Recap

Importance of Damping Ratio and Frequency

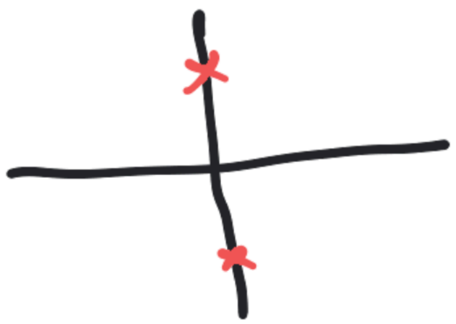
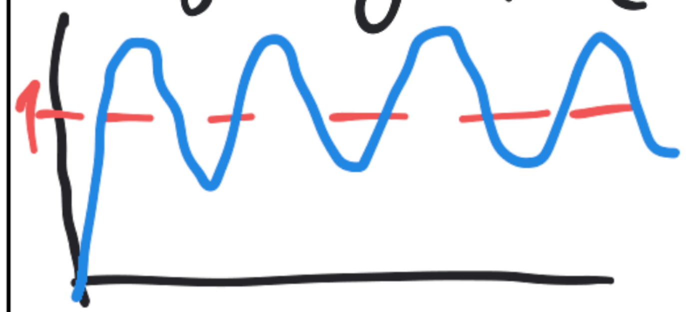
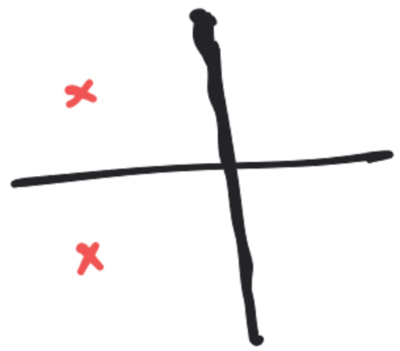
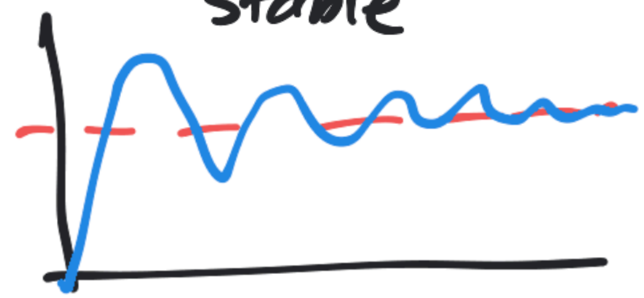
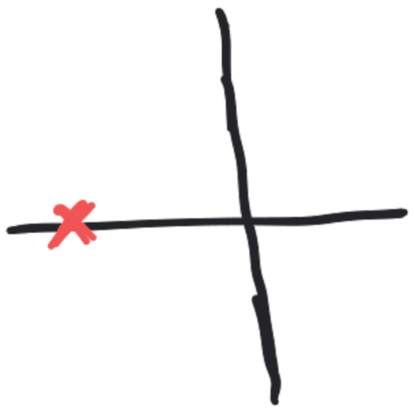
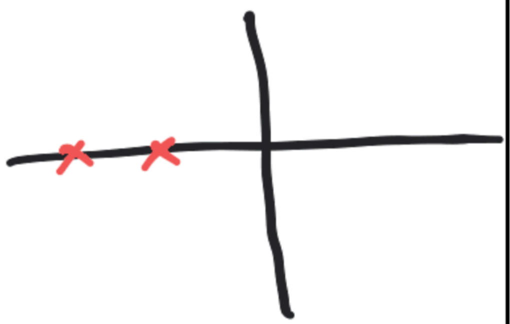
$$G(s) = K \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Change in Damping Ratio and Frequency → Change in Location of Poles on the s-plane → Change in Transient Response.

