Homework 4

1. $[m_1\ddot{y}_1 + (c_1+c_2)\dot{y}_1 + ky_1 - c_2\dot{y}_2] = [F]$

$\mathcal{L}[m_1\ddot{y}_1 + c_1\dot{y}_1 + c_2\dot{y}_1 + ky_1 - c_2\dot{y}_2] = \mathcal{L}[F]$

$Y_1(s)(m_1s^2 + c_1s + c_2s + k) - (c_2s)Y_2(s) = F(s)$

$\mathcal{L}[m_2\ddot{y}_2 + c_2\dot{y}_2] = \mathcal{L}[c_2\dot{y}_1]$

$Y_2(s)(m_2s^2 + c_2s) = Y_1(s)(c_2s)$

$$Y_1(s) = \frac{Y_2(s)(m_2s^2 + c_2s)}{c_2s}$$

$$\left[\frac{Y_2(s)(m_2s^2+c_2s)}{c_2s}(m_1s^2+c_1s+c_2s+k)\right] - Y_2(s)(c_2s) = F(s)$$

$$Y_2(s)\left[\frac{(m_2s^2+c_2s)}{c_2s}(m_1s^2+c_1s+c_2s+k) - c_2s\right] = F(s)$$

$$\frac{Y_2(s)}{F(s)} = \frac{1}{\frac{(m_2s^2+c_2s)(m_1s^2+c_1s+c_2s+k)-(c_2s)^2}{c_2s}}$$

b) $$\boxed{\frac{Y_2(s)}{F(s)} = \frac{c_2s}{(m_2s^2+c_2s)(m_1s^2+c_1s+c_2s+k)-(c_2s)^2}}$$

$$\frac{Y_1(s)}{Y_2(s)} = \frac{\frac{Y_2(s)(m_2s^2+c_2s)}{c_2s}}{Y_2(s)} = \frac{m_2s^2+c_2s}{c_2s}$$

$$\frac{Y_1(s)}{F(s)} = \frac{Y_1(s)}{\cancel{Y_2(s)}}\cdot\frac{\cancel{Y_2(s)}}{F(s)}$$

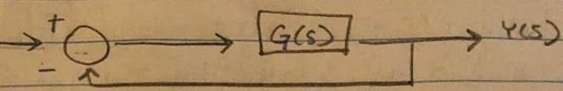
$$\frac{Y_1(s)}{F(s)} = \frac{m_2s^2+c_2s}{\cancel{c_2}s}\cdot\frac{\cancel{c_2}s}{(m_2s^2+c_2s)(m_1s^2+c_1s+c_2s+k)-(c_2s)^2}$$

a) $$\boxed{\frac{Y_1(s)}{F(s)} = \frac{m_2s^2+c_2s}{(m_2s^2+c_2s)(m_1s^2+c_1s+c_2s+k)-(c_2s)^2}}$$

Controls

Homework 4 Cont.

2. $G(s) = \dfrac{2s+1}{s^2}$



$$\frac{Y(s)}{X(s)} = \frac{G}{1+G} = \frac{2s+1}{s^2+2s+1}$$

$X(s) = 1$

$$\frac{Y(s)}{1} = \frac{2s+1}{s^2+2s+1} = \frac{2(s+1)-1}{(s+1)^2}$$

$$\mathcal{L}^{-1}[Y(s)] = \left[\frac{2}{s+1} - \frac{1}{s+1^2}\right]\mathcal{L}^{-1}$$

$$\boxed{y(t) = 2e^{-t} - te^{-t} \quad t \geq 0} \Rightarrow \text{unit-impulse input}$$

Let $x(t) = u(t)$

$X(s) = \frac{1}{s}$

$$\frac{Y(s)}{X(s)} = \frac{2s+1}{s^2+2s+1}$$

$$Y(s) = \frac{(2s+1)}{s(s+1)^2}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$2s+1 = A(s+1)^2 + B(s+1) + Cs$

