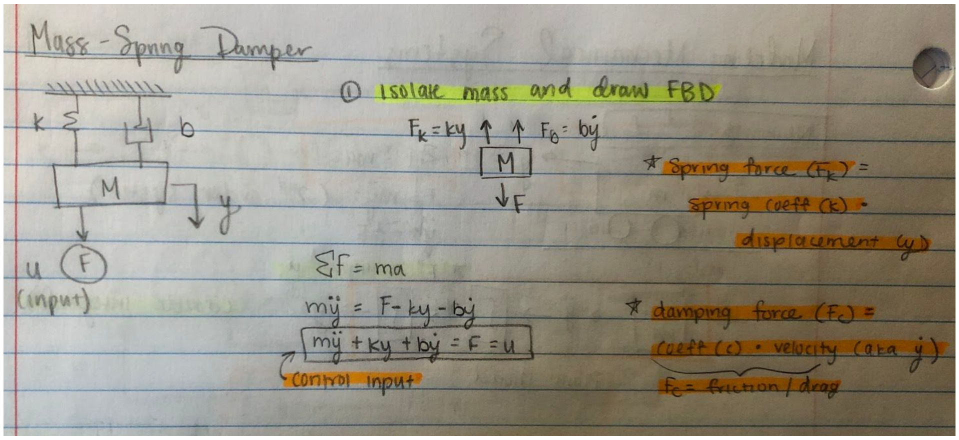
Lab 2: Mathematical Modeling I

Introduction:

In this lab, our focus was on mathematical modeling. The central dogma was to describe/model our physical system mathematically to be able to analyze its behavior. We used free body diagrams to model our robot, then derived torque and force equations, which helped model our robot.

MASS-SPRING-DAMPER SYSTEM

1. Free Body Diagram:



Mass - Spring Damper

k b M y

u (F) (input)

① Isolate mass and draw FBD

$F_k = ky \uparrow \quad \uparrow F_b = b\dot{y}$

M

$\downarrow F$

$\sum F = ma$

$m\ddot{y} = F - ky - b\dot{y}$

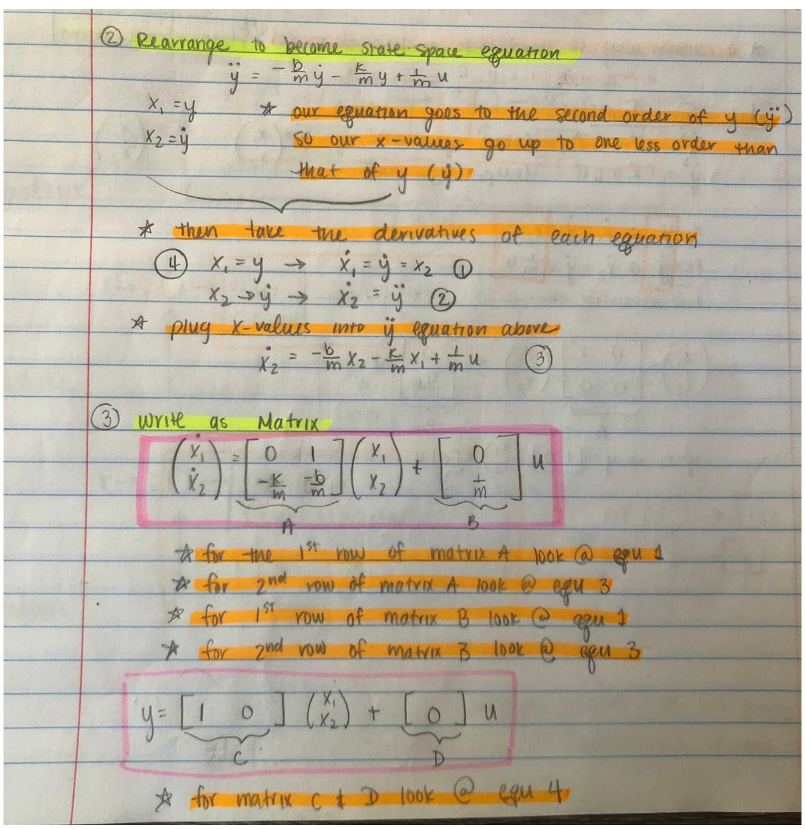
$\boxed{m\ddot{y} + ky + b\dot{y} = F = u}$

control input

☆ Spring force (F_k) = spring coeff (k) • displacement (y)

