Points and Distances

Topic A: Pair Distances

Round One

Ananya Shahrin Promi

Rajshahi College

Grade: 12 (Gap-year)

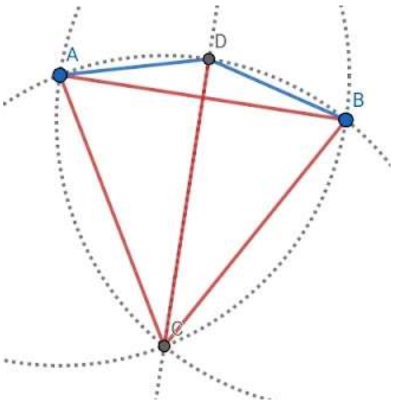
Age: 18

No. 1

Each of the six pairs of four points can't be the same distance apart on a plane. Imagine having three points (A, B, C) that have equal distance, a, from each other on a plane. Now we want to place the fourth point (D) that has a distance from the other three points (who form an equilateral triangle). As we try to place it, it comes out of the plane and it forms a tetrahedron (3D).

Now, if these six distances consist of exactly two distinct values, there are five different conditions.

Condition 1: (4 same, 2 same)

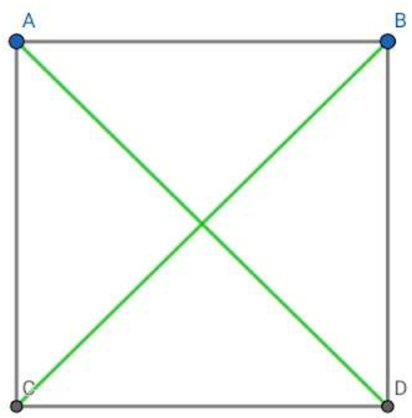


Here,

AC = BC = AB = CD = a

AD = BD = b

Condition 2: (4 same, 2 same)

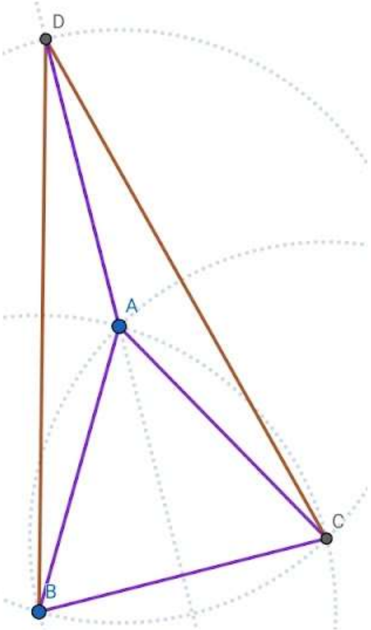


Here,

AB = BD = CD = AC = a

AD = BC = b

Condition 3: (4 same, 2 same)

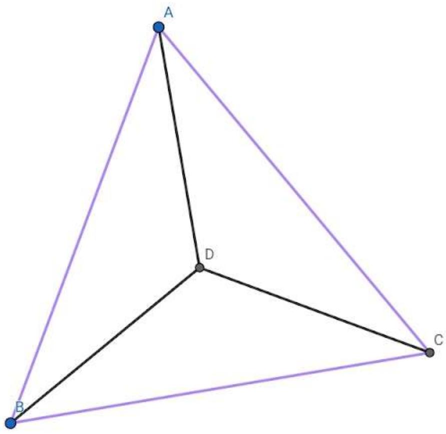


Here,

AB = BC = AC = AD = a

BD = CD = b

Condition 4: (3 same, 3 same)

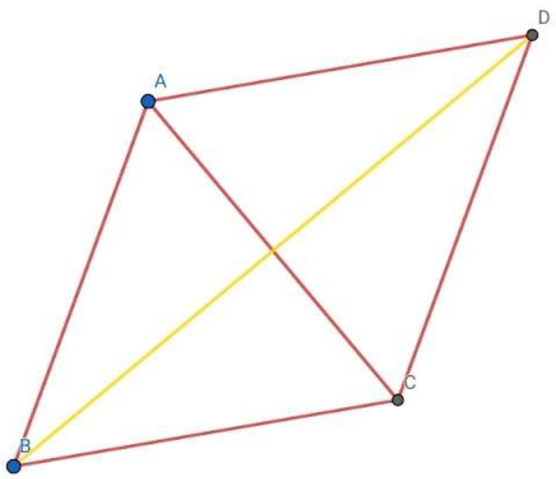


Here,

AB = BC = AC = a

AD = BD = CD = b

Condition 5: (5 same)

