Points and distances.

Round Two: Bangladesh edition.

§0. The contributors.

There were contributions from three sources, all high school students in Bangladesh:

(a) Ananya Shahrin Promi **(ASP)**
Rajshahi College, Rajshahi (founded 1873)

(b) Tahmid Hameen Chowdhury & Nujhat Ahmed Disha **(TCND)**

(c) Adnan Sadik, Mamnoon Siam, & Thamin Nur Zahin **(SSZ)**
AS: Notre Dame College, Dhaka (founded 1949)
MS: Chittagong College, Chittagong (founded 1869)
TNZ: Motijheel Government Boys' High School, Dhaka (founded 1957)

All of these made substantial progress, recorded below. Results obtained by all will not be individually attributed.

§1. Pair distances for four points.

Suppose that we are given four distinct points A, B, C, D in the plane. There are six pairs of points, each with its own distance between them. The first question was whether it was possible for all of these distances to be equal.

In looking at this problem, it is stylistically preferable to avoid additional complexity framing the justification as a contradiction argument. **TCND** did this by noting that, once A and B are selected, the the positions of C and D are determined as the intersections of the circle with centres A and B and radius AB. Another approach is simply adding one point at a time, until the placement of the points is fixed by five distances and there is no option for the sixth. Thus, any three points, ABC determine an equilateral triangle. D is forced to be the reflection of A in the line BC. Therefore $ABCD$ form a $60° - 120\ ^{circ}$ rhombus, whose diagonal AD is unequal to the other five distances. Therefore, a placement of the points for which all six distances are equal is impossible. **ASP** observed that, having chosen A, B, C, the only way you can make $AD = BD = CD$ is to go into the third dimension and get the vertices of a regular tetrahedron.

When we come to placing four points to realize exactly two distinct distances, the issue here is the efficiency of the argument. Some contributors broke down the possibilities into a great many case.

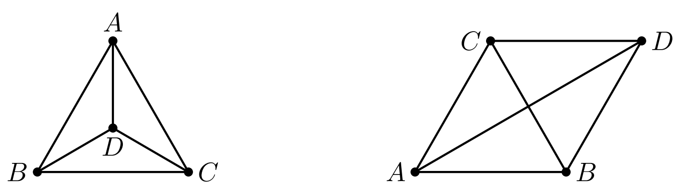
Suppose that the distances are a and b. One of these, say a, must occur at least three times. We make the preliminary observation that if one point, A belongs to three pairs of length a, then either all the pairs not involving A have length b and

BCD is equilateral or the distance between one of the pairs (B, C), (C, D) and (D, B) is a and we have an equilateral triangle one of whose vertices is A.

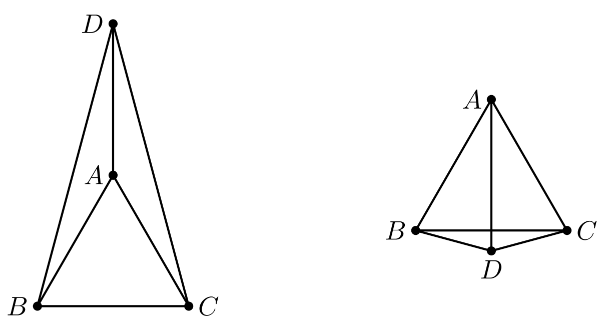
Thus, there are essentially two cases. Either three of the points are vertices of an equilateral triangle with length a or we get a configuration where AB, BC and CD have length a. At least two of the remaining three distances have to be equal to deach other; suppose $DB = DC$. If $DB = DC = a$, then $AD = b$, the length of the diagonal of the rhombus $ABCD$

Otherwise, $DB = DC = b$. If also, $AD = b$, then D must be the centroid of the equilateral triangle ABC. Suppose $AD = a$. Then D must lie on the right bisector of BC passing through A, and as we see in the third and fourth diagrams below, there are two possibilities.

Cases 1 & 2.

