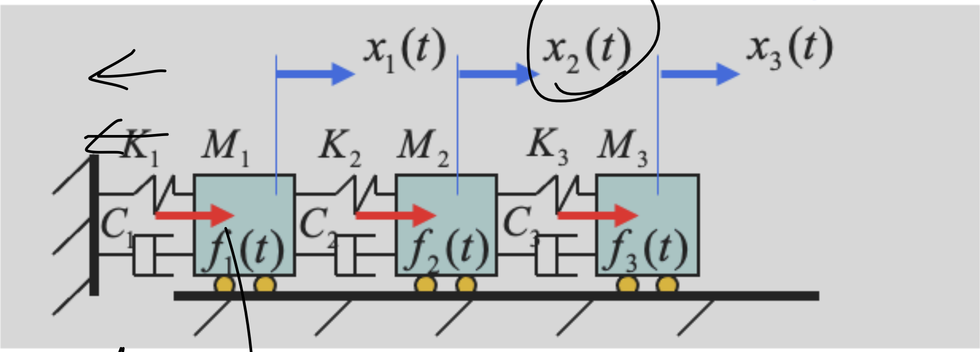
$$L = \frac{1}{2} a t^2$$

$$t = \sqrt{\frac{2L}{(\sin\theta - \mu_k \cos\theta) g}}$$

✓

- (b) Model following system:

Examples: Mass-Spring-Damper systems



3DOF system

Assumptions:
- small horizontal motion only ;
- xi(t), i=1,2,3 measured from the SEP
- No air resistance

