Laplace Transform Tables

Time Function	LaPlace Transform
Unit Impulse, $\delta(t)$	1
Unit step, $u_s(t)$ $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
1) $C\ e^{-\alpha t}$	$C\ \frac{1}{s+\alpha}$
2) $C\ te^{-\alpha t}$	$C\ \frac{1}{(s+\alpha)^2}$
$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$

Time Function	LaPlace Transform
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
3) $C\ e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
3) $C\ e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)$ for $(\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{-\omega_n^2 e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\theta)$ where $\theta=\cos^{-1}(\zeta)$ and $(\zeta<1)$	$\frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

2nd order dynamic response

Properties of Laplace Transform:

1. Superposition:

$$\mathcal{L}[a f_1(t) + b f_2(t)] = a F_1(s) + b F_2(s)$$

2. Time Delay:

$$\mathcal{L}[f(t-\tau)] = \underline{e^{-s\tau}} F(s)$$

3. multiplication of $f(t)$ by $e^{-\alpha t}$

$$[e^{-\alpha t} f(t)] = \underline{F(s+\alpha)}$$

4. Time scaling:

$$\mathcal{L}(f(at)] = \frac{1}{|a|} F\left(\frac{s}{a}\right) \rightarrow$$ useful to switch btw units.

There are 3 possible cases:

1) Poles are real and distinct:
↳ no repeating roots.

Example: $\boxed{F(s) = \dfrac{2}{(s+1)(s+2)}} = \underbrace{\dfrac{K_1}{s+1} + \dfrac{K_2}{s+2}}$

- PFE w/ unknown coefficients.
- 1 term per pole

Find K_1 by multiplying both sides by $(s+1)$

$$\frac{2\,\cancel{(s+1)}}{\cancel{(s+1)}(s+2)} = \frac{K_1\,\cancel{(s+1)}}{\cancel{s+1}} + \frac{K_2\,(s+1)}{s+2}$$

$$\frac{2}{s+2} = \boxed{K_1 + \frac{K_2\,(s+1)}{s+2}}$$

Let $s=-1 \Rightarrow \dfrac{2}{-1+2} = K_1 + \cancel{K_2}\,(0) \Rightarrow \boxed{K_1 = 2}$

$$\boxed{K_i = \Big[\underline{(s+p_i)}\,\underline{F(s)}\Big]_{s=-p_i}}$$

For K_2, multiply by $(s+2)$, evaluate at $s=-2$

$$K_2 = \left[\frac{2}{(s+1)\cancel{(s+2)}}\cancel{(s+2)}\right]_{s=-2} = \frac{2}{-2+1} \Rightarrow \boxed{K_2 = -2}$$

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2} \Rightarrow \boxed{f(t) = 2e^{-t} - 2e^{-2t}}$$

$$\boxed{t>0}$$

2) Poles are real and repeating:

Ex: $$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{s+1} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2}$$

K_1: $$(s+1)F(s) = \left.\frac{2}{(s+2)^2}\right]_{\boxed{s=-1}} \Rightarrow \boxed{K_1 = 2}$$

K_2, K_3: $$(s+2)^2 F(s) = \boxed{\frac{2}{s+1} = \frac{K_1 (s+2)^2}{s+1} + K_2 + K_3 (s+2)} \quad (*)$$

when $\boxed{s=-2}$ $\Rightarrow$ $\frac{2}{-2+1} = K_2 \Rightarrow \boxed{K_2 = -2}$

K_3: differentiate $(*)$:

$$\left.\frac{-2}{(s+1)^2} = \cancel{\frac{d}{ds}\left(\frac{K_1 (s+2)^2}{s+1}\right)}^{\,0} + 0 + K_3 \right]_{\boxed{s=-2}}$$

-2

$$\frac{-2}{(-2+1)^2} = \boxed{K_3 = -2}$$

$$F(s) = \frac{2}{s+1} - \frac{2}{(s+1)^2} - \frac{2}{s+2}$$

$$\boxed{f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}, \quad t > 0}$$

Case J: Poles are complex;

