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Q.1
Please find the graphs of $f(x) = cos(x), [-\pi, \pi]$ in the folder submission/cos.
Please find the graphs of $f(x) = \frac{1}{1+10x^2,[-4,4]}$ in the folder submission.
Color code:
Exact function: blue
Interpolation polynomial: red
Error: green

With more number o fpoints, the interpolation polynomial is supposed to give better results. For the cosine function, we notice that when we increase the number of points till n=20, the interpolation polynomial becomes nicer. but with values ≥ 60, the interpolation polynomial starts behaving abnormally.
In runge's function, With more number points the errors near the edge explode gradually.For odd values the result are symmetrical, whereas for even values they are not.

Q.2
There are three points. So, we can find out the unique polynomial of degree 2.

Lagrange's Interpolation Method
Lagrange's interpolation polynomial of degree 2 is given by,
$P_2(x) = y_0 L_0(x) + y_1 L_(x) + y_2 L_2(x)$
where $L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$, $L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$ and $L_0(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

i)The points are $(x_0, y_0), (x_1, y_1), (x_2, y_2) = (0, 1), (1, 2), (2, 3)$
Hence,the the Lagrange's polynomial
$P_2(x) = 1.\frac{(x-1)(x-2)}{(0-1)(0-2)} + 2.\frac{(x-0)(x-2)}{(1-0)(1-2)} + 3.\frac{(x-0)(x-1)}{(2-0)(2-1)}$
$= \frac{1}{2}(x-1)(x-2) - 2x(x-2) + \frac{3}{2}x(x-1)$
$= \frac{1}{2}(x^2 - x - 2x + 2 - 4x^2 + 8x + 3x^2 - 3x)$
$= \frac{1}{2}(2x + 2)$
$= (x + 1)$
ii) The points are $(x_0, y_0), (x_1, y_1), (x_2, y_2) = (0, 1), (1, 1), (2, 1)$
Hence,the the Lagrange's polynomial
$P_2(x) = 1.\frac{(x-1)(x-2)}{(0-1)(0-2)} + 1.\frac{(x-0)(x-2)}{(1-0)(1-2)} + 1.\frac{(x-0)(x-1)}{(2-0)(2-1)}$
$= \frac{1}{2}(x-1)(x-2) - x(x-2) + \frac{1}{2}x(x-1)$
$= \frac{1}{2}(x^2 - x - 2x + 2 - 2x^2 + 4x + x^2 - x)$
$= \frac{1}{2}(2)$
$= 1$

Newton's Interpolation Method
Newton's interpolation polynomials are given by
$P_0(x) = f(x_0)$
$P_1(x) = f(x_0) + (x - x_0)f[x_0, x_1]$
$P_2(x) = f(x_0) + (x - x_0)f[x_0, x_1] + f(x_0) + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$

Where $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ and $f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$

i)The points are $(x_0, y_0), (x_1, y_1), (x_2, y_2) = (0, 1), (1, 2), (2, 3)$
Hence the finite differences table for Newton's method is:

x_i	$f(x_i)$	$f[,]$	$f[,,]$
0	1	1	0
1	2	1	
2	3		

$P_2(x) = 1 + (x - 0).1 + f(x_0) + (x - 0)(x - 1).0$
$= 1 + x$

ii) The points are $(x_0, y_0), (x_1, y_1), (x_2, y_2) = (0, 1), (1, 1), (2, 1)$
Hence the finite differences table for Newton's method is:

x_i	$f(x_i)$	$f[,]$	$f[,,]$
0	1	0	0
1	1	0	
2	1		

$P_2(x) = 1 + (x - 0).0 + f(x_0) + (x - 0)(x - 1).0$
$= 1$

Q.3
We have:$q(0) = -1, q(1) = -1$ and $q'(1) = 4$
We need to find the quadratic polynomial $q(x)$. Let us assume the quadratic polynomial is:
$q(x) = a_0 + a_1x + a_2x^2$
So, $q'(x) = a_1 + 2a_2x$

