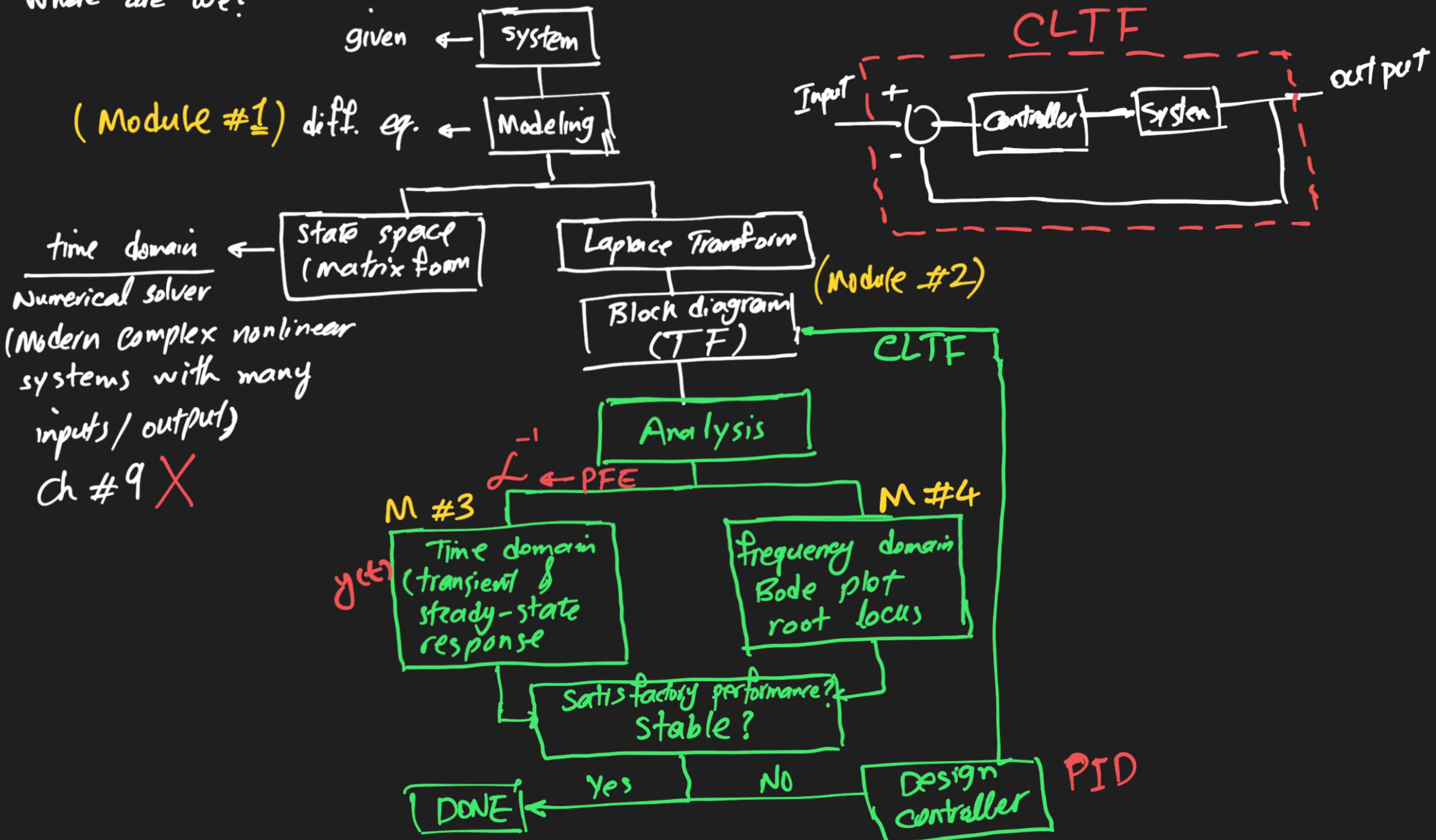
Recap

Where are we?



given
System
(Module #1) diff. eq.
Modeling
time domain
Numerical solver
(Modern complex nonlinear systems with many inputs / outputs)
Ch #9
State space (matrix form)
Laplace Transform
(Module #2)
Block diagram (T F)
CLTF
Analysis
$\mathcal{L}^{-1}$
PFE
M #3
M #4
y(t)
Time domain (transient & steady-state response
frequency domain
Bode plot
root locus
Satisfactory performance?
stable?
Yes
NO
DONE
Design controller
PID
CLTF
Input
+
-
controller
System
output

Two additional properties of Laplace Transform:

1) IVT (Initial Value Theorem):

$$f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$$

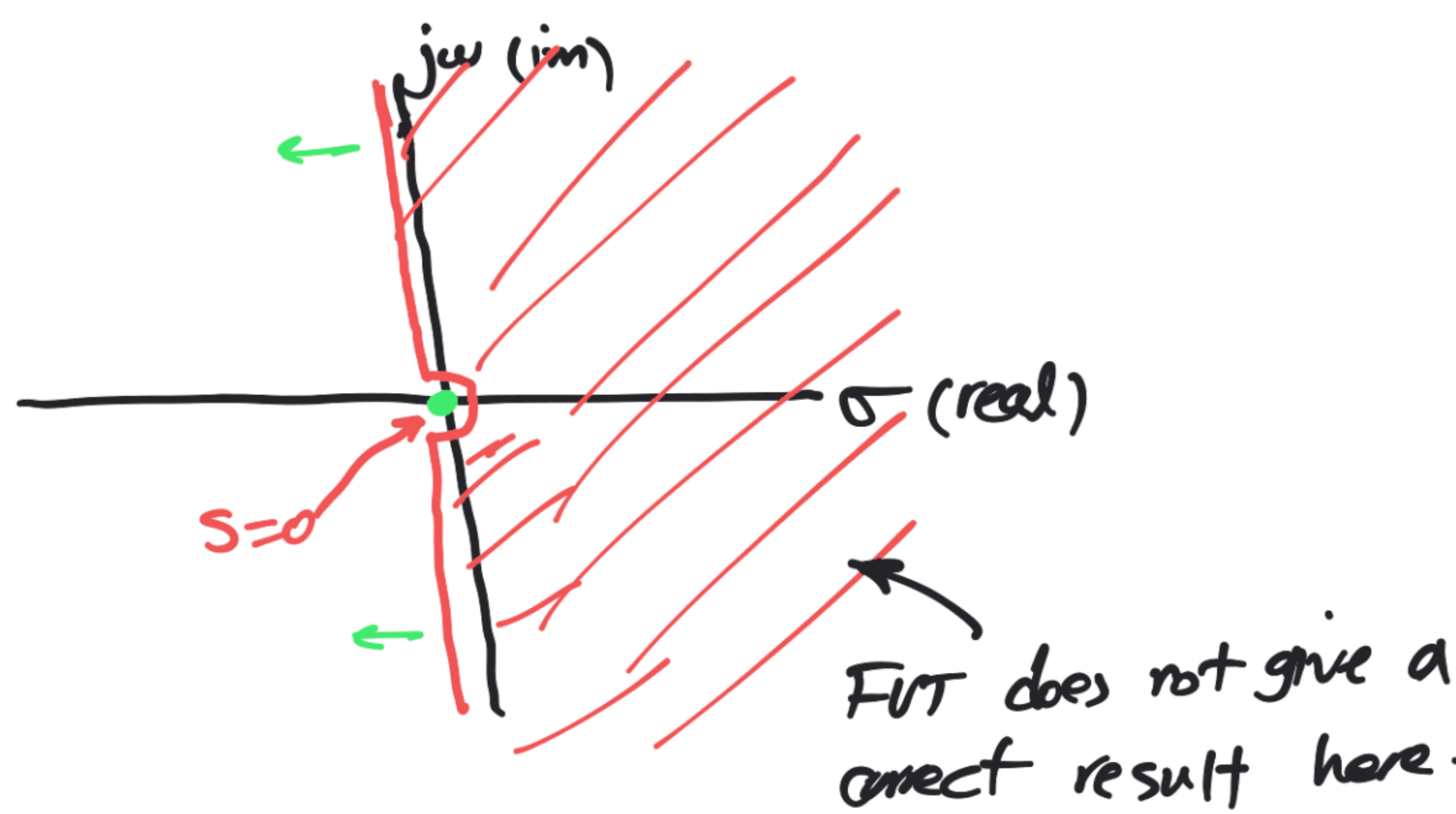
2) FVT (Final Value Theorem)

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$$

} steady-state response

$s = \sigma + j\omega$

Poles:



The final value theorem works for systems with all poles either on the $\underbrace{\text{LHP}}_{\sigma<0}$ or at the origin or combination

$(FV=0)$ $F(s) = \dfrac{1}{s+2}$ $\Rightarrow$ S-S : $\lim_{s \to 0} s \dfrac{1}{s+2} = 0$ (No poles at the origin)

$s=-2$

$(FV=cte)$ $F(s) = \dfrac{1}{s}$ $\Rightarrow$ S-S : $\lim_{s \to 0} \cancel{s} \dfrac{1}{\cancel{s}} = 1$ (One pole at the origin)

$s=0$

$(FV=\infty)$ $F(s) = \dfrac{1}{s^2}$ $\Rightarrow$ S-S : $\lim_{s \to 0} \cancel{s} \dfrac{1}{s^{\cancel{2}}} = \infty$ (Two or more poles at the origin)

$s=0$

Pole : roots of denominator of TF

denominator $=0$ ← characteristic eq.

