Name : Ananya Prasad

Reg No: 20BCE10093

Course : MAT2002

Faculty : Dr Navneet Verma

Exam : Midsem Fall 22-23

Date : 13/05/2022

Show that :

(5) Graph cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges (1)

G is a simple graph with n vertices and k components. Let the number of vertices in i^{th} component be n_i.

$\therefore n_1 + n_2 + n_3 \dots n_k = n$ where $n_i \geq 1$.

$\therefore$ Maximum number of edges is $\frac{n_i(n_i-1)}{2}$

$\therefore$ summing, $= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$

$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - n \longrightarrow (1)$

Consider $= \sum_{i=1}^{k}(n_i - 1) = (n_1 - 1) + (n_2 - 1) + (n_3 - 1) \dots (n_k - 1)$

$= (n_1 + n_2 \dots n_k) - (1 + 1 \dots$ k times)

$= (n_1 + n_2 + n_3 \dots n_k) - k$.

$= (n - k)$

squaring, $\sum_{i=1}^{k}(n_i - 1)^2 = (n-k)^2$

$\sum_{i=1}^{k}(n_i - 1)^2 + 2R = n^2 + k^2 - 2nk$, (R is a non-negative term)

$\sum_{i=1}^{k}(n_i^2 - 1 - 2n_i) \leq n^2 + k^2 - 2nk$

$$\sum_{i=1}^{k} (n_i^2 + 1 - 2n_i) \leq (n^2 + k^2 - 2nk)$$

$$\sum_{i=1}^{k} (n_i^2 + 1 - 2n_i) \leq (n^2 + k^2 - 2nk)$$

separating,

$$\sum_{i=1}^{k} n_i^2 + k - 2n \leq (n^2 + k^2 - 2nk) \qquad \left[\because \sum_{i=1}^{k} 1 = k \ \& \ \sum_{i=1}^{k} n_i = n \right.$$

adding $(n-k)$,

$$\sum_{i=1}^{k} n_i^2 - n \leq (n^2 + k^2 - 2nk + n - k)$$

$$\sum_{i=1}^{k} n_i^2 - n \leq \{ (n^2 + k^2 - 2nk) + (n-k) \}$$

$$\leq \{ (n-k)^2 + (n-k) \}$$

$$\leq \{ (n-k)(n-k+1) \}$$

$$\therefore \sum_{i=1}^{k} n_i (n_i - 1) \leq \{ (n-k)(n-k+1) \}$$

$$\therefore \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) \leq \frac{1}{2} \{ (n-k)(n-k+1) \}$$

$\therefore$ a graph G with n vertices and k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges

(4) $\{[(p\vee q)\longrightarrow r]\wedge(\neg p)\}\longrightarrow(q\to r)$ is a tautology $\longrightarrow$ T.P (3)

Let $\left[\{[(p\vee q)\to r)\wedge(\sim p)]\}\to(q\to r)\right]\equiv F$ (Suppose)

i.e, $((p\vee q)\to r)\wedge\sim p\equiv T$ $\qquad Q\Rightarrow R\equiv F$

$((p\vee T)\Rightarrow F)\wedge\sim p\equiv T$ $\qquad$ i.e, $Q\equiv T,\ R\equiv F$

$(T\Rightarrow F)\wedge\ p\equiv T$

$F\wedge\sim p\equiv T$

$F\equiv T$ $\Rightarrow$ this is a contradiction. So,

$[[((p\vee q)\Rightarrow r)\wedge(\sim p))]\to(q\to r)]\equiv T$

$\therefore [((p\vee q)\to R)\wedge(\sim p))]\longrightarrow(q\to R)$ is a tautology.

Hence proved.

④

negation	$\sim(\sim p \to \sim q)$ / $(p \to \sim q)$
converse	if q then p
Inverse	if not p, then not q
C.P.	if not q, then not p

3 (a) If Neha wins, then Shally loses

(b) If 9 is odd then the square of 9 is odd

(c) If all cats meow, then some dogs bark.

(d) If John wins, then Mary loses and the school closes.

(a) converse = If Shally loses, then Neha wins.

Inverse – If Neha does not win, then Shally does not lose.

Contrapositive – ~~If Neha~~ If Shelly does not lose, then Neha does not win.

Negation – If Neha wins, then Shelly does not lose

(b) If 9 is odd then the square of 9 is odd.

Converse = If square of 9 is odd then 9 is odd.

Inverse = If 9 is not odd then the square of 9 is not odd.

Contrapositive – If the square of 9 is not odd then 9 is not odd.

negation – If 9 is odd then the square of 9 is not odd.

(c) If all cats meow, then some dogs bark.

Converse – If some dogs bark, then all cats meow.

Inverse – If not all cats meow, then some dogs do not bark.

Contrapositive – If not some dogs bark, then all cats dont meow

negation – If all cats meow, then some dogs dont bark.

(d) If John wins, then Mary loses and the school closes

Converse – If Mary loses and the school closes, then John wins.

Inverse – If John does not win, then Mary does not lose and the school does not closes.

Contrapositive – If Mary does not lose and the school does not close, John does not win.

negation – If John wins, then Mary does not lose and school does not close.

