Laplace Transform Tables

Time Function	LaPlace Transform
Unit Impulse, $\delta(t)$	1
Unit step, $u_s(t)$ $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
1) $C\ e^{-\alpha t}$	$C\ \frac{1}{s+\alpha}$
2) → $C\ te^{-\alpha t}$	$C\ \frac{1}{(s+\alpha)^2}$
$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$

Time Function	LaPlace Transform
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
3) $C\ e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
3) $C\ e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)$ for $(\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{-\omega_n^2 e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\theta)$ where $\theta=\cos^{-1}(\zeta)$ and $(\zeta<1)$	$\frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

2nd order dynamic response

1.) Please consider the differential equation below with zero initial conditions $x(0) = \dot{x}(0) = 0$.

impulse function

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = u(t)$$

a) If $u(t) = 3\delta(t)$, $X(s) = \mathrm{L}[x(t)] = ?$

b) Find the solution $x(t)$ to the differential equation by finding the inverse Laplace transform of $X(s)$ for the same input.

a) $\mathcal{L}[\ddot{x} + 5\dot{x} + 6x] = \mathcal{L}[u]$

$s^2 X(s) + 5sX(s) + 6X(s) = 3$

$X(s)(s^2 + 5s + 6) = 3$

$$\boxed{X(s) = \frac{3}{s^2 + 5s + 6}} = \frac{3}{(s+3)(s+2)}$$

$s \quad 3$

$s \quad 2$

b) PFE:

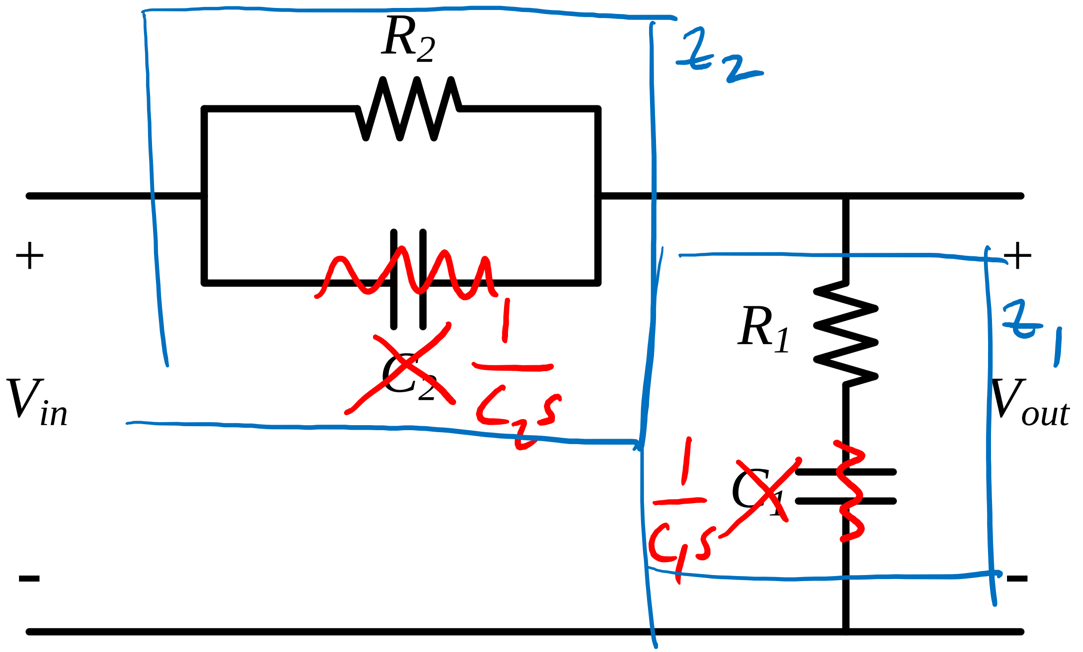
$$X(s) = \frac{3}{(s+2)(s+3)} = \frac{c_1}{s+2} + \frac{c_2}{s+3}$$

$$c_1 = \Big[X(s)(s+2)\Big]_{s=-2} = \left[\frac{3}{\cancel{(s+2)}(s+3)}\cancel{(s+2)}\right]_{s=-2} = \frac{3}{-2+3} = \boxed{3}$$

$$c_2 = \Big[X(s)(s+3)\Big]_{s=-3} = \left[\frac{3}{s+2}\right]_{s=-3} = \boxed{-3}$$

$$\boxed{x(t) = 3e^{-2t} - 3e^{-3t}, \quad t > 0}$$

2) Please find the transfer function Vout(s) / Vin(s) for the following electrical system:



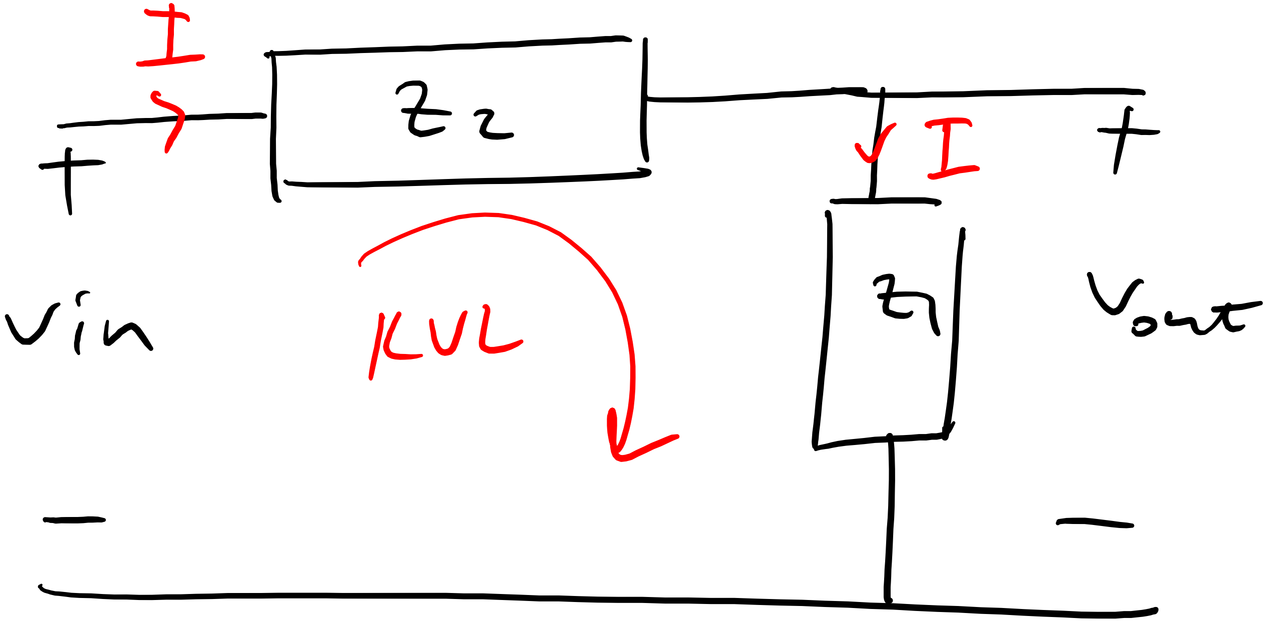
$$\frac{1}{z_2} = \frac{1}{R_2} + C_2 s$$

