Homework 3

1. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = u$

a) $\mathcal{L}[y'' + 6y' + 5y] = \mathcal{L}[u]$

$\mathcal{L}[y''] + 6\mathcal{L}[y'] + 5\mathcal{L}[y] = U(s)$

$[s^2Y(s) - sy(0) - y'(0)] + 6[sY(s) - y(0)] + 5Y(s) = U(s)$

$s^2Y(s) + 6sY(s) + 5Y(s) = U(s)$

$Y(s)[s^2 + 6s + 5] = U(s)$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{1}{s^2+6s+5}}$$

b) $y(0) = 1 \quad y'(0) = 0$

$[s^2(Y(s)) - sy(0) - y'(0)] + 6[sY(s) - y(0)] + 5Y(s) = U(s)$

$s^2Y(s) - s + 6sY(s) - 6 + 5Y(s) = 0$

$s^2Y(s) + 6sY(s) + 5Y(s) = s + 6$

$Y(s)[s^2 + 6s + 5] = s + 6$

$$Y(s) = \frac{s+6}{s^2+6s+5} = \frac{s+6}{(s+5)(s+1)}$$

$$\frac{s+6}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$$

$s+6 = A(s+1) + B(s+5)$

$s+6 = As + A + Bs + 5B$

$s = As + Bs$ $\qquad A + 5B = 6$

$1 = A + B$ $\qquad 1 - B + 5B = 6$

$A = 1 - B$ $\qquad 1 + 4B = 6$

$A = -\frac{1}{4}$ $\qquad B = \frac{5}{4}$

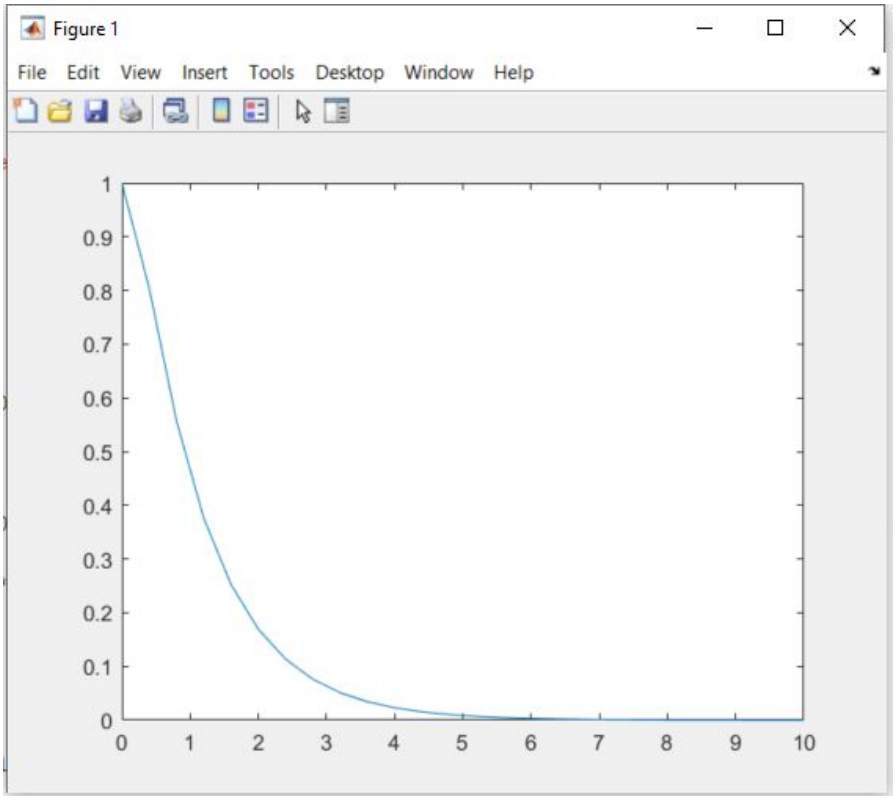
$$Y(s) = -\frac{1}{4}\left(\frac{1}{s+5}\right) + \frac{5}{4}\left(\frac{1}{s+1}\right)$$

$$y(t) = \mathcal{L}^{-1}\left[-\frac{1}{4}\left(\frac{1}{s+5}\right)\right] + \mathcal{L}^{-1}\left[\frac{5}{4}\left(\frac{1}{s+1}\right)\right]$$

$$y(t) = -\frac{e^{-5t}}{4} + \frac{5e^{-t}}{4} \Rightarrow \boxed{\frac{-e^{-5t} + 5e^{-t}}{4} = y(t)}$$

