Homework 7

1. $G(s) = \dfrac{k}{s(s+1)(s+8)}$

for $k = 10$:

$$G(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+8)} = \frac{10}{(j\omega-\omega^2)(j\omega+8)} = \frac{10}{-\omega^2+8j\omega-j\omega^3-8\omega^2}$$

$$= \frac{10}{-9\omega^2+j(8\omega-\omega^3)}$$

@ ω_{pc}, $G(j\omega)=0$

$$8\omega-\omega^3=0$$

$$\omega = \cancel{0},\ 2\sqrt{2} \text{ rad/sec}$$

$$\omega_{pc} = 2\sqrt{2} = 2.8284 \text{ rad/sec}$$

Gain margain = 20log (mag)

$$\text{mag} = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{\frac{10}{|-9(2\sqrt{2})^2|}} = 7.2$$

$$20\log(7.2) = \boxed{17.1466 \text{ dB}}$$

Cross-over frequency:

$$\frac{10}{\sqrt{(8\omega-\omega^3)^2+(-9\omega^2)^2}} = 1$$

$$100 = (8\omega-\omega^3)^2+(-9\omega^2)^2$$

$$100 = \omega^2(8-\omega^2)^2+81\omega^4$$

$$100 = \omega^2(64-16\omega^2+\omega^4)+81\omega^4$$

$$100 = 64\omega^2-16\omega^4+\omega^6+81\omega^4$$

$$100 = \omega^6+65\omega^4+64\omega^2$$

$$\omega^6+65\omega^4+64\omega^2-100=0$$

$$\omega = \boxed{.9158 \text{ rad/sec}}$$

Phase margin:

$$\phi = \angle G(j\omega)|_{\omega=.9158} = -\left[180-\tan^{-1}\left(\frac{8\omega-\omega^3}{9\omega^2}\right)\right]$$

$$= -\left[180-\tan^{-1}\left(\frac{8(.9158)-(.9158)^3}{9(.9158)^2}\right)\right]$$

$$= -139.0139$$

$$180-139.0139 = \boxed{40.986^\circ}$$

Homework 7

1 Cont.

for $K = 1000$:

$$G(j\omega) = \frac{1000}{-9\omega^2 + j(8\omega - \omega^3)}$$

@ ω_{pc} $G(j\omega) = 0$

$$8\omega - \omega^3 = 0$$

$$\omega = 2\sqrt{2}$$

Gain margin:

$$\frac{1}{\left|\frac{1000}{-9\omega^2}\right|} = \frac{1}{\left|\frac{1000}{9(2\sqrt{2})^2}\right|} = .072$$

$$20\log(.072) = \boxed{-22.8533 \text{ dB}}$$

Cross-over frequency:

$$\frac{1000}{\sqrt{(8\omega-\omega^3)^2 + (-9\omega^2)^2}} = 1$$

$$10^6 = (8\omega - \omega^3)^2 + (81\omega^4)$$

$$10^6 = 64\omega^2 - 16\omega^4 + \omega^6 + 81\omega^4$$

$$10^6 = \omega^6 + 65\omega^4 + 64\omega^2$$

$$\omega = \boxed{9.06665 \text{ rad/sec}}$$

Phase Margin:

$$PM = 180 + \phi$$

$$\phi = |G(j\omega)|_{\omega = 9.06665} = -\left[180 + \tan^{-1}\left(\frac{8\omega - \omega^3}{-9\omega^2}\right)\right]$$

$$= -\left[180 + \tan^{-1}\left(\frac{\omega^2 - 8}{9\omega}\right)\right]$$

$$= -\left[180 + \tan^{-1}\left(\frac{9.06665^2 - 8}{9(9.06665)}\right)\right]$$

$$= -[180 + 42.28]$$

$$= -222.2822°$$

$$PM = 180 - 222.2822 = \boxed{-42.2822°}$$

Homework 7

2. $G(s) = \dfrac{K}{s(s^2+s+4)}$

$$G(jw) = \frac{K}{jw((jw)^2+jw+4)} = \frac{K/4}{jw\left(\frac{-w}{4} + \frac{jw}{4} + 1\right)} = \frac{K/4}{jw\left(\left(\frac{jw}{2}\right)^2 + \frac{jw}{4} + 1\right)}$$

$$\left[\begin{array}{l} \star \ s^2 + 2\xi_\varphi w_0 s + w_0^2 \\ \star \ \left(\frac{s}{w_0}\right)^2 + 2\xi_\varphi\left(\frac{s}{w_0}\right) + 1 \end{array}\right]$$

$w_0 = 2$ $\quad 2 \cdot \xi_\varphi \cdot \frac{\cancel{s}}{\cancel{w_0}} = \frac{\cancel{jw}}{\cancel{2}} \cdot \frac{1}{2}$

