Modeling Mechanical Systems:

* "ALL MODELS ARE WRONG, BUT SOME ARE USEFUL!"
G. Box

– Newton's 2nd law:

$$\sum F = ma$$

$$\sum M = J\ddot{\theta}$$

$$F = m\ddot{y} \quad \text{(2nd order system)}$$

$$\ddot{y} = \frac{1}{m}F$$

u

$$u \longrightarrow \boxed{\frac{1}{m}} \xrightarrow{\ddot{y}} \boxed{\int} \xrightarrow{\dot{y}} \boxed{\int} \longrightarrow y \qquad \text{(Double integrator)}$$

Plant model

A common method to standardize: $\boxed{\text{state-space form}}$

State-Space Form

$$\dot{\vec{x}} = A\vec{x} + B\vec{u} \quad \text{(state eqn.)}$$

$$\vec{y} = C\vec{x} + D\vec{u} \quad \text{(output eqn.)}$$

$$\boxed{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

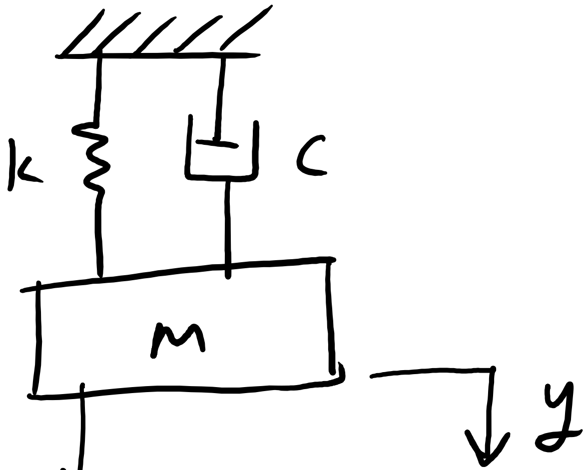
$$\boxed{x_1 = y} \Rightarrow \dot{x}_1 = \dot{y} = x_2$$

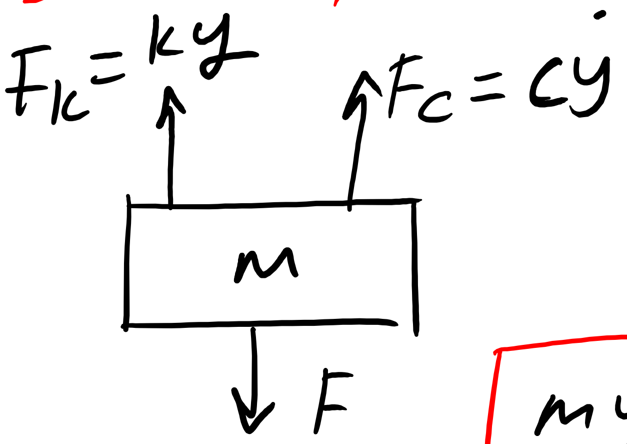
$$x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} = \frac{1}{m} u$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u$$

$$\boxed{y = x_1} \Rightarrow y = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{bmatrix} 0 \end{bmatrix}}^{D} u$$

Mass-Spring-Damper:





$$ma = \Sigma F$$

$$m\ddot{y} = F - ky - c\dot{y}$$

$$\boxed{m\ddot{y} + c\dot{y} + ky = F = u}$$

F u

$$\boxed{(x_1 = y)}$$

$$x_2 = \dot{y}$$

Rearrange:

$$\ddot{y} = \boxed{-\frac{c}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u}$$