First-order system:

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau s+1} \Rightarrow \underbrace{Y(s) = \frac{K}{\tau s+1} U(s)}$$

$$\overset{?}{\boxed{y(t)}}$$

Unit-step Response:

Unit-step $U(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s} \frac{K}{\tau s+1}$

$$\Rightarrow Y(s) = \frac{C_1}{s} + \frac{C_2}{\tau s+1} = \underbrace{\frac{K}{s} - \frac{\tau K}{\tau s+1}} = \underbrace{K\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right)}$$

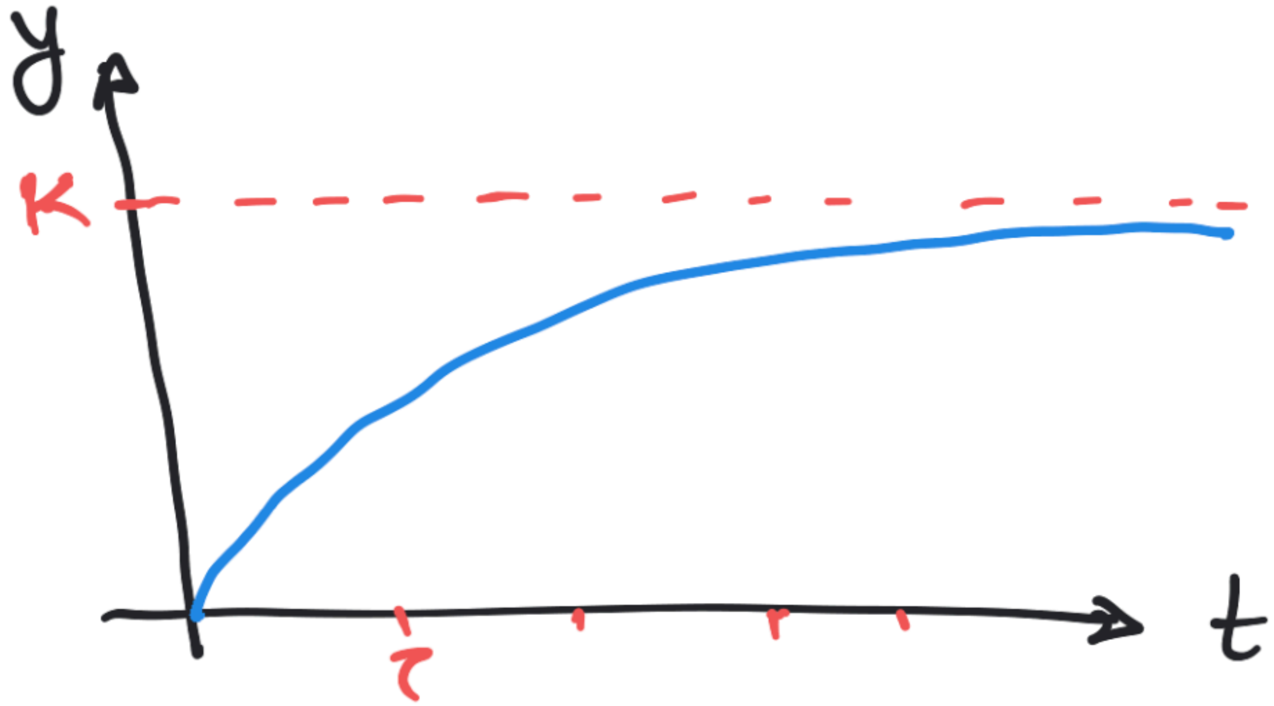
$$C_1 = s \frac{1}{s} \frac{K}{\tau s+1}\Big|_{s=0} = K$$

$$C_2 = (\tau s+1) \frac{1}{s} \frac{K}{\tau s+1}\Big|_{s=-1/\tau} = -\tau K$$

$$\boxed{Y(s) = K\left(\frac{1}{s} - \frac{1}{s+1/\tau}\right)}$$

$$\mathcal{L}^{-1} \Rightarrow \boxed{y(t) = K\left(1 - e^{-t/\tau}\right)}, \ t > 0$$