$0 = A+B \qquad 2 = 2A+B+C \qquad 1 = A$

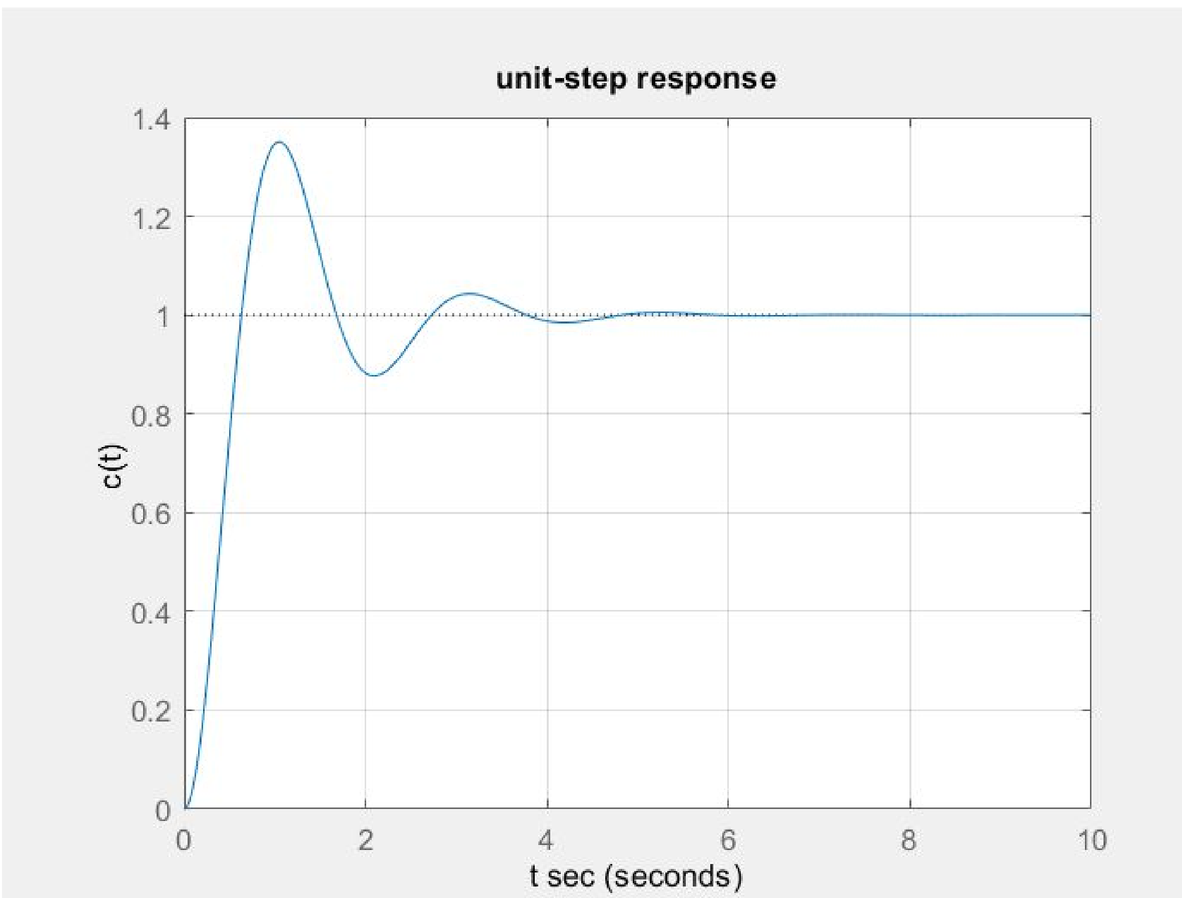
$B = -A \qquad B = -1 \qquad C = 1$

$$\mathcal{L}^{-1}[Y(s)] = \left[\frac{1}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2}\right]\mathcal{L}^{-1}$$

