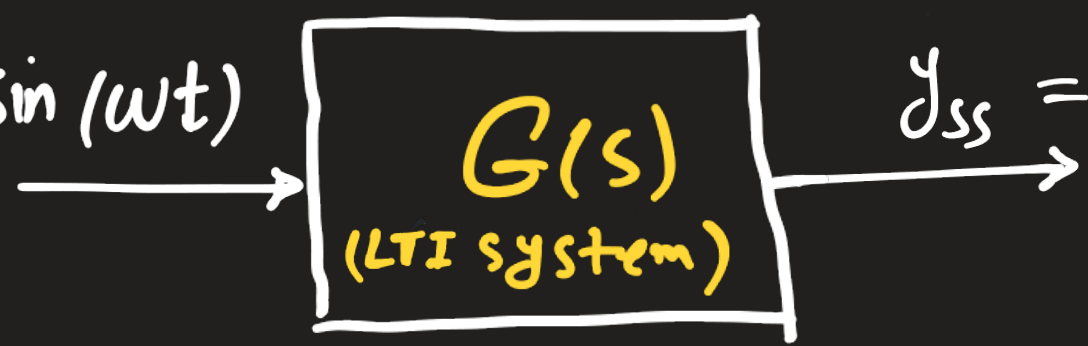
Recap

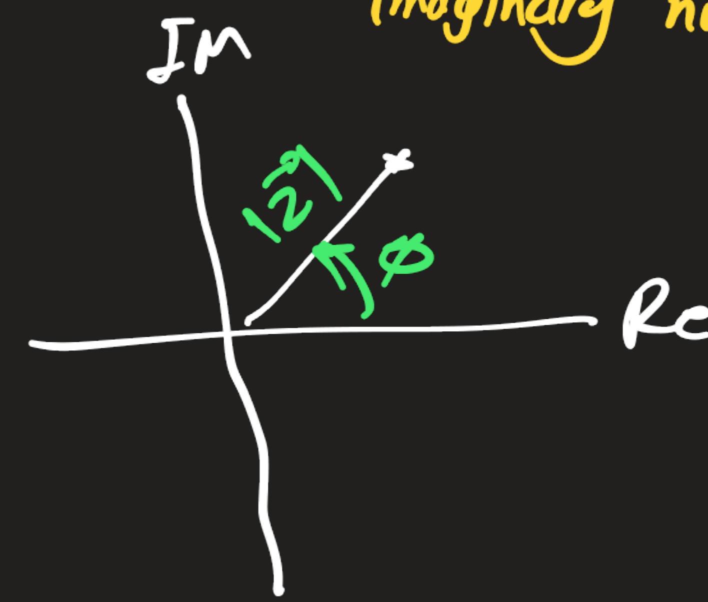
$u(t) = A \sin(\omega t)$



$y_{ss} = A \underbrace{|G(j\omega)|}_{\text{Gain}} \sin(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase}})$

Recap

Reminder: Amplitude & phase of a complex, real, & imaginary number.



$\vec{z}$	Amplitude $\lvert a+jb \rvert$	phase $\angle(a+jb)$	
$a+jb$	$\sqrt{a^2+b^2}$	$\tan^{-1} b/a$	$\Rightarrow$ Complex
$(b=0)$ a	a	$\tan^{-1} 0 = 0$	$\Rightarrow$ real
$(a=0)$ jb	b	$\tan^{-1} \infty = \pi/2$	$\Rightarrow$ Imaginary