$$y(t) = K\left(1 - e^{-t/\tau}\right), \quad t > 0$$



t	% final value	
0	0	
τ	63.2%	
2τ	86.5%	
3τ	95%	$\rightarrow 5\%$
4τ	98.2%	$\rightarrow 2\%$
5τ	99.3%	$\rightarrow 1\%$
$\vdots$		

settling time

time required for unit step response to remain within 2% or 5% of final valu

Unit-Ramp Response: $(K=1)$

$u(t)=t$

$$\frac{Y(s)}{R(s)} = \frac{1}{\tau s+1} = G(s) \qquad U(s) = \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{1}{s^2(\tau s+1)} = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{\tau s+1}$$

$$= \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$$

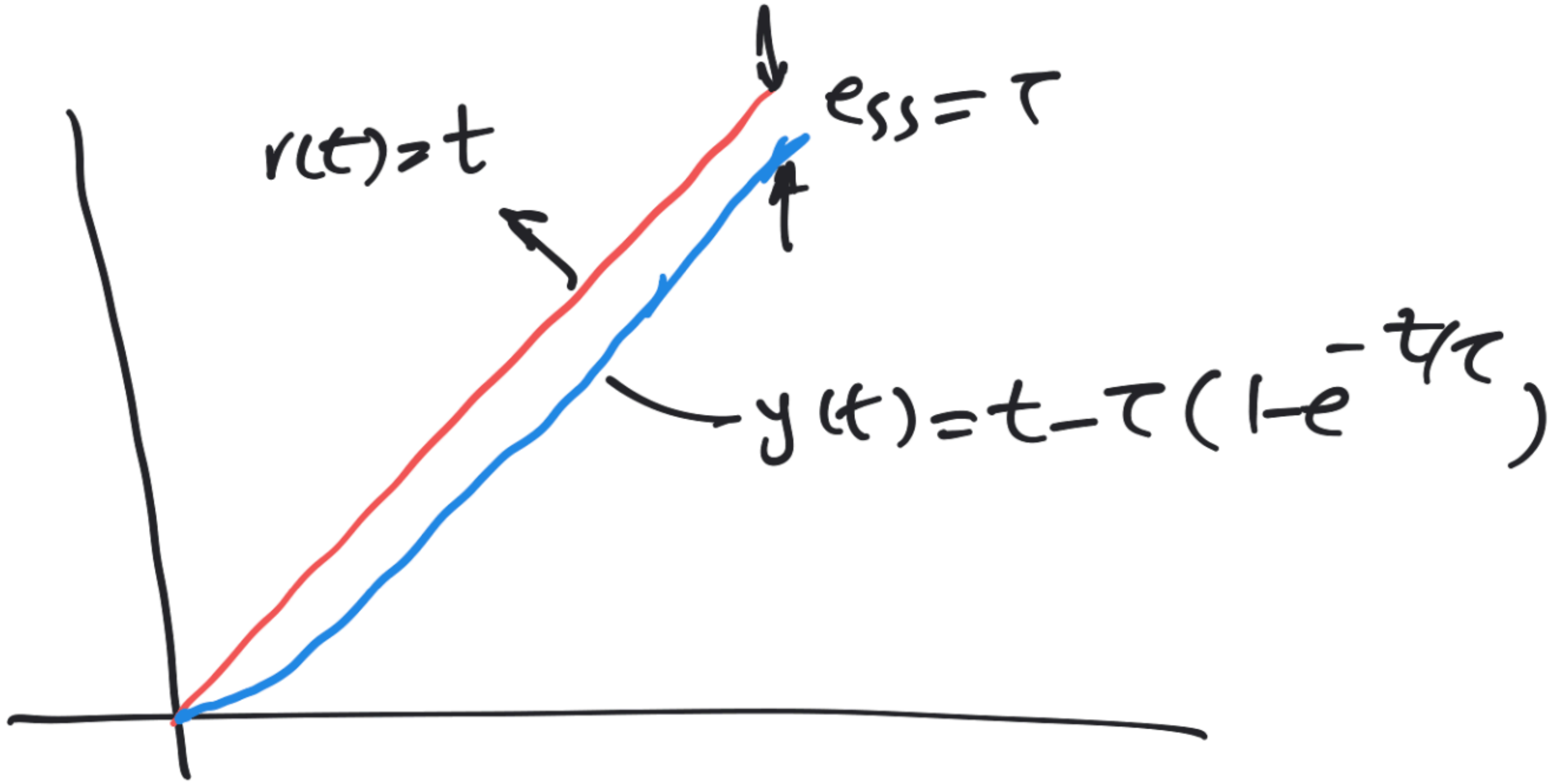
$$\stackrel{\mathcal{L}^{-1}}{\Rightarrow} \boxed{y(t) = t - \tau + \tau e^{-t/\tau}}$$