$$\boxed{P_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}}$$

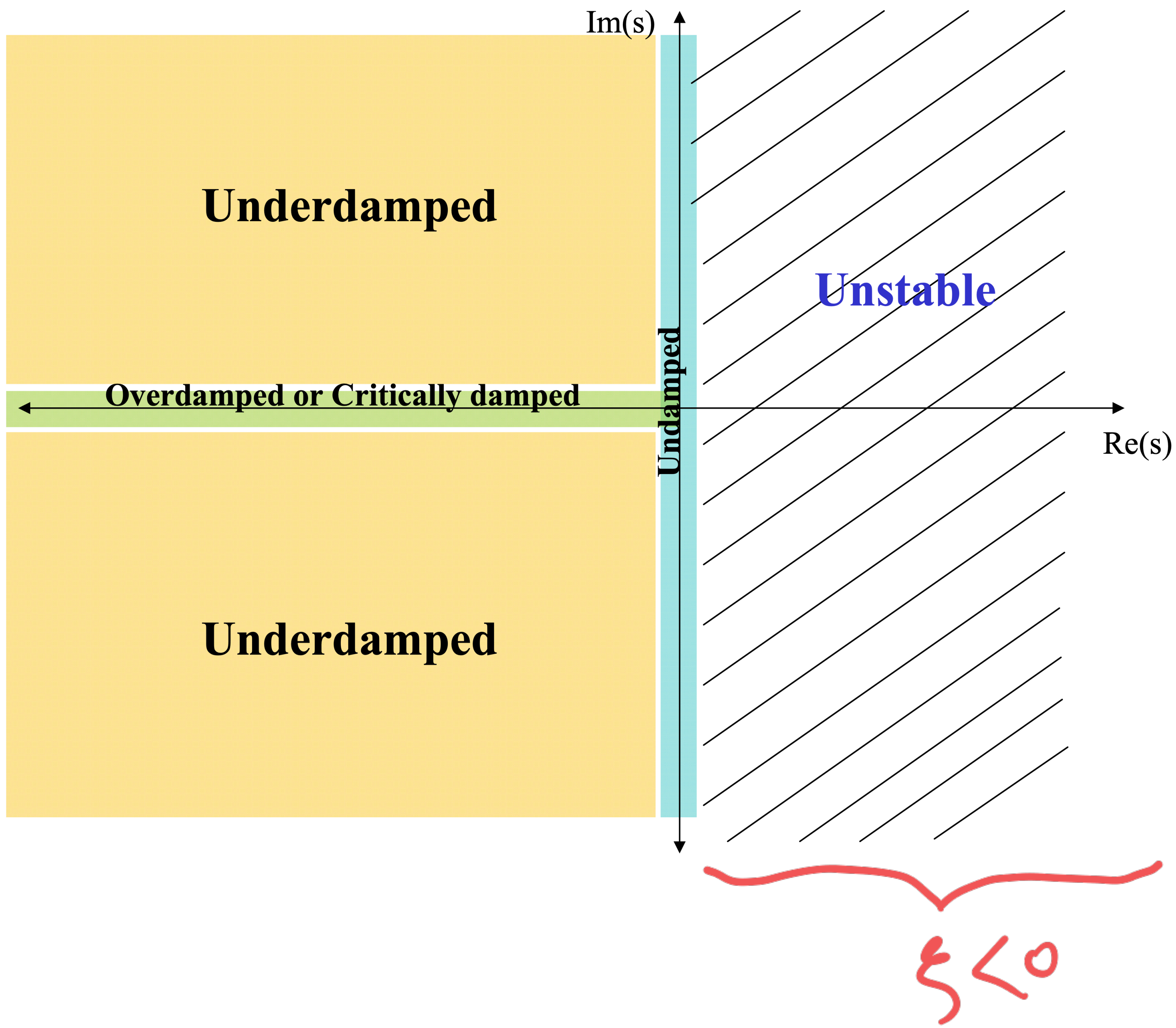
(non-negative ξ)