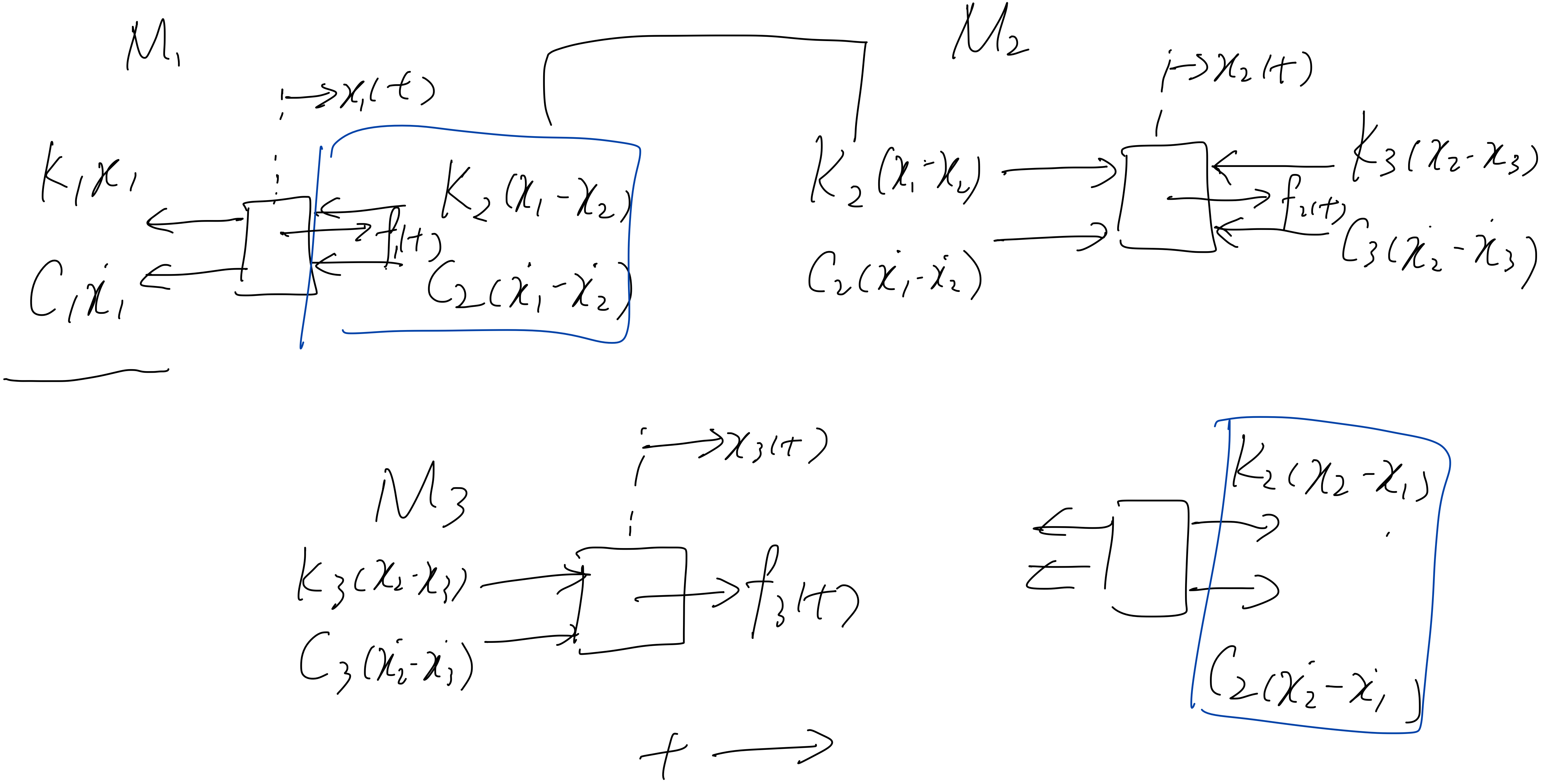
$$\boxed{(x_2 - x_1)}$$

$$F = Kx$$

$$\dot{x} = \frac{dx}{dt}$$

$$F = C \cdot \dot{x} = C \cdot V$$

→ velocity


M_1
$x_1(t)$
$k_1 x_1$
$c_1 \dot{x}_1$
$k_2(x_1 - x_2)$
$f_1(t)$
$c_2(\dot{x}_1 - \dot{x}_2)$
M_2
$x_2(t)$
$k_2(x_1 - x_2)$
$c_2(\dot{x}_1 - \dot{x}_2)$
$k_3(x_2 - x_3)$
$f_2(t)$
$c_3(\dot{x}_2 - \dot{x}_3)$
M_3
$x_3(t)$
$k_3(x_2 - x_3)$
$c_3(\dot{x}_2 - \dot{x}_3)$
$f_3(t)$
$k_2(x_2 - x_1)$
$c_2(\dot{x}_2 - \dot{x}_1)$
t

$$\Sigma F_x = m\underline{a_x} \rightarrow \ddot{x}$$

M_1:

$$M_1 \ddot{x}_1 = f_1(t) - C_1 \dot{x}_1 - K_1 x_1 - C_2(\dot{x}_1 - \dot{x}_2) - K_2(x_1 - x_2)$$

M_2:

$$M_2 \ddot{x}_2 = f_2(t) + K_2(x_1 - x_2) + C_2(\dot{x}_1 - \dot{x}_2) - K_3(x_2 - x_3)$$

$$- C_3(x_2 - x_3)$$

M_3:

$$M_3 \ddot{x}_3 = f_3(t) + K_3(x_2 - x_3) + C_3(\dot{x}_2 - \dot{x}_3)$$

$$
\begin{cases}
M_1 \ddot{x}_1 = f_1(t) - C_1 \dot{x}_1 - K_1 x_1 - C_2(\dot{x}_1 - \dot{x}_2) - K_2(x_1 - x_2) \\
M_2 \ddot{x}_2 = f_2(t) + K_2(x_1 - x_2) + C_2(\dot{x}_1 - \dot{x}_2) - K_3(x_2 - x_3) \\
\qquad - C_3(\dot{x}_2 - \dot{x}_3) \\
M_3 \ddot{x}_3 = f_3(t) + K_3(x_2 - x_3) + C_3(\dot{x}_2 - \dot{x}_3)
\end{cases}
$$