x_2 x_1

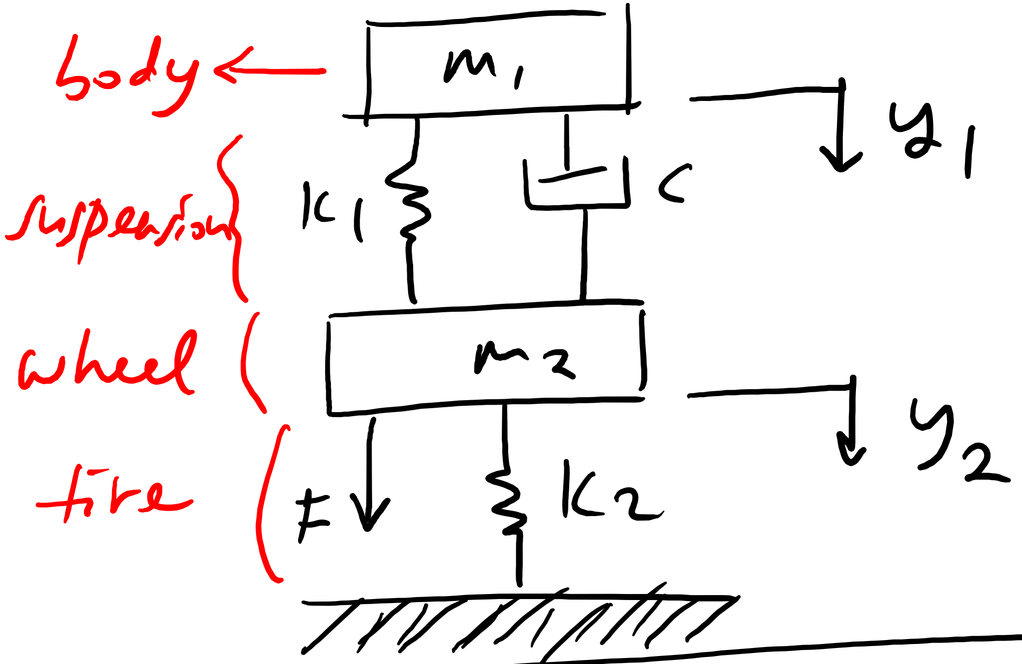
$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u$$

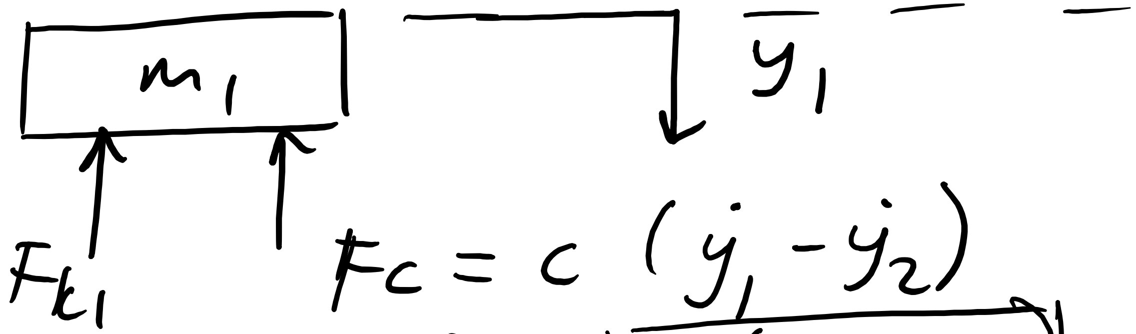
$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

Automobile Suspension:



For n masses $\longrightarrow$ n equations

n degrees of freedom



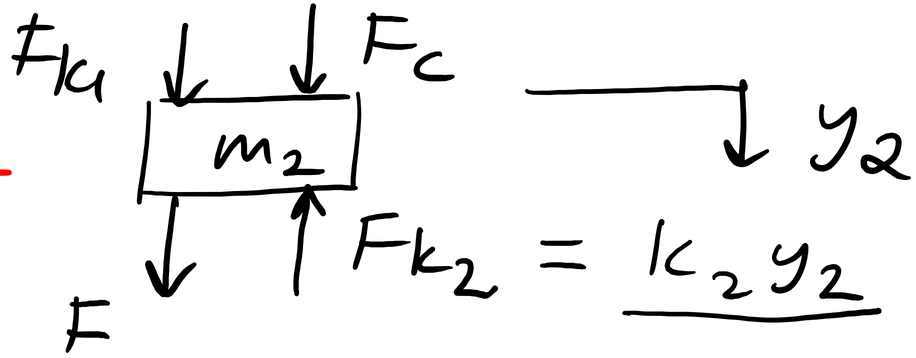
$$F_c = c(\dot{y}_1 - \dot{y}_2)$$

$$F_{k_1} = k_1(\Delta y) = k_1(y_1 - y_2)$$

mass 1:

