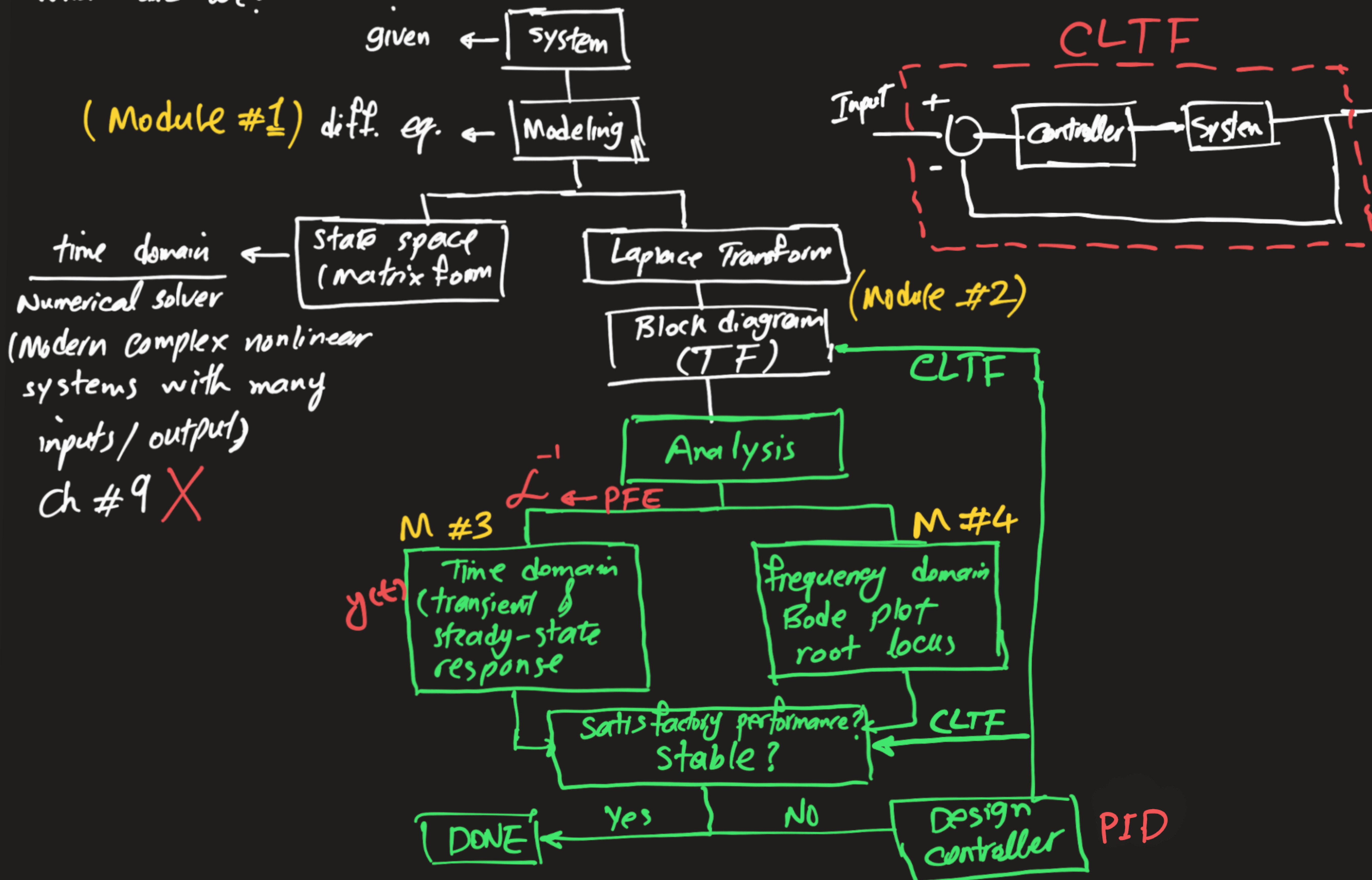
Where are we?



given
system
(Module #1) diff. eq.
Modeling
state space (matrix form)
time domain
Numerical solver
(Modern complex nonlinear systems with many inputs / outputs)
Ch #9
Laplace Transform
(Module #2)
Block diagram (TF)
CLTF
Input
+
-
controller
System
CLTF
Analysis
$\mathcal{L}^{-1}$ ← PFE
M #3
M #4
y(t)
Time domain (transient & steady-state response
frequency domain Bode plot root locus
Satisfactory performance? Stable?
CLTF
Yes
NO
DONE
Design controller
PID

Frequency Response of LTI systems : Chapter 7

LTI : Linear Time Invarient: systems are described by linear equations w/ coefficents that do not change w/ time

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Transient Response : How the output evolves w/ time.