Substituting the values of x, we get
$q(0) = a_0 + a_1.0 + a_2.0^2 = a_0$, Hence $a_0 = -1$
$q(1) = a_0 + a_1.1 + a_2.1^2 = a_0 + a_1 + a_2$, Hence $a_0 + a_1 + a_2 = -1$

$q'(1) = a_1 + 2a_2.1^2 = a_1 + 2a_2$, Hence $a_1 + 2a_2 = 4$
Solving the equations for a_0, a_1 and a_2 we get, $a_0 = -1$, $a_1 = -4$ and $a_2 = 4$.
Hence $q(x) = 4x^2 - 4x - 1$

Q.4
For an interval $[a, b]$ define $h = (b-a)/n$ and the evenly spaced points $x_j = a+jh, j = 0, 1, , n$.
Consider the polynomial, $\Omega_n(x) = (x - x_0)(x - x_1).....(x - x_n), a \leq x \leq b$
We need to show that $|\Omega_n(x)| \leq n!h^{n+1}$

Let us consider the case where $n = 2$. There are total three points in this case $x_0 = a$, $x_1 = a + h$ and $x_2 = b = a + 2h$. Let's assume x is a point lying between x_0 and x_1. x is very close to x_0 and $x - x_0 = k$.Hence $x = a + k$
$|\Omega_n(x)| = |(x - x_0)(x - x_1)(x - x_2)|$
$= |k(k - h)(k - 2h)|$
$\leq n!h^n$ when k is very small
$\leq n!h^{n+1}$ when k is very small

Now, let's assume x is not so close to x_0, rather is is of the order of h, $x-x_0 = h/2$(say).
So, $|\Omega_n(x)| = |(x - x_0)(x - x_1)(x - x_2)|$
$= |(h/2)(-h/2)(-3h/2)|$
$\leq n!h^{n+1}$

Similarly, we can prove these bounds when x is a point between x_1 and x_2

We can generalize it in the same way. We have $n + 1$ evenly spaced points x_0, x_1, ..., x_n. We take a point x such that $x_0 \leq x \leq x_1$. x is very close to x_0. Let's say, $x - x_0 = k$.
Hence, $|\Omega_n(x)| = |(x - x_0)(x - x_1).....(x - x_n)|$
$= |k(k - h)(k - 2h).....(k - nh)|$
$\leq n!h^n$ when k is very small
$\leq n!h^{n+1}$ when k is very small

Now, let's assume x is not so close to x_0, rather is is of the order of h, $x-x_0 = h/2$(say).
So, $|\Omega_n(x)| = |(x - x_0)(x - x_1)......(x - x_n)|$
$= |(h/2)(-h/2)(-3h/2)(-5h/2).....((1 - n)h/2)|$
$\leq n!h^{n+1}$

We can show it to be true for any position of the point x, where $a \leq x \leq b$. Hence the proof.

The error bound of the polynomial is given by
$\max\limits_{a \leq x \leq b} |f(x) - p_n(x)| \leq \max\limits_{a \leq x \leq b} \frac{|\Omega_n(x)|}{(n+1)!} \max\limits_{a \leq x \leq b} \left| f^{(n+1)}(x) \right|$.

For equidistant points, as seen from the first part of the problem, $|\Omega_n(x)| \leq n! h^{n+1}$. Hence substituting th value of $|\Omega_n(x)|$, we get the error bound to be
$\max\limits_{a \leq x \leq b} |f(x) - p_n(x)| \leq \max\limits_{a \leq x \leq b} \frac{|h^{n+1}|}{(n+1)} \max\limits_{a \leq x \leq b} \left| f^{(n+1)}(x) \right|$.

From the error bound, we notice that there are two terms which influence the error. They are h^{n+1} and $f^{(n+1)}(x)$. If the number of points is large, then h becomes very small and h^{n+1} becomes even smaller. But if the $(n+1)th$ derivative explodes, it dominates over h^{n+1} and makes the error very large. In the Runge function, this happens near the edges. The higher derivatives gradually explode, making the error near the end points very large.