Error: $e(t) = r(t) - y(t)$

$$e(t) = \cancel{t} - \cancel{t} + \tau(1 - e^{-t/\tau})$$

$$\Rightarrow \boxed{e_{ss} = \tau}$$

$e(t)$ when $t \to \infty$


$e_{ss} = \tau$
$r(t) = t$
$y(t) = t - \tau(1 - e^{-t/\tau})$

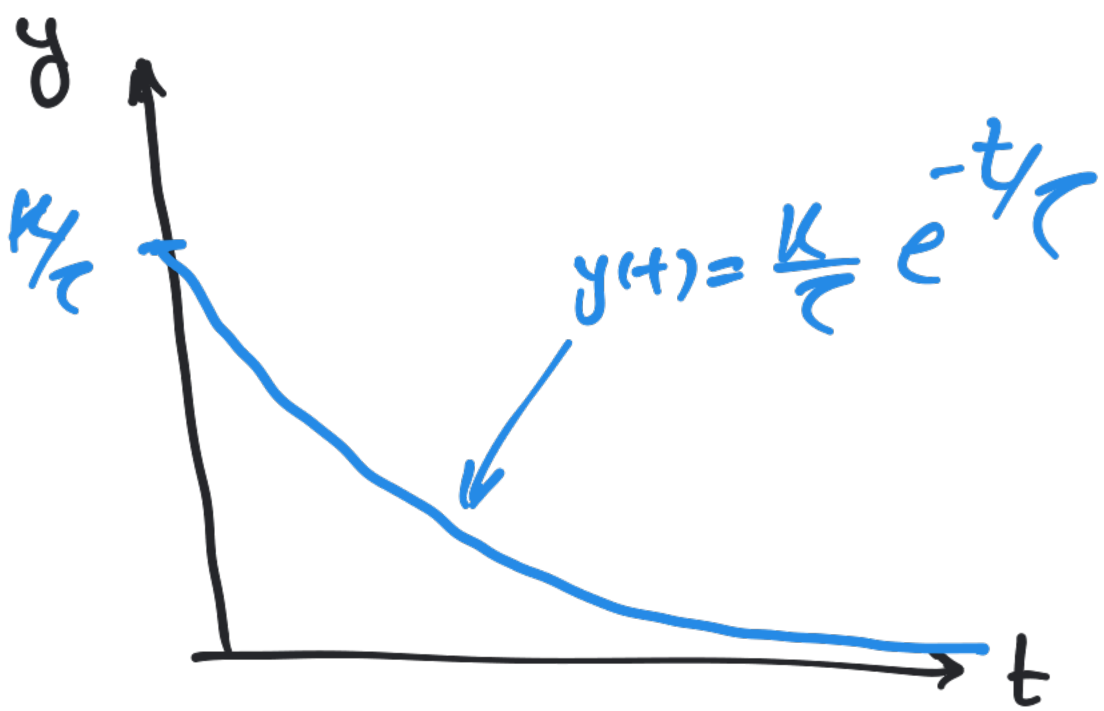
Continue

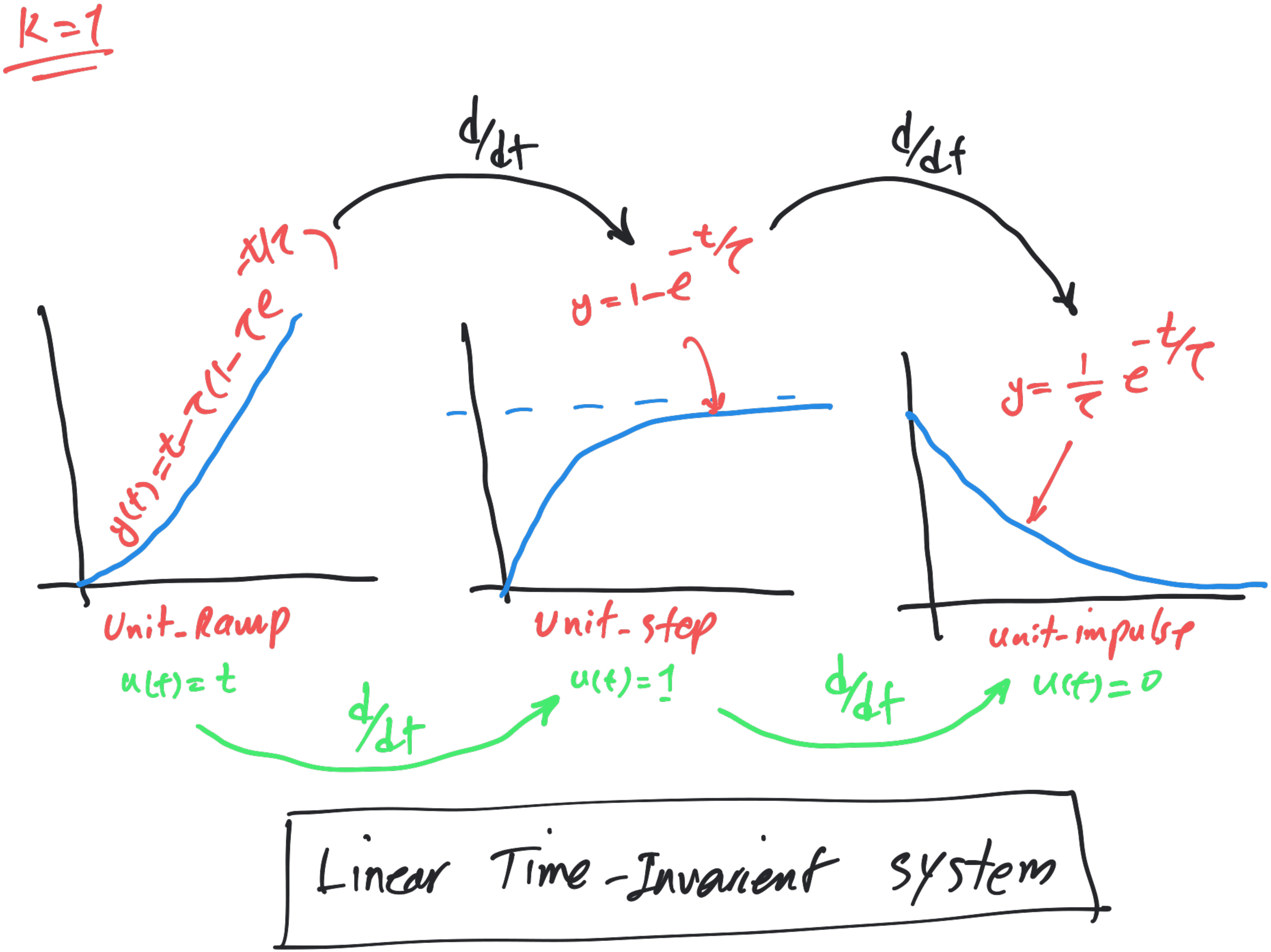
First-order system Response to Impulse Input: $U(s)=1$

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau s+1} \xrightarrow{U(s)=1} Y(s) = \frac{K}{\tau s+1} \Rightarrow \boxed{y(t) = \frac{K}{\tau} e^{-t/\tau}}$$

char. eq. $\tau s+1=0 \Rightarrow s=-1/\tau$ (LHP)

(No poles at the origin) $\Rightarrow$ $FV = y_{ss} = 0$





K=1
d/dt
d/dt
$y(t)=t-\tau(1-e^{-t/\tau})$
$y=1-e^{-t/\tau}$
$y=\frac{1}{\tau}e^{-t/\tau}$
Unit_Ramp
u(t)=t
Unit_step
u(t)=1
unit-impulse
u(t)=0
d/dt
d/dt
Linear Time-Invariant system

Second-order systems:

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0} \Rightarrow G(s) = K \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

standard form

K: DC gain

ω_n : Natural undamped frequency

ξ : damping ratio

$\omega_d = \omega_n\sqrt{1-\xi^2}$: damped frequency

$\alpha = \xi\omega_n$: Exponential Decay coefficient

We are looking for poles:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\Rightarrow \underbrace{s^2 + 2\xi\omega_n s + (\xi\omega_n)^2}_{(s+\underbrace{\xi\omega_n}_{\alpha})^2} + \underbrace{\omega_n^2 - (\xi\omega_n)^2}_{(\underbrace{\omega_n\sqrt{1-\xi^2}}_{\omega_d})^2}$$

$$\Rightarrow G(s) = \frac{\omega_n^2}{(s+\alpha)^2 + \omega_d^2} = \boxed{C \frac{\omega_d}{(s+\alpha)^2 + \omega_d^2}} \xrightarrow{\mathcal{L}^{-1}} e^{-\alpha t} \sin \omega_d t$$