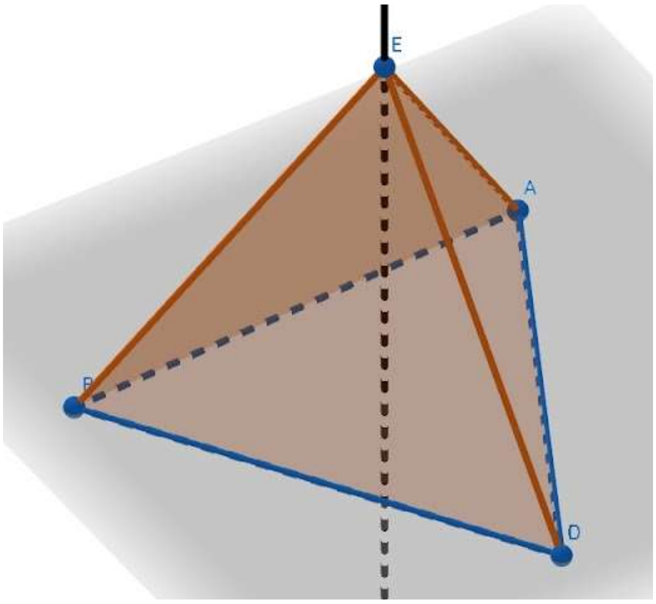


Here,

AB = BC = CD = AD = a

BD = b

In the case of three-dimensional space, it will be a triangular pyramid.

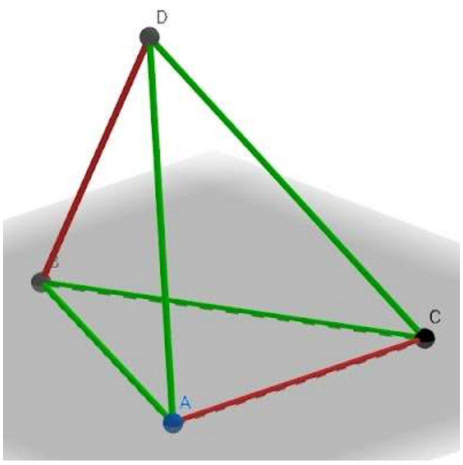


Case 1

The pyramid has one equilateral $\triangle ABC$. The fourth point D moves along the line perpendicular to the ABC plane, passing through the centroid of triangle ABC. Here, this perpendicular line is the symmetric line of this pyramid.

In this case, when all the three triangles become congruent to each other (all become equilateral triangles), it forms a tetrahedron.

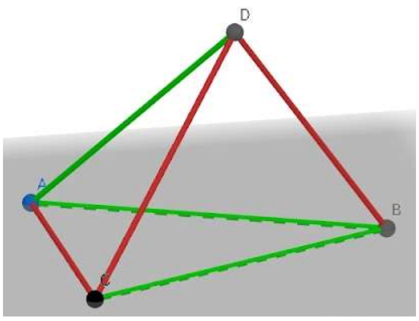
Case 2



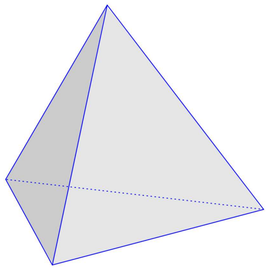
The pyramid consists of four isosceles triangles. In the first picture, four congruent isosceles triangles form a pyramid. Here the sides AC and BD are the symmetric lines of this pyramid. Also, the plane passing through BD perpendicular to AC is the symmetric plane of this pyramid. Similarly, another plane through AC perpendicular to BD is a symmetric plane for the pyramid.

Case 2

The pyramid consists of two pairs of congruent triangles. This pyramid is not symmetric at all.

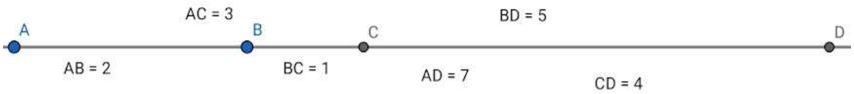


In three-dimensional space, for the above cases, all six distances can be positive integers. Because the pyramids are formed by three congruent equilateral or isosceles triangles, that we can choose by ourselves. When the pyramid is tetrahedral, all the six pair distances are equal.



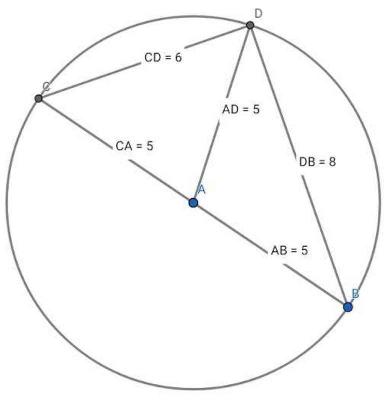
So, in three-dimensional space, the minimum number of distinct positive integers is 1.