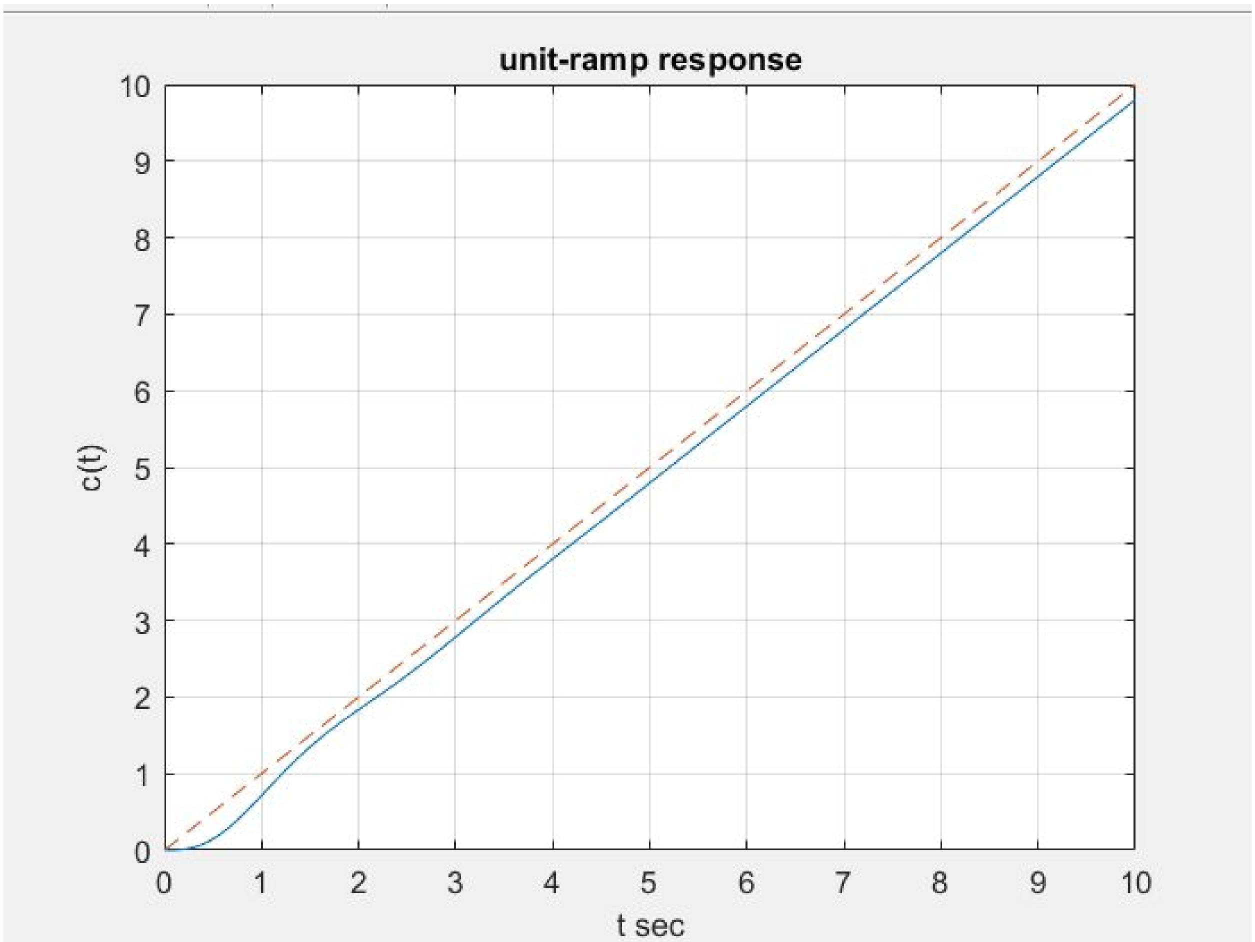
$$\boxed{y(t) = 1 + te^{-t} - e^{-t}} \Rightarrow \text{unit-step input} \quad t \geq 0$$

3.