There are four possible cases in the 2^{nd}-order System Response

Damping Ratio	Poles $P_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$	Poles on the s-Plane	How it affects the output	Converge?
$\xi = 0$	$P_{1,2} = \pm j\omega_n$ undamped		marginally stable 	NO! Oscillation
$0 < \xi < 1$	$P_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ under damped		stable 	Yes
$\xi = 1$	$P_{1,2} = -\omega_n \pm 0$ Repeated real poles		critically damped  stable	Yes No oscil.
$\xi > 1$	$P_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ Real distinct poles		No oscillations (over damped)  stable	Yes But very slowly

Poles and Stability

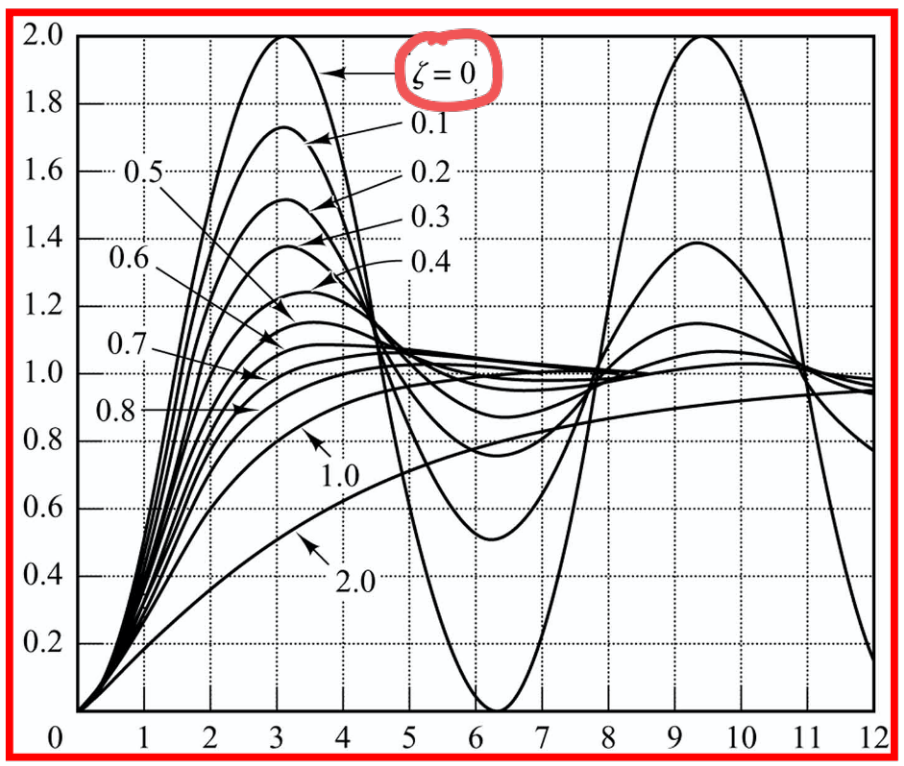
(stable, marginally stable, and unstable systems)