Ex: $F(s) = \dfrac{s+2}{s^2+2s+5}$ → use $b^2-4ac<0$

$(-1 \mp j2)$ → $(s+1+j2)$, $(s+1-j2)$

$\underbrace{s^2+2s+5}$

$\underbrace{s^2+2s+1}+4$

$(s+1)^2 + 2^2$

$(s+1)^2 + 2^2$

$F(s) = \dfrac{s+2}{(s+1)^2+2^2}$

$\alpha = 1 \qquad \omega = 2$

Remember:

$\to \mathcal{L}\left[e^{-\alpha t}\sin\omega t\right] = \dfrac{\omega}{(s+\alpha)^2+\omega^2}$

$\to \mathcal{L}\left[e^{-\alpha t}\cos\omega t\right] = \dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

$F(s) = \dfrac{c_1(s+\alpha)+c_2\omega}{(s+\alpha)^2+\omega^2}$

1 2

$F(s) = \dfrac{c_1(s+1)+2c_2}{\cancel{(s+1)^2+2^2}} = \dfrac{s+2}{\cancel{(s+1)^2+2^2}}$

$c_1 s + c_1 + 2c_2 = 1s + 2 \Rightarrow \begin{array}{l} c_1 = 1 \\ c_1 + 2c_2 = 2 \end{array} \to \boxed{c_2 = 1/2}$

$$F(s) = 1 \underbrace{\frac{s+1}{(s+1)^2+2^2}}_{e^{-\alpha t}\cos\omega t} + \frac{1}{2} \underbrace{\frac{2}{(s+1)^2+2^2}}_{e^{-\alpha t}\sin\omega t}$$

$$\boxed{f(t) = 1 \cdot e^{-t}\cos 2t + \frac{1}{2} e^{-t}\sin 2t \quad , t > 0}$$

Solving ODEs using LT:

Remember: $\mathcal{L}[\dot f(t)] = (sF(s) - f(0))$

$$\mathcal{L}[\ddot f(t)] = s^2F(s) - sf(0) - \dot f(0)$$

Example: $\overset{m}{1}.\underbrace{\ddot y(t)} + \overset{k}{1}.y(t) = 0 \qquad y(0)=3 \qquad \dot y(0)=5$

$$s^2Y(s) - sy(0) - \dot y(0) + Y(s) = 0$$

$\sin \longrightarrow c_1 \frac{\omega}{s^2+\omega^2}$

insert IC's:

$\cos \longrightarrow c_2 \frac{s}{s^2+\omega^2}$

$$s^2Y(s) - s(3) - 5 + Y(s) = 0$$

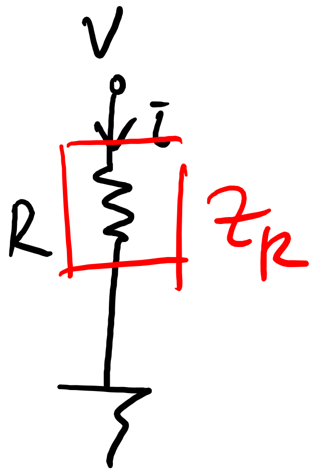
$$Y(s)(s^2+1) = 3s+5$$

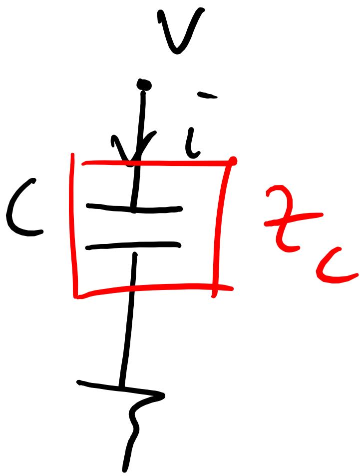
$$Y(s) = \frac{3s+5}{s^2+(1)} = \frac{\overset{3}{c_2}s}{s^2+1} + \frac{\overset{5}{c_1}(1)}{s^2+1} \Rightarrow Y(s) = 3\frac{s}{s^2+1} + 5\frac{1}{s^2+1}$$