Recap

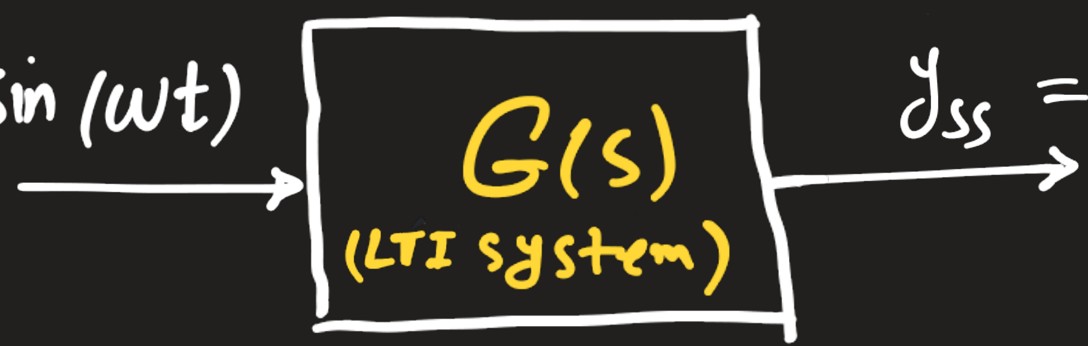
Reminder (Cont.) : Amplitude & phase of a fraction

$$\text{Amplitude} = \left|\frac{a+jb}{c+jd}\right| = \frac{|a+jb|}{|c+jd|} = \boxed{\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}}$$

$$\phi = \angle\left(\frac{a+jb}{c+jd}\right) = \angle(a+jb) - \angle(c+jd)$$

$$\boxed{\phi = \tan^{-1}\frac{b}{a} - \tan^{-1}\frac{d}{c}}$$

$u(t) = A \sin(\omega t)$



$$y_{ss} = A \underbrace{|G(j\omega)|}_{\text{Gain}} \sin(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase}})$$

$$G(j\omega) = G(s) \text{ evaluated at } s = j\omega = G(s)\Big|_{s=j\omega}$$ ← complex quantity

→ If we are able to find Gain and Phase of a system for a given frequency (ω), we have y_{ss}.

Example: $G(s) = \frac{1}{s+1}$ $u(t) = \sin t$ $y_{ss} = ?$

$u(t) = \underset{A}{1} \sin \underset{\omega}{1} t$

Gain: $|G(j\omega)| = \frac{1}{j\underset{1}{\omega}+1} = \frac{1}{j+1} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

Phase: $\angle G(j\omega) = \angle\left(\frac{1}{j\omega+1}\right) = \angle\left(\frac{1}{j+1}\right) =$

$\underbrace{\cancel{\angle(1)}}_{0} - \angle(j+1) = -\tan^{-1}(1) = -45^\circ$

$$\boxed{y_{ss} = \frac{1}{\sqrt{2}} \sin(t - 45)}$$

$$u(t) = A \sin(\omega t) \longrightarrow \boxed{\begin{matrix} G(s) \\ \text{(LTI system)} \end{matrix}} \longrightarrow y_{ss} = A \underbrace{|G(j\omega)|}_{\text{Gain}} \sin(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase}})$$

$$G(j\omega) = G(s) \text{ evaluated at } s = j\omega = G(s)\Big|_{s=j\omega} \leftarrow \text{complex quantity}$$

→ If we are able to find Gain and Phase of a system for a given frequency (ω), we have y_{ss}.

→ Can we solve for spectrum of frequencies ($0 < \omega < \infty$) ?

Example: $G(s) = \frac{2}{s+3}$ $\qquad u(t) = A\sin(\omega t)$

Gain: $|G(j\omega)| = \frac{2}{j\omega+3} = \frac{2}{3+j\omega} = \frac{2}{\sqrt{9+\omega^2}}$: Gain