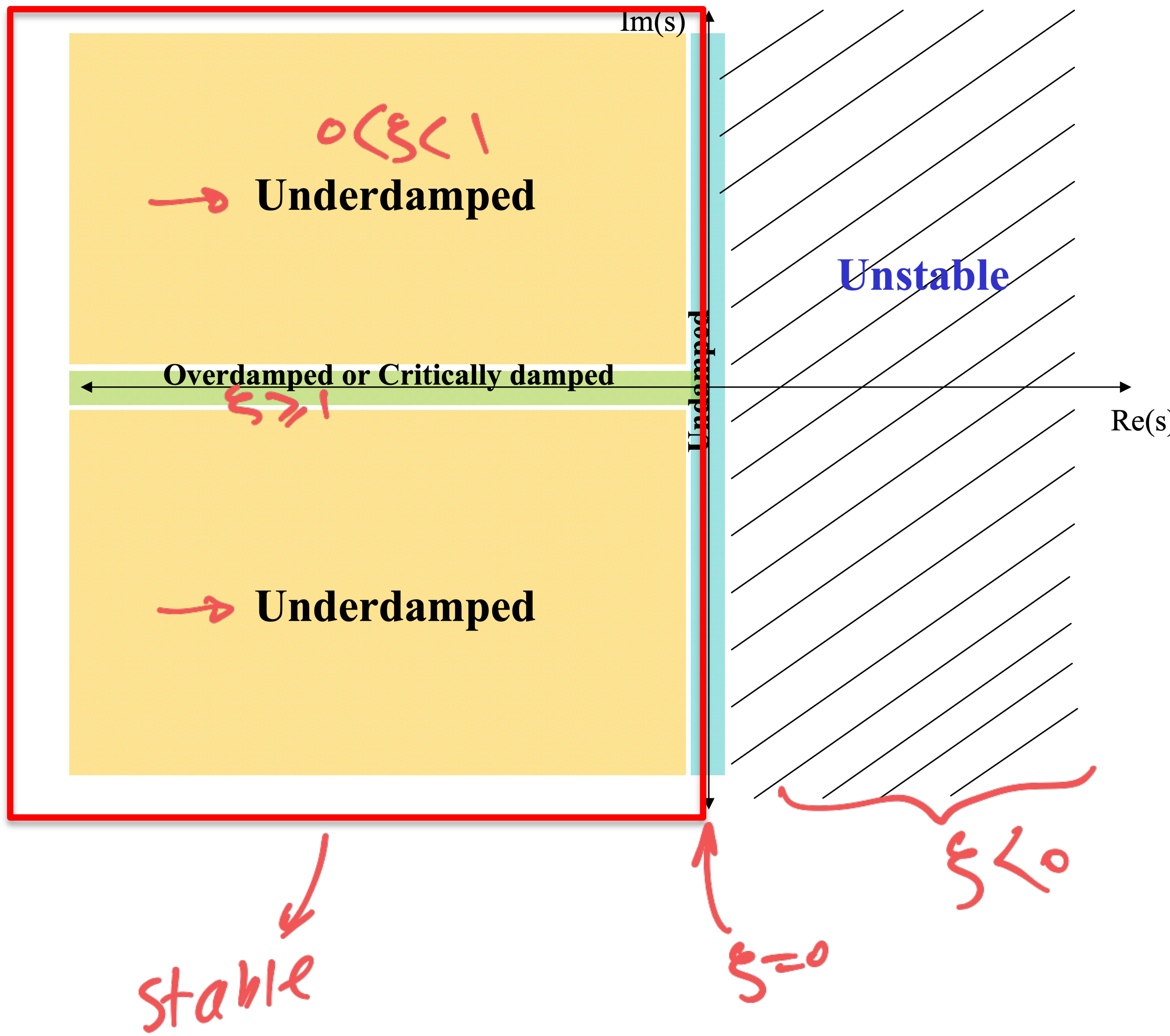


https://www.egr.msu.edu/classes/me451/jchoi/2007/handouts/ME451_S07_lecture17.pdf

Poles and Stability

(stable, marginally stable, and unstable systems)





