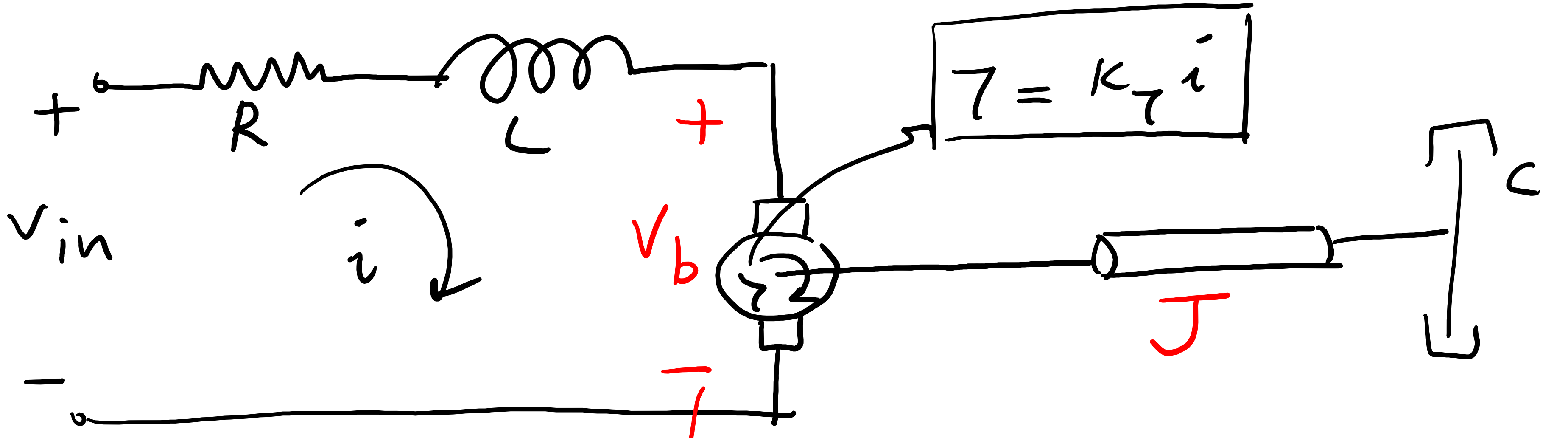
Dynamics of a DC Motor:

↳ Apply a voltage → Rotational motion



"Load" voltage due to the rotation of the armature (AKA Back EMF)

$$\boxed{V_b = K_b \frac{d\theta}{dt}}$$

Elec: (KVL)

$$\boxed{V_{in} = Ri + L\frac{di}{dt} + V_b}$$

$V_b = K_b\dot\theta$

Mech: (Newton's 2nd law)

$$J\ddot\theta = \tau - c\dot\theta$$

$\tau = K_\tau i$

$$K_\tau i = J\ddot\theta + c\dot\theta$$

$$\boxed{i = \frac{J}{K_\tau}\ddot\theta + \frac{c}{K_\tau}\dot\theta}$$

$$V_{in} = \underbrace{\frac{RJ}{K_\tau}\ddot\theta + \frac{RC}{K_\tau}\dot\theta}_{Ri} + \underbrace{\frac{LJ}{K_\tau}\theta^{(3)} + \frac{LC}{K_\tau}\ddot\theta}_{L\frac{di}{dt}} + K_b\dot\theta$$

$$\omega = \dot\theta$$

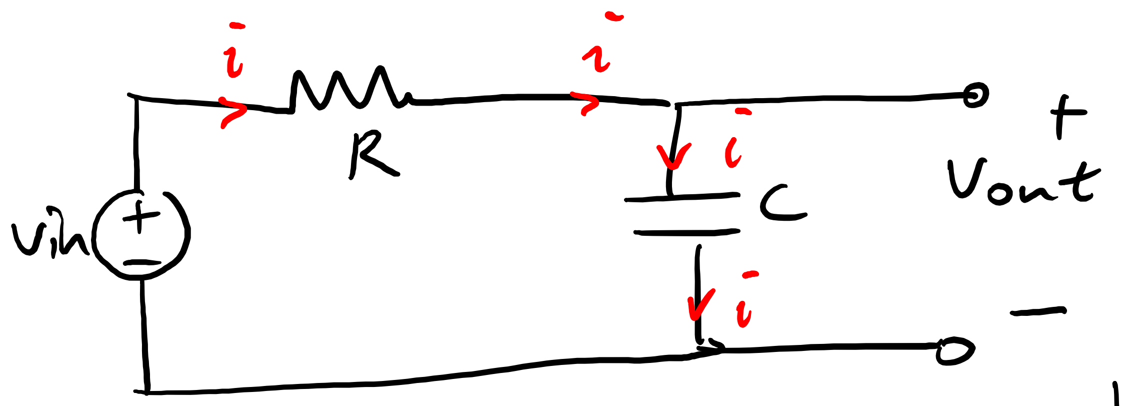
$$\boxed{V_{in}} = \cancel{\frac{LJ}{K_\tau}}^{\approx 0}\,\ddot{\omega} + \cancel{\left(\frac{LC+RJ}{K_\tau}\right)}^{\approx 0}\dot{\omega} + \left(\frac{RC}{K_\tau} + K_b\right)\omega$$

