Assignment 5
Ananya Jana, netID: aj611

Q.2
The modified Newton's program for 2 variables is *modifiednewtontwovar.m.*
a)
The solution is: Value of x and y:
x = ±1.581138830084190
y = ±1.224744871391589

Below, we have shown the results for the initial point(x, y) = (1, 1).

Attaching the matlab output snippet for the modifiednewtontwovar.m
Choose 1 for part a, Choose 2 for part b: 1
Number of iterations: 15
1.750000000000000
1.250000000000000

0.790569415042095

1.589285714285714 1.225000000000000

0.162647107667653

1.581159711075441
1.224744897959184

0.008130006471375

1.581138830222068
1.224744871391589

2.088087027479742e-05

1.581138830084190
1.224744871391589

Iteration number:
3

Value of x and y:
1.581138830084190
1.224744871391589

When we take other starting values of (x, y) as (1, -1), (-1, 1) and (-1, -1), we can obtain the other solutions using the same program.

b)
The solution is: Value of x and y:
x = 1.945026819131982
y = 0.673007169617913

Attaching the matlab output snippet for the modifiednewtontwovar.m
Choose 1 for part a, Choose 2 for part b: 2
Number of iterations: 15
2.250000000000000
0.375000000000000

0.976281209488332

1.993518518518518
0.624537037037037

0.357842819120573

1.946743715555195
0.671290273402707

0.066134312598847

1.945029130257479
0.673004858492416

0.002424789634791

1.945026819136181
0.673007169613714

3.268419084416970e-06

1.945026819131982
0.673007169617913

Iteration number:
4

Value of x and y:
1.945026819131982
0.673007169617913

c) The generalized program for 3 variables is *generalnewtonthreevar.m*
We use 4 initial starting points and obtain the corresponding roots of the given

equation for those.
Initial point: (1, 1, 1)
Roots: x = 2.449489742783178, y = 2.449489742783178, z = 2

Initial point: (1, -2, -1)
Roots: x = 0.585786437626905, y = -3.414213562373095, z = -2

Initial point: (-1, -1, 1)
Roots: x = -2.449489742783178, y = -2.449489742783178, z = 2

Initial point: (2, -1, 11)
Roots: x = 3.414213562373095, y = -0.585786437626905, z = -2

Q.1
a) We have, $z^{(0)} = \sum_{j=1}^{n} a_j x_j$ where x_j are the Eigen vectors corresponding to Eigen values λ_j.

$A^m z^{(0)} = A^m \sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{n} a_j A^m x_j$
Proof:
We will prove that $A^m z^{(0)} = \sum_{j=1}^{n} a_j {\lambda_j}^m x_j$ by using Induction and then move on to the next part of the proof.
Basis($m = 1$):
$A^m z^{(0)} = A \sum_{j=1}^{n} a_j x_j$
$= \sum_{j=1}^{n} a_j A x_j$
$= \sum_{j=1}^{n} a_j \lambda_j x_j$
This is because of the fact that for the jth Eigen value and Eigen vector, we have $Ax_j = \lambda_j x_j$

Let us assume that $A^m z^{(0)} = \sum_{j=1}^{n} a_j {\lambda_j}^m x_j$ holds good for $m = k$. Hence, $A^k z^{(0)} = \sum_{j=1}^{n} a_j {\lambda_j}^k x_j$ -Eqn.(1)

Let us check this for $m = k + 1$.
$A^{k+1} z^{(0)} = A.A^k z^{(0)}$
$= A \sum_{j=1}^{n} a_j {\lambda_j}^k x_j$ [From Eqn. (1)]
$= \sum_{j=1}^{n} a_j {\lambda_j}^k A x_j$
$= \sum_{j=1}^{n} a_j {\lambda_j}^k \lambda_j x_j$ [As we already know $Ax_j = \lambda_j x_j$]
$= \sum_{j=1}^{n} a_j {\lambda_j}^{k+1} x_j$

Hence we proved by Induction that $A^m z^{(0)} = \sum_{j=1}^{n} a_j {\lambda_j}^m x_j$ is true.
We have $A^m z^{(0)} = \sum_{j=1}^{n} a_j {\lambda_j}^m x_j$
$= a_1 {\lambda_1}^m x_1 + a_2 {\lambda_2}^m x_2 + ... + a_n {\lambda_n}^m x_n$
Let us assume, λ_1 is the largest Eigen value. Hence on dividing by ${\lambda_1}^m$ the expression,we get

$= {\lambda_1}^m(a_1x_1 + \frac{{\lambda_2}^m}{{\lambda_1}^m}a_2x_2 + ... + \frac{{\lambda_n}^m}{{\lambda_1}^m}a_nx_n)$ -Eqn.(2)
Since λ_1 is the largest Eigen value, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for all $2 \leq j \leq n$
As $m \to \infty$, $\left|\frac{{\lambda_j}^m}{{\lambda_1}^m}\right| \to 0$ -Eqn.(3) for all $2 \leq j \leq n$