☆ damping force (F_c) = $\underbrace{\text{coeff } (c) \cdot \text{velocity (aka } \dot{y})}_{F_c = \text{friction / drag}}$

2. Differential equations of the system



② Rearrange to become state space equation

$$\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$$

$x_1 = y$

$x_2 = \dot{y}$

☆ our equation goes to the second order of y $(\ddot{y})$ so our x-values go up to one less order than that of y $(\dot{y})$

☆ then take the derivatives of each equation

④ $x_1 = y \rightarrow \dot{x}_1 = \dot{y} = x_2$ ①

$x_2 \rightarrow \dot{y} \rightarrow \dot{x}_2 = \ddot{y}$ ②

☆ plug x-values into $\ddot{y}$ equation above

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u \quad ③$$

③ write as Matrix

$$\boxed{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u}$$

☆ for the 1st row of matrix A look @ equ 1

☆ for 2nd row of matrix A look @ equ 3

☆ for 1st row of matrix B look @ equ 1

☆ for 2nd row of matrix B look @ equ 3

$$\boxed{y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u}$$

☆ for matrix C & D look @ equ 4

3. MATLAB Code:

```
>> k =1;
>> m =5;
>> b =.5;
>> f =2;
>> a = [0 1; -k/m -b/m];
>> b = [0 ; 1/m];
>> c = [1 0];
>> d = 0;
>> sys = ss(a,b,c,d)

sys =

  A =
          x1    x2
   x1      0     1
   x2   -0.2  -0.1

  B =
        u1
   x1    0
   x2  0.2

  C =
       x1  x2
   y1   1   0

  D =
       u1
   y1   0

Continuous-time state-space model.
```

MOTOR POSITION

Lab 2

Part 1 - Motor Position

$x_1 = \theta$

$x_2 = \dot{\theta}$

$x_3 = i$

$\dot{x}_1 = \dot{\theta} = x_2$

$$\dot{x}_2 = \ddot{\theta} = \frac{.025}{3\times10^{-6}} i - \frac{3.5\times10^{-6}}{3\times10^{-6}} \dot{\theta}$$

$$\dot{x}_2 = \frac{.025}{3\times10^{-6}} x_3 - \frac{3.5\times10^{-6}}{3\times10^{-6}} x_2$$

$$\dot{x}_3 = \frac{V}{3\times10^{-6}} - \frac{.025}{3\times10^{-6}} \dot{\theta} - \frac{5}{3\times10^{-6}} i$$

$$\dot{x}_3 = \frac{V}{3\times10^{-6}} - \frac{.025}{3\times10^{-6}} x_2 - \frac{5}{3\times10^{-6}} x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-3.5\times10^{-6}}{3\times10^{-6}} & \frac{.025}{3\times10^{-6}} \\ 0 & \frac{-.025}{3\times10^{-6}} & \frac{-5}{3\times10^{-6}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3\times10^{-6}} \end{bmatrix} V$$

$$\theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] V$$

MATLAB Code:

1.

```
>> var1 = 3*10^-6

var1 =

   3.0000e-06

>> var2 = -3.5*10^-6

var2 =

  -3.5000e-06

>> a = [0 1 0; 0 var2/var1 .025/var1; 0 -.025/var1 -5/var1];
>> b = [0;0;1/var1];
>> c = [1 0 0];
>> d = 0;
>> sys = ss(a,b,c,d)

sys =

  A =
                x1           x2           x3
   x1            0            1            0
   x2            0       -1.167         8333
   x3            0        -8333   -1.667e+06

  B =
              u1
   x1          0
   x2          0
   x3  3.333e+05

  C =
       x1  x2  x3
   y1   1   0   0

  D =
       u1
   y1   0

Continuous-time state-space model.
```

2.

