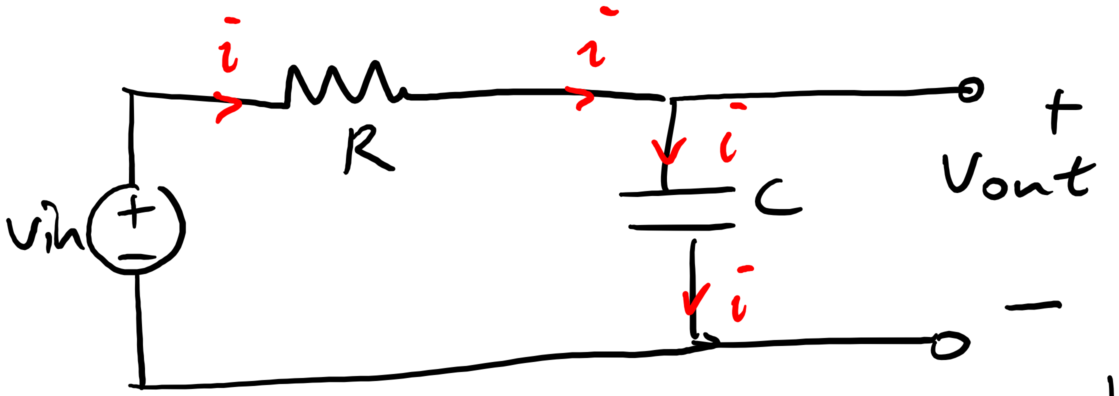


$$\underline{V_{in}} = iR + V_{out}$$

$$i = C\frac{dV_{out}}{dt}$$

$$\boxed{V_{in} = RC\dot{V}_{out} + V_{out}}$$

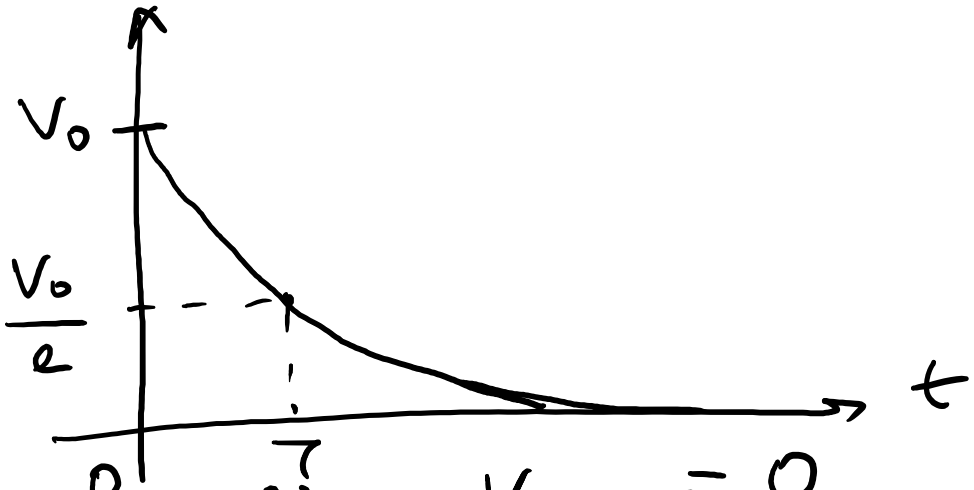
Natural (Free) Response:

$v_{in} = 0$

$$(RC)\dot{V}_{out} + V_{out} = 0 \quad \text{(1st order ODE)}$$

$$V_{out} = K e^{-t/RC} \Rightarrow \boxed{V_{out} = V_0 e^{-t/RC}}$$

$$V_0 = V_{out}(0)$$



$$\lim_{t \to \infty} V_{out} = 0$$

$$\frac{V_0}{e} = V_0 e^{-\tau/RC} \Rightarrow \boxed{\tau = RC}$$

$\tau = RC$: <u>time constant</u>

the time required for the voltage to fall to V_0/e.