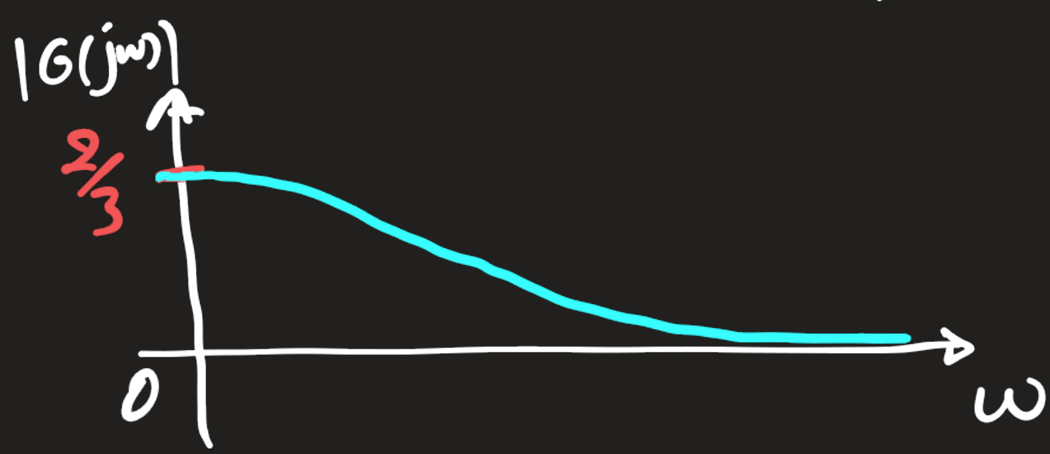
Phase: $\angle\left(\frac{2}{3+j\omega}\right) = \underbrace{\cancel{\angle 2}}_{0} - \angle(3+j\omega) = -\tan^{-1}\frac{\omega}{3} = \phi$

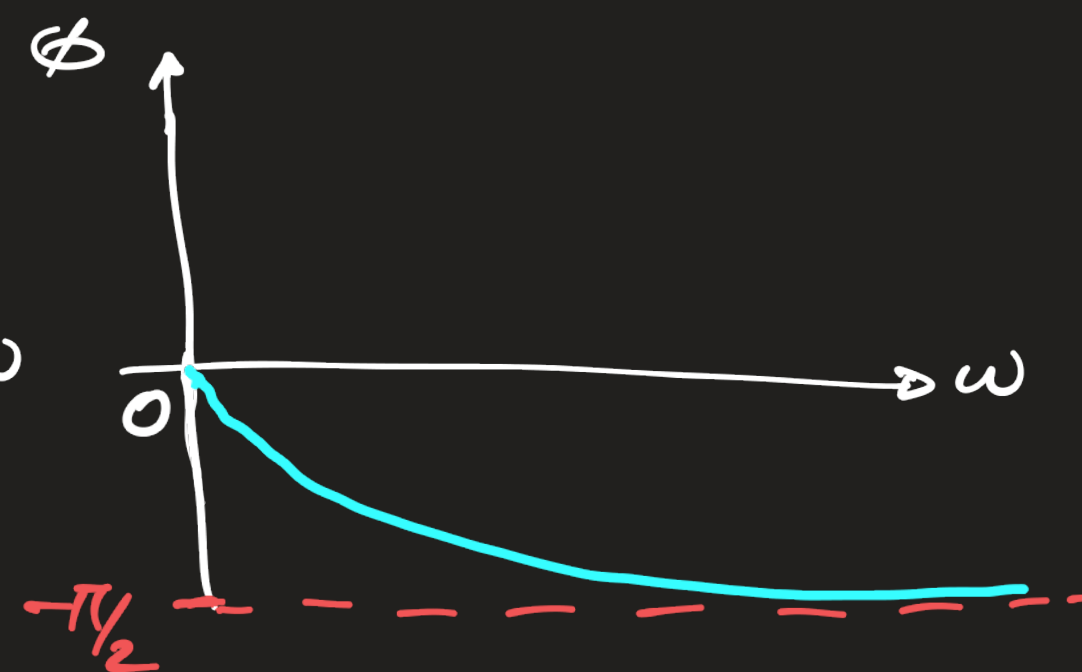
Let's plot Gain & ϕ w.r.t ω (frequency)

$\underline{\underline{a+bj = bj+a}}$

$\omega = 0 \Rightarrow G = \frac{2}{3}$ $\qquad \omega \to \infty \Rightarrow G \to 0$

$\omega = 0 \Rightarrow \phi = 0$ $\qquad \omega \to \infty \Rightarrow \phi \to -\pi/2$





→ linear scale plots are difficult to draw & read.

