For ECS417 Course Credit; Instructed by Dr Ankur Raina, IISERB

Solving a specific 2-dimensional Non-Hermitian Linear System by Simulation in PennyLane of HHL algorithm

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Any Non-hermitian Matrix can be made hermitian when put it into an anti-diagonal square matrix. Harrow–Hassidim–Loyd algorithm is a Quantum algorithm for solving hermitian linear systems. We use PennyLane to modify the Quantum circuit given by Liu et al.[?] to simulate the algorithm for a specific 2 dimensional non-linear system.

I. PROBLEM STATEMENT

Consider the non-Hermitian linear system

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

We change this system to a hermitian System by putting it in the form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \\ 0 \end{bmatrix}$$

. Note the eigenvalues of the new matrix are 1,-2,2,-2. Thus the magnitudes of eigenvalues can be simulated by two bits. We modify and simulate the program of liu et al.[?] in PennyLane to solve for the system using HHL algorithm. non-Hermitian

II. IMPLIMENTING THE HHL ALGORITHM

For a given Hermitian Matrix A in the Linear System $A|x\rangle = |b\rangle$, we find the spectral decomposition of the hermitian matrix and express $|b\rangle$ as a linear combination of eigenvectors $|\nu_i\rangle$ as $|b\rangle = \sum_{i=1}^{N} \beta_i |\nu_i\rangle$. The inverse of the hermitian matrix is $A^{-1} = \sum_{i=1}^{N} \frac{1}{\lambda_i} |\nu_i\rangle\langle\nu_i|$, where λ_i are the eigenvalues. Thus,

$$|x\rangle = \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i} |\nu_i\rangle$$

The HHL Algorithm provides the solution up to a constant as Quantum Measurement are insensitive to a phase factor. Below we present an overview of how this may be implemented on a Quantum Computer using HHL Algorithm with specifics on how we modified it for solving the given linear equation.

1. The circuit consists of three registers the first being an ancilla (one qubit), the second register is used to encode the eigenvalues(here, 2 qubits) while the third register (here, two qubits) is where we obtain the Solution whenever the ancilla is measured to be in $|x\rangle$ state.

2. The first two registers are initialised in $|0\rangle$ while the third register is initialised as $|0\rangle$. In our case since we use a two qubit second register to encode the eigenvalue magnitudes 1 and 2.

3. The phase estimation algorithm with e^{iA} in oracle is utilised to encode the eigenvalues $|\lambda_i\rangle$ in the second register. At this stage the state of the system is in $\sum_{i=1}^{i=2} \beta_i |0\rangle|\lambda_i\rangle|\nu_i\rangle$

4. At this point we wish to convert our state to $\sum_{i=1}^{i=2} \beta_i (\sqrt{1-\frac{1}{\lambda_i^2}}|0\rangle + \frac{1}{\lambda_i}|1\rangle)|\frac{1}{\lambda_i}\rangle|\nu_i\rangle$ We do this approximately by using the $R_y(\theta)$ operation controlled by $|\frac{2}{\lambda_i}\rangle$. The SWAP operation conveniently changes our state to $|\frac{2}{\lambda_i}\rangle^1$. After the Controlled rotation operators our state is $\sum_{i=1}^{i=2} \beta_i (\cos\frac{1}{\lambda i}|0\rangle + \sin\frac{1}{\lambda_i}|1\rangle)|\frac{1}{\lambda_i}\rangle|\nu_i\rangle$. Approximating $\sin\theta \approx \theta$,we obtain the state that we set out to.[2]

5. We invert Step three to obtain $\sum_{i=1}^{i=2} \beta_i (\sqrt{1-\frac{1}{\lambda_i^2}}|0\rangle + \frac{1}{\lambda_i}|1\rangle)|0\rangle|\nu_i\rangle$. and if the Ancilla is measure to be in $|1\rangle$ the third register gives us the solution state, $\sum_{i=1}^{i=2} \frac{\beta_i}{\lambda_i}|\nu_i\rangle$

6. Thus as we simulate the algorithm on PennyLane we should find our solution as the square root probabilities of states $|10010\rangle$ and $|10011\rangle$

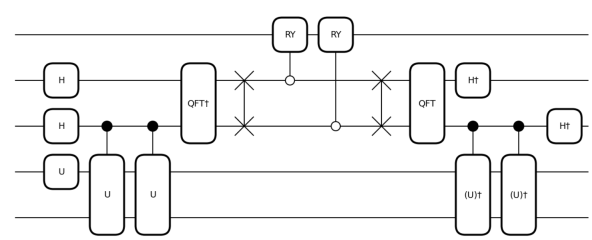

FIG. 1. The Circuit Representation

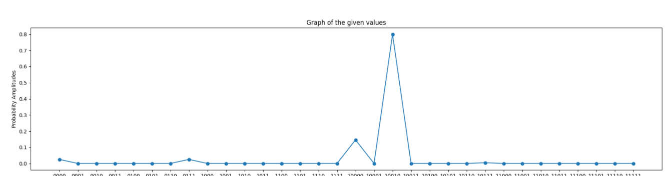

FIG. 2. $P(|10010\rangle) = 0.8$ and $P(|10011\rangle) = 0$

III. INFERENCES

HHL Algorithm for sparse matrices solves the linear system in $O(\log N)$ as compared to $O(N)$. The 'speeding' occurs as a Quantum Computer can solve for all eigenvalues simultaneously. The downsides include the calculation of the oracle (for which we used Mathematica) and the fact that we have to make do with approximations at step 4 thereby increasing the errors. We obtained the normalized solution set $[1,0]$ as opposed the normalised expected solution set $[0.707, 0.707]$ which is to be expected. This shows that there is further need to improve upon the results.

[1] L. H. Xiaonan Liu1, Lina Jing and J. Gao1, "Hhl analysis and simulation verification based on origin quantum platform by xiaonan liu1 , lina jing , lin han , and jie gao1," Journal of Physics **2113** (2021), 10.1088/1742-6596/2113/1/012083.
[2] M. C. B. Jr., "Using quantum algorithms for solving linear systems," (2019).