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ASSIGNMENT - 1

1) $f: R \rightarrow R$ defined as $f(x) = 2x+3 \ \forall \ x \in R$. both one-one, onto f^n.

For one-one,

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$2x_1 + 3 = 2x_2 + 3$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

f is one-one

For onto,

Let $y \in R$ and $f(x) = y$

$$y = 2x+3$$

$$\frac{y-3}{2} = x \in R$$

$\therefore$ for each $y \in R$, there exists $x \in R$ such that $f\left(\frac{y-3}{2}\right) = y$

$\therefore$ f is onto

(2) $f, g : R \to R$ defined by $f(x) = 2x+1$ $\quad g(x) = \frac{x}{3}$ $\forall x \in R$.

For f to be one-one,

Let $x_1, x_2 \in R$.

$\therefore f(x_1) = 2x_1 + 1$; $f(x_2) = 2x_2 + 1$

$\therefore \cancel{2}x_1 + \cancel{1} = \cancel{2}x_2 + \cancel{1}$

$x_1 = x_2$

$\therefore f(x)$ is one-one

For f to be onto,

Let $f(x) = y$

$y = 2x + 1$

$\frac{y-1}{2} = x \in R$

$\therefore$ For every $y \in R$, there exists a $x \in R$ such that $f\left(\frac{y-1}{2}\right) = y$

$\therefore f(x)$ is onto.

For g to be one-one,

Let $x_1, x_2 \in R$.

$g(x_1) = \frac{x_1}{3}$; $g(x_2) = \frac{x}{3}$

$\therefore x_1 = x_2$

$g(x)$ is one-one.

For $g(x)$ to be onto,

Let $g(x) = y$

$y = \frac{x}{3}$

$x = 3y \in R$

$\therefore$ For every $y \in R$, there exists $x \in R$ such that $f(3y) = y$

$\therefore g(x)$ is onto.

$\because$ Both of them are one-one and onto functions, both the inverse functions exist.

2) $f, g : R \rightarrow R$ defined by $f(x) = 2x+1$, $g(x) = \frac{x}{3}$ $\forall$ $x \in R$.

To verify : $(gof)^{-1} = f^{-1} o g^{-1}$

LHS $g(f(x)) = \frac{2x+1}{3}$

$\therefore f(g(f(x)))^{-1} = y = \frac{2x+1}{3}$

$x = \frac{2y}{3} + \frac{1}{3}$

$\therefore (gof)^{-1} = \frac{3x-1}{2}$

RHS $\Rightarrow$ $f^{-1}o g^{-1}$

$f(x) = 2x+1$

$f^{-1}(x) = \cancel{2x}\ \cancel{y} = \frac{x-1}{2}$

$g^{-1}(y) = 3x$

$\therefore f^{-1}(g(x))^{-1} = \frac{3x-1}{2} =$ RHS

$\therefore$ LHS = RHS

Hence proved.

3) $(a+1)^n \geq 1+na$ for $a > -1$ & $n \geq 2$

Let $P(n) = (1+a)^n \geq (1+na)$, for $a > -1$

For $n = 2$

LHS $= (1+a)^2 = (1+2a)$

$= \cancel{1} + a^2 + 2\cancel{a} = \cancel{1} + 2a$

$a^2 \neq 0$

~~LHS ≠ RHS~~

$a^2 \geq 0$

$\therefore$ true $\therefore$ P(1) is true)

Let P(k) be true

$(1+a)^k \geq 1+ka$

We have to prove that P(k+1) is true.

ie, $(1+a)^{k+1} \geq 1+(k+1)a$

now, $(1+a)^{k+1} \geq 1+ka$ ($\because$ P(k) is true.