→ useful way is to use logarithmic scale ⇒ Bodé's plots

* why log scale ?

⇒ 1) Equal emphasis to small + large ω's

2) Easier to find log (Gain)

$$\text{e.g. } \log\left(\frac{ab}{cd}\right) = \underbrace{\log(a) + \log(b) - \log(c) - \log(d)}$$

Polynomial factors of TF can be split and evaluated seperately.

Bode's Diagrams: (Section 7.2)

A graphical representation of the freq. response of a system $G(s)$.

→ two graphs drawn together $\left\{ \begin{array}{l} \text{1) Gain} \Rightarrow dB = 20\log_{10}(\text{Gain}) \text{ vs. } \omega \\ \text{2) Phase} \Rightarrow \phi \text{ vs. } \omega \end{array} \right.$

General approach:

✓ Transfer function is written in terms of its factors

→ put each factor in standard form (unity DC gain)

✓ For each factor of TF, calculate dB & ϕ.

A TF ($G(s)$) may have $\underline{\underline{4}}$ kinds of factors:

1- Constant Gain

2- First-order

3- second-order

4- $s=0$

} any factor of higher-order can be reduced to a multiplication of first- and second-order terms.

✓ Draw dB & ϕ curves for each factor

✓ Add the dB curves for Gain plot

✓ Add the ϕ curves for phase plot

Example:

$$G(S) = \frac{\overset{A}{10}\,\overset{C}{(S+3)}}{\underset{B}{S}\,\underset{D}{(S^2+4S+5)}}$$

$$dB = 20 \log |G(j\omega)| = dB(A) + dB(C) - dB(S) - dB(D)$$

$$\phi = \angle(G(j\omega)) = \phi(A) + \phi(C) - \phi(B) - \phi(D)$$

Reminder: (log & dB)

$$\log(a) = \log_{10}(a) = b \Rightarrow \boxed{10^b = a} \Rightarrow \begin{cases} a = 0 \Rightarrow b = -\infty \leftarrow 10^{-\infty} = \frac{1}{10^{\infty}} = 0 \\ 0 < a < 1 \Rightarrow b < 0 \\ a = 1 \Rightarrow b = 0 \leftarrow 10^{0} = 1 \\ a > 1 \Rightarrow b > 0 \end{cases}$$

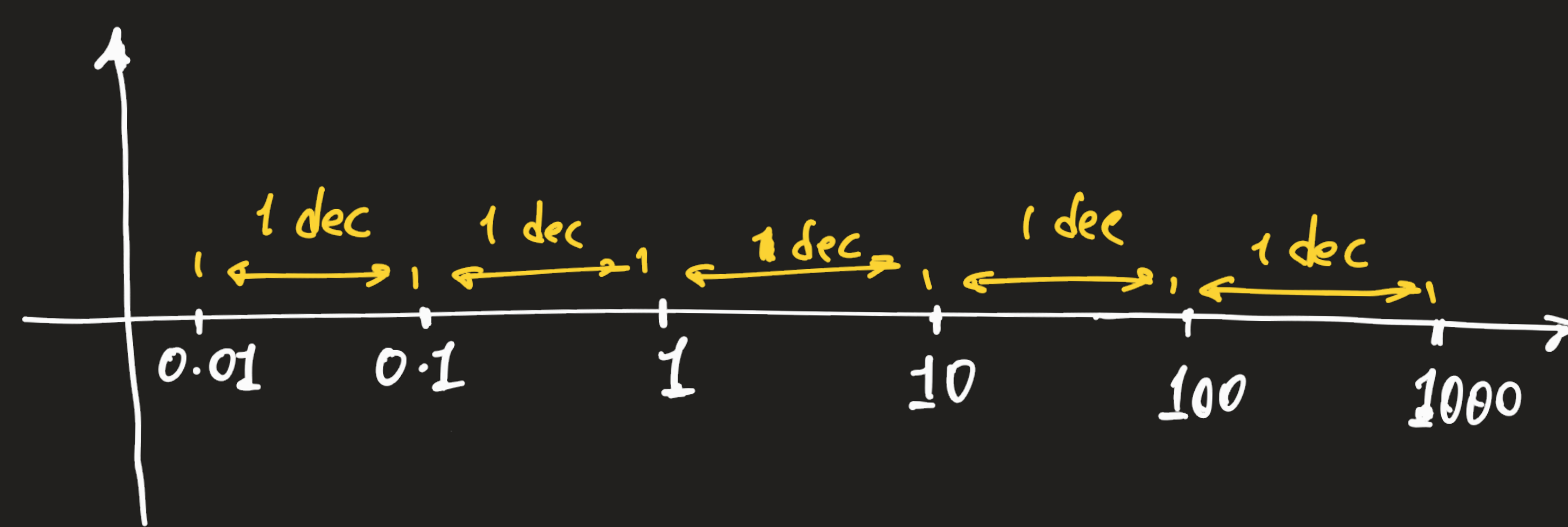
$$\left.\begin{aligned} \log(0.1) &= \log(10^{-1}) = -1 \\ \log(1) &= \log(10^{0}) = 0 \\ \log(10) &= \log(10^{1}) = 1 \\ \log(100) &= \log(10^{2}) = 2 \\ \log(1000) &= \log(10^{3}) = 3 \end{aligned}\right\} \Rightarrow \log(10^{n}) = n \log(10) = \underline{\underline{n}}$$