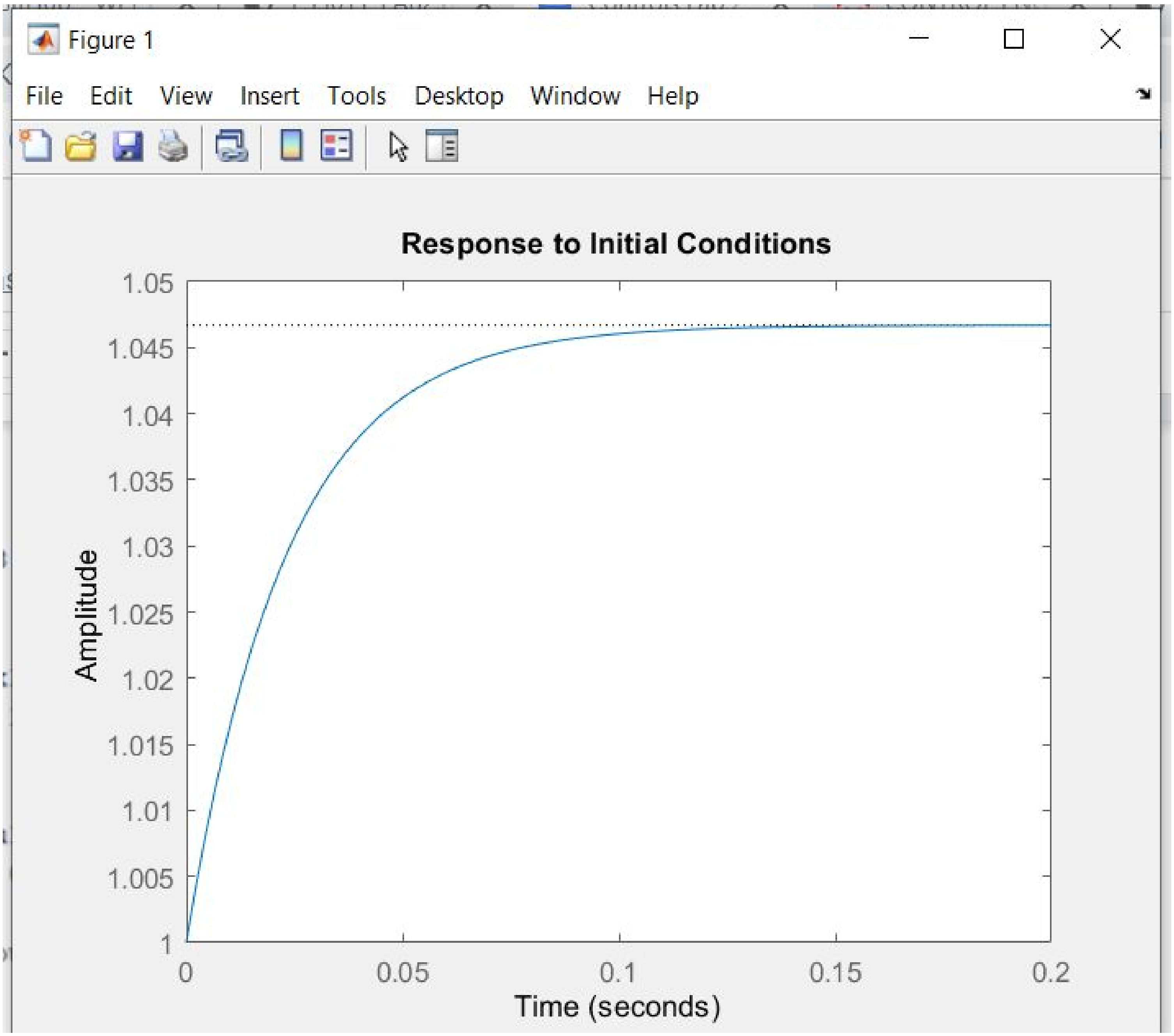


Figure 1
File
Edit
View
Insert
Tools
Desktop
Window
Help
Response to Initial Conditions
Amplitude
1.05
1.045
1.04
1.035
1.03
1.025
1.02
1.015
1.01
1.005
1
0
0.05
0.1
0.15
0.2
Time (seconds)

$\boxed{\text{Part 2}}$

1. $T = k_T \cdot i \quad (N \cdot m)$

$i = \dfrac{V - k_v \cdot w}{R}$

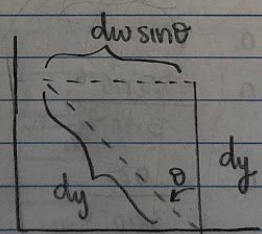
$$\boxed{T_L = k_T \left(\frac{V_L - k_v w_L}{R_L} \right)} \qquad \boxed{T_R = k_T \left(\frac{V_R - k_v w_R}{R_R} \right)}$$

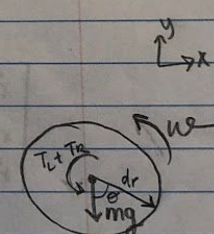
2.