For small L:

$$\left[V_{in} = \frac{RJ}{K_\tau}\ddot{\theta} + \left(\frac{RC}{K_\tau} + K_b\right)\dot{\theta}\right]\frac{K_\tau}{R}$$

$$J\ddot{\theta} + \underbrace{\left(C + \frac{K_b K_\tau}{R}\right)}\dot{\theta} = K_\tau \frac{V_{in}}{R}$$

Back EMF acts as viscous friction! / damping



$$\underline{V_{in}} = iR + V_{out}$$

$$i = C\frac{dV_{out}}{dt}$$

$$\boxed{V_{in} = RC\dot{V}_{out} + V_{out}}$$

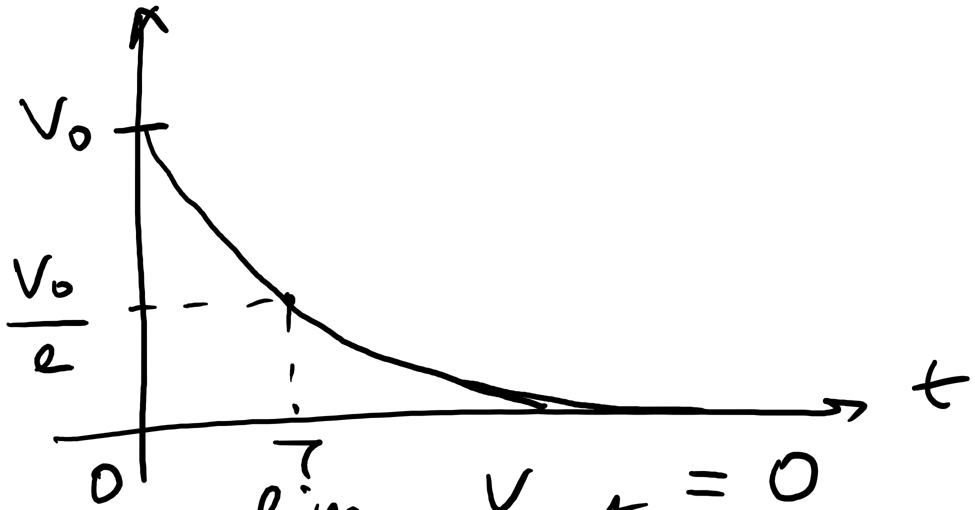
Natural (free) Response:

$v_{in} = 0$

$$(RC)\dot{V}_{out} + V_{out} = 0 \quad \text{(1st order ODE)}$$

$$V_{out} = K e^{-t/RC} \Rightarrow \boxed{V_{out} = V_0 e^{-t/RC}}$$

$$V_0 = V_{out}(0)$$



$$\lim_{t \to \infty} V_{out} = 0$$

$$\frac{V_0}{e} = V_0 e^{-\tau/RC} \Rightarrow \boxed{\tau = RC}$$

$\tau = RC$: time constant

the time required for the voltage to fall to V_0/e.