$$\boxed{dB(a) = 20 \log(a)}$$

→ dec : logarithmic scale/unit w/ ratio of 10:1

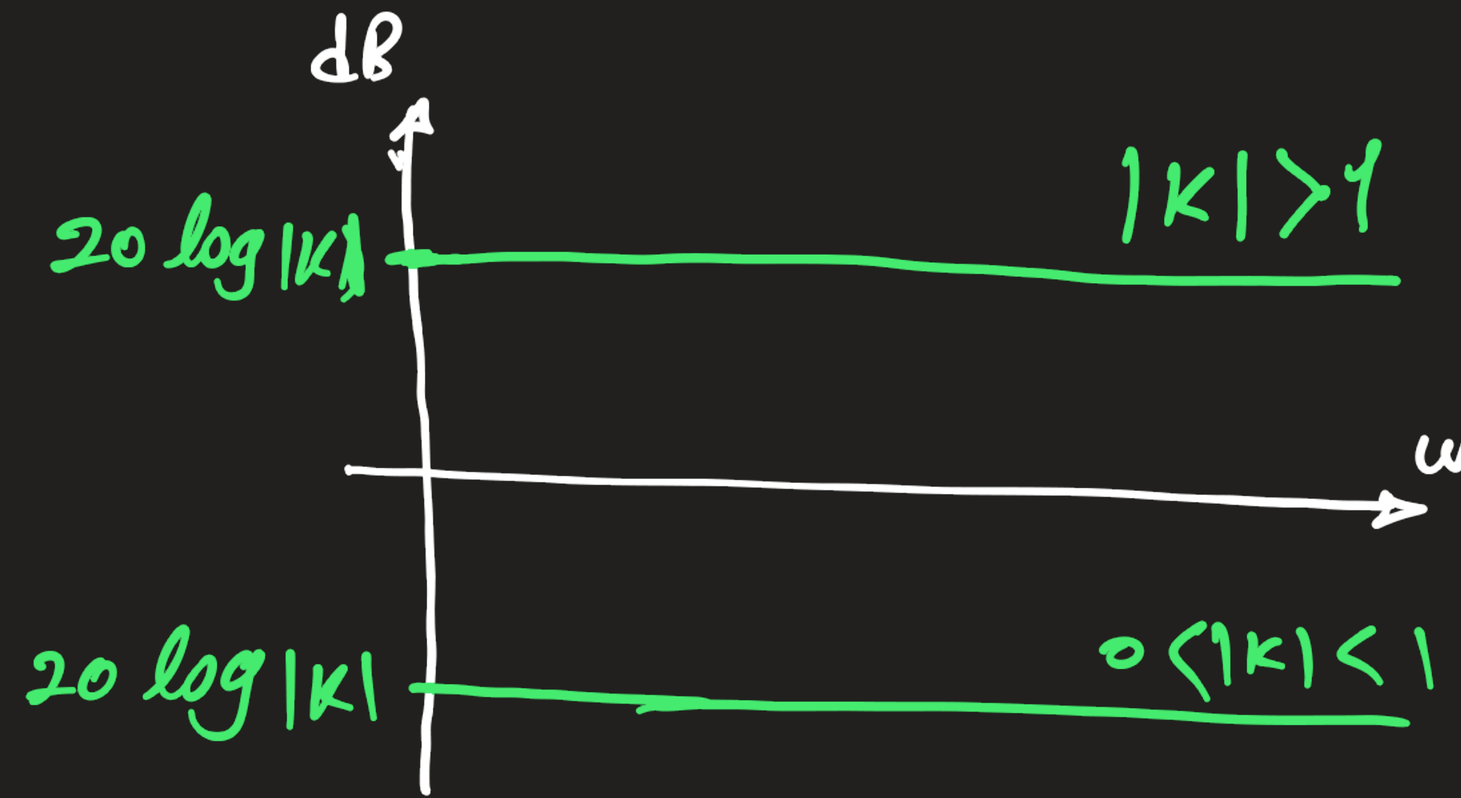
→ slope : $N\ ^{dB}/_{dec}$:

