Mini Assignment - Lambda Calculus

Ananya Mantravadi - CS19B1004

April 2021

Contents

1 Helper Functions

Some common helper functions used are:

$\mathbf{true} = \lambda xy.x$
$\mathbf{false} = \lambda xy.y$
$\mathbf{and} = \lambda xy.(x \rightarrow y|\mathbf{false})$
$\mathbf{not} = \lambda t.t$ **false true**
$\mathbf{snd} = \lambda p.p$ **false**
$\mathbf{suc} = \lambda nfx.nf(fx)$
$\mathbf{iszero} = \lambda n.n(\lambda x.\ \mathbf{false})\ \mathbf{true}$
$\mathbf{add} = \lambda mnfx.mf(nfx)$
$\mathbf{pre} = \lambda nfx.\ \mathbf{snd}(n\ (\mathbf{prefn}f)\ (\mathbf{true}, x))$
$\mathbf{S} = \lambda fgx.(fx)(gx)$
$\mathbf{K} = \lambda xy.x$
$\mathbf{I} = \lambda x.x$

2 Exercise 9

Show that $(\lambda fgx.fx(gx))(\lambda xy.x)(\lambda xy.x) = \lambda x.x$

$$
\begin{aligned}
&(\lambda fgx.fx(gx))(\lambda xy.x)(\lambda xy.x)\\
&= (\lambda gx.(\lambda xy.x)x(gx))(\lambda xy.x)\\
&= (\lambda gx.x)(\lambda xy.x)\\
&= (\lambda g.\lambda x.x)(\lambda xy.x)\\
&= \lambda x.x
\end{aligned}
$$

3 Exercise 10

Work out:

1. $(\lambda y.x(\lambda x.x))\ [(\lambda y.yx)/x]$

   Using $\lambda V'.E_1[E'/V] = \lambda V'.E[E'/V]$, for $V' \neq V$ and V' not free in E':
   $(\lambda y.x(\lambda x.x))\ [(\lambda y.yx)/x] \equiv (\lambda y.(\lambda y.yx)(\lambda x.x)) \equiv (\lambda y.((\lambda x.x)x)) \equiv \lambda y.x$

2. $(y(\lambda z.xz))\ [(\lambda y.zy)/x]$

   Using $\lambda V'.E_1[E'/V] = \lambda V''.E_1[V''/V'][E'/V]$, for $V' \neq V$ and V' free in E':
   $(y(\lambda z.xz))\ [(\lambda y.zy)/x] \equiv y(\lambda z.xz)\ [y/z]\ [(\lambda y.zy)/x] \equiv y(\lambda y.xz)\ [(\lambda y.zy)/x] \equiv y(\lambda y.(\lambda y.zy)z) \equiv y(\lambda y.zz)$

4 Exercise 13

Let **and** be the $\lambda - expression$ $\lambda xy.(x \rightarrow y|\mathbf{false})$. Show that:

and true true = **true**
and true false = **false**
and false true = **false**
and false false = **false**

- **and true true** = **true**
  = $\lambda xy.(x \rightarrow y$ | **false**) **true true**
  = **true** → **true** | **false**
  = **true true false**
  = $(\lambda xy.x)$ **true false**
  = **true**

- **and true false** = **false**
  = $\lambda xy.(x \rightarrow y$ | **false**) **true false**
  = **true** → **false** | **false**
  = **true false false**
  = $(\lambda xy.x)$ **false false**
  = **false**

- **and false true** = **false**
  = $\lambda xy.(x \rightarrow y$ | **false**) **false true**

$= \textbf{false} \rightarrow true \mid \textbf{false}$
$= \textbf{false true false}$
$= (\lambda xy.y)$ **true false**
$=$ **false**

- **and false false** $=$ **false**
$= \lambda xy.(x \rightarrow y \mid \textbf{false})$ **false false**
$= \textbf{false} \rightarrow false \mid false$
$=$ **false false false**
$= (\lambda xy.y)$ **false false**
$=$ **false**

5 Exercise 14

Devise a $\lambda - expression$ **or** such that:

or true true $=$ **true**
or true false $=$ **true**
or false true $=$ **true**
or false false $=$ **false**