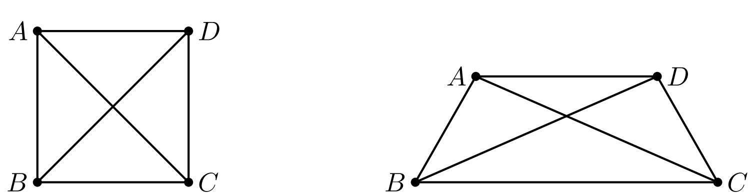


Cases 3 & 4.

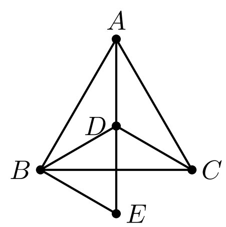


If $AB = BC = CD = a$, there are two possibilities, according as $AD = a$ or $AD = b$. In the former case, $AC = BD = b$ and we get a square, and in the latter, $AC = BD = BC = b$ and we get an equilateral $72° - 108°$ trapezoid (as a little angle-chasing will determine).

Cases 5 & 6.



Since all of these figures are unique, up to scale, there is a relationship between a and b. While this relationship was found using trigonometry, one can use simpler techniques. For example, in Case 1, when ABC is an equilateral triangle with centroid D and $AB = BC = CA = a$, $AD = BD = CD = b$, we can see from the following diagram



that, with BC right-bisecting DE, the areas of triangles BCD and BED are equal, and each is one-third of the area of triangle ABC. Since the areas of similar figures are proportional to the squares of their linear dimensions, $a^2 = 3b^2$. Similarly, in the trapezoidal case, with side a and long diagonal lenth b, $b^2 = 3a^2$.

In the case of the middle two cases where $AB = AC = BC = AD = a$ and $BD = CD = b$, the Law of Cosines leads to $a^2 = b^2(2+\sqrt{3})$ and the first and $b^2 = a^2(2-\sqrt{3})$. Disposing of the surd yields the equation $a^4 - 4a^2b^2 + b^4 = 0$ for both diagrams. Suppose that we consider the two values of b separately in Cases 3 and 4, so that the dimensions for Case 3 are a and b_1, and for Case 4 are a and b_2. Then, from $b_1 = a(2+\sqrt{3})^{1/2}$ and $b_2 = a(2-\sqrt{3})^{1/2}$, it follows that $a^2 = b_1b_2$, $b_1 : a = a : b_2$.

Exercise. Justify the relationship $b_1 : a = a : b_2$ using a geometric construction.

In Case 5 (the square), if the diagonals intersect at E, then the fact that the triangle ACD has twice the area of the similar triangle ADE leads to $b^2 = 2a^2$. As for Case 6, let E be located on the diagonal BD so that $BE = BA = a$. Use the fact that the triangles ABE and EDA are similar to obtain $(b-a) : a = a; b$ or $b^2 - ab - a^2 = 0$ (s0 that b/a is the golden ratio).

Each of these configurations exhibit symmetry. The equilateral triangle with its centroid (Case 1) is stable under reflections in the right bisectors of its sides and a 120° ratation about its centroid.

The rhombus (Case 2) is stable under the two reflections whose axes are the diameter as well as a 180° rotation about its centre, which is also the reflection through the centre point and the product of the two reflections.

The configurations of Cases 3 and 4 are stable under the reflection whose axis is the right bisector of A.

The square (Case 5) is stable under reflections whose axes are the diagonals and midlines of the square, and under rotations about the centre through angles of 90°, 180° and 270°.

The trapezoid (Case 6) is stable under the reflection about the axis that right bisects its parallel sides.

§2. Distances, all integers.

The next task is to locate six points for which all the pair distances are integers. Quite a bit of progress was realized here with some striking examples.

The simplest situation is to have four collinear points suitably spaced to give betweeen 3 and distinct integer values. The next approach is to put together two right triangles along a common side. For example, putting together two $3-4-5$ triangles gives four points forming a rectangle with sides 3 and 4 (**ASP, TCND**).