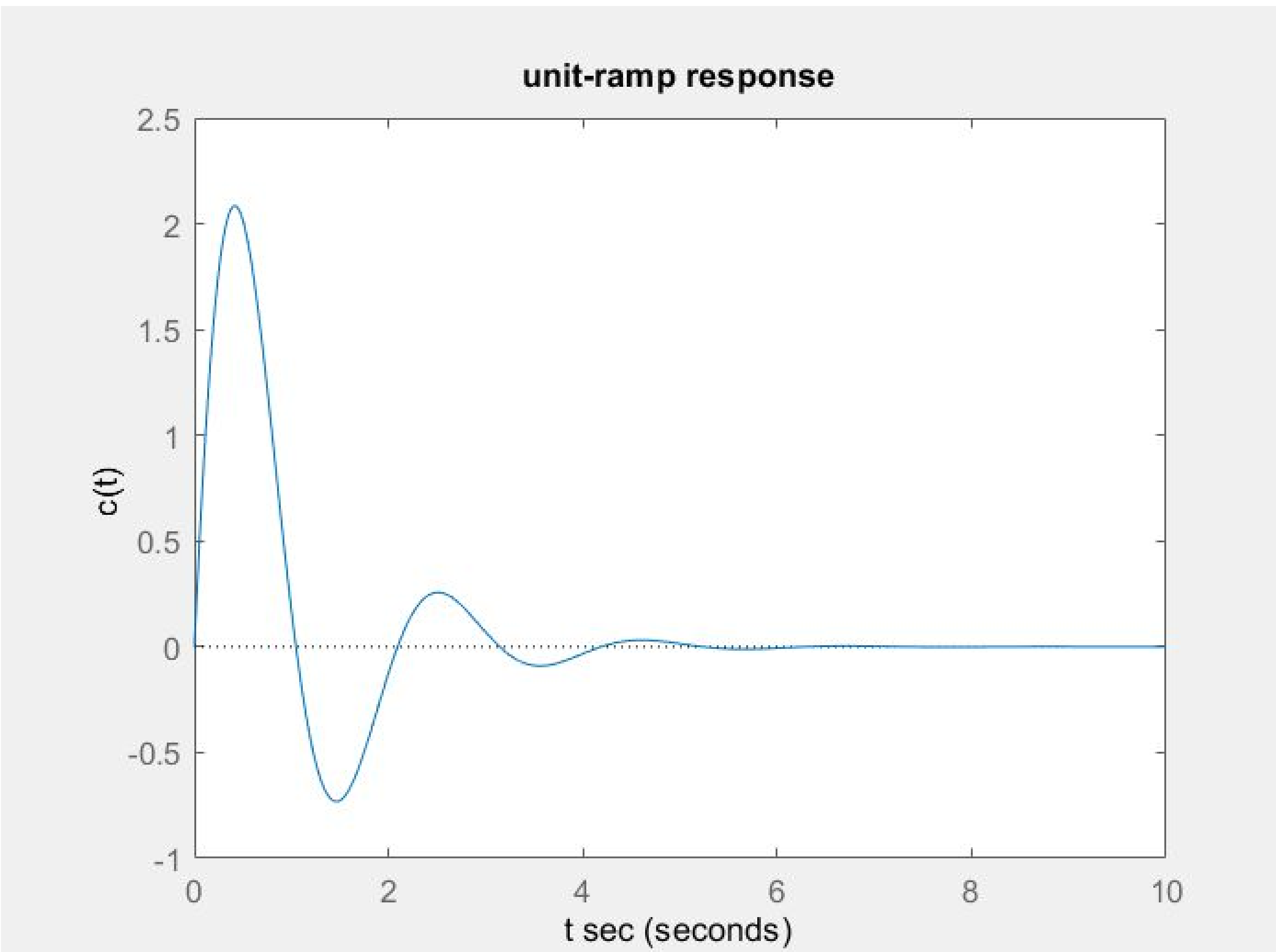
```
>> num = [0 0 10];
>> den = [1 2 10];
>> t = 0:0.02:10;
>> step(num, den, t)

grid
>> title('unit-step response')
>> xlabel('t sec')
>> ylabel('c(t)')
```



```
>> numr = [0 0 0 10];
denr = [1 2 10 0];
c = step(numr, denr, t);
plot(t,c,'-',t,t,'--')
grid
title('unit-ramp response')
xlabel('t sec')
ylabel('c(t)')
```



```
>> impulse(num,den,t)
title('unit-ramp response')
xlabel('t sec')
ylabel('c(t)')
```



4. $G(s) = \dfrac{4}{s(s+2)}$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{\frac{4}{s(s+2)}}{1+\frac{4}{s(s+2)}} = \frac{C(s)}{R(s)} = \frac{4}{s^2+2s+4}$$

$s^2 + 2\xi\omega_n s + \omega_n^2$

$\omega_n^2 = 4 \Rightarrow \omega_n = 2$

$2\xi\omega_n = 2 \Rightarrow 4\xi = 2 \Rightarrow \xi = \frac{1}{2}$

$$\begin{aligned} \omega_d &= \omega_n\sqrt{1-\xi^2} \\ &= 2\sqrt{1-\tfrac{1}{4}} \\ &= 2\sqrt{\tfrac{3}{4}} \\ &= \frac{2\sqrt{3}}{2} = \sqrt{3} \end{aligned}$$