poles? $(s+\alpha)^2+\omega_d^2=0$

$$*\ a^2+b^2=(a+jb)(a-jb)$$

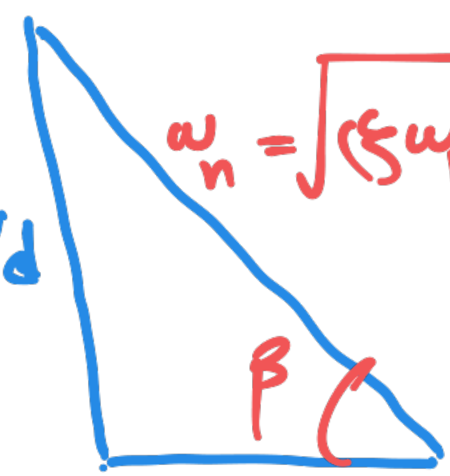
$a=s+\alpha$

$b=\omega_d$

$\alpha=\xi\omega_n$

$$\Rightarrow (s+\alpha+j\omega_d)(s+\alpha-j\omega_d)=0$$

$$\Rightarrow \boxed{s_{1,2}=P_{1,2}=-\alpha\pm j\omega_d}$$ Poles



$$\omega_n=\sqrt{(\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

$$\omega_n\sin\beta=\underline{\omega_n\sqrt{1-\xi^2}}=\omega_d$$

$$\underline{\xi}\omega_n=\underline{\omega_n}\cos\beta \rightarrow \beta=\cos^{-1}\xi$$

For a given ξ, poles will be on a line making $\pm\beta$ from real axis

$$\Rightarrow P_{1,2}=-\omega_n\cos\beta\pm j\omega_n\sin\beta$$

$$\Rightarrow P_{1,2}=\omega_n\cos(\pi-\beta)\pm j\overset{\omega_n}{\sin}(\pi-\beta)$$

$$\Rightarrow P_{1,2}=\omega_n\left(\underbrace{\cos(\pi-\beta)\pm j\sin(\pi-\beta)}\right)\rightarrow e^{\pm j(\pi-\beta)}\Rightarrow \boxed{P_{1,2}=\omega_n e^{\pm j(\pi-\beta)}}$$

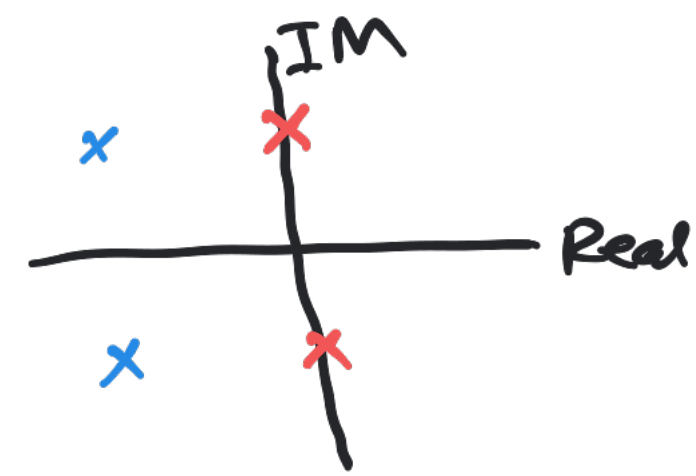
Impulse Response of 2nd order system :