Using this, we can rewrite the Eqn.(2) as
$A^m z^{(0)} = {\lambda_1}^m(a_1x_1 + \frac{{\lambda_2}^m}{{\lambda_1}^m}a_2x_2 + ... + \frac{{\lambda_n}^m}{{\lambda_1}^m}a_nx_n)$
$= {\lambda_1}^m(a_1x_1 + \sum_{j=2}^n \frac{{\lambda_j}^m}{{\lambda_1}^m}a_jx_j)$
$= {\lambda_1}^m(a_1x_1 + \sum_{j=2}^n 0.a_jx_j)$
$= {\lambda_1}^m a_1x_1$

Here a is an arbitrary constant which doesn't affect the vector x_1. Hence we can say, $A^m z^{(0)} \to {\lambda_1}^m x_1$

b) ${\lambda_1}^{(m)} = \frac{{w_k}^{(m)}}{{z_k}^{(m)}}$ -Eqn.(1) and λ_1 is the largest Eigen value. i.e $|\lambda_1| > |\lambda_2| > ..... > |\lambda_n|$.
$z^{(0)} = c_1v^{(1)} + c_2v^{(2)} + ... + c_nv^{(n)}$ where $c_1, c_2, ...c_n$ are constants and $v^{(1)}, v^{(2)}, ..v^{(n)}$ are the Eigen vectors
$z^{(m-1)} = A^{m-1}z^{(0)}$
$= \sum_{i=1}^n c_i{\lambda_i}^{(m-1)}v^{(i)}$
$= {\lambda_1}^{(m-1)}(c_1v^{(1)} + \sum_{i=2}^n \frac{{\lambda_i}^{(m-1)}}{{\lambda_1}^{(m-1)}}c_iv^{(i)})$ -Eqn.(3)

$w^m = Az^{(m-1)}$
$= A\sum_{i=1}^n c_i{\lambda_i}^{(m-1)}v^{(i)}$
$= \sum_{i=1}^n c_i{\lambda_i}^{(m-1)}Av^{(i)}$
$= \sum_{i=1}^n c_i{\lambda_i}^{(m-1)}\lambda_iv^{(i)}$ As we know from the Eigen value property that $Av^{(i)} = \lambda_iv^{(i)}$
$= \sum_{i=1}^n c_i{\lambda_i}^{(m)}v^{(i)}$
$= c_1{\lambda_1}^{(m)}v^{(1)} + c_2{\lambda_2}^{(m)}v^{(2)} + ... + c_n{\lambda_n}^{(m)}v^{(n)}$
$= {\lambda_1}^{(m)}(c_1v^{(1)} + \frac{{\lambda_2}^{(m)}}{{\lambda_1}^{(m)}}c_2v^{(2)} + ... + \frac{{\lambda_n}^{(m)}}{{\lambda_1}^{(m)}}c_nv^{(n)})$ -Eqn.(3)

From the Eqns (1), (2),(3), we can write,
${\lambda_1}^{(m)} = \frac{{\lambda_1}^{(m)}(c_1v^{(1)} + \frac{{\lambda_2}^{(m)}}{{\lambda_1}^{(m)}}c_2v^{(2)} + ... + \frac{{\lambda_n}^{(m)}}{{\lambda_1}^{(m)}}c_nv^{(n)})}{{\lambda_1}^{(m-1)}(c_1v^{(1)} + \frac{{\lambda_2}^{(m-1)}}{{\lambda_1}^{(m-1)}}c_2v^{(2)} + ... + \frac{{\lambda_n}^{(m-1)}}{{\lambda_1}^{(m-1)}}c_nv^{(n)})}$ -Eqn.(4)
Since $|\lambda_1| > |\lambda_2| > ..... > |\lambda_n|$, as $m \to \infty$, we have, $\left|\frac{{\lambda_j}^m}{{\lambda_1}^m}\right| \to 0$ and $\left|\frac{{\lambda_j}^{(m-1)}}{{\lambda_1}^{(m-1)}}\right| \to 0$ for all $2 \leq j \leq n$

Hence Eqn.(4) is reduced to ${\lambda_1}^{(m)} = \lambda_1 \frac{(c_1v^{(1)} + \frac{{\lambda_2}^{(m)}}{{\lambda_1}^{(m)}}c_2v^{(2)} + ... + \frac{{\lambda_n}^{(m)}}{{\lambda_1}^{(m)}}c_nv^{(n)})}{(c_1v^{(1)} + \frac{{\lambda_2}^{(m-1)}}{{\lambda_1}^{(m-1)}}c_2v^{(2)} + ... + \frac{{\lambda_n}^{(m-1)}}{{\lambda_1}^{(m-1)}}c_nv^{(n)})}$
$\approx \lambda_1 \frac{(c_1v^{(1)} + 0.c_2v^{(2)} + ... + 0.c_nv^{(n)})}{(c_1v^{(1)} + 0.c_2v^{(2)} + ... + 0.c_nv^{(n)})}$
$\approx \lambda_1 \frac{(c_1v^{(1)})}{(c_1v^{(1)})}$
$\approx \lambda_1$