N dB increase per decade

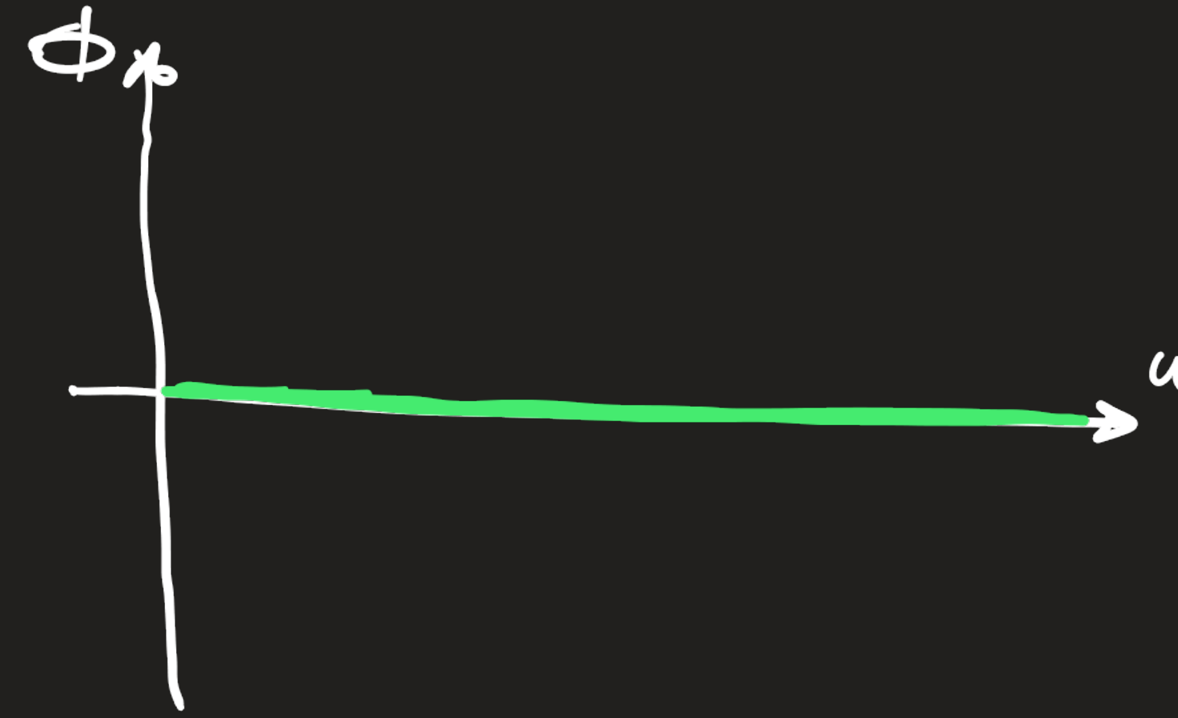


A) Constant Gain (K)