ASP glued together triangles that were not right-angled. Let A, B, C be three points on a line in that order, and D be a point off the line. Suppose that $\theta = \angle DBA$. Suppose $AB = a$, $DB = b$, $CB = c$, $AD = d$ and $CD = e$. Then, by the Law of Cosines,

$$\cos\theta = \frac{a^2 + b^2 - d^2}{-2ab},$$

so that

$$e^2 = b^2 + c^2 + 2bc\cos\theta = b^2 + c^2 + \frac{c(d^2 - a^2 - b^2}{d}.$$

This works when $(a, b, c, d, e) = (2, 5, 4, 5, 7)$ in which case $\cos\theta = 1/5$.

TCND produced other examples. They put together four $3-4-5$ right triangles to form a rhombus $ABCD$ for which $AB = BC = CD = DA = 5$, $BD = 8$ and $AC = 6$. Pasting a $6-8-10$ and $8-15-17$ triangle together yields four points A, B, C, D with A, B, C collinear for which we get the six distinct integer values

$$AB = 15, BC = 6, AC = 21, BD = 8, AD = 17, CD = 10.$$

However, their *pièce de resistance* was the example of four points at the vertices of a quadrilateral $ABCD$ for which

$$AB = 315, BC = 165, CD = 280, DA = 80, AC = 312, BD = 325.$$

There are right angles at A and C.

TCND got these as particular cases of the more general construction. In the first instance, let $p > q$, integers of the same parity, be chosen arbitrarily, and define $2x = p + q$, $2y = p - q$, $2u = pq + 1$, $2v = pq - 1$, so that $x^2 - y^2 = u^2 - v^2$. Then make A, B, C collinear and

$$AB = 2xy, BC = 2uv, AC = 2(xy + uv), AD = x^2 + y^2,$$
$$CD = u^2 + v^2, BD = x^2 - y^2 = u^2 - v^2.$$

For the second example, pick x, y, u, v in the same way. Form two right triangles ABD and BCD using x, y, u, v as parameters so that the diagonal BD is equal to $x^2 - y^2 = u^2 - v^2$. Since $ABCD$ is a convex quadrilateral with right angles at A

and C, it is concyclic, so that Ptolemy's Theorem applies and

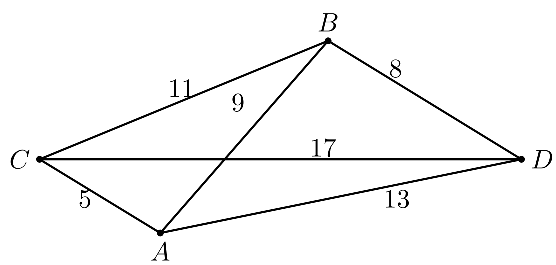
$$AC = \frac{AB \cdot CD + BC \cdot AD}{BD}.$$

Since AC is rational, we can scale up to make it an integer.

SSZ found other examples. The first is again a convex quadrilateral. In the diagram below, we have the lengths

$$AB = 9, AC = 5, BC = 11, AD = 13, BD = 8, CD = 17.$$

Using Herons's formula, we find that the areas of triangles ABC and ACD are both $(15/4)\sqrt{35}$, and the areas of triangles ABD and BCD are both $6\sqrt{35}$, so that AC and BD are parallel.



SSZ's second example is interesting in that it is a triangle ABC with the point D strictly in its interior. Let

$$BC = a, AC = b, AB = c, AD = d, BD = e, CD = f.$$

We select integer values for the lengths and check that they work by using Heron's formula to find the areas $\alpha = [DBC]$, $\beta = [DAC]$, $\gamma = [DAB]$ and $\delta = [ABC]$ and check that $\alpha+\beta+\gamma = \delta$, or more precisely that $\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha) = \delta^2$.