$\xi_\varphi = \frac{1}{4}$

$$\phi = -90° - \tan^{-1}\left[\frac{2\xi_\varphi \frac{w}{w_0}}{1-\left(\frac{w}{w_0}\right)^2}\right]$$

$$\phi = -90° - \tan^{-1}\left[\frac{2 \cdot \frac{1}{4} \cdot \frac{w}{2}}{1-\left(\frac{w}{2}\right)^2}\right]$$

$PM = 180 + \phi \Rightarrow 50 = 180 + \phi \Rightarrow -130 = \phi$

$$-130 = -90° - \tan^{-1}\left[\frac{\frac{w}{4}}{1-\frac{w^2}{4}}\right]$$

$$40° = \tan^{-1}\left[\frac{\frac{w}{4}}{1-\frac{w^2}{4}}\right]$$

$w = \cancel{-2.68276}$, $\underline{1.49}$

Cross-over frequency:

$$|G(jw)|_{w=1.49} = \left|\frac{K}{jw((jw)^2+jw+4)}\right| = \left|\frac{K}{1.49j((1.49j)^2 + 1.49j + 4}\right|$$

$$= \left|\frac{K}{1.49j(-2.22 + 1.49j + 4)}\right|$$

$$= \left|\frac{K}{1.49j(1.77 + 1.49j)}\right|$$

$$1 = \frac{K}{1.49\sqrt{1.77^2 + (1.49)^2}}$$

$$1.49^2(1.77^2) + 1.49^4 = K^2$$

$$\boxed{K = 3.46}$$

Homework 7

2 Cont. Gain margin:

gain cross-over frequency:

$$\phi = -90° - \tan^{-1}\left(\frac{w_{pc}}{4-w_{pc}^2}\right) = -180°$$

$$\tan^{-1}\left(\frac{w_{pc}}{4-w_{pc}^2}\right) = 90°$$

$$4 - w_{pc}^2 = 0$$

$$4 = w_{pc}^2$$

$$w_{pc} = 2$$

$$k_g = \frac{1}{|G(jw_{pc})|}$$

$$= \frac{1}{\left|\frac{3.46}{jw((jw)^2+jw+4)}\right|_{w=2}}$$

$$= \frac{1}{\left|\frac{3.46}{2j(\cancel{(2j)^2}+j2\cancel{+4})}\right|}$$

$$= \frac{|4j^2|}{3.46}$$

$$= 1.156$$

$$20\log(1.156) = \boxed{1.26\ dB}$$

Homework 7

3. $G(s) = k\left(\frac{s+.1}{s+.5}\right)\left(\frac{10}{s(s+1)}\right) = \frac{10k(s+.1)}{s(s+.5)(s+1)} = \frac{2k(1+10s)}{s(s+1)(2s+1)}$

$G(jw) = \frac{2k(10jw+1)}{jw(jw+1)(2jw+1)}$

magnitude:

$|G(jw)| = \frac{2\sqrt{(10w)^2+1}}{w\sqrt{w^2+1}(\sqrt{4w^2+1})}$

gain $= 20\log(2) + 20\log(\sqrt{100w^2+1}) - 20\log(w) - 20\log(\sqrt{w^2+1}) - 20\log(\sqrt{4w^2+1})$

$\angle G(jw) = \angle \frac{(10jw+1)}{jw(jw+1)(2jw+1)}$

$\phi = \tan^{-1}(10w) - 90° - \tan^{-1}(w) - \tan^{-1}(2w) = -130$

$PM = 180 + \phi = 50 \Rightarrow \phi = -130$

$w = 1.44$ rad/sec

magnitude @ $w = 1.44$:

gain $= 20\log(2) + 20\log(\sqrt{100(1.44)^2+1}) - 20\log(1.44) - 20\log(\sqrt{1.44^2+1}) - 20\log(\sqrt{1.44^2(4)+1})$

$= 11.5$ dB

$20\log(k) = -11.5$ dB

$\boxed{k = .2666}$

or

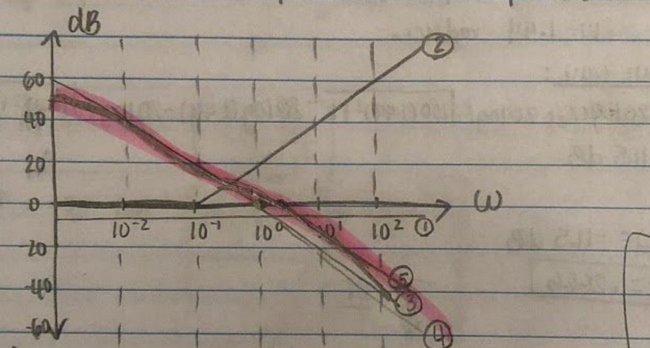
$\left|\frac{2k(10j(1.44)+1)}{j(1.44)(1.44j+1)(2(1.44)j+1)}\right| = 1$