Forced Response

V_{in} = constant

$$RC\dot{V}_{out} + V_{out} = V_{in} \equiv \text{constant}$$

$$V_{out} = \underbrace{K_1 e^{-K_2 t}}_{\substack{\text{free response} \\ \text{(transient)}}} + \underbrace{K_3}_{\substack{\text{forced response} \\ \text{(steady-state)}}}$$

$$\dot{V}_{out} = -K_1 K_2 e^{-K_2 t}$$

$$RC\underbrace{\left(-K_1K_2e^{-K_2t}\right)}_{\dot{v}_{out}} + \underbrace{K_1e^{-K_2t}+K_3}_{V_{out}} = V_{in}$$

$$\boxed{K_3 = V_{in}}$$

$$RC\left(-K_1K_2e^{-K_2t}\right) + K_1e^{-K_2t} = 0$$

$$(K_1)\left(e^{-K_2t}\right)\left(1 - RCK_2\right) = 0$$

$$\cancel{K_1 = 0}$$

$$\cancel{e^{-K_2t} = 0}$$

$$1 - RCK_2 = 0 \Rightarrow \boxed{K_2 = \frac{1}{RC} = \frac{1}{\tau}}$$

$K_1 = ?$ use initial value

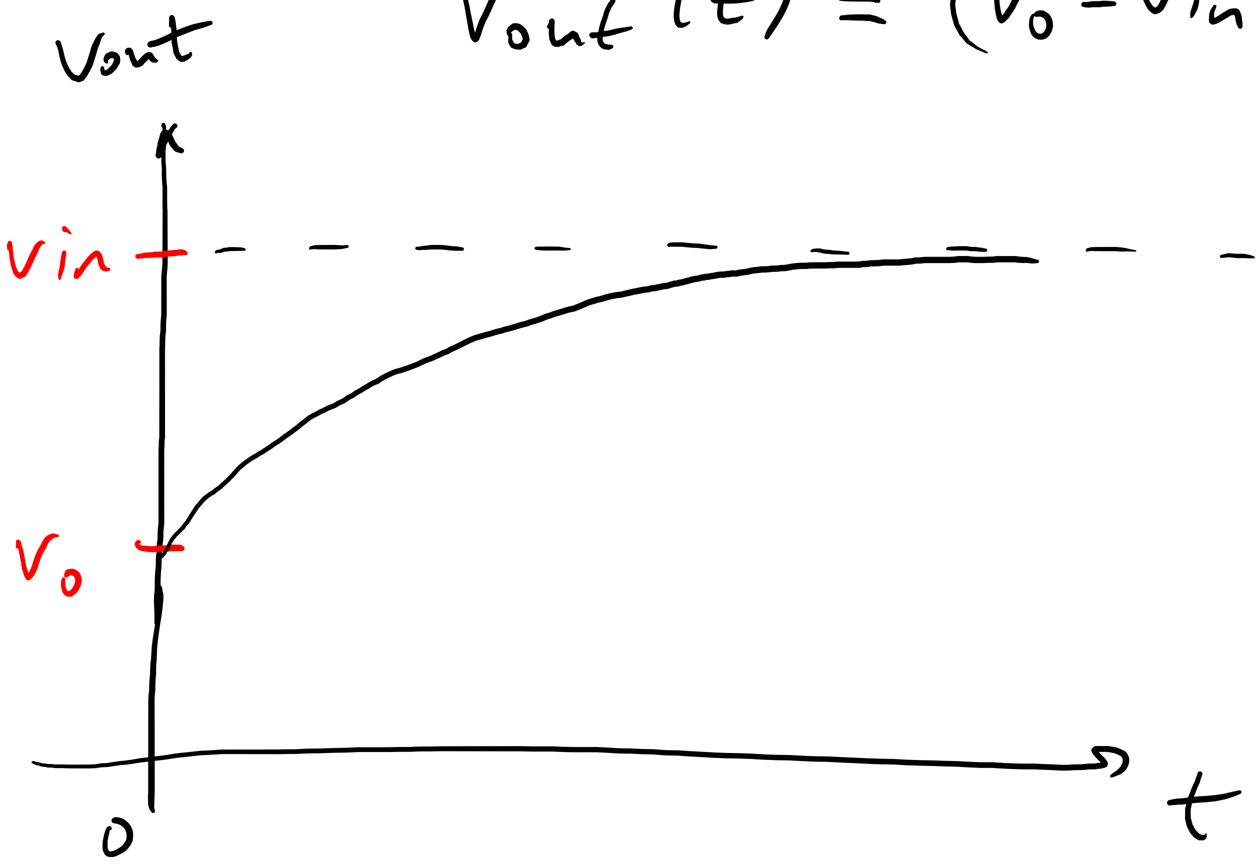
$$V_{out}(0) = V_0$$

$$V_{out} = K_1e^{-t/RC} + V_{in}$$

$$V_{out}(0) = K_1 + V_{in} = V_0$$

$$\boxed{K_1 = V_0 - V_{in}}$$

$$V_{out}(t) = (V_0 - V_{in})\underbrace{e^{-t/RC}}_{1} + V_{in}$$



when $V_{in} = 0$

$$\Rightarrow \underline{V_{out}(t) = V_0\, e^{-t/RC}}$$

time constant : $\underline{\underline{\tau = RC}}$ transient

$$\lim_{t \to \infty} V_{out} = V_{in}$$

steady-state

* We have 2 significant problems:
  1) We don't know how to use diff. eqn's as system models in block diagrams.
  2) It is very difficult to solve these eqn's in time domain for GENERAL input func's.