$$dB = 20 \log_{10} |K|$$



$$\phi = \angle(K + 0j) = \tan^{-1} 0 = 0$$



Not function of ω

$\Rightarrow$ straight horizontal line

Side note

$\log_{10}(10) = 1$ / $\log_{10}(100) = 2$ / $\log_{10}(1000) = 3$, - - / $\log_{10}(0.1) = (-1)$

$\Rightarrow \log_{10}(1) = 0 \Leftarrow 10^0 = 1$

$dB = 0 = 20 \log|\omega| \Rightarrow \log|\omega| = 0 \Rightarrow \boxed{\omega = 1}$

B) $S = 0$ (Zero or pole at origin)

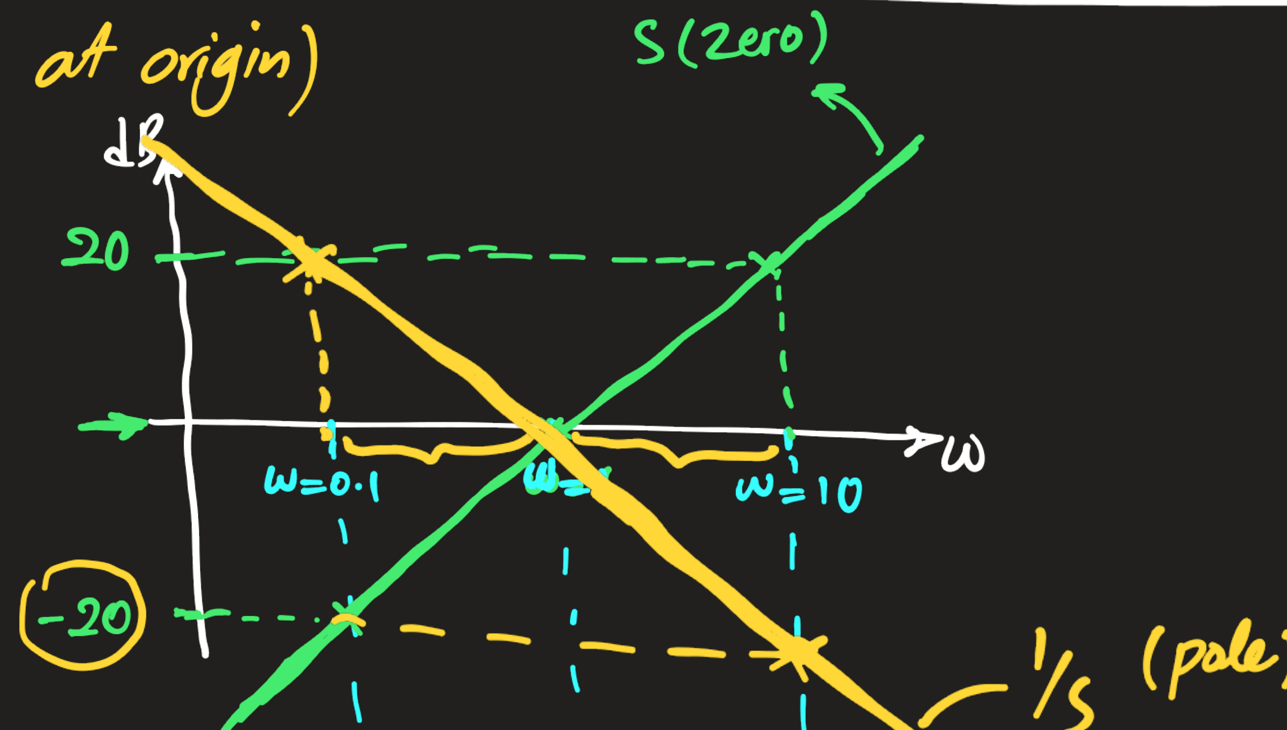
$G(s) = s$ (zero) $G(j\omega) = j\omega$

