Homework 3

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1. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = u$

a) $\mathcal{L}[y'' + 6y' + 5y] = \mathcal{L}[u]$

$\mathcal{L}[y''] + 6\mathcal{L}[y'] + 5\mathcal{L}[y] = U(s)$

$[s^2Y(s) - s\cancel{y(0)} - \cancel{y'(0)}] + 6[sY(s) - \cancel{y(0)}] + 5Y(s) = U(s)$

$s^2Y(s) + 6sY(s) + 5Y(s) = U(s)$

$Y(s)[s^2 + 6s + 5] = U(s)$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{1}{s^2+6s+5}}$$

b) $y(0) = 1 \quad y'(0) = 0$

$[s^2(Y(s)) - sy(0) - \cancel{y'(0)}] + 6[sY(s) - y(0)] + 5Y(s) = U(\cancel{s})$

$s^2Y(s) - s + 6sY(s) - 6 + 5Y(s) = 0$

$s^2Y(s) + 6sY(s) + 5Y(s) = s + 6$

$Y(s)[s^2 + 6s + 5] = s + 6$

$Y(s) = \frac{s+6}{s^2+6s+5} = \frac{s+6}{(s+5)(s+1)}$

$\frac{s+6}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$

$s + 6 = A(s+1) + B(s+5)$

$s + 6 = As + A + Bs + 5B$

$s = As + Bs \qquad A + 5B = 6$

$1 = A + B \qquad 1 - B + 5B = 6$

$A = 1 - B \qquad 1 + 4B = 6$

$A = -\frac{1}{4} \qquad B = \frac{5}{4}$

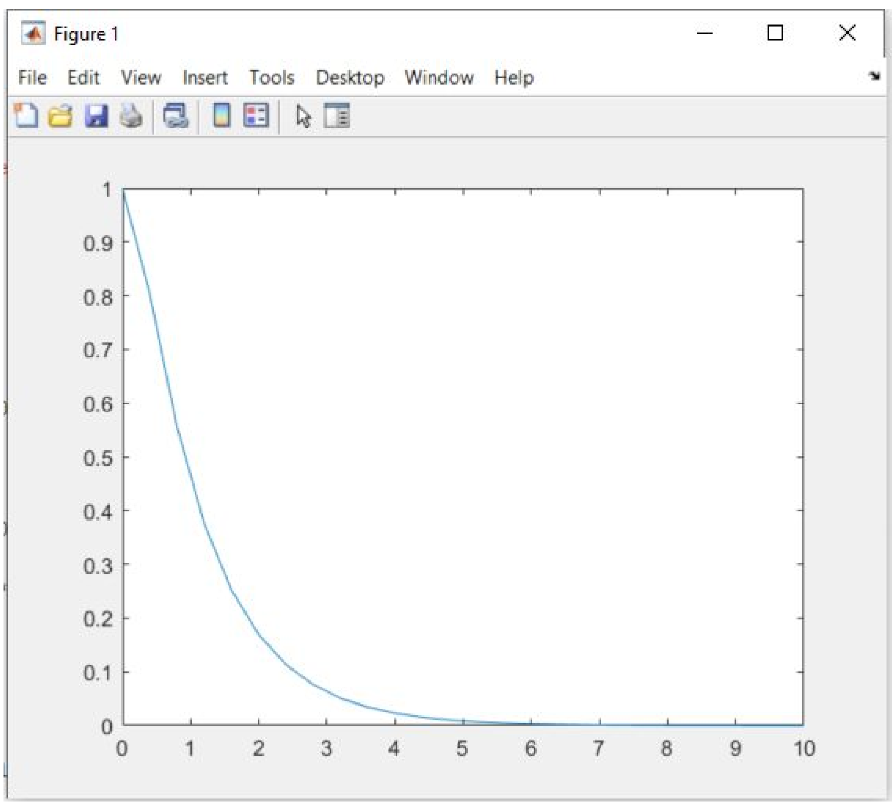
$Y(s) = -\frac{1}{4}\left(\frac{1}{s+5}\right) + \frac{5}{4}\left(\frac{1}{s+1}\right)$