The numbers that work here are given by

$$(a, b, c, d, e, f) = (49, 154, 147, 130, 35, 28).$$

We find that $\alpha^2 = 230496 = 2^5 \cdot 3 \cdot 7^4$, $\beta^2 = 1038336 = 2^{11} \cdot 3 \cdot 13^2$, $\gamma^2 = 4416984 = 2^3 \cdot 3^3 \cdot 11^2 \cdot 13^2$, $\alpha\beta = 2^8 \cdot 3 \cdot 7^2 \cdot 13$, $\alpha\gamma = 2^4 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 13$, and $\beta\gamma = 2^7 \cdot 3^2 \cdot 11 \cdot 13^2$.

SSZ arrived at such examples by using the software C++ to present some examples. Let a, b, c, d, e be a choice of positive integers representing possible distances that lie in the interval $[1, 40]$. Construct configurations of four points that realize these distance and calculate possible values for the sixth distance f. Check for close rational approximations p^2/q^2 for f; if you find one, substitute and scale up to get integer values for the distances.

§3. When do six numbers represent six distances?

Suppose that we have a collection of six, not necessarily distinct, postive numbers. Can we determine necessary and sufficient conditions for which there is a set of four points that have these values as pair distance? If there is such a set of four points, it the configuration unique?

At this point, we have only a few observations. **TCND** noted that if we choose any five numbers that satisfy the pair of triangle inequalities $a+b \geq e$ and $c+d \geq e$, we can glue two triangles together along the sides of length e to get two possible configurations and thus two possible values of f.

TCND noted that a choice of a, b, c, d, e, f is in effect constructing a tetrahedron with these sides whose volume is 0. There is a formula for this volume (given in the Wikipedia entry for Tetrahedron), that will provide an equation satisfied by the lengths.

$4. The situation with n points.

There are a number of questions we can ask about n points.

(a) What is the number of minimum distinct values for the $\binom{n}{2}$ distances between pairs of points? What are the corresponding configurations?

(b) What is the maximum number of times that a distance can occur among pairs of points?

(c) What are the configurations for which all distances can be integers?

For question (a), let $L(n)$ be the least number of distinct pair distances for n points, where $n \geq 3$. It is clear that $L(n) \leq L(n+1)$ for each n. We know that $L(3) = 1$ and $L(4) = 2$. The fact that $L(5) = 2$ is established by the example of a regular pentagon. **TCND** makes additional obervations.

In fact, the regular pentagon is the only configuration with two distinct pair distances. This can be verified by looking at each configuration with four points and two distances and trying to append an additional point; only the trapezoid (case 6) can admit an addtional point. Is it possible to find a configuration of 6 points with two distinct distances? If so, any 5 must constitute a regular pentagon. A sixth point must be equidistant from at least three of these five points and therefore by the centre of a circle through them. Such a circle must be the circumcircle of the pentagon and we are led to an impossible situation. Hence $L(6) = 3$.

The number of pair distances for a regular $n-$gon is given by $\lfloor n/2 \rfloor$, so that this serves as an upper bound for the values of $L(n)$. However, there is at least one additional configuration that gives this value for the number of distinct distances. If n is even, we get another example by removing one vertex of a regular $(n+1)-$gon. If n is odd, we can take the vertices of a regular $(n-1)-$gon along with its centre.

It is still an open equation whether $L(n) = \lfloor n/2 \rfloor$ and whether other examples of the minimizing configuration exists.

For question (b), if we let $P(n)$ denote the maximum number of times a pair distances can be repeated, we know that $P(3) = 3$ and $P(4) = 5$. **TCND** establishes the result:

$$2n - 4 + \left\lfloor \frac{n-1}{3} \right\rfloor \leq P(n) \leq \left\lfloor \frac{n^2}{3} \right\rfloor .$$

To establish the left inequality, we need only display a configuration that realizes the number of points. These points will belong to three consecutive rows of a lattice whose cells are equilateral triangles. Suppose $n = 3k$. Let $A_1, \ldots, A_k$ be equally spaced points in the bottom row. Let $B_1, \cdots, B_k$ be equally spaced points in the next row such that $B_1B_2A_1$ form an equilateral triangle, and $C_1, \ldots, C_k$ be equally spaced points in the top row such that $B_1B_2C_1$ form an equilateral triangle. We count the number of sides in these triangles: $3(k-1)$ horizontal sides and $4k-2$ slant sides for a total of $7k-5$.