Symbol	Unit
I_x	$kg\,m^2$
I_z	$kg\,m^2$
m	kg
g	N
d_g	m
d_w	m
d_r	m
θ_x	rad
w_z	rad/sec
$w_{R/L}$	rad/sec
V_y	m/s

3.1 Torque = angular acceleration (Inertia)

$T = \alpha I$



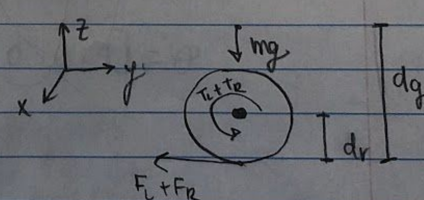


★ 3.2 $T = I\alpha$

$\alpha = \dfrac{d\vec{v}}{dt} = \ddot{\theta}$

$T_x = I_x \ddot{\theta}_x$

$$\boxed{T_x = I_x \frac{d^2\theta_x}{dt^2}}$$



$T = Fr\sin\theta$

$\underbrace{T = mg \cdot dg \cdot \sin\theta}_{\text{torque of machine excluding wheels}} \Rightarrow T = (m \cdot g \cdot dg)\,\theta_x$

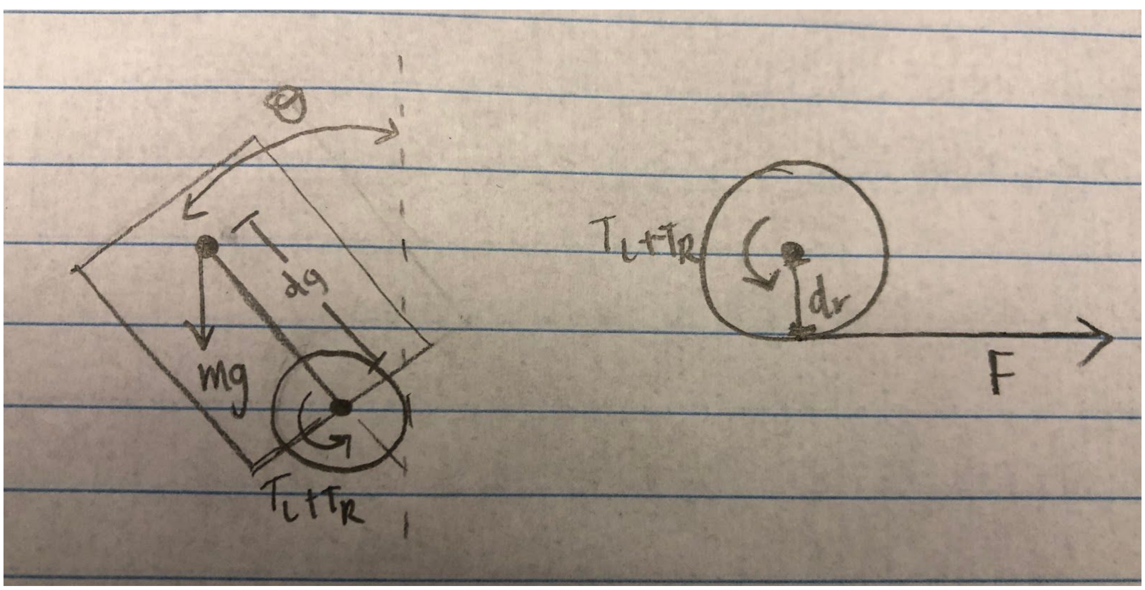
$$\sum T_x = \left(\frac{T_L}{d_r} + \frac{T_R}{d_r} \right) dg + mg\,dg\,\theta = \underbrace{\frac{dg}{d_r}(T_R + T_L) + (mgdg)\,\theta_x}_{\text{Torque of machine including wheels}}$$

3.3 $F = m\dot{v}_y$ $\quad F_R = \frac{T_R}{d_r}$ $\quad F_L = \frac{T_L}{d_r}$

$$F_{friction} = \frac{T_L}{d_r} + \frac{T_R}{d_r}$$

3.4 $T_z = I_z \cdot \dot{\omega}_z$ $\quad T_z = \frac{T_r - T_L}{d_r}$

FBD of Robot:



4.1 $V_y = d_r \cdot \frac{(w_r + w_L)}{2}$

$V = r\omega \Rightarrow$

$\omega = \frac{V}{r}$

$\omega = \frac{V_y}{r}$

$\omega = \frac{\frac{d_r(w_r + w_L)}{2}}{d_w}$ ★ d_w = radius = r (wheel track)

$\omega_z = \frac{(w_R - w_L)\, d_r}{2\, d_w}$

4.2 $V_y = \frac{w_r + w_L}{2} \cdot d_r$

4.3

$$W_z = \frac{d_r (w_R - w_L)}{2 d_w}$$

$$\frac{2 d_w W_z}{d_r} = W_R - W_L \Rightarrow \left[\frac{2 d_w W_z}{d_r} + W_L = W_R \right] \left[-\frac{2 d_w W_z}{d_r} + W_R = W_L \right]$$

$$V_y = \frac{W_r + W_L}{2} (d_r)$$

$$\left[\frac{2 V_y}{d_r} - W_L = W_R \right] \qquad \left[\frac{2 V_y}{d_r} - W_R = W_L \right]$$

$$\frac{2 d_w W_z}{d_r} - W_R + \frac{2 V_y}{d_r} = W_R \qquad \frac{2 V_y}{d_r} - \frac{2 d_w W_z}{d_r} - W_L = W_L$$

$$\boxed{\frac{d_w W_z + V_y}{d_r} = W_R} \qquad \boxed{\frac{V_y - d_w W_z}{d_r} = W_L}$$

5.1

$$T_x = \frac{d_g}{d_r} (T_R + T_L) + (m g d_g) \theta_x$$

$$T_x = \frac{d_g}{d_r} \left[\frac{k_T (V_R - k_v W_R)}{R_R} + \frac{k_T (V_L - k_v W_L)}{R_L} \right] + m g d_g \theta_x$$

$$T_x = \frac{d_g k_T}{d_r R} \left[V_R - k_v \left(\frac{d_w W_z + V_y}{d_r} \right) + V_L - k_v \left(\frac{V_y - d_w W_z}{d_r} \right) \right] + m g d_g \theta_x = I_x \ddot{\theta}_x$$

$$\frac{I_x \ddot{\theta}_x}{d_g} = \frac{k_T}{R d_r} \left[V_R + V_L - k_v \frac{(2 V_y)}{d_r} \right] + m g d_g \theta_x$$

$$\frac{I_x \ddot{\theta}_x d_r}{d_g} = \frac{k_T}{R} \left[V_R + V_L - k_v \frac{(2 V_y)}{d_r} \right] + m g d_g \theta_x$$

5.2

$$m \dot{V}_y = \frac{T_L + T_R}{d_r}$$

$$d_r m \dot{V}_y = T_L + T_R$$

$$d_r m \dot{V}_y = k_T \frac{(V_L - k_v W_L)}{R_L} + k_T \frac{(V_R - k_v W_R)}{R_R}$$

$$d_r m \dot{V}_y = \frac{k_T}{R} (V_L + V_R - k_v (W_L + W_R))$$

$$d_r m \dot{V}_y = \frac{k_T}{R} \left(V_L + V_R - k \frac{(2 V_y)}{d_r} \right)$$

5.3

$$I_z \dot{\omega}_z = \frac{T_r - T_L}{d_r}$$

$$d_r (I_z \dot{\omega}_z) = T_r - T_L$$

$$d_r (I_z \dot{\omega}_z) = k_T \frac{(V_R - k_V W_R)}{R_R} - k_T \frac{(V_L - k_V W_L)}{R_L}$$

$$d_r (I_z \dot{\omega}_z) = \frac{k_T}{R} (V_R - k_V W_R) + \frac{k_T}{R} (-V_L + k_V W_L)$$

$$d_r (I_z \dot{\omega}_z) = \frac{k_T}{R} (V_R - V_L - k_V (W_R - W_L))$$

$$d_r (I_z \dot{\omega}_z) = \frac{k_T}{R} \left(V_R - V_L - k_V \frac{(2 W_z d_w)}{d_r}\right)$$

Conclusion:

The goals of this lab include understanding, describing, and modeling the physical dynamic system mathematically using differential equations. We converted the differential equations to state-space model form by hand and using MATLAB functionality. From this lab we were able to solidify the basics of MATLAB modeling in regards to the state-space model. We also gained an in-depth understanding of the torques, velocities, accelerations, and forces acting on our machine.