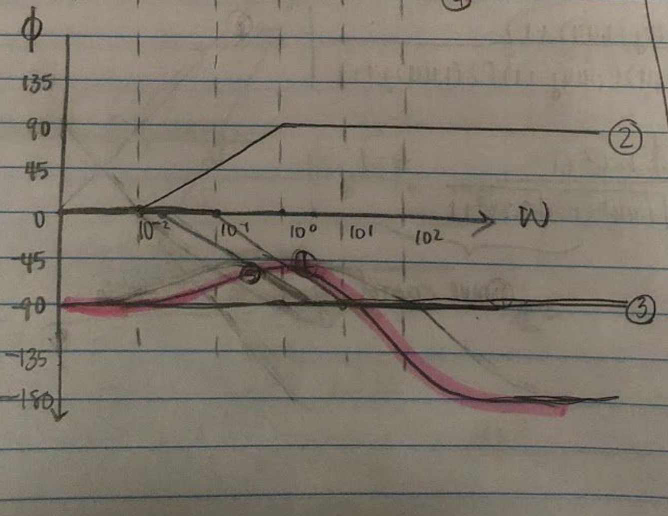
$\frac{2k\sqrt{14.4^2+1}}{1.44\underbrace{\sqrt{1.44^2+1}\sqrt{2.88^2+1}}} = 1$

Solve and get k = .2666

3 cont. $G(s) = \frac{2(.2666)(1+10s)}{s(1+s)(1+2s)} = \frac{.532(1+10s)}{s(1+s)(1+2s)} = \frac{.532(\frac{s}{.1}+1)}{s(s+1)(\frac{s}{.5}+1)}$

	factor	gain	phase	
①	.532	20log(.532) = −5.482	0	
②	$w_0 = .1$	0 until .1 (10^{-1}) then 20	.1(.1) = .01 10(.1) = 1	> +90° = 1×10^{-2} 1×10^{0}
③	$\frac{1}{s}$	−20, go thru 0 @ w=1 10^{0}	−90°	
④	$w_0 = 1$	0 until 1 then −20	.1(1) = .1 10(1) = 10	> −90° 10^{-1} 10^{1}
⑤	$w_0 = .5$	0 until .5 then −20	.1(.5) = .05 10(.5) = 5	> −90°





Since the values for all the frequencies are greater than 180°, the gain margin is $\boxed{+\infty}$

Homework 7

4. (Bonus)

OLTF: $G(s) = \frac{1}{s(s+1)}$

$PM = 180° + \angle G(j\omega) H(j\omega)|_{\omega} = \omega_{gc}$

$60° = 180° + \angle \underbrace{G(j\omega)}_{\phi_c} \cdot G_c(j\omega) H(j\omega)|_{\omega = \omega_{gc}}$

$\phi_c = -120°$

TF of lead compansator:

$G_c(s) = \frac{1+TS}{1+\alpha TS}$

phase of $G(s) = \frac{1}{S(1+S)}$

$\angle G(j\omega) = \angle \frac{1}{j\omega(1+j\omega)}|_{\omega = \omega_{gc}}$

$= -90° - \tan^{-1}(\omega_{gc})$

where $\omega_{gc} = 10$ rad/sec

$= -90° - \tan^{-1}(10) = -174.28°$

Desired phase of compansator:

$\angle G(j\omega) + \angle G_c(j\omega) = -120°$

$\angle G_c(j\omega) = -120° + 174.28°$

$\angle G_c(j\omega) = 54.28°$

$\angle G_c(j\omega) = \angle \frac{1 + j\omega_{gc} T}{1 + j\omega_{gc}\alpha T} = 54.28°$

$\tan\left[\tan^{-1}(\omega_{gc} T) - \tan^{-1}(\omega_{gc}\alpha T)\right] = [54.28°] \tan$

$= \frac{\omega_{gc}T - \omega_{gc}\alpha T}{1 + \omega_{gc}T \cdot \omega_{gc}\alpha T}$

$= \frac{\omega_{gc}T - \omega_{gc}\alpha T}{1 + \omega_{gc}^2 T^2 \alpha} = 1.39$

$\frac{10T - 10\alpha T}{1 + 100T^2\alpha} = 1.39$ Let $\alpha = .1$

$\frac{9T}{1 + 10T^2} = 1.4$

$9T = 1.39 + 13.9T^2$

$13.9T^2 - 9T + 1.39 = 0 \Rightarrow T = .254$ or $.393$

$$\boxed{G_c(s) = \frac{1 + .254S}{1 + .0254S}}$$

where $T = .254$

$\alpha = .1$