Q.5.a
Lagrange's interpolation polynomial is given by
$P_n(x) = \sum_{j=0}^{n} L_j(x) f_j$ where $L_j(x) = \prod_{i=0, i\neq j}^{n} (x - x_i) / \prod_{i=0, i\neq j}^{n} (x_j - x_i)$

We can take a special case where we want to fit the polynomial $P_n(x)$ to a function $f(x) = 1$. Here, all f_j values are equal to 1. Then, $P_n(x) = \sum_{j=0}^{n} L_j(x).1 = \sum_{j=0}^{n} L_j(x)$. Again $P_n(x) = f(x) = 1$.
Hence, $\sum_{j=0}^{n} L_j(x) = 1$.

Q.5.b
We need to show that, $\sum_{j=0}^{n} {x_j}^m L_j(x) = x^m$
Here the polynomial is $P_n(x) = {x_0}^m L_0(x) + {x_1}^m L_1(x) + ... + {x_n}^m L_n(x)$ where $m < n$.
This is another special case of the polynomial where $f_0 = {x_0}^m$, $f_1 = {x_1}^m$, ... $f_n = {x_n}^m$ etc. This is nothing but the function $f(x) = x^m$.
So, $P_n(x) = \sum_{j=0}^{n} {x_j}^m L_j(x) = f(x) = x^m$
$\sum_{j=0}^{n} {x_j}^m L_j(x) = x^m$

Q.6.A
Newton's interpolation polynomials upto degree 4 are given below:
Newton's interpolation polynomials are given by
$P_0(x) = f(x_0)$
$P_1(x) = P_0(x) + (x - x_0) f[x_0, x_1]$
$P_2(x) = P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$

$P_3(x) = P_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$
$P_4(x) = P_3(x) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$
Where $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ and $f[x_0, .., x_n] = \frac{f[x_1,..,x_n]-f[x_0,..,x_{n-1}]}{x_n-x_0}$

i)Newton's divided difference table for the given points is given below:

x_i	$f(x_i)$	$f[,]$	$f[,,]$	$f[,,,]$	$f[,,,,]$
0	1	0	3	1	0
1	1	6	6	1	
2	7	18	9		
3	25	36			
4	61				

ii) The values of the polynomials are calculated next.
$P_0(x) = f(x_0) = 1$

$P_1(x) = P_0(x) + (x - x_0)f[x_0, x_1] = 1 + (x - 0).0 = 1$

$P_2(x) = P_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2] = 1 + (x - 0)(x - 1).3 = 1 + 3x^2 - 3x$

$P_3(x) = P_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$
$= 1 + 3x^2 - 3x + x(x^2 - 3x + 2) =$
$= 1 + 3x^2 - 3x + x^3 - 3x^2 + 2x$
$= x^3 - x + 1$

$P_4(x) = P_3(x) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]$
$= x^3 - x + 1 + 0.(x - 0)(x - 1)(x - 2)(x - 3) = x^3 - x + 1$ Where $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ and $f[x_0, .., x_n] = \frac{f[x_1,..,x_n]-f[x_0,..,x_{n-1}]}{x_n-x_0}$

iii) Yes, these data come from a polynomial of degree 3. Hence, any 4 of the given 5 points are sufficicent to find out the polynomial. Here, we had 5 points, so we tried calculating the polynoial of degree 4 as well. But that yielded the same polynomial as the data is from a polynomial of degree 3 and hence the coefficient of x^4 is 0.

Q.6.B
The points are $(x_0, y_0), (x_1, y_1)$
i)Lagrange's interpolation polynomial of degree 1 is given by,
$P_1(x) = y_0L_0(x) + y_1L_(x)$
where $L_0(x) = \frac{x-x_1}{x_0-x_1}$ and $L_1(x) = \frac{x-x_0}{x_1-x_0}$
ii)

$$L_0(x) + L_1(x) = \frac{x-x_1}{x_0-x_1} + \frac{x-x_0}{x_1-x_0}$$
$$= \frac{x-x_1}{x_0-x_1} - \frac{x-x_1}{x_0-x_1}$$
$$= \frac{x-x_1-x+x_0}{x_0-x_1}$$
$$= \frac{x_0-x_1}{x_0-x_1}$$
$$= 1$$