$dB = 20 \log|j\omega| = 20 \log_{10}|\omega|$

$\Rightarrow$ $\underbrace{\text{cross } \omega\text{-axis}}^{dB=0}$ at $\omega = 1$

$\Rightarrow$ slope : 20 dB/dec

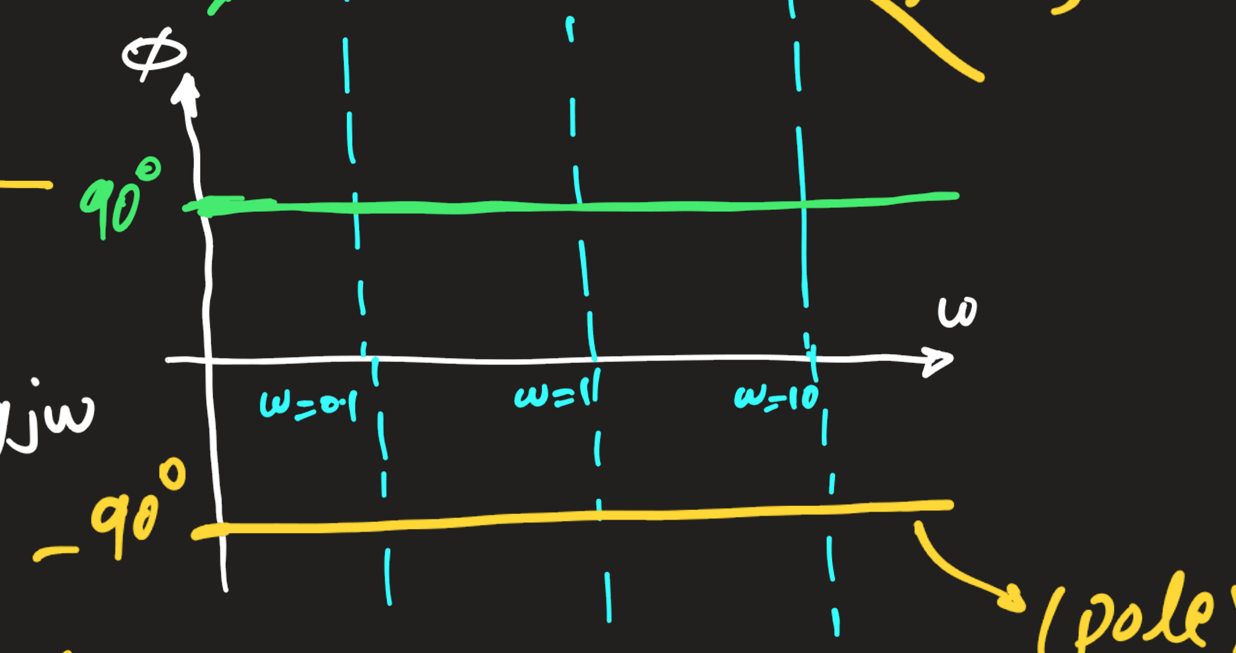
$\phi = \angle(0 + j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = \pi/2$



$G(s) = \frac{1}{s}$ (pole)

$dB = 20 \log\left|\frac{1}{j\omega}\right| = 20 \log 1 - 20 \log j\omega$

$dB = -20 \log j\omega$



dB $\Rightarrow$ cross ω-axis at $\underset{dB=0}{\omega = 1}$ and slope -20 dB/de

$\phi = \angle(1) - \angle(j\omega) = 0 - 90° = -90°$

C) First_order :