Points and distances.

This is an invitation for you to help build a mathematics publication. We will proceed in a number of rounds. In this first round, a situation is introduced and we are looking for you to explore it and make a contribution. After about a month, any contributions will be recorded and published as a second round. Some questions may still need to be answered and new questions introduced.

Here are the rules. While anyone is welcome to work at the situation, we want to hear from school students, teachers of any school grade, and amateurs – people who have not had advanced mathematical training, but who like to try things out and enjoy the thrill of the chase. Professional mathematicians are strictly spectators. Some of you may want to work alone, but you can also get together with a friend and make a joint contribution. Collaboration can bring about good mathematics. We want to know what **you** have done with the topic; go to the net only if you need to get some backgroubnd.

An important thing to remember when you are doing mathematics is that often you do not understand everything at once. Solving a problem often means to look at something in a different way, to see it from a different perspective. So, if you are getting "stuck", sometimes it is better to be patient, take a break so that you can return to your investigation with fresh eyes.

Here is how you can contribute. Send the results of your investigation to Ed Barbeau at *barbeau@math.utoronto.ca*. Any received by November 1 will be acknowledged in the next round. If you are a student or a student group, let us know your name(s), age(s), school grade(s), as well as whatever help you have received from friends, teachers, and books or the web. If you are a teacher, let us know at what grade level and your educational background in mathematics. Any others are encouraged to provide whatever information they wish about their careers and the basis for their mathematical interest.

TOPIC A: PAIR DISTANCES

Round One

1. If you put a pair of two distinct points on a page, you can measure the distance between them. If you put three distinct points on a page, then there are three pairs, each with its own distance. Generally, these distances will be distinct. But sometimes, as in the case of an isosceles triangle or three points on a line with one of them halfway between the other two, the three distances will consist of exactly two distinct numbers. And if the three points are vertices of an equilateral triangle, the distances will be the same for all three pairs.

Now let us look at four distinct points placed on a flat page or plane. Is it possible for each of the six pairs of points to be the same distance apart? Suppose we ask that the six distances consist of exactly two distinct values. What are the possible configurations?

We can also ask the same question for four points in three-dimensional space. Another interesting angle to pursue is how symmetrical the configuration is and how you would describe the symmetry. (A configuration is symmetrical if it can be carried to itself by a reflection in a line or point, a rotation or any combination of these.)

Can you find any configurations of four points for which all six distances are positive integers for some scale of measurement? What is the minimum number of distinct positive integers? Can all the six distances be distinct positive integers?

2. Suppose that you have four points with exactly two distinct values for the pair distances, call them a and b. What is the algebraic relationship between a and b? If you are a high school student, you can answer this question using algebra and trigonometry. However, students in elementary school can look at this question even without these advanced tools.

Suppose you choose you choose six numbers. Are the any restrictions that have to be satisfied in order that you can find four points for which these are the pair distances?

3. Of course, we can expand our research to look at more than four point in a plane. We can imagine n distinct points where n is any one of the numbers $5, 6, 7, \dots$.

Perhaps a good place to start is the situation where the n points are the vertices of a regular polygon. In this case, they are points on the circumference of a circle such that the distance between each of the n adjacent pairs of them is the same. (A regular polygon with n vertices, called a regular $n-$gon, is a convex figure where each adjacent pair of vertices is joined by an edge, all the edges have the same length and the angle between each adjacent pair of edges is the same.) The pair distances are the lengths of the sides and the lengths of the diagonals of the $n-$gon.

Look at five points that are the vertices of a regular $5-$gon, or pentagon. How many distinct pairs of points are there? How many distinct distances between pairs of points are there? Answer the same question for regular $6-$gons (hexagons) or $7-$gons (heptagons)? Can you find a formula that works for general values of n?

What is the minimum number of distinct values for pair distances when there are 5 points, 6 points, 7 points, ... What are the possible configurations that realize them? Is it necessarily the case that the points are vertices of a regular polygon? What sort of symmetry is displayed by the configurations?

If you have n distinct points, what is the maximum number of times that a pair of points can be the same distance apart?

Can you find any configurations with at least four points not all on the same line where all the pair distances are integers?

If you have a configuration of n point in which the pair distances take a small number of values, say $a, b, c, \ldots$, can you find any algebraic relations connecting these numbers. Look in particular at the situation where the points are vertices of a regular $n-$gon, where a is the length of a side, b is the length of a diagonal joining two vertices with one in between, c is the length of the next longest diagonal, and so on.

4. You may have some other questions that you would like to explore on this topic. Send them in and we can add them to our forum.