Adv.
- → useful for non-periodi inputs (impulse, step, ramp)
- → Explicily shows closed-loop poles, which indicate trade-offs in transient specs. (t_r, M_p, etc.)

Disadv.
- → stability is inferred <u>separately</u> using Routh criterion.
- → Difficult to infer performance for higher-order systems
- → Requires explicit knowledge of TF
- → Does not work for periodic inputs (sinusoidal)

Frequency Response : How a system responds to a sinusoidal input in steady state.

» supplements transient response analysis.

Adv.

» performance and stability inferred from same analysis.

» graphical $\longrightarrow$ easier

» can use measured data instead of a model (TF)

» independent of system order.

» any periodic input can be analyzed.

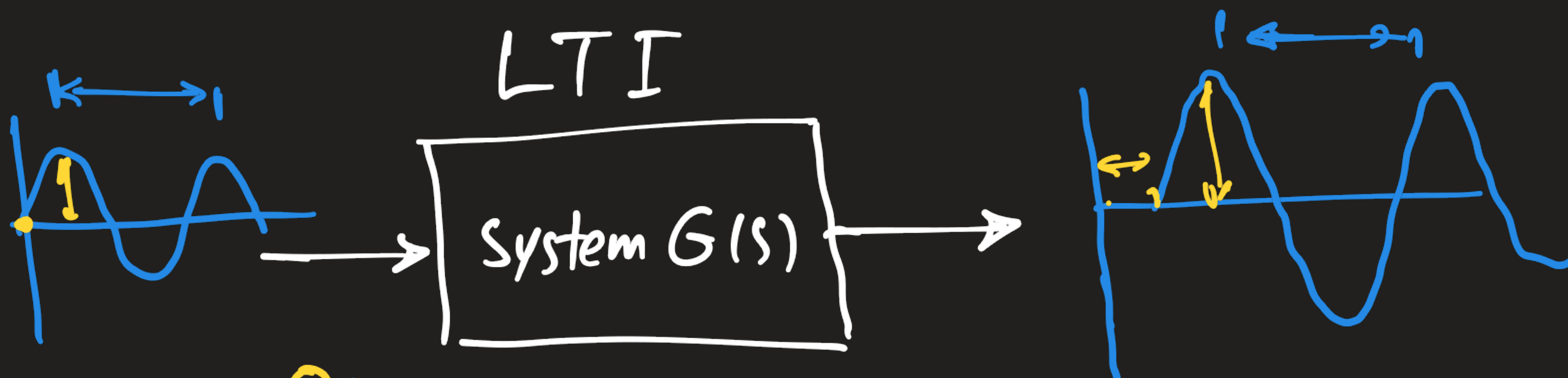
Disadv.

→ Less explicit about poles and time behavior.

→ does not work for non-periodic inputs

system: $U(s) \longrightarrow \boxed{G(s)} \xrightarrow{} Y(s)$ $\qquad Y(s) = G(s)\,U(s)$

Objective: Show that a periodic input to a LTI system will resul in a periodic steady-state output with the same frequency while amplitude & phase can change.



$u(t) = A \sin(\omega t)$

A → amplitude, ω → frequency

$y_{ss} = B \sin(\omega t + \phi)$

B → Amplitude, ϕ → phase

$$Y(s) = G(s)\,U(s) \qquad u(t) = A\sin(\omega t)$$

$$\mathcal{L}\sin(\omega t) = U(s) = \frac{A\omega}{s^2+\omega^2}$$

$$Y(s) = G(s)\cdot\frac{A\omega}{s^2+\omega^2}$$

$$\rightarrow s^2+\omega^2 = (s+j\omega)(s-j\omega) \leftarrow$$

$$\text{PFE}: Y(s) = \frac{c_1}{s+\alpha_1} + \frac{c_2}{s+\alpha_2} + \cdots + \frac{c_n}{s+\alpha_n} + \frac{k_1}{s+j\omega} + \frac{k_2}{s-j\omega}$$

$$\mathcal{L}^{-1} \Rightarrow y(t) = \underbrace{c_1 e^{-\alpha_1 t} + c_2 e^{-\alpha_2 t} + \cdots + c_n e^{-\alpha_n t}}_{\text{they all decay to zero at s-s } (t\to\infty)} + k_1 e^{-j\omega t} + k_2 e^{j\omega t}$$