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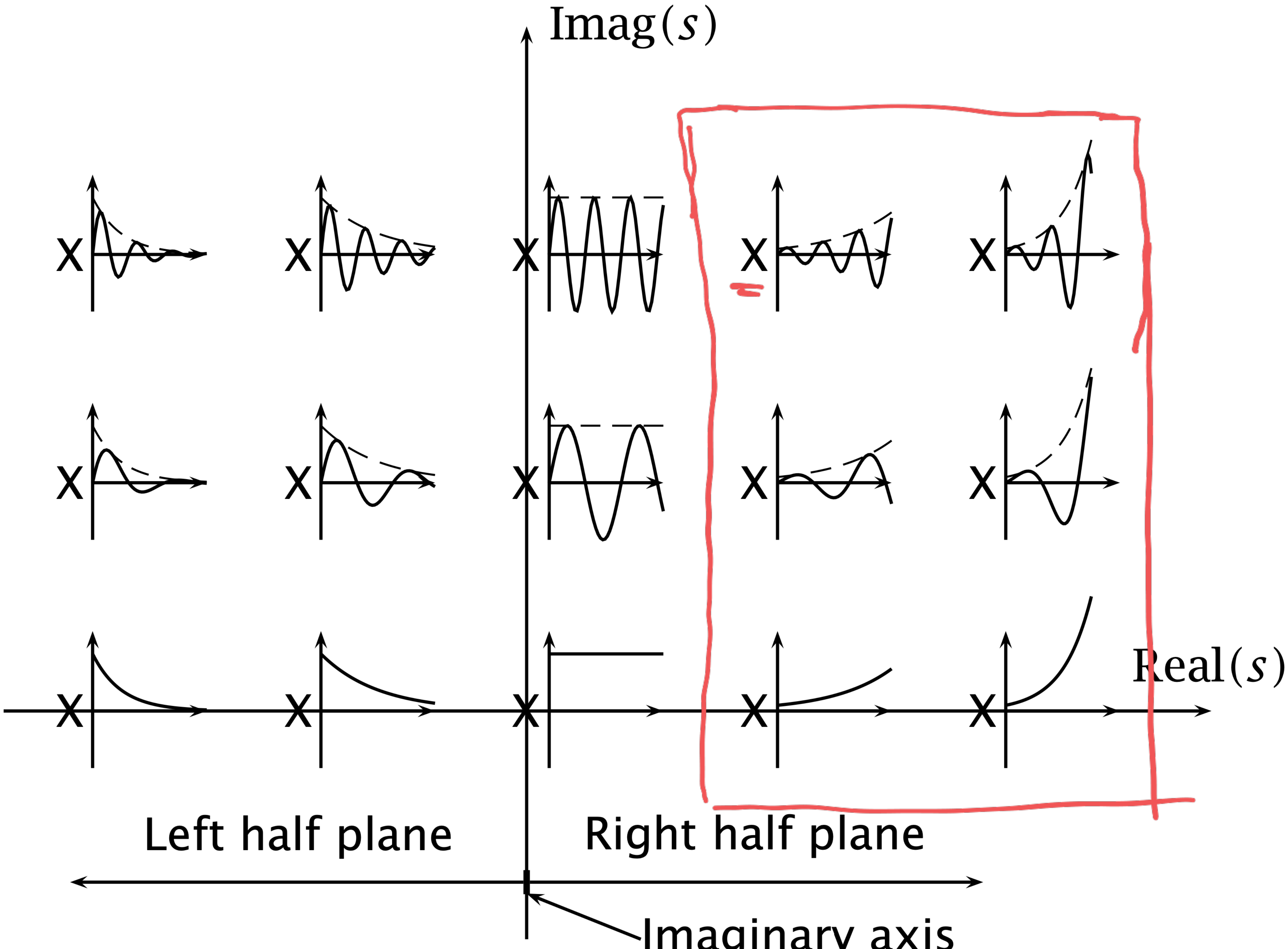
Poles and Stability

(stable, marginally stable, and unstable systems)