If $n = 3k+1$, augment to the middle row by one node to the right, so that we now have $3(k-1)+1$ horizontal sides and $4k$ slant sides for a total of $7k-2$. Finally, when $n = 3k+2$, augment the bottom row by one node to the right to get $3(k-1)+2$ horizontal sides and $4k+1$ slant sides for a total of $7k$. In each case, it can be checked that the number of equal segments is the left member of the inequality. When $2 \leq n \leq 6$, we have $P(n) = 2n-3$.

To establish the right inequality, we define a graph G with n points where two points are joined by an edge if and only if they are a given distance d apart. A clique is a set of points in G for which each pair is joined by an edge. We know that there is no clique with four points.

Recall *Turan's Theorem:* If a graph does not contain a clique with $r+1$ points, then the number of edges in the graph does not exceed

$$\left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

The right inequality follows from this.

For question (c), we have these results:

ASP gives this example of a set of 5 points, all of whose pair differences are integers:

Let $ABCD$ be an equilateral trapezoid and E be the midpoint of AD with the following pair distances:

$14 = BC; 25 = AE = BE = CE = DE; 30 = AB = CD; 40 = AC = BD; 50 = AD.$

The height of the trapezoid (the length of the segment joining the midpoints of AD and BC) is 24.

This can be modified to give six points for which all the pair distances are equal. The points $A, B, C.D$ are lie on a circle of radius 25 and centre E. Let F be the reflection of C in the diameter AD, so that BF is itself a diameter. The length of

CF is $2 \times CD \times \sin \angle CDA = 60 \times (4/5) = 48$. Thus we have the lengths

$$14 = BC; 25 = AE = BE = CE = DE = FE;$$

$$30 = AB = CD = DF; 40 = AC = AF = BD; 48 = CF; 50 = AD = BF.$$

Let A, B, C, D, E be points in a line and F off the line for which $CF \perp AE$ and

$$7 = AB = DE\ ; 9 = BC = CD; 12 = CF; 15 = BF = DF; 16 = AC = CE;$$

$$18 = BD; 20 = AF = EF; 25 = AD = BE; 32 = AE.$$

$5. The regular pentagon and heptagon.

If a and b are respectively the side and diagonal lengths of a regular heptagon, we know that

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} = 3,$$

whereupon the two members of the left side are the roots of the quadratic equation $x^2 - 3x + 1 = 0$. Since, for each integer k,

$$\frac{a^{2k}}{b^{2k}} + \frac{b^{2k}}{a^{2k}}$$

is a symmetric function of the roots of this equation, it can be expressed as a polynomial of the coefficients with integer terms, and so is also an integer. Indeed, using Newton's formula for the sum of powers of roots, this quantity satisfies the recursion

$$x_{n+1} = 3x_n - x_{n-1}$$

with intial conditions $x_0 = 2$, $x_1 = 3$.

When we come to the regular heptagon, things get more interesting. Let A be its side length, b the length of its shorter diagonal, and c the length of its longer diagonal. Picking any four vertices and using Ptolemy's theorem relating the lengths of the sides and diagonals of a concyclic quadrilateral, we obtain the relations:

$$a^2 + ac = b^2; \tag{1}$$

$$a^2 + bc = c^2; \tag{2}$$

$$b^2 + ab = c^2; \tag{3}$$

$$ab + ac = bc. \tag{4}$$

It is easy to check that (4) is a consequence of the other three, and can be rewritten as $1/a = (1/b) + (1/c)$.

Let $u = a^2/b^2$, $v = b^2/c^2$, and $w = c^2/a^2$. Then $uvw = 1$ and $vw = b^2/a^2, wu = c^2/b^2$, and $uv = a^2/c^2$. We investigate the sum of $u + v + w$ and $vw + wu + uv$.

