Final

1 a) $y_{ss} = \cancel{A}\, |G(s)| \sin(\omega t + \phi)$ $\qquad u(t) = .5\sin(20t)$

$y_{ss} = .5\, |G(s)| \sin(20t + \phi)$

$$G(j\omega) = \frac{100 j\omega + 100}{j\omega((j\omega)^2 + 16 j\omega + 100)} = \frac{1\cancel{00}(j\omega + 1)}{j\omega\left(\frac{(j\omega)^2}{100} + \frac{16}{100} j\omega + 1\right)} \cdot \frac{1}{1\cancel{00}} = \frac{j\omega + 1}{j\omega\left(\left(\frac{j\omega}{10}\right)^2 + \frac{16}{100} j\omega + 1\right)}$$

$$\frac{1}{\left|\frac{100 j\omega + 100}{j\omega(j\omega)^2 + 16 j\omega + 100}\right|} \Bigg|_{\omega = 20}$$

$$= \frac{1}{\left|\frac{100(20)j + 100}{20j((20j)^2 + 16(20)j + 100)}\right|} \qquad 20^2 = -400$$

$$= \frac{1}{\left|\frac{2000j + 100}{20j(320j - 300)}\right|}$$

$$= \left|\frac{64\cancel{00}j^2 - 60\cancel{00}j}{200\cancel{0}j + 1\cancel{00}}\right|$$

$$= \left|\frac{64j^2 - 60}{20j + 1}\right|$$

$$= \frac{|-124|}{|20j + 1|}$$

$$|G(j\omega)| = \frac{\sqrt{\omega^2 + 1}}{\omega\sqrt{\left(1 - \left(\frac{\omega}{10}\right)^2\right)^2 + \left(\frac{16}{10}\right)^2}}$$

$$\text{gain} = 20\log\left(\sqrt{\omega^2 + 1}\right) - 20\log(\omega) - 20\log\left(\sqrt{\left(1 - \left(\frac{\omega}{10}\right)^2\right)^2 + \left(\frac{16\omega}{10}\right)^2}\right)$$

$$= 20\log\left(\sqrt{20^2 + 1}\right) - 20\log(20) - 20\log\sqrt{\left(1 - \left(\frac{20}{10}\right)^2\right)^2 + \left(\frac{16(20)}{10}\right)^2}$$

$$= 21.911$$

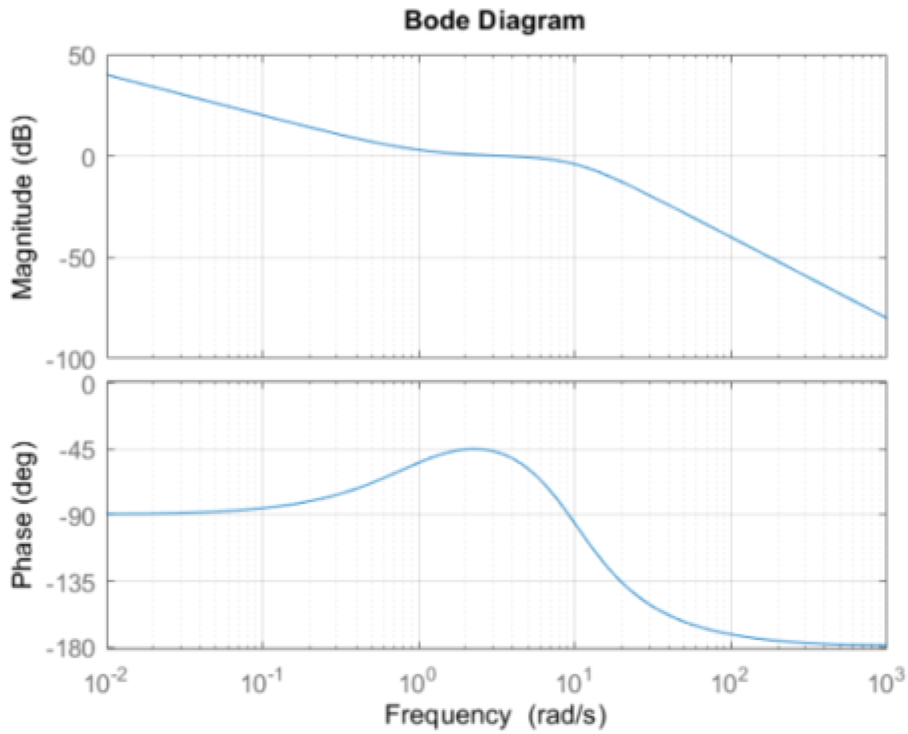
$$\text{phase} = \angle G(j\omega) = \angle \frac{(j\omega + 1)}{j\omega((j\omega)^2 + 16 j\omega + 100)}$$

$$= \tan^{-1}(\omega) - 90^\circ - \tan^{-1}\left(\frac{16\omega/10}{1 - \left(\frac{\omega}{10}\right)^2}\right) = \tan^{-1}(20) - 90 - \tan\left(\frac{16(20)/10}{1 - \left(\frac{20}{10}\right)^2}\right) = 81.7818^\circ$$

