1) Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 5 & 1 & 2 & 3 \\ 2 & 11 & 5 & 0 & -1 \\ 3 & -1 & 4 & 0 & 2 \end{array}\right]$$

2) Solve system from (1).

$$\left[\begin{array}{cccc|c} 1 & 5 & 1 & 2 & 3 \\ 2 & 11 & 5 & 0 & -1 \\ 3 & -1 & 4 & 0 & 2 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 5 & 1 & 2 & 3 \\ 0 & 1 & 3 & -4 & -7 \\ 0 & -16 & 1 & -6 & -7 \end{array}\right] \xrightarrow[R_3+16R_2]{R_1-5R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -14 & 22 & 38 \\ 0 & 1 & 3 & -4 & -7 \\ 0 & 0 & 49 & -70 & -119 \end{array}\right] \xrightarrow[\frac{1}{49}R_3]{} \left[\begin{array}{cccc|c} 1 & 0 & -14 & 22 & 38 \\ 0 & 1 & 3 & -4 & -7 \\ 0 & 0 & 1 & -10/7 & -17/7 \end{array}\right] \xrightarrow[R_2-3R_3]{R_1+14R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 2/7 & 2/7 \\ 0 & 0 & 1 & -10/7 & -17/7 \end{array}\right].$$

$$x_1 + 2x_4 = 4$$
$$x_2 + 2/7\, x_4 = 2/7$$
$$x_3 - 10/7\, x_4 = -17/7.$$

Solution $\begin{bmatrix} 4 \\ 2/7 \\ -17/7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -2/7 \\ 10/7 \\ 1 \end{bmatrix}$.

3) Since the RREF of A has 3 pivots, the rank of A is 3. So $Ax=y$ has a solution for any $y \in \mathbb{R}^3$.

4) The dimension of the nullspace is 1, by (3) and the Rank-Nullity Theorem. A basis for the nullspace is given by $\begin{bmatrix} -2 \\ -2/7 \\ 10/7 \\ 1 \end{bmatrix}$.

5) A linear relation between columns of A is

$$-2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2/7 \begin{bmatrix} 5 \\ 11 \\ -1 \end{bmatrix} + 10/7 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

6) Yes, for example $A\begin{bmatrix}-2\\-2/7\\10/7\\1\end{bmatrix} = A\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

(There exist 2 such vectors if and only if Nullspace of A is non-trivial)

7) $\det\begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\\-1 & 0 & 1\end{bmatrix} = 1\cdot\det\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = -2.$ (Expand along col 3!)

Because $\det(B)\neq 0$, B has full rank. So the rank of B is 3.

8) char poly of B is $\det\begin{bmatrix}1-\lambda & 1 & 0\\1 & -1-\lambda & 0\\-1 & 0 & 1-\lambda\end{bmatrix} = (1-\lambda)\left[(1-\lambda)(-1-\lambda)-1\right]$

$= (1-\lambda)(\lambda^2-2)$.

Eigenvalues are $\lambda_1 = 1$, $\lambda_2=\sqrt{2}$, $\lambda_3 = -\sqrt{2}$.

9) Columns of B are lin. ind. because the determinant of B is non-zero.

10) B^{-1} exists because the determinant is non-zero.

11) $\lambda_1 = 1$, we compute an eigenvector, i.e. vector in nullspace of $B-I$

$$\left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\1 & -2 & 0 & 0\\-1 & 0 & 0 & 0\end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right].$$ Eigenvector $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

$\lambda_2 = \sqrt{2}$, we compute a vector in Nullspace $B-\sqrt{2}I$.

$$\left[\begin{array}{ccc|c}1-\sqrt{2} & 1 & 0 & 0\\1 & -1-\sqrt{2} & 0 & 0\\-1 & 0 & 1-\sqrt{2} & 0\end{array}\right]$$

$(1-\sqrt{2})(-1-\sqrt{2}) = -1+2 = 1.$

write α for $1-\sqrt{2}$.

$$\left[\begin{array}{ccc|c} \alpha & 1 & 0 & 0 \\ 1 & \alpha^{-1} & 0 & 0 \\ -1 & 0 & \alpha & 0 \end{array}\right] \to \begin{bmatrix} 1 & \alpha^{-1} & 0 \\ 1 & \alpha^{-1} & 0 \\ -1 & 0 & \alpha \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha^{-1} & \alpha & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \alpha^{-1} & 0 & 0 \\ 0 & \alpha^{-1} & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\alpha & 0 \\ 0 & 1 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \quad \begin{aligned} x_1 &= \alpha x_3 \\ x_2 &= -\alpha^2 x_3 \end{aligned}$$

So eigenvector is $\begin{bmatrix} 1-\sqrt{2} \\ -(1-\sqrt{2})^2 \\ 1 \end{bmatrix}$.