Forced Response

$V_{in} =$ constant

$$RC\dot{V}_{out} + V_{out} = V_{in} \equiv \text{constant}$$

$$V_{out} = \underbrace{K_1 e^{-K_2 t}}_{\text{free response (transient)}} + \underbrace{K_3}_{\text{forced response (steady state)}}$$

$$\dot{V}_{out} = -K_1 K_2 e^{-K_2 t}$$

$$RC\underbrace{\left(-K_1 K_2 e^{-K_2 t}\right)}_{\dot{V}_{out}} + \underbrace{K_1 e^{-K_2 t} + K_J}_{V_{out}} = V_{in}$$

$$\boxed{K_J = V_{in}}$$

$$RC\left(-K_1 K_2 e^{-K_2 t}\right) + K_1 e^{-K_2 t} = 0$$

$$(K_1)\left(e^{-K_2 t}\right)\left(1 - RCK_2\right) = 0$$

$$\cancel{K_1 = 0}$$

$$\cancel{e^{-K_2 t} = 0}$$

$$1 - RCK_2 = 0 \Rightarrow \boxed{K_2 = \frac{1}{RC} = \frac{1}{\tau}}$$

$K_1 = ?$ use initial value

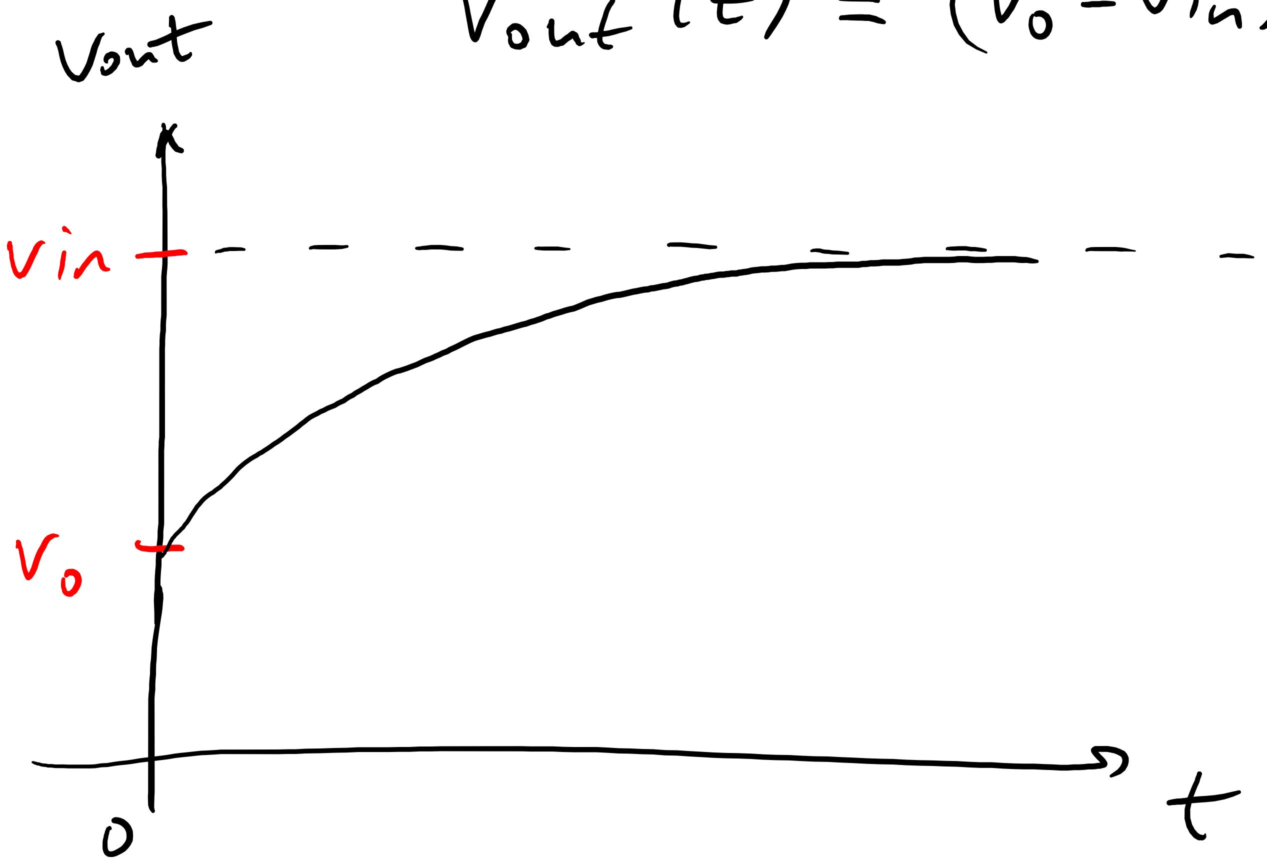
$$V_{out}(0) = V_0$$

$$V_{out} = K_1 e^{-t/RC} + V_{in}$$

$$V_{out}(0) = K_1 + V_{in} = V_0$$

$$\boxed{K_1 = V_0 - V_{in}}$$

$$V_{out}(t) = (V_0 - V_{in})\underbrace{e^{-t/RC}}_{1} + V_{in}$$



when $V_{in} = 0$

$$\Rightarrow \underline{V_{out}(t) = V_0 e^{-t/RC}}$$

time constant: $\tau = RC$

transient

$$\lim_{t \to \infty} V_{out} = V_{in}$$