$y(t) = \mathcal{L}^{-1}\left[-\frac{1}{4}\left(\frac{1}{s+5}\right)\right] + \mathcal{L}^{-1}\left[\frac{5}{4}\left(\frac{1}{s+1}\right)\right]$

$y(t) = -\frac{e^{-5t}}{4} + \frac{5e^{-t}}{4} \Rightarrow \boxed{\frac{-e^{-5t} + 5e^{-t}}{4} = y(t)}$

1B) Matlab graph:

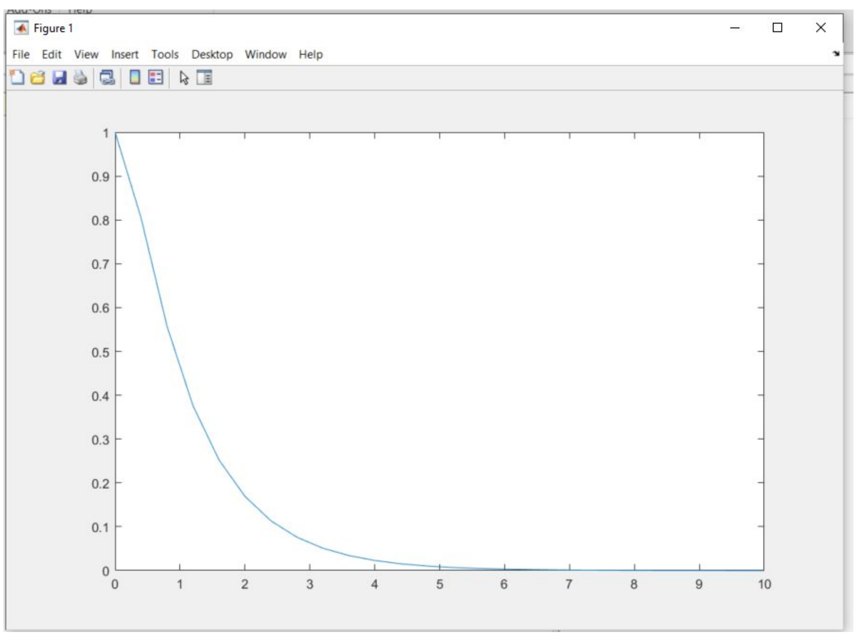
```
>> t = (0:0.4:10)

t =

  Columns 1 through 13

         0    0.4000    0.8000    1.2000    1.6000    2.0000    2.4000    2.8000    3.2000    3.6000    4.0000    4.4000    4.8000

  Columns 14 through 26

    5.2000    5.6000    6.0000    6.4000    6.8000    7.2000    7.6000    8.0000    8.4000    8.8000    9.2000    9.6000   10.0000

>> y = (-1/4)*exp(-5*t)+(5/4)*exp(-t)

y =

  Columns 1 through 13

    1.0000    0.8041    0.5571    0.3759    0.2523    0.1692    0.1134    0.0760    0.0510    0.0342    0.0229    0.0153    0.0103

  Columns 14 through 26

    0.0069    0.0046    0.0031    0.0021    0.0014    0.0009    0.0006    0.0004    0.0003    0.0002    0.0001    0.0001    0.0001

>> plot(t,y)
```