<u>Solution:</u>

LAPLACE TRANSFORM

BIG PICTURE

Module 2

Differential Eqn's

Laplace Transform

State-space equations

Transfer functions

Numerical solver

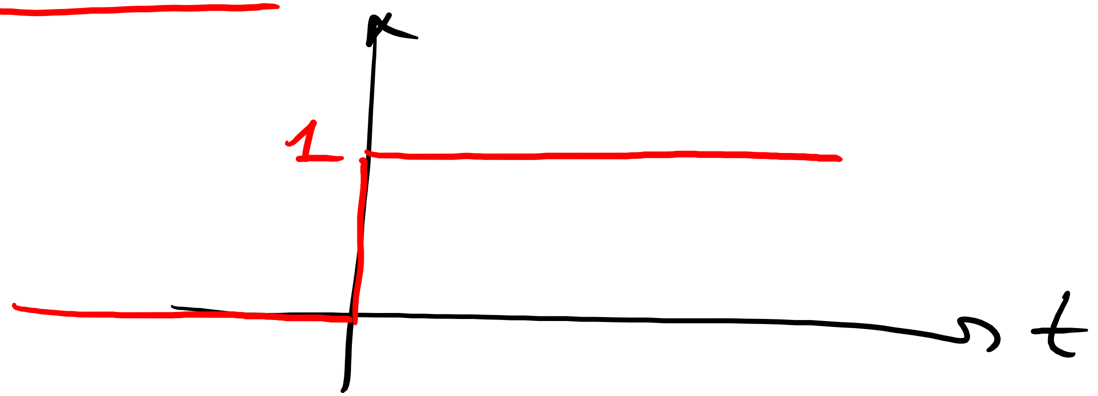
Block Diagrams

Control Design and Analysis

Standard Input Functions:

Unit Step Function:

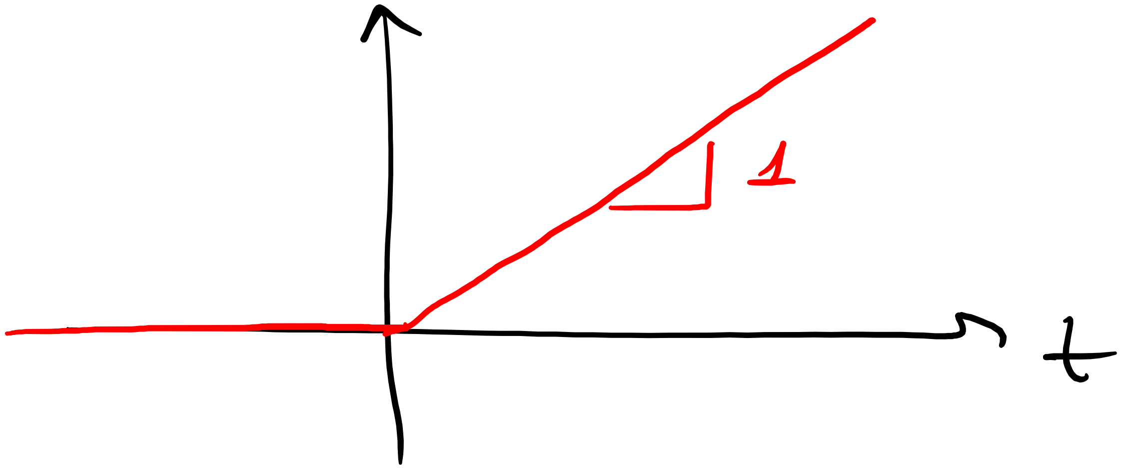
$$u(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$V_{in} \cdot 1(t) \longrightarrow$ A step input of V_{in} at $t=0$.

Unit Ramp Function:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Unit Impulse Function (Dirac Delta)

Defined only $t=0$

$\delta(t) = 0 \qquad t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$