Observe that

$$\frac{a^2}{b^2} = 1 - \frac{ac}{b^2}; \quad \frac{b^2}{c^2} = 1 - \frac{ab}{c^2}; \quad \frac{c^2}{a^2} = 1 + \frac{bc}{a^2}.$$

We note that

$$\begin{aligned}
\frac{bc}{a^2}-\frac{ac}{b^2}-\frac{ab}{c^2} &= \frac{b^3c^3-a^3c^3-a^3b^3}{a^2b^2c^2} = \frac{(bc-ac)(b^2c^2-abc^2+a^2c^2)-a^3b^3}{a^2b^2c^2} \\
&= \frac{b^2c^2-abc^2+a^2c^2-a^2b^2}{abc^2} = \frac{b^2(c^2-a^2)+ac^2(b+a)}{abc^2} = \frac{b^3c+ac^2(b+a)}{abc^2} \\
&= \frac{b^3+abc+a^2c}{abc} = \frac{2abc+b(b^2-ac)+a^2c}{abc} \\
&= \frac{2abc+a(ab+ac)}{abc} = \frac{3abc}{abc} = 3.
\end{aligned}$$

Adding the three equations yields

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=6.$$

Similarly

$$\frac{b^2}{a^2}=1+\frac{c}{a}; \quad \frac{c^2}{b^2}=1+\frac{a}{b}; \quad \frac{a^2}{c^2}=1-\frac{b}{c}.$$

In addition,

$$\begin{aligned}
\frac{c}{a}+\frac{a}{b}-\frac{b}{c} &= \frac{bc^2+ca^2-ab^2}{abc} = \frac{c^3-ab^2}{abc} \\
&= \frac{c^3-b^2c+abc}{abc} = \frac{abc+abc}{abc} = 2.
\end{aligned}$$

Thus we obtain

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=5.$$

We conclude that u, v, w are the roots of the cubic equation

$$x^3-6x^2+5x-1=0,$$

and $vw=1/u, wu=1/v, uv=1/w$ are the roots of the cubic equation

$$x^3-5x^2+6x-1=0.$$

As in the pentagon case, we see that the sequence

$$\{x_k=u^k+v^k+w^k : k\ge 0\}=\{3,6,26,129,650,328,\dots\}$$

satisfies the recursion $x_0=3$, $x_1=6$, $x_2=26$ and

$$x_{k+1}=6x_k-5x_{k-1}+x_{k-2}.$$

The sequence

$$\{x_k=(1/u)^k+(1/v)^k+(1/w)^k : k\ge 0\}=\{3,5,13,38,117,370,\dots\}$$

satisfies the recursion $x_0=3$, $x_1=5$, $x_2=13$ and

$$x_{k+1}=5x_k-6x_{k-1}+x_{k-2}.$$

MS has done an extensive investigation of this question for polygons with more sides. Numerical evidence indicates that, were $a_1 < a_2 < \cdots < a_m$ are the side and diagonal lengths for a regular $(2m+1)-$gon when $2m+1$ is prime, then

$$\frac{a_1^{2k}}{a_2^{2k}} + \frac{a_2^{2k}}{a_3^{2k}} + \cdots + \frac{a_m^{2k}}{a_1^{2k}}$$

and

$$\frac{a_2^{2k}}{a_1^{2k}} + \frac{a_3^{2k}}{a_2^{2k}} + \cdots + \frac{a_1^{2k}}{a_m^{2k}}$$

are integers for every positive integer k. In fact, it seems that if we take certain permutation of the order of the a_i in these expressions, then we still obtain integers. He has found a couple of publications that might be relevant. It is likely that the sequences of integers so found form a linear recursion of the mth order.

$6. Points in space.

ASP investigated situations of four points in space in which there are only two distinct pair distances. There are a great many. For example, start with an equilateral triangle ABC and place D on the perpendicular to ABC through its centroid. Also, let ABC and DBC be two congruent isosceles triangles hinged along BC, the unequal side of length a and place D so that $AD = a$. Finally, we can have $AB = BC = CD$ and $BD = DA = AC$.