If four points lie on a straight line, they all can have integer pair distances. Here all six distances can be distinct positive integers.



Again, we can choose four points on a straight line so that they have three distinct pair distances. Here four points A, B, C, D are placed such that C and D divide DB into three equal parts. (We took DB divisible by 3 for some scale of measurement). Now, DC=CA=AB=a, DA=CB=$2a$, and DC=$3a$, where a is a positive integer. Hence, 3 is the minimum number of distinct positive integer pair distances in a two-dimensional space.

We find another configuration of four points on a plane for which all six distances are positive integers. And here the number of distinct distances is 4.



If A, B, C, *D* are four points, three of them would lie on a straight line and three of them would form a right angle triangle.

We know from the sine law, $\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{d}{\sin D} = \frac{d}{\sin(B+C)} = 2R$, where *R* is the circumradius.

As $R \in Z$, if we take *b* and *c* integers, $\sin B$ and $\sin C$ would be integers. For *d* to be an integer, $\sin(B + C) = \sin B \cos C + \sin C \cos B = \sin B\sqrt{1 - \sin^2 C} + \sin C\sqrt{1 - \sin^2 B}$ must be an integer. And it concludes that *b* and *c* must be the two side lengths of a right angle triangle other than the hypotenuse. So, *b*, *c*, *d* are Pythagorean triple.

<u>No. 2</u>

If we have four points with two distinct values (a and b) for the paired distances, we get 5 conditions mentioned in the previous question.

<u>For condition 1,</u> (a, a, a, a, b, b)

ABCD is a cyclic quadrilateral. Hence, $\angle CDA = \frac{\pi}{3}$.

And, $\sin \angle CDA = \frac{\frac{a}{2}}{b} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

So, we obtain, $a = \sqrt{3}b$

<u>For condition 2,</u> (a, a, a, a, b, b)

We easily get, from Pythagoras's theorem, $a^2 + a^2 = b^2$

And so, $b = \sqrt{2}a$

<u>For condition 3,</u> (a, a, a, a, b, b)

Here, $\angle DAC = \frac{1}{2} \cdot (2\pi - \frac{\pi}{3})$

Applying cosine law in $\triangle DAC$, $b^2 = a^2 + a^2 - 2 \cdot a \cdot a \cdot \cos \frac{5 \cdot \pi}{6}$

After calculation, we get, $b = \frac{\sqrt{6}+\sqrt{2}}{2} a$

<u>For condition 4,</u> (a, a, a, b, b, b)

Here, $\angle ADB = \frac{1}{3} \cdot 2\pi$

Applying cosine law in $\triangle ADB$, $a^2 = b^2 + b^2 - 2 \cdot b \cdot b \cdot \cos \frac{2\pi}{3} = 3b^2$

Hence, $a = \sqrt{3}b$

<u>For condition 5,</u> (a, a, a, a, a, b)

Here, $\angle BAD = 2 \cdot \frac{\pi}{3}$

Applying cosine law in $\triangle BAD$, $b^2 = a^2 + a^2 - 2 \cdot a \cdot a \cdot \cos\frac{2\pi}{3} = 3a^2$

Hence, $b = \sqrt{3}a$

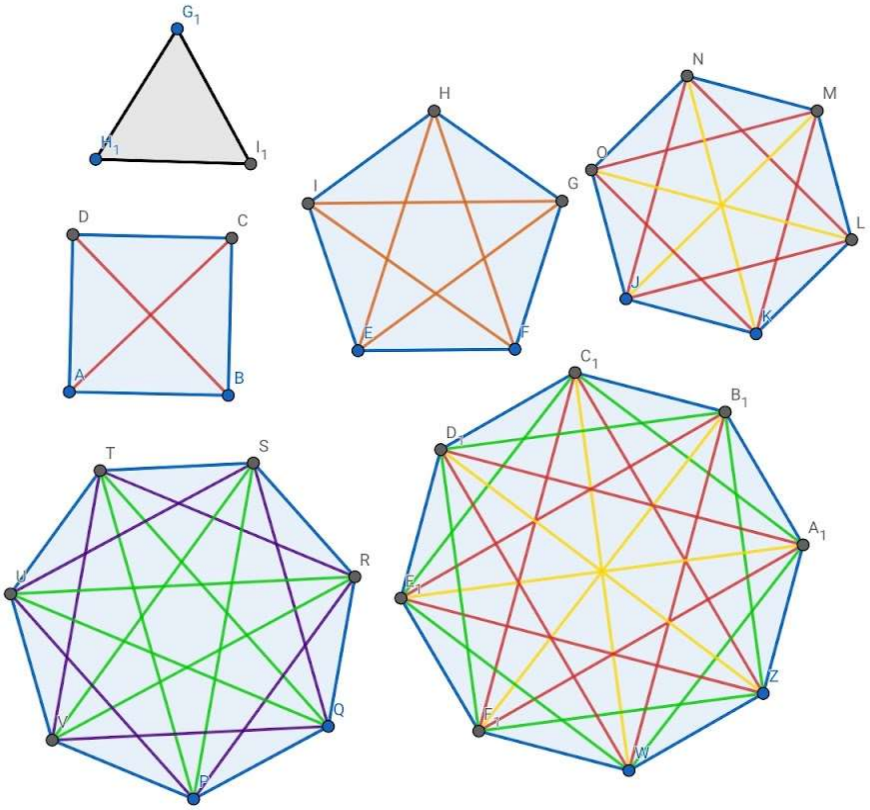
We cannot choose six numbers randomly to be six pair distances of four points. We know, from the triangle property that to form a triangle, the sum of any of the two side lengths must be greater than the third one. In case they are equal, three points lie on a straight line. When we choose any three points among the four, they either form a triangle or lie on the same line. Suppose, A, B, C, D are four points on some plane and AB, AC, AD, BC, BD, CD are six pair distances.