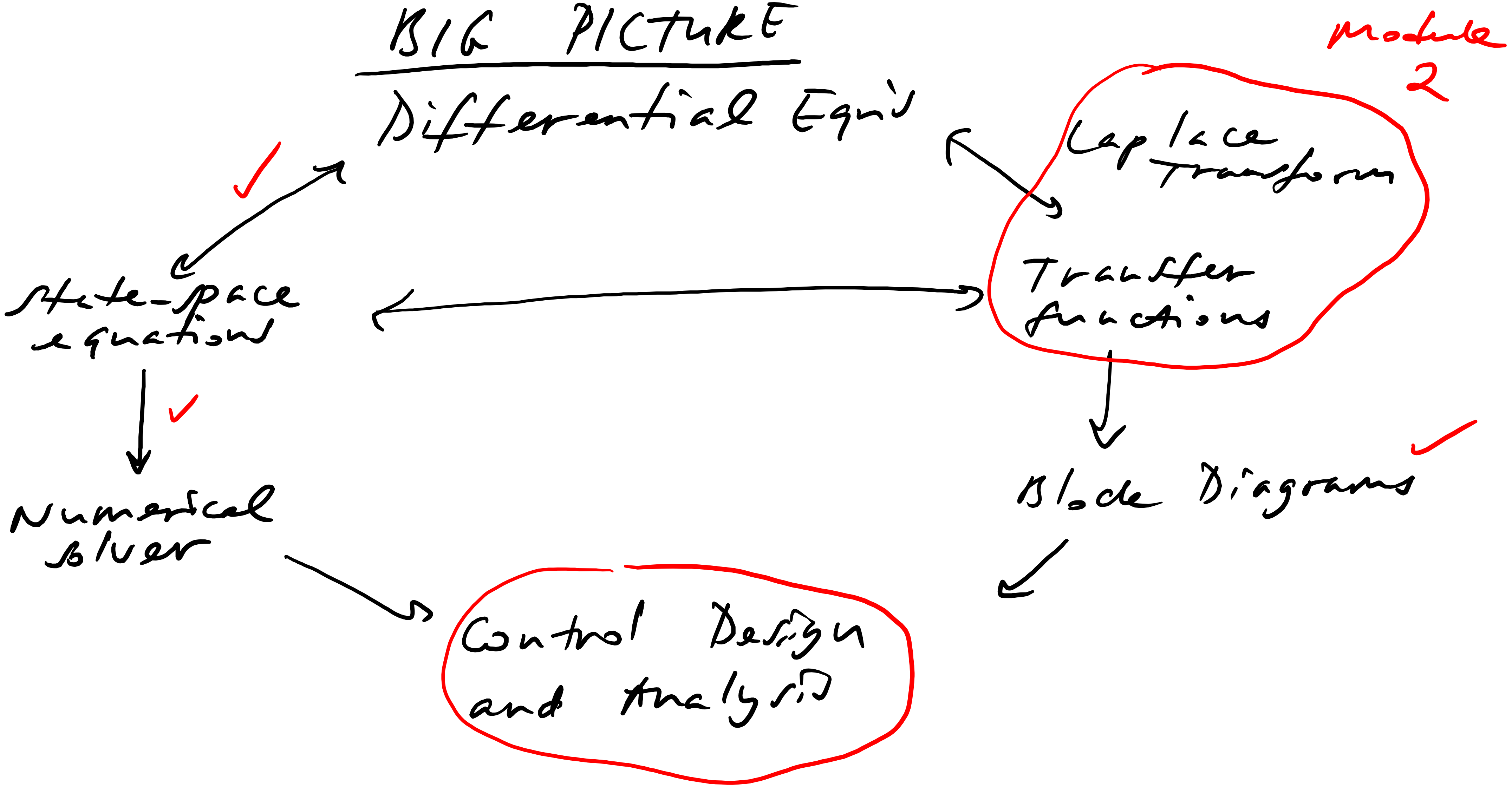
steady-state

* We have 2 significant problems:
  1) We don't know how to use diff. eqn's as system models in block diagrams.
  2) It is very difficult to solve these eqn's in time domain for GENERAL input func's.

<u>Solution:</u>

LAPLACE TRANSFORM

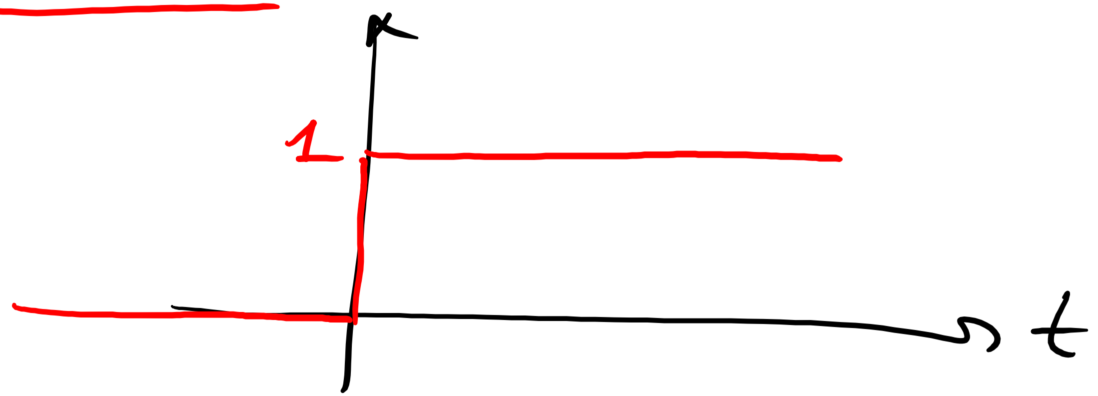
BIG PICTURE


Module 2
Differential Eqn's
Laplace Transform
State-space equations
Transfer functions
Numerical solver
Block Diagrams
Control Design and Analysis

Standard Input Functions:

Unit Step Function:

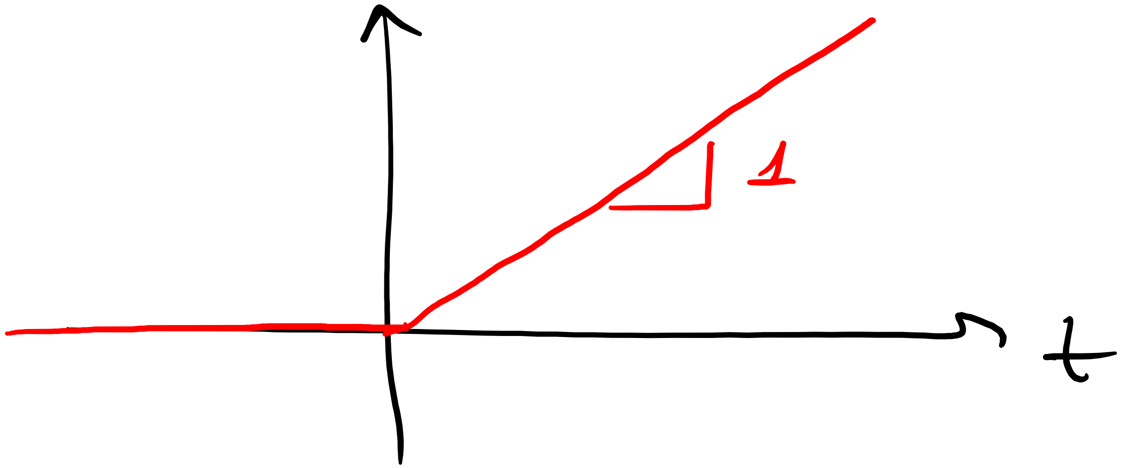
$$u(t) = \underline{1(t)} = \begin{cases} 1, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$V_{in} \cdot 1(t) \longrightarrow$ A step input of V_{in} at $t=0$.

Unit Ramp Function:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

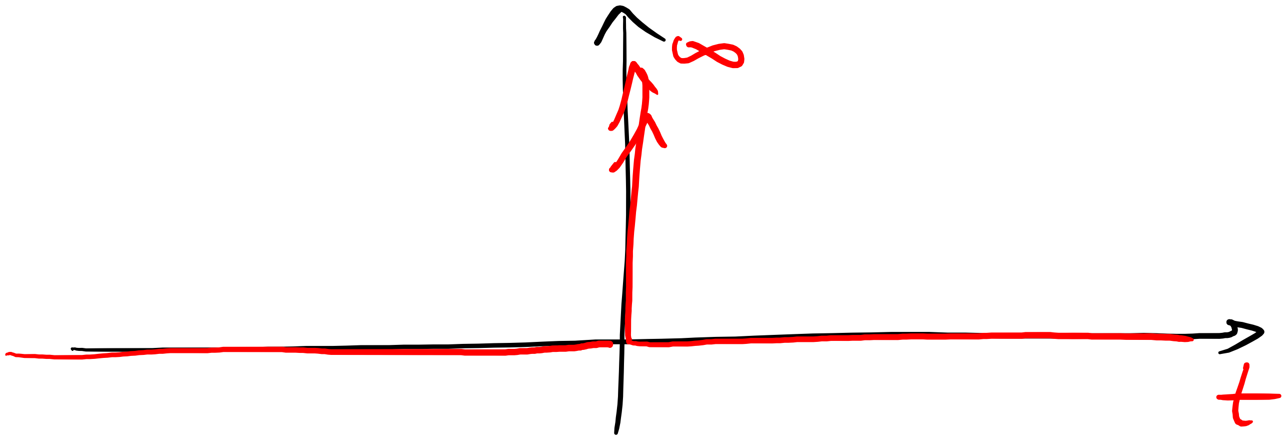


Unit Impulse Function (Dirac Delta)

Defined only $t=0$

$\delta(t) = 0 \quad t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



* The output func. of a dynamic system as a result of an input function is called its <u>RESPONSE</u> to these inputs.

step response, impulse response, etc.

- Impulse response can be used to find the response to <u>GENERAL</u> inputs.

- Define impulse response: $g(t)$
- The generic response to any input $u(t)$ is found using the convolution of its impulse response with the input $u(t)$.

$$\boxed{y(t) = g(t) * u(t)}$$

$$y(t) = \underbrace{\int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau}$$

convolution operation

you never have to take this integral!