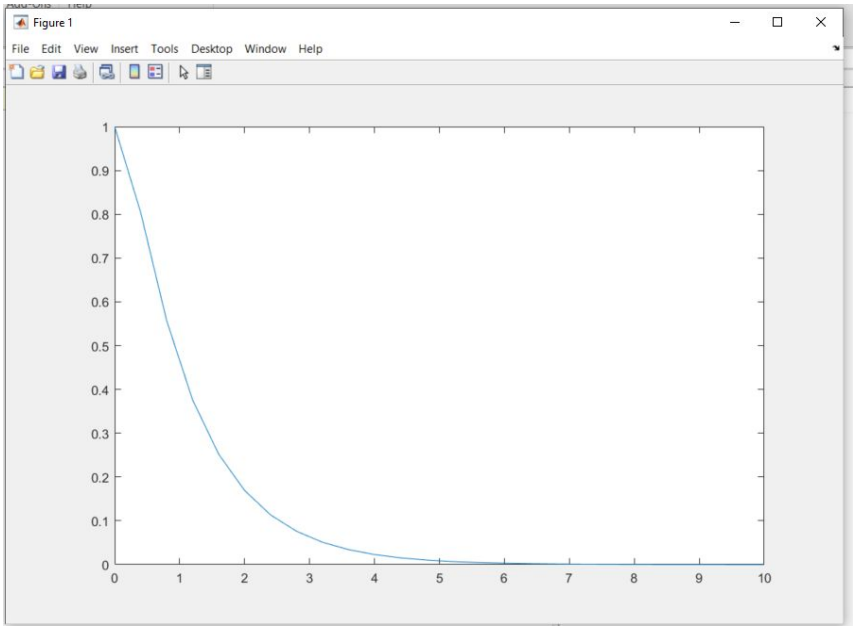
1B) Matlab graph:

```
>> t = (0:0.4:10)

t =

  Columns 1 through 13

         0    0.4000    0.8000    1.2000    1.6000    2.0000    2.4000    2.8000    3.2000    3.6000    4.0000    4.4000    4.8000

  Columns 14 through 26

    5.2000    5.6000    6.0000    6.4000    6.8000    7.2000    7.6000    8.0000    8.4000    8.8000    9.2000    9.6000   10.0000

>> y = (-1/4)*exp(-5*t)+(5/4)*exp(-t)

y =

  Columns 1 through 13

    1.0000    0.8041    0.5571    0.3759    0.2523    0.1692    0.1134    0.0760    0.0510    0.0342    0.0229    0.0153    0.0103

  Columns 14 through 26

    0.0069    0.0046    0.0031    0.0021    0.0014    0.0009    0.0006    0.0004    0.0003    0.0002    0.0001    0.0001    0.0001

>> plot(t,y)
```



1C)

```
>> syms t s
>> t = (0:0.4:10)

t =

  Columns 1 through 13

         0    0.4000    0.8000    1.2000    1.6000    2.0000    2.4000    2.8000    3.2000    3.6000    4.0000    4.4000    4.8000

  Columns 14 through 26

    5.2000    5.6000    6.0000    6.4000    6.8000    7.2000    7.6000    8.0000    8.4000    8.8000    9.2000    9.6000   10.0000

>> f = (s+6)/((s+5)*(s+1))

f =

(s + 6)/((s + 1)*(s + 5))

>> y = ilaplace(f)

y =

(5*exp(-t))/4 - exp(-5*t)/4

>> y = (5*exp(-t))/4 - exp(-5*t)/4

y =

  Columns 1 through 13

    1.0000    0.8041    0.5571    0.3759    0.2523    0.1692    0.1134    0.0760    0.0510    0.0342    0.0229    0.0153    0.0103

  Columns 14 through 26

    0.0069    0.0046    0.0031    0.0021    0.0014    0.0009    0.0006    0.0004    0.0003    0.0002    0.0001    0.0001    0.0001

>> plot(t,y)
```



Both the plots from part b and part c look the same. The first part is just graphing the equation after manually solving for the free response differential equation. The second part takes in the original function and solves for the Laplace transform of that function in MATLAB and then graphs it. Because both graphs are the same, we can verify that manual calculations were completed correctly.

2. $M_1\ddot{x}_1 = -k_1(x_1 - 0) - k_3(x_1 - x_2) - b_1(\dot{x}_1 - 0) + u$

$M_2\ddot{x}_2 = k_3(x_1 - x_2) - k_2(x_2 - 0) - b_2(\dot{x}_2 - 0)$

$M_1\ddot{x}_1 = -k_1x_1 - k_3x_1 + k_3x_2 - b_1\dot{x}_1 + u$

$M_2\ddot{x}_2 = k_3x_1 - k_3x_2 - k_2x_2 - b_2\dot{x}_2$

$\mathcal{L}[M_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_3x_1] = [k_3x_2 + u]\mathcal{L}$

$\mathcal{L}[M_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 + k_3x_2] = [k_3x_1]\mathcal{L}$

$(M_1s^2 + b_1s + k_1 + k_3)X_1(s) = k_3X_2(s) + U(s)$

$(M_2s^2 + b_2s + k_2 + k_3)X_2(s) = k_3X_1(s)$

$$X_2(s) = \frac{k_3X_1(s)}{M_2s^2 + b_2s + k_2 + k_3}$$

$$(M_1s^2 + b_1s + k_1 + k_3)X_1(s) = \frac{(k_3)^2X_1(s)}{(M_2s^2 + b_2s + k_2 + k_3)} + U(s)$$

$$\left[(M_1s^2 + b_1s + k_1 + k_3) - \frac{(k_3)^2}{M_2s^2 + b_2s + k_2 + k_3}\right]X_1(s) = U(s)$$

$$\boxed{\frac{X_1(s)}{U(s)} = \frac{M_2s^2 + b_2s + k_2 + k_3}{(M_2s^2 + b_2s + k_2 + k_3)(M_1s^2 + b_1s + k_1 + k_3) - (k_3)^2}}$$