Pole locations and corresponding transient responses

http://www-control.eng.cam.ac.uk/gv/p6/Handout3.pdf

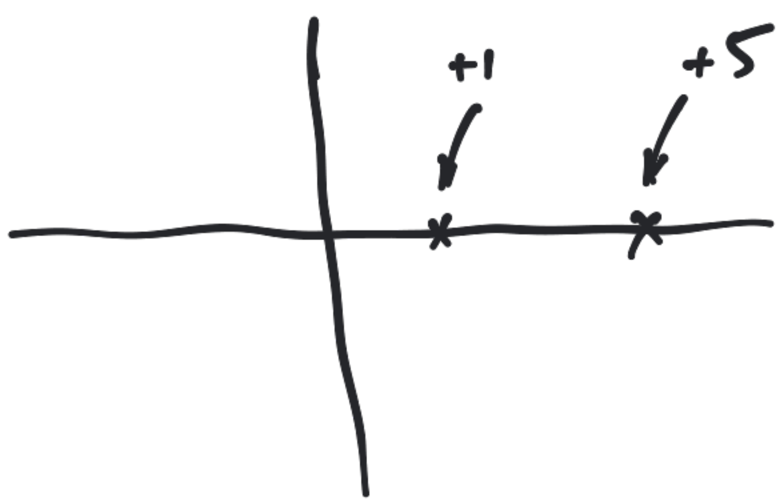
Poles and Stability

$$Y(s) = \frac{K}{\boxed{s^2 - 6s + 5}}$$

$$(s-1)(s-5) = 0$$

$$\Longrightarrow s = +1 \;\&\; s = +5$$

$$\underbrace{y(t) = C_1 e^{5t} + C_2 e^{t}}_{\text{unstalbe}}$$

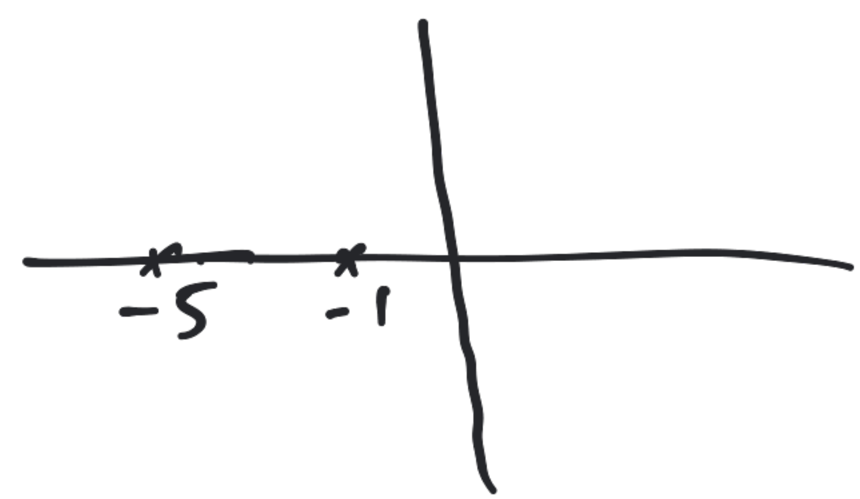


$$Y(s) = \frac{K}{\boxed{s^2 + 6s + 5}}$$

$$(s+5)(s+1) = 0$$

$$\Longrightarrow s = -5 \;\&\; s = -1$$

$$\underbrace{y(t) = C_1 e^{-5t} + C_2 e^{-t}}_{\text{stable}}$$



Poles and Stability

(stable, marginally stable, and unstable systems)

Summary

If you are to conclude about stability of a system whose closed-loop transfer function is given, simply find the poles of the system (roots of the CLTF's denominator a.k.a. characteristic equation)

- **Any pole on the RHP? (positive real part) → Unstable**
- **Any pair of poles on the imaginary axis? ($\pm$jw) → Marginally Stable**
- **All poles on the LHP? (negative real part) → Stable**

Poles and Stability (Second-order Systems)

$$G(s) = K \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Finding Poles in a second order system, and therefore, concluding about the stability of the system is simple.

$$P_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Poles and Stability (Higher-order Systems)

What if we have higher order systems?

What if we have higher order systems?

Routh Stability Criterion

(conclusion about stability without the need to solve for poles)

process for general CLTF : (n-th order system)

1- $a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^1 + a_n s^0 = 0$

$a_n \neq 0$

if $a_n = 0 \Rightarrow$ Divide the rest by s

$\rightarrow$ Get rid of the pole at $\underline{\underline{s=0}}$

2- Check if all a_i have the same sign.