Hence ${\lambda_1}^{(m)} \to \lambda_1$ as $m \to \infty$.

c) Refer to the program *q1test.m*.

d) Using the program we found out the the largest Eigen value to be: 9.6235.
Attaching below the output snippet from matlab
Enter the matrix: [1 2 3; 2 3 4; 3 4 5]
Enter vector z: [1 1 1]'
Enter value of n: 50
The largest eigen value for this matrix is:
9.6235

The corresponding eigen vector for this matrix is:
0.5247
0.7623
1.0000

Total number of iterations:
8

e) A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

For Eigen value ad Eigen vector property, we know that $Ax = \lambda x$ and $Av = \lambda v$ where λ is the Eigen value and v is the Eigen vector.
The determinant of $(A - \lambda I)$ must be equal to 0 for the matrix to have non-zero Eigen vectors.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\implies \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{bmatrix} = 0$$

$$\implies (1-\lambda)[(3-\lambda)(5-\lambda) - 16] + 2[12 - 2(5-\lambda)] + 3[8 - 3(3-\lambda)] = 0$$
$$\implies (1-\lambda)[15 + \lambda^2 - 8\lambda - 16] + 2[12 - 10 + 2\lambda] + 3[8 - 9 + 3\lambda] = 0$$
$$\implies (1-\lambda)[\lambda^2 - 8\lambda - 1] + 4[1+\lambda] + 3[3\lambda - 1] = 0$$
$$\implies \lambda^2 - 8\lambda - 1 - \lambda^3 + 8\lambda^2 + \lambda + 4[1+\lambda] + 3[3\lambda - 1] = 0$$
$$\implies -\lambda^3 + 9\lambda^2 + 6\lambda = 0$$
$$\implies \lambda^3 - 9\lambda^2 - 6\lambda = 0$$
$$\implies \lambda[\lambda^2 - 9\lambda - 6] = 0$$

Solving this, we get the roots: $\lambda = 0, -0.6235, 9.6235$. There are total 3 Eigen values $\lambda_1 = 9.6235, \lambda_2 = -0.6235, \lambda_3 = 0$.

We need to find out the normalized Eigen vectors. We have given the procedure below:
For $\lambda_1 = 9.6235$, $(A - \lambda_1 I)v^{(1)} = 0$

$$\implies \begin{bmatrix} 1-9.6235 & 2 & 3 \\ 2 & 3-9.6235 & 4 \\ 3 & 4 & 5-9.6235 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Let's take $v_1 = 1$, We get three equations from the matrix above. The equations are:
$2v_2 + 3v_3 = 8.6235$ -Eqn.(1)
$-6.6235v_2 + 4v_3 = -2$ -Eqn.(2)
$4v_2 - 4.6235v_3 = -3$ -Eqn.(3)

Solving this we get, $v_2 = 1.4529$, $v_3 = 1.9059$

$$\text{Hence, } v^{(1)} = \begin{bmatrix} \frac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \\ \\ \frac{v2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \\ \\ \frac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \end{bmatrix} = \begin{bmatrix} 0.3851 \\ 0.5595 \\ 0.7339 \end{bmatrix}$$

For $\lambda_2 = -0.6235$, let's solve for corrsponding eigen vector $v^{(2)}$ using $(A - \lambda_2 I)v^{(2)} = 0$
We have the matrix,

$$\begin{bmatrix} 1+0.6235 & 2 & 3 \\ 2 & 3+0.6235 & 4 \\ 3 & 4 & 5+0.6235 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

we have the following equations after setting v_1 to 1,

$2v_2 + 3v_3 = -1.6235$ -Eqn.(4)
$3.6235v_2 + 4v_3 = -2$ -Eqn.(5)
$4v_2 + 5.6235v_3 = -3$ -Eqn.(6)

Solving which we get $v_2 = 0.1723$ and $v_3 = -.6560$.Using these values We have the normalized Eigen vector

$$v^{(2)} = \begin{bmatrix} \frac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \\ \\ \frac{v2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \\ \\ \frac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \end{bmatrix} = \begin{bmatrix} 0.8277 \\ 0.1424 \\ -0.5428 \end{bmatrix}$$

For $\lambda_3 = 0$, let's solve for corrsponding eigen vector $v^{(3)}$ using $(A - \lambda_3 I)v^{(3)} = 0$
We have the matrix,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

we have the following equations after setting v_1 to 1,
$2v_2 + 3v_3 = -1$
$3v_2 + 4v_3 = -2$
$4v_2 + 5v_3 = -3$

Solving which we get $v_2 = -2$ and $v_3 = 1$.Using these values we figure out that for the Eigen value $\lambda_3 = 0$, the corresponding normlized Eigen vector

$$v^{(3)} == \begin{bmatrix} \frac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \\ \\ \frac{v2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \\ \\ \frac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}} \end{bmatrix} = \begin{bmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{bmatrix}$$