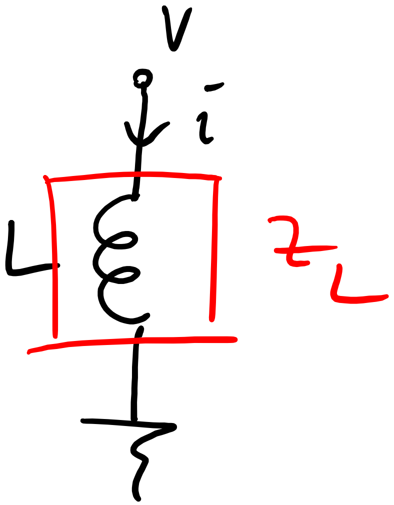
$\omega=1$

$$\boxed{y(t) = 3\cos t + 5\sin t} \quad \boxed{t>0}$$

Complex Impedance:



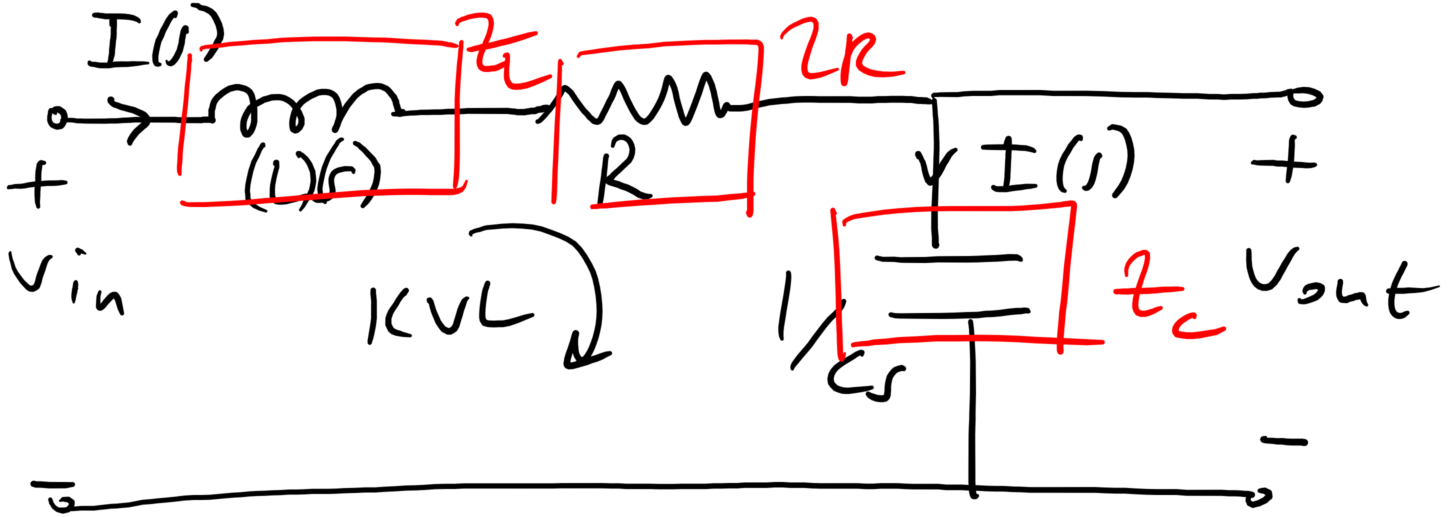




$$v = R i \qquad i = C\frac{dv}{dt} \qquad v = L\frac{di}{dt} \qquad : t$$

$$V(s) = R\,I(s) \qquad I(s) = Cs\,V(s) \qquad V(s) = L(s\,I(s)) \qquad : \mathcal{L}$$

$$\frac{V(s)}{I(s)} = \underbrace{R}_{Z_R} \qquad \frac{V(s)}{I(s)} = \frac{1}{Cs} \; Z_C \qquad \frac{V(s)}{I(s)} = \underbrace{\boxed{Ls}}_{Z_L}$$



$$V_{in}(s) = I(s)\,Ls + I(s)\,R + I(s)\,\frac{1}{Cs}$$

$$V_{in}(s) = I(s)\left(Ls + R + \frac{1}{Cs}\right)$$

$$V_{out}(s) = \frac{1}{Cs}\,I(s)$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{(1/Cs)\cancel{I(s)}}{(Ls + R + 1/Cs)\cancel{I(s)}}$$

$$\boxed{\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{LCs^2 + RCs + 1}}$$