Let $u(t) = e^{st}$, where $s = \sigma + j\omega$

$j = \sqrt{-1}$

$e^{st} = e^{(\sigma + j\omega)t} = (e^{\sigma t})(e^{j\omega t})$

some real value

$A(t) \quad (\cos\omega t + j\sin\omega t)$

$$y(t) = \int_{-\infty}^{\infty} g(\tau)\, e^{s(t-\tau)}\, d\tau$$

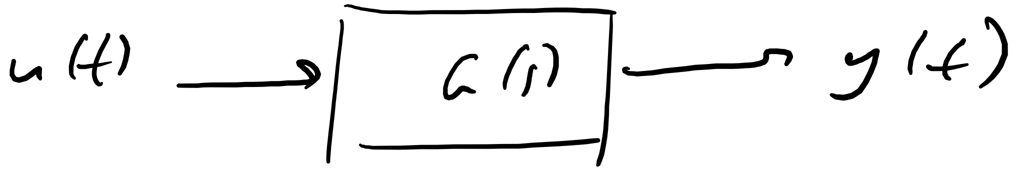
$$y(t) = \int_{-\infty}^{\infty} g(\tau)\,(e^{st})\,e^{-\tau s}\, d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau}\, d\tau}_{(G(s))}$$

$y(t) = u(t)\,(G(s))$: Laplace Transform of $g(t)$

aka: Transfer func.

$y(t) = u(t)\, G(s) \Rightarrow e^{st}$ decouples the convolution into a multiplication.

$\dfrac{y(t)}{u(t)} = G(s)$: transfer function from input $\to$ output



LAPLACE TRANSFORM

Def. $$\mathcal{L}[g(t)] = G(s) = \int_{\boxed{0}}^{\infty} g(t) e^{-st}\, dt$$

$\boxed{0}$ → this is where time starts.

→ Converts functions from t-domain to s-domain.

→ Gets rid of convolution integral and converts differential eqn's into algebraic functions.

Laplace Transform of Common Functions:

Dirac Delta (impulse):

$$\mathcal{L}[\delta(t)] = 1$$

Key to find response to general inputs

$$y(t) = g(t) * u(t)$$

$$Y(s) = G(s)\, U(s)$$

Unit Step Function:

$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\mathcal{L}[1(t)] = \int_0^\infty 1\, e^{-st}\, dt}$$

$$\left[\frac{e^{-st}}{-s}\right]_0^\infty = 0 + \frac{1}{+s} = \boxed{\frac{1}{s}}$$

Unit Ramp Function:

$$r(t) = t \quad , t \geq 0$$

$$\mathcal{L}[r(t)] = \int_0^\infty t\, e^{-st}\, dt$$

IBP: $\int_a^b u\, dv = (uv)_a^b - \int_a^b v\, du$

$u = t \longrightarrow du = 1\, dt$

$dv = e^{-st}\, dt \longrightarrow v = e^{-st}/(-s)$

$$= \underbrace{\left[(t)\left(-\frac{e^{-st}}{s}\right)\right]_0^\infty}_{0} - \int_0^\infty \underbrace{-\frac{e^{-st}}{s}}_{v}\, \underbrace{dt}_{du} = \boxed{\frac{1}{s^2}}$$

Exponential:

$$f(t) = e^{-\alpha t}$$

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-\alpha t} e^{-st}\, dt = \underline{\int_0^\infty e^{-(s+\alpha)t}\, dt} = \frac{1}{s+\alpha}$$

Sinusoidal:

$$f(t) = \sin \omega t$$

Euler identities:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$- \quad e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$e^{j\omega t} - e^{-j\omega t} = 2j \sin \omega t$$

$$\boxed{\sin \omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)}$$

$$\mathcal{L}[\sin \omega t] = \frac{1}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right) e^{-st}\,dt$$

$$= \frac{1}{2j}\int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right) dt$$

$$\frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \boxed{\frac{\omega}{s^2+\omega^2}}$$

$$\boxed{\mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}}$$

Properties of Laplace Transform:

1. Superposition:

$$\mathcal{L}[a f_1(t) + b f_2(t)] = a F_1(s) + b F_2(s)$$

2. Time Delay:

$$\mathcal{L}[f(t-\tau)] = \underline{e^{-s\tau}} F(s)$$

3. multiplication of $f(t)$ by $e^{-\alpha t}$

$$[e^{-\alpha t} f(t)] = \underline{F(s+\alpha)}$$