We can define **or** $= \lambda xy.(x \rightarrow \textbf{true} \mid y)$

- **or true true** $=$ **true**
$= \lambda xy.(x \rightarrow \textbf{true} \mid y)$ **true true**
$= \textbf{true} \rightarrow \textbf{true} \mid \textbf{true}$
$=$ **true true true**
$= (\lambda xy.x)$ **true true**
$=$ **true**

- **or true false** $=$ **true**
$= \lambda xy.(x \rightarrow \textbf{true} \mid y)$ **true false**
$= \textbf{true} \rightarrow \textbf{true} \mid \textbf{false}$
$=$ **true true false**
$= (\lambda xy.x)$ **true false**
$=$ **true**

- **or false true** $=$ **true**
$= \lambda xy.(x \rightarrow \textbf{true} \mid y)$ **false true**
$= \textbf{false} \rightarrow \textbf{true} \mid \textbf{true}$
$=$ **false true true**
$= (\lambda xy.y)$ **true true**
$=$ **true**

- **or false false** $=$ **false**
$= \lambda xy.(x \rightarrow \textbf{true} \mid y)$ **false false**
$= \textbf{false} \rightarrow \textbf{true} \mid \textbf{false}$
$=$ **false true false**
$= (\lambda xy.y)$ **true false**
$=$ **false**

6 Booleans

$\mathbf{NAND} = \lambda xy.\ (x \to \mathbf{not}\ y)\ \mathbf{true}$
$\mathbf{NOR} = \lambda xy.\ (x \to \mathbf{false} \mid \mathbf{not}\ y)$
$\mathbf{XOR} = \lambda xy.\ (x \to \mathbf{not}\ y \mid y)$

7 Exercise 15

Show that $\mathbf{snd}\ (E_1, E_2) = E_2$

$\mathbf{snd}(E_1, E_2) = (\lambda p.p\ \mathbf{false})(E_1, E_2)$
$= (E_1, E_2)\ \mathbf{false}$
$= (\lambda f.f\ E_1 E_2)\ \mathbf{false}$
$= \mathbf{false}\ E_1 E_2$
$= (\lambda xy.y) E_1 E_2$
$= E_2$

8 Exercise 17

Show for all numbers m and n:

1. $\mathbf{suc}\ \underline{n} = \underline{n+1}$
   $\mathbf{suc}\ n = (\lambda n.\lambda fx.nf(fx))(\lambda fx.f^n x)$
   $= \lambda fx.(\lambda fx.f^n x) f(fx)$
   $= \lambda fx.f^n(fx)$
   $= \lambda fx.f^{n+1} x$
   $= n + 1$

2. $\mathbf{iszero}\ (\mathbf{suc}\ \underline{n}) = \mathbf{false}$
   $= (\lambda n.n(\lambda x.\ \mathbf{false})\ \mathbf{true})\ (\lambda fx.f^{n+1}(fx))$
   $= (\lambda fx.f^{n+1} x)(\lambda x.\ \mathbf{false})\ \mathbf{true}$
   $= (\lambda x.\ \mathbf{false})^{n+1}\ \mathbf{true}$
   $= \mathbf{false}$

3. $\mathbf{add}\ \underline{0}\ \underline{n} = \underline{n}$
   $= (\lambda mnfx.mf(nfx))(\lambda fx.x)(\lambda fx.f^n x)$
   $= \lambda fx.(\lambda fx.x) f((\lambda fx.f^n x) fx)$
   $= \lambda fx.(\lambda fx.x) f(f^n x)$
   $= \lambda fx.f^n x$
   $= n$

4. $\mathbf{add}\ \underline{m}\ \underline{0} = \underline{m}$
   $= (\lambda mnfx.mf(nfx))(\lambda fx.f^m x)(\lambda fx.x)$
   $= \lambda fx.(\lambda fx.f^m x) f((\lambda fx.x) fx)$
   $= \lambda fx.(\lambda fx.f^m x) f(x)$
   $= \lambda fx.f^m x$
   $= m$

5. $\mathbf{add}\ \underline{m}\ \underline{n} = \underline{m+n}$
   $= (\lambda mnfx.mf(nfx))(\lambda fx.f^m x)(\lambda fx.f^n x)$