$\theta = \cos^{-1}(\xi) = \cos^{-1}(\frac{1}{2}) = \pi/3$

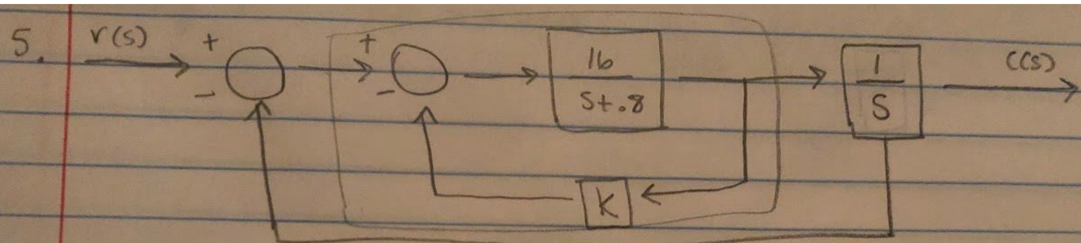
Risetime $= \dfrac{\pi - \pi/3}{\sqrt{3}} = \boxed{1.2092 \text{ sec}}$

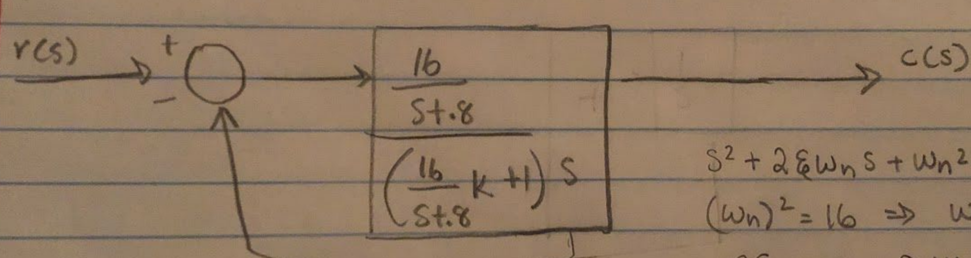
Peaktime $= \dfrac{\pi}{\sqrt{3}} = \boxed{1.81379 \text{ sec}}$

Maximum Overshoot $= M_p = e^{-\pi(.5)/\sqrt{1-.5^2}} = \boxed{.163}$

Setting time = 2% tolerance $= \dfrac{4}{.5(2)} = \boxed{4 \text{ sec.}}$

5.





$s^2 + 2\xi\omega_n s + \omega_n^2$

$(\omega_n)^2 = 16 \Rightarrow \omega_n = 4$ $\qquad \xi = .5$

$2\xi\omega_n = .8 + 16k$

$\dfrac{2(.5)(4) - .8}{16} = k = .2$

$$\frac{C(s)}{r(s)} = \frac{16}{s^2 + (.8 + 16k)s + 16}$$

$\omega_d = 4\sqrt{1-(.5)^2} = 3.46$

$\theta = \cos^{-1}(\frac{1}{2}) = \pi/3$

Risetime $= \dfrac{\pi - \theta}{\omega_d} = \dfrac{\pi - \pi/3}{3.46} = \boxed{.605 \text{ sec}}$

5 Cont.

peaktime $= \frac{\pi}{w_d} = \frac{\pi}{3.46} = \boxed{.907 \text{ sec}}$

Maximum Overshoot $= M_p = e^{\frac{-.5(\pi)}{\sqrt{1-.25}}} = e^{-1.814} = \boxed{.163}$

Settling Time $= \frac{4}{\xi w_n} = \frac{4}{.5(4)} = \boxed{2 \text{ sec}}$

6. $\frac{C(s)}{R(s)} = \frac{36}{s^2+2s+36}$

$s^2 + 2\xi w_n s + w_n^2$

$(w_n)^2 = 36 \Rightarrow w_n = 6$

$2\xi w_n s = 2s$

$6\xi = 1 \Rightarrow \xi = \frac{1}{6}$

$w_d = w_n\sqrt{1-\xi^2} = 6\sqrt{1-\frac{1}{36}} = \sqrt{35}$

$\theta = \cos^{-1}(\xi) = \cos^{-1}(\frac{1}{6}) = 1.4034$ rad

Risetime: $\frac{\pi-\theta}{w_d} = \frac{\pi - 1.4034}{\sqrt{35}} = \boxed{.2938 \text{ sec}}$

Peaktime: $\frac{\pi}{w_d} = \frac{\pi}{\sqrt{35}} = \boxed{.5310 \text{ sec}}$

Maximum Overshoot $= M_p = e^{\frac{-\pi(1/6)}{\sqrt{1-1/36}}} = e^{\frac{-\pi}{\sqrt{35}}} = e^{-.5310} = \boxed{.5880}$

Settling time $= \frac{4}{\xi w_n} = \frac{4}{\frac{1}{6}(6)} = \boxed{4 \text{ sec}}$