1C)

```
>> syms t s
>> t = (0:0.4:10)

t =

  Columns 1 through 13

         0    0.4000    0.8000    1.2000    1.6000    2.0000    2.4000    2.8000    3.2000    3.6000    4.0000    4.4000    4.8000

  Columns 14 through 26

    5.2000    5.6000    6.0000    6.4000    6.8000    7.2000    7.6000    8.0000    8.4000    8.8000    9.2000    9.6000   10.0000

>> f = (s+6)/((s+5)*(s+1))

f =

(s + 6)/((s + 1)*(s + 5))

>> y = ilaplace(f)

y =

(5*exp(-t))/4 - exp(-5*t)/4

>> y = (5*exp(-t))/4 - exp(-5*t)/4

y =

  Columns 1 through 13

    1.0000    0.8041    0.5571    0.3759    0.2523    0.1692    0.1134    0.0760    0.0510    0.0342    0.0229    0.0153    0.0103

  Columns 14 through 26

    0.0069    0.0046    0.0031    0.0021    0.0014    0.0009    0.0006    0.0004    0.0003    0.0002    0.0001    0.0001    0.0001

>> plot(t,y)
```

use ss() and initial()



Both the plots from part b and part c look the same. The first part is just graphing the equation after manually solving for the free response differential equation. The second part takes in the original function and solves for the Laplace transform of that function in MATLAB and then graphs it. Because both graphs are the same, we can verify that manual calculations were completed correctly.

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2. $M_1\ddot{x}_1 = -k_1(x_1-0) - k_3(x_1-x_2) - b_1(\dot{x}_1-0) + u$

$M_2\ddot{x}_2 = k_3(x_1-x_2) - k_2(x_2-0) - b_2(\dot{x}_2-0)$

$M_1\ddot{x}_1 = -k_1x_1 - k_3x_1 + k_3x_2 - b_1\dot{x}_1 + u$

$M_2\ddot{x}_2 = k_3x_1 - k_3x_2 - k_2x_2 - b_2\dot{x}_2$

$\mathcal{L}[M_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_3x_1] = [k_3x_2 + u]\mathcal{L}$

$\mathcal{L}[M_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 + k_3x_2] = [k_3x_1]\mathcal{L}$

$(M_1s^2 + b_1s + k_1 + k_3)X_1(s) = k_3X_2(s) + U(s)$

$(M_2s^2 + b_2s + k_2 + k_3)X_2(s) = k_3X_1(s)$

$$X_2(s) = \frac{k_3X_1(s)}{M_2s^2+b_2s+k_2+k_3}$$

$$(M_1s^2+b_1s+k_1+k_3)X_1(s) = \frac{(k_3)^2X_1(s)}{(M_2s^2+b_2s+k_2+k_3)} + U(s)$$

$$\left[(M_1s^2+b_1s+k_1+k_3) - \frac{(k_3)^2}{M_2s^2+b_2s+k_2+k_3}\right]X_1(s) = U(s)$$

$$\boxed{\frac{X_1(s)}{U(s)} = \frac{M_2s^2+b_2s+k_2+k_3}{(M_2s^2+b_2s+k_2+k_3)(M_1s^2+b_1s+k_1+k_3)-(k_3)^2}}$$

$$\frac{X_2(s)}{X_1(s)} = \frac{k_3}{M_2s^2+b_2s+k_2+k_3}$$

$$\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{\cancel{X_1(s)}}\cdot\frac{\cancel{X_1(s)}}{U(s)}$$

$$\boxed{\frac{X_2(s)}{U(s)} = \frac{k_3}{(M_1s^2+b_1s+k_1+k_3)(M_2s^2+b_2s+k_2+k_3)-(k_3)^2}}$$

Homework 3 Cont.