$\lambda_3 = -\sqrt{2}$, we compute a vector in Nullspace of $B + \sqrt{2} I$.
Similar to the above, get $\begin{bmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2})^2 \\ 1 \end{bmatrix}$.

12) P has eigenvectors of B as columns, so

$$P = \begin{bmatrix} 0 & 1-\sqrt{2} & 1+\sqrt{2} \\ 0 & -(1-\sqrt{2})^2 & -(1+\sqrt{2})^2 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1+\sqrt{2} \end{bmatrix}.$$

And $B = PDP^{-1}$.

13) For any integer n, we have

$$B^n = P D^n P^{-1}.$$

Since B is invertible, the formula holds for negative values of n as well.

Alternative solution for 13.

Since $B = PDP^{-1}$, it follows from the socks-shoes prop:

$$B^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$$
$$= PD^{-1}P^{-1}.$$

14) $$\begin{bmatrix} 1-\sqrt{2} \\ -(1-\sqrt{2})^2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2})^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}.$$

So $$B^n \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = B^n \left(\begin{bmatrix} 1-\sqrt{2} \\ -(1-\sqrt{2})^2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2})^2 \\ 1 \end{bmatrix} \right)$$
$$= (1-\sqrt{2})^n \begin{bmatrix} 1-\sqrt{2} \\ -(1-\sqrt{2})^2 \\ 1 \end{bmatrix} + (1+\sqrt{2})^n \begin{bmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2})^2 \\ 1 \end{bmatrix}.$$

Note $|1-\sqrt{2}|<1$ so $(1-\sqrt{2})^n$ tends to 0 as $n \to \infty$, so

$$B^n \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \sim (1+\sqrt{2})^n \begin{bmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2})^2 \\ 1 \end{bmatrix}$$ for large n.

15) $$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 1\cdot 2 + 1\cdot(-1) + (-1)\cdot 3$$
$$= 2 - 1 - 3 = -2.$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 + 2 - 1 = 2.$$

$$c_1 \cdot (c_2 + c_3) = c_1 \cdot c_2 + c_1 \cdot c_3$$

So $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = 0.$

16) $\|c_1\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}.$

Unit vector in direction of c_1 is $d_1 = 1/\sqrt{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$

$\|c_2 + c_3\| = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}.$

Unit vector in direction of $c_2 + c_3$ is $d_2 = 1/\sqrt{26} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

17) We need a vector orthogonal to c_1 and to $c_2 + c_3$

Let's start with $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

$x \cdot d_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot 1/\sqrt{3} = 1/\sqrt{3}.$

So $x' = x - (x \cdot d_1) d_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (1/\sqrt{3})(1/\sqrt{3}) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 1/3 \end{bmatrix}$

Now $x' \cdot d_1 = 0.$

Next we compute $x' \cdot d_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot 1/\sqrt{26}.$

$$= (2 - 1/3 + 4/3) \cdot 1/\sqrt{26}$$

$$= 3/\sqrt{26}.$$

So $x'' = x' - (x' \cdot d_2) d_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 1/3 \end{bmatrix} - 3/26 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 25/78 \\ -35/78 \\ -10/78 \end{bmatrix}$

For convenience, we multiply by 78/5

So the vector $x''' = \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix}$ satisfies

$$x''' \cdot d_1 = 0 \text{ and } x''' \cdot d_2 = 0.$$

Finally, we normalize to get $d_3 = 1/\sqrt{78} \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix}$.

So an orthonormal basis for $\mathbb{R}^3$ is

$$1/\sqrt{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad 1/\sqrt{26} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad 1/\sqrt{78} \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix}.$$

18) $$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right) d_1 + \left(\frac{1}{\sqrt{26}} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right) d_2 + \left(\frac{1}{\sqrt{78}} \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right) d_3$$

$$= \frac{-2}{\sqrt{3}} d_1 + \frac{17}{\sqrt{26}} d_2 + \frac{11}{\sqrt{78}} d_3.$$

Use formula for projections onto an orthonormal basis!

19) Orthogonal projection formula gives

$$\frac{c_1 \cdot c_2}{\|c_2\|^2} c_2 = \frac{-1}{7} c_2$$

So the projection of c_1 onto $\mathrm{Span}(c_2)$ is $\begin{bmatrix} -2/7 \\ 1/7 \\ -3/7 \end{bmatrix}$.

Formulae get messy for the c_i.

Instead we compute with the d_i.

Projection of c_2 onto $Sp\{d_1\}$ is $\frac{-2}{\sqrt{3}} d_1$

" " " " $Sp\{d_2, d_3\}$ is $\frac{17}{\sqrt{26}} d_2 + \frac{11}{\sqrt{78}} d_3$

The distance of c_2 from $Sp\{d_2, d_3\}$ is

$$\left\| \frac{-2}{\sqrt{3}} d_1 \right\| = \frac{2}{\sqrt{3}} \sim 1.15$$