So, we can form $3 \times {}^4C_3 = 12$ inequalities, i.e $AB + BC \geq CA$, $BC + CD \geq BD$, etc. If any of them does not hold, they cannot be six distances of four points on the plane.

From this idea, we can conclude, if a, b, c, d, e, f are six pair distances, the sum of any five must be greater than or equal to the sixth one.

No. 3

As we experiment, we see, in the case of regular 3-gon (equilateral triangle) we get only one distance. In the case of regular 4-gon and 5-gon, there are two distinct distances. Similarly, in regular 6-gon and 7-gon, there are three distinct distances. So, we can come to a general formula for the number of distinct pairs of distances, that is $floor(\frac{n}{2})$.

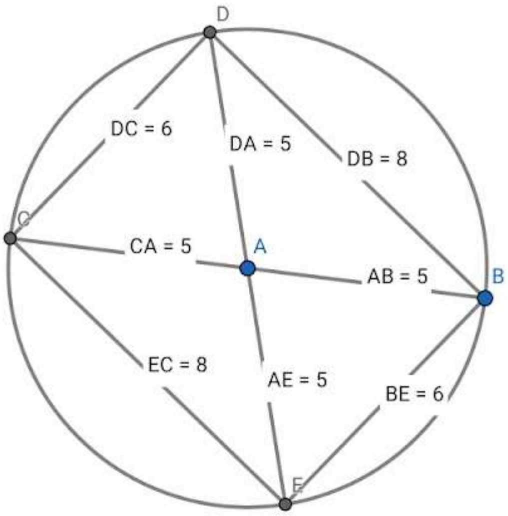


And it is the minimum number of distinct values. A regular polygon is a configuration that ensures the minimum number of distinct pair distances.

And here, n is the maximum number that a pair of points can be the same distance apart among n points.

We can visualize it easily.

And it is not always necessary to be a regular polygon. We can take a regular n-gon that is symmetric in respect to some diameter of that circle passing through any of the points.



And yes we can find a configuration of four points, not all lying on the same line, where all the pair distances are integers. It is the case we discussed in question number 2, where 3 of the four points form a right-angled triangle and the fourth point is the circumcenter of that triangle.

In the case of five points, we get such configuration too. What we need to do is take the fifth point, *E*, diametrically opposite to point *D*.

Now, if we take *a*, *b*, *c*... as pair distances of a regular n-gon, we see a relation between them. We can easily deduce that each angle of a regular n-gon equals $\frac{(n-2)\times\pi}{n}$.

If *a* is the length of each side of that polygon, for *b* being the next diagonal adding two adjacent nodes, we get $b^2 = a^2 + a^2 - 2 \times a \times a \times \cos(\frac{(n-2)\times\pi}{n}) = 2a^2(1 - \cos\frac{(n-2)\pi}{n})$.

For c being the next diagonal distance, $c^2 = b^2 + a^2 - 2 \times b \times a \times \cos(\frac{(n-2)\times\pi}{n} - \frac{(\pi - \frac{(n-2)\times\pi}{n})}{2}) = b^2 + a^2 - 2ab\cos\frac{(n-3)\pi}{n}$.

For *d* being the next diagonal distance, after calculation we get, $d^2 = a^2 + c^2 - 2ac\cos(\frac{(n-4)\pi}{n})$

And here we see a general formula already...

So, $A_1A_2A_3......A_n$ being a regular n-gon on inscribed in a circle, and a being their side length, we can write, $A_iA_{i+k} = a^2 + A_iA_{i+k-1}{}^2 + 2aA_iA_{i+k-1}\cos\frac{(n-k)\pi}{n}$.