$$G(s)=\frac{Y(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$R(s)=1$

$y(t)=?$

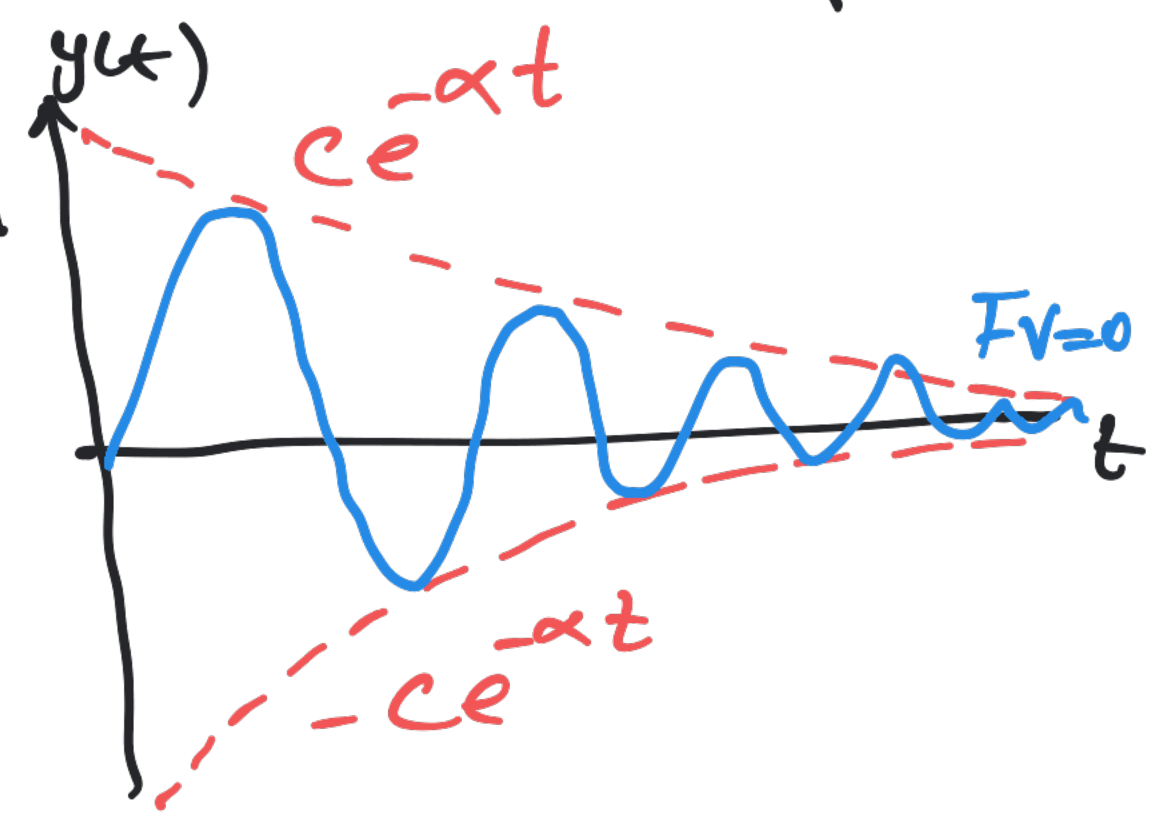
$y_{ss}=?=y(t)$
$t\to\infty$



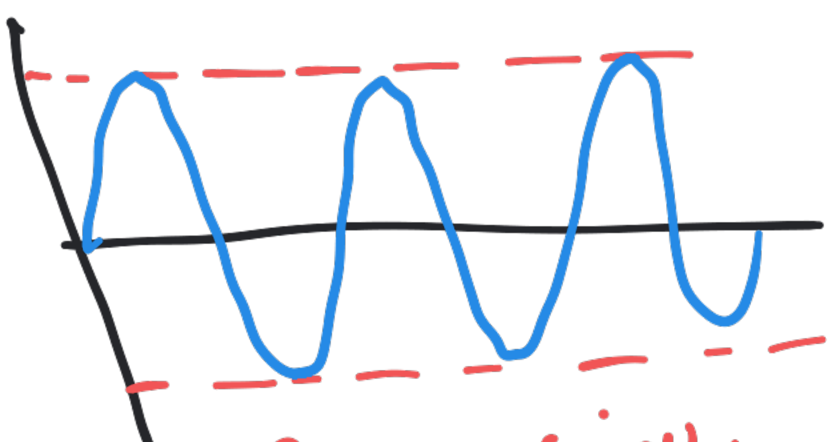
$$\Rightarrow Y(s)=\underbrace{\frac{\omega_n^2}{\omega_d}}_{C}\,\frac{\omega_d}{(s+\alpha)^2+\omega_d^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \boxed{y(t)=\underbrace{C\,e^{-\alpha t}}\ \underbrace{\sin\omega_d t}_{-1<\ <1}\ ,\ t>0}$$

$Ce^{-\alpha t}$ is the envelope of oscilation



If $\alpha=0$



$P_{1,2}=\pm j\omega_d$

Unit-step response of a 2nd order system :

$$G(S) = \frac{Y(S)}{R(S)} = K \, \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2} \qquad R(S) = \frac{1}{S}$$

$$\Rightarrow Y(S) = \frac{1}{S} \cdot K \cdot \frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2}$$

$$y_{ss} = \lim_{s \to 0} s\,Y(s) = \lim_{s \to 0} \cancel{s} \cdot \frac{1}{\cancel{s}} \cdot K \, \cancelto{1}{\frac{\omega_n^2}{s^2 + \xi \omega_n s + \omega_n^2}} = \underline{\underline{K}}$$

$1 = K$