2(a) $(a+1)^n \geq 1+na$ for $a \geq -1$ and $n \geq 2$

Let $P(n) = (1+a)^n \geq (1+na)$ for $a \geq -1$

For $n = 2$, $LHS = (1+a^2) = (1+2a)$

$a^2 \geq 0$

$\therefore$ true, $P(1)$ is true

Let $P(k)$ be true. (assume)

$(1+a)^k \geq 1+ka$

We have to prove $P(k+1)$ is true.

i.e, $(1+a)^{k+1} \geq 1+(k+1)a$

now, $(1+a)^{k+1} \geq 1+ka$ $(\because P(k)$ is true$)$

$(1+a)^k(1+a) \geq (1+ka)(1+a)$

$= (1+a)^{k+1} \geq 1+ka+a+ka^2$

$(1+a)^{k+1} \geq 1+(k+1)a+ka^2 \longrightarrow (1)$

now, $1+(k+1)a+ka^2 \geq 1+(k+1)a \longrightarrow (2)$

$\therefore$ from (1) & (2),

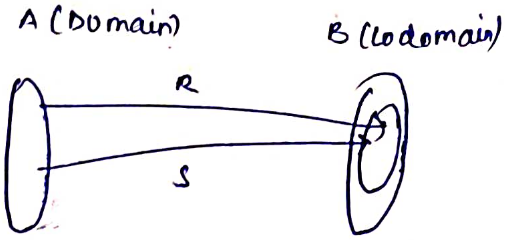
$(1+a)^{k+1} \geq 1+(k+1)\cancel{x}$

$\therefore$ $P(k+1)$ is true if $P(k)$ is true

Hence, $P(n)$ is true for $a \geq -1$ and $n \geq 2$

2(b)

$$\begin{tikzcd} \text{set } A \arrow[r, "R"] \arrow[r, bend right, "S"'] & \text{set } B \end{tikzcd}$$



TO PROVE $\Rightarrow$ i) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

ii) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(i) $x \in (R \cup S)^{-1}$ iff $f(x) \in R \cup S$

iff $f(x) \in R$ OR $f(x) \in S$

iff $x \in R^{-1}$ or $x \in S^{-1}$

iff $x \in R^{-1} \cup S^{-1}$

$\therefore \boxed{(R \cup S)^{-1} = R^{-1} \cup S^{-1}}$

(ii) $x \in (R \cap B)^{-1}$, it means $x \in R \cap S$

and $f(x) \in R$ and $f(x) \in S$ so,

$x \in f^{-1}(R)$ and $x \in f^{-1}(S)$

so, $x \in R^{-1} \cap S^{-1}$

$\therefore \boxed{(R \cap B)^{-1} = R^{-1} \cap S^{-1}}$

Hence proved.

1(a) we have, Z: set of integers

R: Relation over Z aRb gives $a^b = b^a$

Reflexive: Let us assume aRa

$\therefore a^a = a^a \ \forall \ a \in Z$

$\therefore R$ is reflexive.

Symmetric: Let $aRb \ \forall \ a, b \in Z$

$\therefore a, b \in Z$.

$a^b = b^a$ and $b^a = a^b \ \forall \ a, b \in Z$.

$\therefore R$ is symmetric

Transitive: Let aRb and $bRc \ \forall \ a, b, c \in Z$.

then, $a^b = b^a$ and $b^c = c^b$

$\Rightarrow a^{b/a} = b$ and $b = c^{b/c}$

equating, $a^{b/a} = c^{b/c}$

equating the roots by eliminating b from exponential

$a^{1/a} = c^{1/c}$

$\therefore a^c = c^a$ (~~property~~) (raising both to the power ac)

$\therefore aRc$.

R is transitive.

$\therefore R$ is an equivalance relation as it is reflexive, symmetric and transitive.

1 (b) Verify $\overline{A \cup B} \cup \overline{(\bar{A} \cap \bar{B} \cap C)} = U$

LHS $\Rightarrow$ $\overline{A \cup B} \cup \overline{(\bar{A} \cap \bar{B} \cap C)}$

$\Rightarrow$ by de-morgan's law, $\because (\bar{A} \cap \bar{B})$ $\quad (\overline{A \cup B} = \bar{A} \cap \bar{B})$

$\therefore (\bar{A} \cap \bar{B}) \cup \overline{(\bar{A} \cap \bar{B} \cap C)}$

$\Rightarrow$ again by de-morgan's law, $\overline{\bar{A} \cap \bar{B} \cap C} = \bar{\bar{A}} \cup \bar{\bar{B}} \cup \bar{C}$

$\therefore (\bar{A} \cap \bar{B}) \cup (\bar{\bar{A}} \cup \bar{\bar{B}} \cup \bar{C})$

$\Rightarrow$ by double negation . $\bar{\bar{A}} = A$ and $\bar{\bar{B}} = B$

$\therefore (\bar{A} \cap \bar{B}) \cup (A) \cup (B) \cup \bar{C}$

$\Rightarrow$ commutative law : $(\bar{A} \cap \bar{B}) \cup A \cup B \cup \bar{C}$

$= (A \cup (\bar{A} \cap \bar{B}) \cup B \cup \bar{C})$

$\Rightarrow$ $\because$ redundancy law, $(A \cup (\bar{A} \cap \bar{B})) \cup B \cup \bar{C} = (A \cup \bar{B}) \cup B \cup \bar{C}$

$\Rightarrow$ Commutative law, $A \cup \bar{B} \cup B \cup \bar{C} = A \cup B \cup \bar{B} \cup \bar{C}$

$\Rightarrow$ Complement law $(X \cup \bar{X} = 1)$.

$\therefore A \cup (B \cup \bar{B}) \cup \bar{C} = A \cup \bar{C} \cup (U) = A \cup (U) \cup \bar{C}$ universal set

and $X \cup (U) = U$,

$\therefore (A \cup (U)) \cup \bar{C} = U \cup \bar{C}$

and $((U) \cup \bar{C}) = U = \text{RHS}$.

$\therefore$ LHS = RHS hence proved

—— x —— x —— x ——