$$\boxed{y_{ss} = .5(21.911)\sin(20t + 81.7818^\circ)}$$

1b)

```
sys = tf([100,100],[1,16,100,0])
bode(sys)
grid on
```



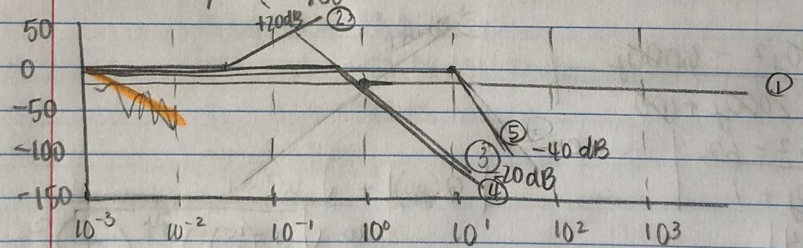
Final

2. $G(s) = \dfrac{200s + 10}{s(s+1)(s^2+16s+100)} = \dfrac{\cancel{10}(20s+1)}{s(s+1)\left(\left(\frac{s}{10}\right)^2 + \frac{16s}{100} + 1\right)} \cdot \dfrac{1}{10\cancel{0}} = \dfrac{.1(\overbrace{20s+1}^{\frac{s}{.05}+1})}{s(s+1)\left(\left(\frac{s}{10}\right)^2 + \frac{16s}{100} + 1\right)}$

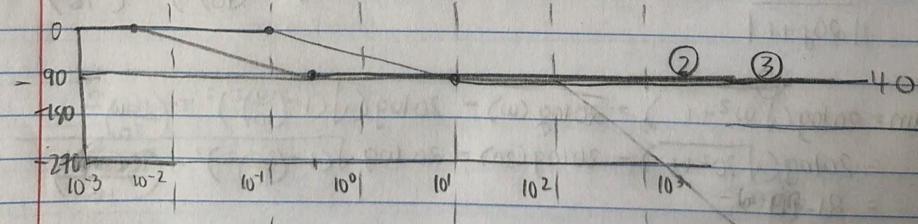
	factor	gain	phase
1	$.1$	$20\log(.1) = -20$	0
2	$\frac{s}{.05} + 1$ [$\omega_0 = .05$]	0 dB until .05 then +20dB	[0 until] $.1(.05) = .005$; $10(.05) = .5$ [$90°$ after]; low $0°$ then high $-90°$
3	$\frac{1}{s}$	-20dB goes thru 0 @ $\omega = 1$	$-90°$
4	$\frac{1}{\frac{s+1}{1}}$ $\omega_0 = 1$	0 until $\omega = 1$ then -20	[0 until] $.1(1) = .1$ 1×10^{-1}; $10(1) = 10$ [$-90°$ after]
5	$\left(\frac{s}{10}\right)^2 + \frac{16s}{100} + 1$ $\omega_0 = 10$	0 dB until 10 then -40	$\frac{10}{10^8} = 1\times10^{-7}$ $\underline{0°}$; $10(10^8) = 10^9$ $\underline{-180°}$