$(1+a)^k(1+a) \geq (1+ka)(1+a)$

$= (1+a)^{k+1} \geq 1+ka+a+ka^2$

$(1+a)^{k+1} \geq 1+(k+1)a+ka^2 \longrightarrow$ 1

now, $1+(k+1)a+ka^2 \geq 1+(k+1)a \longrightarrow$ 2

$\therefore (1+a)^{k+1} \geq 1+(k+1)x$

$\therefore$ P(k+1) is true if P(k) is true

P(n) is true $\forall n$

(d) $1.2 + 2.3 + 3.4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

Let $P(n) = 1.2 + 2.3 + 3.4 + \dots n(n+1) = \frac{n(n+1)(n+2)}{3}$

For $n = 1$,

LHS = 2

RHS = $\frac{1.2.3}{3} = 2$

LHS = RHS

$\therefore$ P(n) is true for $n = 1$

Assume P(k) to be true

$1.2 + 2.3 + 3.4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3} \rightarrow (1)$

for P(k+1) to be true,

$1.2 + 2.3 + 3.4 + \dots + (\cancel{k+k}).((k+1)+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$

$1.2 + 2.3 + 3.4 + \dots (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$

$1.2 + 2.3 + 3.4 + \dots k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$

We have to prove P(k+1) from P(k)

From (1), adding (k+1)(k+2) on both sides,

$1.2 + 2.3 + 3.4 + \dots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1).(k+2)$

$= \frac{(k+1)(k+2)(k+3)}{3}$

which is same as P(k+1)

$\therefore$ P(k+1) is true $\forall$ P(k) is true

$\therefore$ By principle of mathematical induction, P(n) is true for n, where n is a natural number.

5) $F_0 = 1 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad n \geq 2 \longrightarrow (1)$

$G_1 = 1 \quad G_2 = 3 \quad G_n = G_{n-1} + G_{n-2} + 1 \quad n \geq 3 \longrightarrow (2)$

$P(n) = G_n = 2F_n - 1 \quad \forall\, n \geq 1 \longrightarrow (3)$

Let $n = 2$ in (1),

$F_2 = 2$

Let $n = 3$ in (3)

$G_3 = G_2 + G_1 + 1 = 5$

Putting $n = 1$ in eq (3)

$G_1 = 2F_1 - 1$

LHS $G_1 = 1 \quad F_1 = 1$

RHS $G_1 = 2 - 1 = 1$

LHS = RHS

$\therefore$ P(n) is true for $n = 1$

Assume P(k) is true.

$G_k = 2F_k - 1 \quad \forall\, n \geq 1 \longrightarrow (4)$

Now to prove that P(k+1) is true.

LHS = G_{k+1}

RHS = $2F_{(k+1)} - 1$

LHS from (2)

$G_{k+1} = G_k + G_{k-1} + 1$

Substituting G_k from 4

$= 2F_k - 1 + G_{k-1} + 1$

$G_{k-1} = 2G_{k-1} - 1$

$= 2F_k + 2F_{k-1} - 1 \longrightarrow (5)$

RHS from (1)

$F_{k+1} = F_k + F_{k-1}$

$2(F_k + F_{k-1}) - 1 = 0$

$2F_k + 2F_{k-1} - 1 \longrightarrow (6)$

From (5) and (6) LHS = RHS

$\therefore$ P(k+1) is true when P(k) is true

$\therefore$ By principle of mathematical induction, P(n) is true $\forall\, n \geq 1$

$P(n): G_n = 2F_n - 1$.

6 Proof by strong induction :

* First define $P(n)$

$P(n) = n$ and can be written as the product of primes

* Basic step : Show $P(2)$ is true

2 can be written as the product of one prime itself

so $P(2)$ is true)

* Inductive step : Show $\forall k \geq 2. ([P(2) \wedge \ldots \wedge P(k)] \longrightarrow P(k+1))$ is true

- Hypothesis :

  j can be written as the product of primes when $2 \leq j \leq k$.

- Show $P(k+1)$ true.

  Case 1 : $(k+1)$ is prime

  If $(k+1)$ is prime, $k+1$ can be written as the product of one prime, itself. So, $P(k+1)$ is true.

  Case 2 : $(k+1)$ is composite

  $k+1 = a \cdot b$ with $2 \leq a \leq b \leq k$

  By inductive hypothesis, a and b can be written as the product of primes.

  So, $k+1$ can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

$\therefore$ $P(k+1)$ is true

$\therefore$ by strong induction $\forall n$ $P(n)$ is true