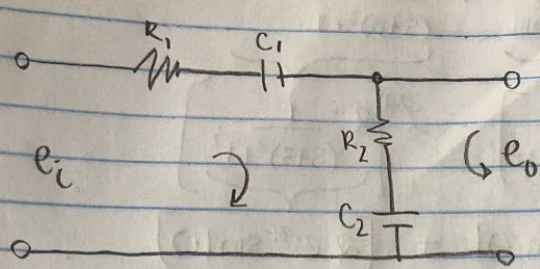
$$\frac{X_2(s)}{X_1(s)} = \frac{k_3}{M_2s^2 + b_2s + k_2 + k_3}$$

$$\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)}$$

$$\boxed{\frac{X_2(s)}{U(s)} = \frac{k_3}{(M_1s^2 + b_1s + k_1 + k_3)(M_2s^2 + b_2s + k_2 + k_3) - (k_3)^2}}$$

Homework 3 Cont.

3.



$$e_i = i_1 R_1 + \frac{1}{C_1}\int i_1\,dt + e_o$$

$$e_o = i_1 R_2 + \frac{1}{C_2}\int i_1\,dt$$

$$\mathcal{L}[e_i] = \mathcal{L}[i_1 R_1] + \mathcal{L}\left[\frac{1}{C_1}\int i_1\,dt\right] + \mathcal{L}[e_o]$$

$$\mathcal{E}_i(s) = I_1(s)R_1 + \frac{1}{C_1 s} I_1(s) + \mathcal{E}_o(s)$$

$$\mathcal{L}[e_o] = \mathcal{L}[i_1 R_2] + \mathcal{L}\left[\frac{1}{C_2}\int i_1\,dt\right]$$

$$\mathcal{E}_o(s) = I_1(s) R_2 + \frac{1}{C_2 s} I_1(s)$$

$$\mathcal{E}_o(s) = I_1(s)\left[R_2 + \frac{1}{C_2 s}\right]$$

$$I_1(s) = \frac{\mathcal{E}_o(s)}{\left[R_2 + \frac{1}{C_2 s}\right]}$$

$$\mathcal{E}_i(s) = I_1(s)\left[R_1 + \frac{1}{C_1 s}\right] + \mathcal{E}_o(s)$$

$$\mathcal{E}_i(s) = \frac{\mathcal{E}_o(s)}{\left[R_2 + \frac{1}{C_2 s}\right]} \cdot \left[R_1 + \frac{1}{C_1 s}\right] + \mathcal{E}_o(s)$$

$$\mathcal{E}_i(s) = \mathcal{E}_o(s)\left[\frac{R_1 + \frac{1}{C_1 s}}{R_2 + \frac{1}{C_2 s}} + 1\right]$$

$$\boxed{\frac{\mathcal{E}_i(s)}{\mathcal{E}_o(s)} = \frac{R_1 + \frac{1}{C_1 s}}{R_2 + \frac{1}{C_2 s}} + 1}$$

4. $G(s) = \frac{2s+4}{s^2+6s+10} = \underbrace{\frac{2s}{(s+3)^2+1}} + \underbrace{\frac{4}{(s+3)^2+1}}$

$$\mathcal{L}^{-1}\left[2 \cdot \underbrace{\frac{s+3}{(s+3)^2+1^2}}\right] - \mathcal{L}^{-1}\left[2 \cdot \underbrace{\frac{1}{(s+3)^2+1^2}}\right]$$

$$\left[2e^{-3t}\cos(t)\right] - \left[2e^{-3t}\sin(t)\right]$$

$$\boxed{\mathcal{L}^{-1}[G(s)] = g(t) = 2e^{-3t}\cos(t) - 2e^{-3t}\sin(t)}$$

5. $G(s) = \frac{s^2+s+2}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$

$$s^2+s+2 = A(s+1)^2 + B(s+1) + C$$

$$s^2+s+2 = As^2 + 2As + A + Bs + 1B + C$$

$A = 1$ $\quad 2A + B = 1$ $\quad B = -1$ $\quad A + B + C = 2$ $\quad 1 - 1 + C = 2$ $\quad C = 2$

$$\mathcal{L}^{-1}[G(s)] = \underbrace{\left[\frac{1}{s+1}\right]\mathcal{L}^{-1}}_{e^{-t}} + \underbrace{\left[\frac{-1}{(s+1)^2}\right]\mathcal{L}^{-1}}_{-te^{-t}} + \underbrace{\left[\frac{2}{(s+1)^3}\right]\mathcal{L}^{-1}}_{t^2e^{-t}}$$

$$\boxed{\mathcal{L}^{-1}[G(s)] = g(t) = e^{-t} - te^{-t} + t^2e^{-t}}$$