impedances in parallel

$$\boxed{z_2 = \frac{1}{1/R_2 + C_2 s}}$$

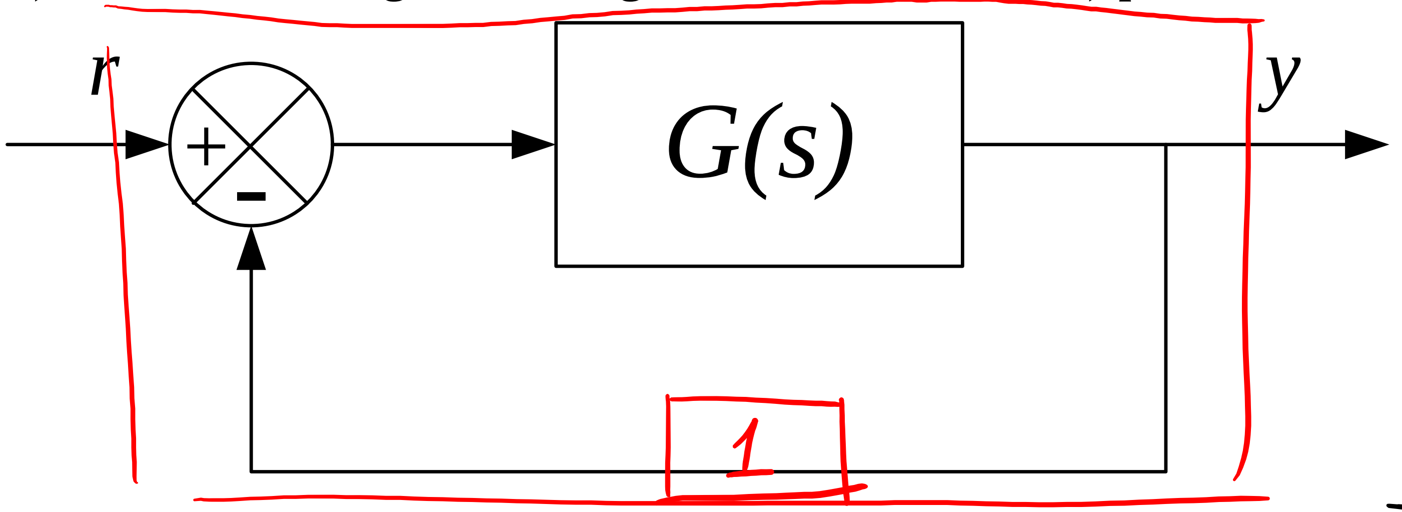
Impedances in series:

$$\boxed{z_1 = R_1 + \frac{1}{C_1 s}}$$



$$\boxed{\frac{V_{out}}{V_{in}} = \frac{Z_1}{Z_1 + Z_2}}$$

3) In the following block diagram, if G(s) = 1/(s+2) please calculate the closed-loop transfer function Y(s)/R(s).



$$\frac{Y(s)}{R(s)} = \frac{G}{1+G} = \frac{\frac{1}{s+2}}{1+\frac{1}{s+2}} = \frac{\frac{1}{\cancel{s+2}}}{\frac{s+3}{\cancel{s+2}}} = \boxed{\frac{1}{s+3}} = \frac{Y(s)}{R(s)}$$

4.) Please find the time constant of the following equation if $x(0) = 10$:

$$5\frac{dx}{dt} + 6x = 0$$

$$\mathcal{L}(\dot{x}) = sX(s) - x(0)$$

$$5\left(sX(s) - 10\right) + 6X(s) = 0$$

$$X(s)\left(5s + 6\right) = 50$$

$$\boxed{X(s) = \frac{10}{s + 6/5}} = \frac{50}{5s+6}$$

$$x(t) = 10e^{-6/5 t}, \quad t > 0$$

$$= Ce^{-t/\tau}$$

$$\frac{5}{6}\dot{x} + x = 0$$

$$\boxed{\tau = \frac{5}{6}}$$

$$-\frac{6}{5}t = -t/\tau \Rightarrow \boxed{\tau = \frac{5}{6}}$$

5.) Please find the free response $x(t)$ of the following differential equation using Laplace transform. Initial conditions are $x(0) = 2,\ \dot{x}(0) = -2$.

$1\ddot{x}$ $\dot{x}$

$$\mathcal{L}\left[\cancel{\frac{d^2x}{dt^2}} + 3\cancel{\frac{dx}{dt}} + 2x = u(t)\right]$$

$$1\left(s^2 X(s) - s\,\underset{2}{x(0)} - \underset{-2}{\dot{x}(0)}\right) + 3\left(s X(s) - \underset{2}{x(0)}\right) + 2X(s) = 0$$

$$\underline{X(s)(s^2+3s+2) - 2s + 2 - 6} = 0$$

$$X(s)(s^2+3s+2) = 2s+4$$

$$X(s) = \frac{2s+4}{s^2+3s+2} = \frac{2\cancel{(s+2)}}{\cancel{(s+2)}(s+1)} = \frac{2}{s+1} \Rightarrow \boxed{x(t) = 2e^{-1t} \quad t>0}$$

$s \quad 2$

$s \quad 1$