$\left(\frac{s}{10}\right)^2 + 16\frac{s}{100} + 1$

$2\zeta\cancel{\frac{s}{100}} = 16\cancel{\frac{s}{100}} \Rightarrow \zeta = 8$

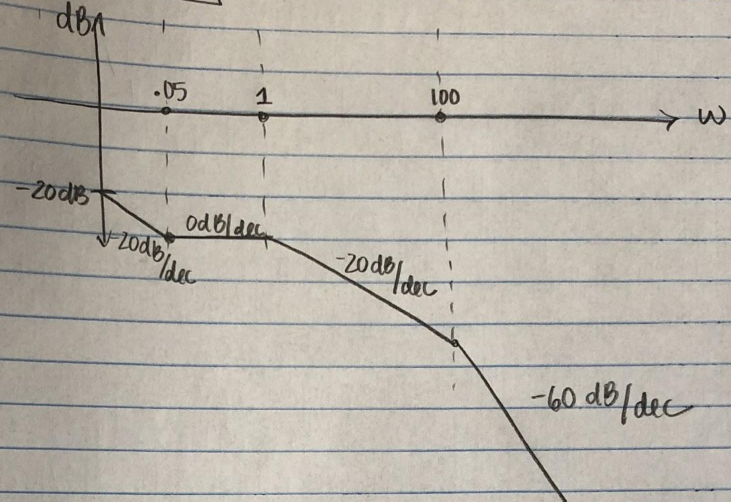


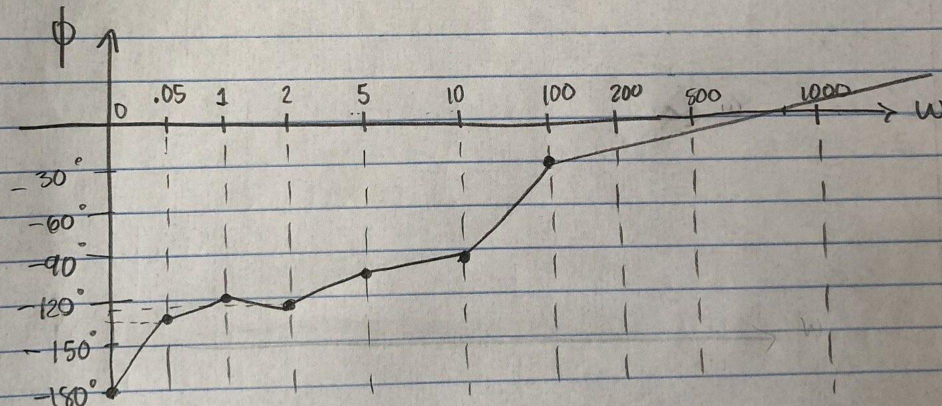
$\frac{x}{.05} =$



Final

2 cont. Final plots:





Final

3. a) gain crossover frequency: Magnitude (in dB) = 0

$\Rightarrow \boxed{2 \text{ rad/sec}}$

phase crossover frequency: phase ∠ (in deg) = 180°

$\Rightarrow$ phase ∠ is −180° between 3 rad/s & 4 rad/sec

$\Rightarrow$ so phase crossover frequency $= \sqrt{3 \cdot 4} = \sqrt{12} = \boxed{3.46 \text{ rad/s}}$

gain margin = − Mag @ cross over frequency

Slope from frequency 2 rad/sec onward is −40 dB/dec

Mag @ phase crossover frequency = Mag @ 2 rad/s − Slope (log (phase crossover freq) − log(2))

$= 0 - 40\log(3.46/2)$

$= \boxed{-9.54 \text{ rad/sec}}$

Phase margin = 180 + phase ∠ @ cross over

$= 180 + (-135°)$

$= \boxed{45°}$

b) both the gain and phase margins are positive so the system is $\boxed{\text{stable}}$

c) OLTF for bode plot is $\dfrac{k}{s(s+2)}$

@ .2 rad/s mag = 20 dB

$20 \log\left(\dfrac{k}{.2(2)}\right) = 20$

$k = .2(2)(10) = 4$

$$\text{CLTF} = \frac{4}{s^2+2s+4}$$

damping factor = .5

max overshoot $= e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} = e^{-\left(\frac{.5\pi}{\sqrt{1-.5^2}}\right)} = \boxed{.1630}$

Final

3 cont. d) Initial value of slope is equal to -20 dB/dec, because of the pole @ the origin. From the OLTF we can see that initial (before 2 rad/s) only effects the pole @ origin. So initial slope is due to pole @ origin.

4. phase margin = 30°
gain crossover freq = 15 rad/sec

Safty factor ∠ = 15°
$\phi = 30 + 15 = 45°$

lead compensator: $G_c(s) = \dfrac{(s\tau + 1)}{(1 + \alpha \tau s)}$

phase ∠: $\tan^{-1}(w\tau) - \tan^{-1}(\alpha w \tau) = \phi$

$$\phi = \tan^{-1}\left(\frac{w\tau - \alpha w \tau}{1 + w^2 \alpha \tau^2}\right)$$

$$\sin\phi = \frac{1-\alpha}{1+\alpha}$$

$$\alpha = \frac{1 - \sin\phi}{1 + \sin\phi}$$

$$\alpha = \frac{1 - \sin(45°)}{1 + \sin(45°)} = \frac{1 - .707}{1 + .707} = \underline{.1716}$$

$$\tau = \frac{1}{w_c\sqrt{\alpha}} = \frac{1}{15\sqrt{.1716}} = .1610$$

gain:

$$\left|\frac{k_c\sqrt{(w_c\tau)^2 + 1}}{\sqrt{(w_c\alpha\tau)^2 + 1}}\right| = 1$$

$$\left|\frac{k_c\sqrt{(15(.1610))^2 + 1}}{\sqrt{(15(.1716)(.1610))^2 + 1}}\right| = 1$$

$\underline{k = .5}$

Final

4 cont. System design:

$PM = 45°$

gain = 15 rad/sec

$\phi = 60°$

$$\frac{1-\alpha}{1+\alpha} = \sin\phi$$

$$\alpha = \frac{1-\sin\phi}{1+\sin\phi} = \frac{1-\sqrt{3}/2}{1+\sqrt{3}/2} = \frac{.134}{1.866} = .071$$

$$T = \frac{1}{w_c\sqrt{\alpha}} = \frac{1}{15\sqrt{.071}} = .25$$

final lead Compensator function:

$$G_c(s) = \frac{k_c(sT+1)}{(sT\alpha+1)}$$

$$\boxed{G_c(s) = \frac{.5(.25s+1)}{1+.0175}}$$