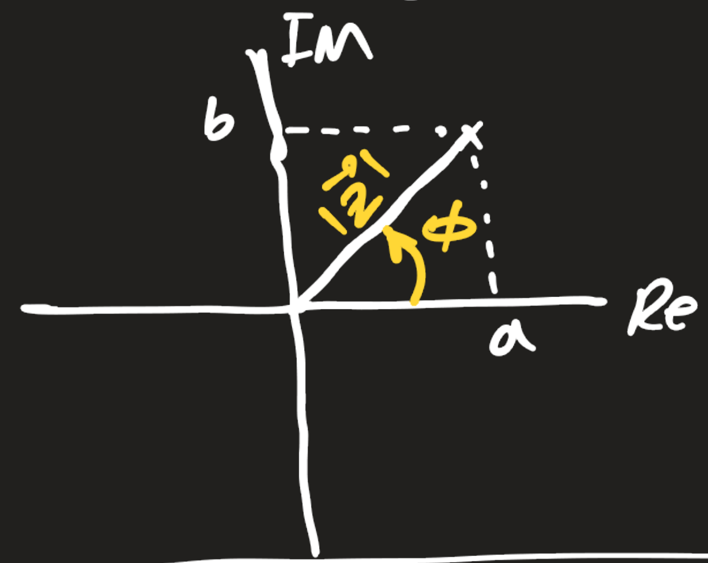
$$\text{Steady-state}: \boxed{y_{ss} = \lim_{t\to\infty} y(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t}}$$

$$k_1 = \Big[Y(s)\cdot(s+j\omega)\Big]_{s=-j\omega} = \left[G(s)\,\frac{A\omega}{\cancel{(s+j\omega)}(s-j\omega)}\cdot\cancel{(s+j\omega)}\right] = -\frac{AG(-j\omega)}{2j}$$

$s=-j\omega$

$$k_2 = \Big[Y(s)\cdot(s-j\omega)\Big]_{s=j\omega} = \frac{AG(j\omega)}{2j}$$

Reminder: Any complex quantity like $\vec{z}=a+jb$ can be written in the exp. form: $\boxed{\vec{z}=|\vec{z}|\,e^{j\phi}}$



$$|\vec{z}| = |a+jb| = \sqrt{a^2+b^2}$$

$$\phi = \angle \vec{z} = \angle(a+jb) = \tan^{-1}\frac{b}{a} = \phi$$

$G(j\omega)$ is a complex quantity: $G(j\omega) = |G(j\omega)|\,e^{j\phi}$

$$\boxed{\phi = \angle G(j\omega)}$$

$$G(-j\omega) = |G(j\omega)|\,e^{-j\phi}$$

$$\Rightarrow K_1 = -\frac{A\,|G(j\omega)|\,e^{-j\phi}}{2j}, \quad K_2 = \frac{A\,|G(j\omega)|\,e^{j\phi}}{2j}$$

$$\Rightarrow y_{ss} = K_1 e^{-j\omega t} + K_2 e^{j\omega t} = A\,|G(j\omega)| \boxed{\left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right]}$$

$$\boxed{y_{ss} = A\,|G(j\omega)|\sin(\omega t+\phi)}$$

Euler formula: $\sin(\omega t+\phi)$

$$u(t) = A\sin(\omega t + 0)$$

$$y_{ss} = \underbrace{A\,|G(j\omega)|}_{B}\,\sin(\omega t + \overbrace{\angle G(j\omega)}^{\phi})$$

$$\text{Gain} = \frac{B}{A} = \frac{A\,|G(j\omega)|}{A} = \boxed{|G(j\omega)| = \text{Gain}} \; *$$

$$\boxed{\text{phase} = \phi = \angle G(j\omega)} \; *$$

$$y_{ss} = A\,\underbrace{|G(j\omega)|}_{\text{Gain}}\,\sin(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase}})$$

Reminder: Amplitude & phase of a complex, real, & imaginary number.

$\vec{z}$	Amplitude $\lvert a+jb \rvert$	phase $\angle(a+jb)$	
$a+jb$	$\sqrt{a^2+b^2}$	$\tan^{-1} b/a$	$\Rightarrow$ Complex
$(b=0)$ a	a	$\tan^{-1} 0 = 0$	$\Rightarrow$ real
$(a=0)$ jb	b	$\tan^{-1} \infty = \pi/2$	$\Rightarrow$ Imaginary

Reminder (Cont.) : Amplitude & phase of a fraction

$$\text{Amplitude} = \left|\frac{a+jb}{c+jd}\right| = \frac{|a+jb|}{|c+jd|} = \boxed{\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}}$$

$$\phi = \angle\left(\frac{a+jb}{c+jd}\right) = \angle(a+jb) - \angle(c+jd)$$

$$\boxed{\phi = \tan^{-1}\frac{b}{a} - \tan^{-1}\frac{d}{c}}$$