↳ if not $\Rightarrow$ Unstable!

↳ if all negative $\rightarrow$ multiply by (-1)

3) Construct the Routh Array:

s^n	a_0	a_2	a_4	a_6	.... until you run out.
s^{n-1}	a_1	a_3	a_5	a_7	----
s^{n-2}	b_1	b_2	b_3	----	
s^{n-3}	c_1	c_2	c_3	----	
$\vdots$	d_1				
$\vdots$	$\vdots$				
s^2	e_1	e_2 ...			
s^1	f_1				
s^0	g_1				

Look at the signs of the first column

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

Note: can multiply any row w/ a constant to simplify

TWO outcomes :

1) If all the terms in the first column same sign

$\Rightarrow$ STABLE

2) if not, the number of sign changes is the number of POLES ON RHP (unstable pole)

$\Rightarrow$ Unstable

Example:

$$-s^5 - 3s^4 - 3s^3 - 6s^2 - 5s = 0$$

1) No coefficient for s^0 $\rightarrow$ Divide by s.

$$\cancel{s}\,\underbrace{(-s^4 - 3s^3 - 3s^2 - 6s - 5)}\,= 0$$

2) All coefficients are negative $\rightarrow$ multiply by (-1)

$$1.s^4 + 3s^3 + 3s^2 + 6s^1 + 5 = 0$$

3)

s^4	1	3	5
s^3	$\cancel{3}\ 1$	$\cancel{6}\ 2$	0
s^2	$b_1 = \underline{1}$	$b_2 = 5$	
s^1	$c_1 = -3$	0	
s^0	$d_1 = 5$		

2 sign changes

$\Rightarrow$ 2 poles on RHP

Unstable!

$$b_1 = \frac{(1)(3) - (1)(2)}{1} = \frac{(3)(3) - 1(6)}{3}$$
$$= \underline{1}$$

$$b_2 = \frac{(1)(5) - 1(0)}{1} = 5$$

$$c_1 = \frac{(1)(2) - (1)(5)}{1} = -3$$

$$d_1 = \frac{(-3)(5) - 0}{(-3)} = 5$$

Special cases:

A) First Column is 0, remaining terms in that row are non-zero.

$\Rightarrow$ Replace 0 with small positive number (ϵ)

$$1s^3+2s^2+s+2=0$$

$$\begin{array}{c|c c c}
s^3 & 1 & 1 & 0 \\
s^2 & 2 & 2 & 0 \\
s^1 & b_1=\epsilon & 0 & \\
s^0 & c_1=2 & &
\end{array}$$

$$b_1=\frac{(2)(1)-(1)(2)}{(2)}=0\rightarrow\epsilon$$

$$c_1=\frac{(\epsilon)(2)-0}{\epsilon}=2$$

$\hookrightarrow$ no sign change

$\hookrightarrow$ stable! (marginally stable)

$\hookrightarrow$ a pair of $\pm j\omega$ (on the imaginary axis)

B) All terms in a row are 0.

Example: $1\cdot s^5 + 2s^4 + 5s^3 + 10s^2 + 3s + 6 = 0$

s^5	1	5	3
s^4	2	10	6
s^3	$\cancel{8}\ \cancel{0}\ 2$	$\cancel{0}\ \cancel{20}\ 5$	0
s^2	5	6	
s	$d_1 = 2.6$	0	
s^0	$c_1 = 6$		

Stable!

Auxiliary polynomial:

$$P(s):\ 2s^4 + 10s^2 + 6 = 0$$

$$\frac{dP(s)}{ds} = 8s^3 + 20s + 0 = 0$$

$$d_1 = \frac{(5)(5) - (2)(6)}{5} = 2.6$$

$$c_1 = \frac{(2.6)(6) - (5)(0)}{2.6}$$