$$m_1 \ddot{y}_1 = -F_{k_1} - F_c$$

$$m_1 \ddot{y}_1 = -k_1(y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2)$$



$$F_{k_2} = k_2 y_2$$

mass 2:

$$m_2 \ddot{y}_2 = \underbrace{F}_{u} + F_{k_1} + F_c - F_{k_2}$$

$$m_2 \ddot{y}_2 = F + k_1(y_1 - y_2) + c(\dot{y}_1 - \dot{y}_2) - k_2 y_2$$

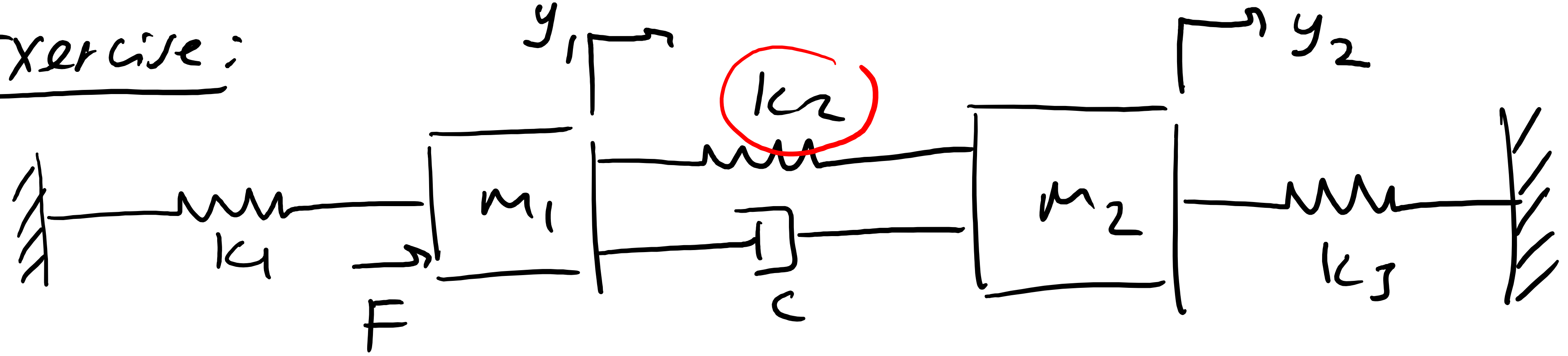
$$m_1 \ddot{y}_1 = -k_1 (y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2)$$

$$m_2 \ddot{y}_2 = F + k_1 (y_1 - y_2) + c(\dot{y}_1 - \dot{y}_2) - k_2 y_2$$

$$m_2 \ddot{y}_2 = F - k_1 (y_2 - y_1) - c(\dot{y}_2 - \dot{y}_1) - k_2 (y_2 - 0)$$

RULE: ALWAYS $-$, always starting w/ the variable of that mass.

Exercise:



Mass 1:

$$m_1 \ddot{y}_1 = F - k_1(y_1 - 0) - k_2(y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2)$$

Mass 2:

$$m_2 \ddot{y}_2 = -k_2(y_2 - y_1) - k_3(\underline{\underline{y_2}} - 0) - c(\dot{y}_2 - \dot{y}_1)$$

$x_1 = y_1$

$x_2 = \dot{y}_1$

$x_3 = y_2$

$x_4 = \dot{y}_2$