$= \lambda fx.(\lambda fx.f^m x) f((\lambda fx.f^n x) fx)$
$= \lambda fx.(\lambda fx.f^m x) f(f^n x)$
$= \lambda fx.f^m f^n x$
$= \lambda fx.f^{m+n} x$
$= m + n$

9 Exercise 19

Show that:

1. $\textbf{pre}\ (\textbf{suc}\ \underline{n}) = \underline{n}$
   $= \lambda nfx.\ \textbf{snd}(n\ (\textbf{prefn}f)\ (\textbf{true}, x))\ n+1$
   $= \lambda fx.\ \textbf{snd}((\textbf{prefn}f)^{n+1}(\textbf{true}, x))$
   $= \lambda fx.\ \textbf{snd}(\textbf{false}, f^{n+1-1}x)$
   $= \lambda fx.f^n x$
   $= n$

2. $\textbf{pre}\ \underline{0} = \underline{0}$
   $= \lambda nfx.\ \textbf{snd}(n\ (\textbf{prefn}f)\ (\textbf{true}, x))\ 0$
   $= \lambda fx.\ \textbf{snd}(0\ (\textbf{prefn}f)\ (\textbf{true}, x))$
   $= \lambda fx.\ \textbf{snd}(\textbf{true}, x)$
   $= \lambda fx.x$
   $= 0$

10 Exercise 22

Show that if $\mathbf{Y_1}$ is defined by:

LET $\mathbf{Y_1} = \mathbf{Y}(\lambda yf.f(yf))$. Then $\mathbf{Y_1}$ is a fixed-point operator, i.e. for any E: $\mathbf{Y_1}$ E = E ($\mathbf{Y_1}$ E)

$\mathbf{Y_1}E = \mathbf{Y}(\lambda yf.f(yf))E$
$= (\lambda yf.f(yf))(\mathbf{Y}(\lambda yf.f(yf)))E$ (Since $\mathbf{Y}$ = E($\mathbf{Y}$E))
$= E(\mathbf{Y}(\lambda yf.f(yf))E)$
$= E(\mathbf{Y_1}E)$

Since $\mathbf{Y_1}$ E = E ($\mathbf{Y_1}E$), $\mathbf{Y_1}$ is a fixed-point operator.

11 Exercise 23

Show that $(\lambda xy.y(xxy))(\lambda xy.y(xxy))$ is a fixed-point operator.

A λ-expression **Fix** with the property that **Fix** E = E(**Fix** E) for any E is called a fixed-point operator.
Let $\mathbf{Y}$ =($\lambda xy.y(xxy))(\lambda xy.y(xxy))$

$\mathbf{Y}E = (\lambda xy.y(xxy))(\lambda xy.y(xxy))E$
$= E((\lambda xy.y(xxy))(\lambda xy.y(xxy))E)$
$= E(\mathbf{Y}E)$

Since $\mathbf{Y}$ = E($\mathbf{Y}$E), $\mathbf{Y}$ is a fixed-point operator.

12 Exercise 34

Show that $\mathbf{I} = \mathbf{S}\ \mathbf{K}\ \mathbf{K}$

Using definitions of **S** and **K** we have:
$\mathbf{S}\ \mathbf{K}\ \mathbf{K} = (\lambda fgx.(fx)(gx))(\lambda xy.x)(\lambda xy.x)$
$= \lambda x.((\lambda xy.x)x)((\lambda xy.x)x)$
$= \lambda x.((\lambda x.\lambda y.x)x)((\lambda x.\lambda y.x)x)$
$= \lambda x.(\lambda y.x)(\lambda y.x)$
$= \lambda x.x$
$= \mathbf{I}$

13 Combinators and Functions using λ expressions

1. **Y** combinator
   $\mathbf{Y} = \lambda y.(\lambda x.y(xx))(\lambda x.y(xx))$

2. Θ combinator
   $\Theta = (\lambda xy.y(xxy))(\lambda xy.y(xxy))$

3. ω combinator
   $\omega = \lambda x.xx$

4. Factorial function
   $\mathbf{factorial} = \mathbf{Y}(\lambda fx.\ \mathbf{iszero}\ x\ \underline{1}\ (\mathbf{mult}\ x\ (f\ (\mathbf{pre}\ x))))$