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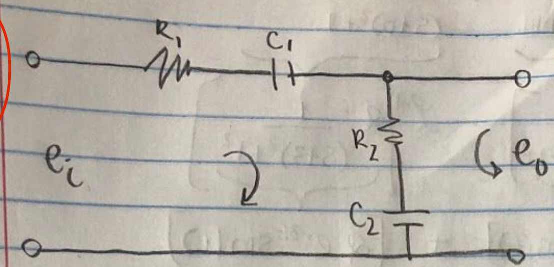
3.



$$e_i = i_1 R_1 + \frac{1}{C_1}\int i_1\,dt + e_o$$
$$e_o = i_1 R_2 + \frac{1}{C_2}\int i_1\,dt$$

$$\mathcal{L}[e_i] = \mathcal{L}[i_1 R_1] + \mathcal{L}\left[\frac{1}{C_1}\int i_1\,dt\right] + \mathcal{L}[e_o]$$
$$\mathcal{E}_i(s) = I_1(s)R_1 + \frac{1}{C_1 s} I_1(s) + \mathcal{E}_o(s)$$

$$\mathcal{L}[e_o] = \mathcal{L}[i_1 + R_2] + \mathcal{L}\left[\frac{1}{C_2}\int i_1\,dt\right]$$
$$\mathcal{E}_o(s) = I_1(s) + R_2 + \frac{1}{C_2 s} I_1(s)$$

$$\mathcal{E}_o(s) = I_1(s)\left[R_2 + \frac{1}{C_2 s}\right]$$

$$I_1(s) = \frac{\mathcal{E}_o(s)}{\left[R_2 + \frac{1}{C_2 s}\right]}$$

$$\mathcal{E}_i(s) = I_1(s)\left[R_1 + \frac{1}{C_1 s}\right] + \mathcal{E}_o(s)$$

$$\mathcal{E}_i(s) = \frac{\mathcal{E}_o(s)}{\left[R_2 + \frac{1}{C_2 s}\right]} \cdot \left[R_1 + \frac{1}{C_1 s}\right] + \mathcal{E}_o(s)$$

$$\mathcal{E}_i(s) = \mathcal{E}_o(s)\left[\frac{R_1 + \frac{1}{C_1 s}}{R_2 + \frac{1}{C_2 s}} + 1\right]$$

$$\boxed{\frac{\mathcal{E}_i(s)}{\mathcal{E}_o(s)} = \frac{R_1 + \frac{1}{C_1 s}}{R_2 + \frac{1}{C_2 s}} + 1}$$

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4. $G(s) = \dfrac{2s+4}{s^2+6s+10} = \dfrac{2s}{(s+3)^2+1} + \dfrac{4}{(s+3)^2+1}$

$$\mathcal{L}^{-1}\left[2 \cdot \underbrace{\frac{s+3}{(s+3)^2+1^2}}\right] - \mathcal{L}^{-1}\left[2 \cdot \underbrace{\frac{1}{(s+3)^2+1^2}}\right]$$

$$\left[2e^{-3t}\cos(t)\right] - \left[2e^{-3t}\sin(t)\right]$$

$$\boxed{\mathcal{L}^{-1}[G(s)] = g(t) = 2e^{-3t}\cos(t) - 2e^{-3t}\sin(t)}$$

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5. $G(s) = \dfrac{s^2+s+2}{(s+1)^3} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{C}{(s+1)^3}$

$s^2+s+2 = A(s+1)^2 + B(s+1) + C$

$s^2+s+2 = As^2 + \underline{2As} + A + \underline{Bs} + 1B + C$

$A = 1$ $\qquad 2As + Bs = s \qquad A + B + C = 2$

$2 + B = 1 \qquad 1 + 1 + C = 2$

$B = -1 \qquad C = 2$

$$\mathcal{L}^{-1}[G(s)] = \underbrace{\left[\frac{1}{s+1}\right]\mathcal{L}^{-1}}_{e^{-t}} + \underbrace{\left[\frac{-1}{(s+1)^2}\right]\mathcal{L}^{-1}}_{-te^{-t}} + \underbrace{\left[\frac{2}{(s+1)^3}\right]\mathcal{L}^{-1}}_{t^2e^{-t}}$$

$$e^{-t} \quad -te^{-t} \quad + \quad t^2e^{-t}$$

$$\boxed{\mathcal{L}^{-1}[G(s)] = g(t) = e^{-t} - te^{-t} + t^2e^{-t}}$$