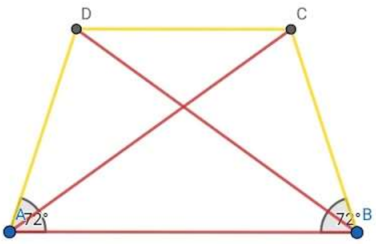
You have given five conditions of four points with two distinct distances. Do you know whether this covers all the possibilities? If so why?

I found another condition with the stated condition. Here, AD = CD = BC = a
And AB = AC = BD = b.



I think these six conditions of four points with two distinct distances cover all possibilities.
We can think like this: there are two distinct values for six pair distances, so at least 3 of them are equal by pigeonhole principle.
Also, we can say that they would form 4C3=4 triangles. We are not considering any three of the four points lying on the same line. It is easy to check that they can never have two distinct values of distances.

Suppose, they form no equilateral triangle. So, we place four points, A, B, C, D with three a-lengths adjacent so they do not form a line. Now, we can join the two endpoints with an a-length or a b-length. If it is a, it forms a square. (condition 2) Otherwise, it would be a trapezoid (condition 6) like in the picture. We need the two diagonals equal to b. Let, $\angle DAB = \angle CBA = \angle ADB = \angle ACD = x$.
So we get, $\angle DBA = \angle CAB = \angle DCA = \angle DAC = \angle DBC = \angle CAB = \pi - 2x$
So, $\angle DAC + \angle CAB = (\pi - 2x) + (\pi - 2x) = 2\pi - 4x = \angle DAB = x$
From here we obtain, $x = \frac{2\pi}{5}$.
Now we suppose, 3 equal distances form an equilateral triangle. Now, we want to place the fourth point so it is either a-away or b-away from the other three.
So, the fourth point could be the triangle's centroid. b-away from the other three. (condition 4)
Or, a-away from two, b-away from one. (That was my condition 5)
Or, a-away from one, b-away from two. (condition 1 and 3)
That covers all the possibilities.

In condition 1, you claim that angle CDA is pi/3. If this is so, then this makes ACD a right triangle whose hypotenuse equals one of its sides.

$\angle CDA \neq \frac{\pi}{3}$ (Sorry, I made some mistake here)

$\angle ACD = \frac{\angle ACB}{2} = \frac{\pi}{6}$

Hence, $b^2 = a^2 + a^2 - 2a^2 \cos \frac{\pi}{6} = a^2(2 - \sqrt{3})$

So, $b = \frac{\sqrt{6}-\sqrt{2}}{2} a$

In conditions 2, 4 and 5 you get a relationship between a and b expressed using a surd. However, you can also express these relationships in algebraic equations whose coefficients are all integers: (2) b^2 = 2a^2; (4) a^2 = 3b^2; (5) b^2 = 3a^2. Is there a similar equation relating a and b for Condition (3)?

When we square both sides, we get $b^2 = a^2(2 + \sqrt{3})$

$\Rightarrow b^2 = 2a^2 + \sqrt{3}a^2$

$\Rightarrow b^2 - 2a^2 = \sqrt{3}a^2$

$\Rightarrow (b^2 - 2a^2)^2 = (\sqrt{3}a^2)^2$

$\Rightarrow b^4 + 4a^4 - 4a^2b^2 = 3a^4$

$\Rightarrow a^4 + b^4 = 4a^2b^2$

So, the same implies for our condition 1 as well.

The same question can be applied for your regular polygons further down. In the case of the pentagon, there are two distances, side a and diagonal b. Is there an algebraic equation with integer coefficients connect these two numbers. For a regular hexagon or heptagon, let a be a side, b the shorter diagonal and c the longer diagonal. Can you find algebraic equations relating these?

Pentagon

Here every angle is $\frac{3\pi}{5}$.

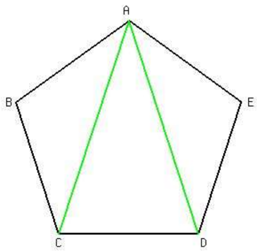
So, $b^2 = a^2 + a^2 - 2a^2 \cos \frac{3\pi}{5}$

$b^2 = 2a^2(1 - \cos \frac{3\pi}{5}) = 2a^2(1 - \frac{1-\sqrt{5}}{4}) = a^2(\frac{3+\sqrt{5}}{2})$

$\Rightarrow 2b^2 = 3a^2 + \sqrt{5}a^2 \Rightarrow 2b^2 - 3a^2 = \sqrt{5}a^2$

$\Rightarrow (2b^2 - 3a^2)^2 = (\sqrt{5}a^2)^2$

$\Rightarrow 4b^4 + 9a^4 - 12a^2b^2 = 5a^4 \Rightarrow a^4 + b^4 = 3a^2b^2$



Hexagon

Every angle is $\frac{2\pi}{3}$.

So, $b^2 = a^2 + a^2 - 2a^2 \cos \frac{2\pi}{3} = 2a^2 + a^2$

So we get, $b^2 = 3a^2$

Now for the longer diagonal, c

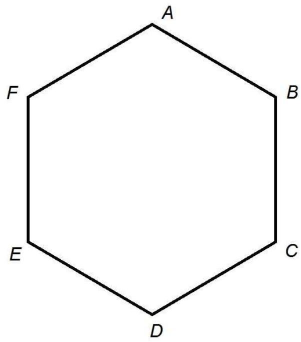
$c = a + 2 \times h$ *where* h = *height of* $\triangle AFB$

Area of $\triangle AFB = \frac{1}{2} \times bh = \frac{1}{2} \times a^2 \sin \frac{2\pi}{3}$

$\Rightarrow bh = a^2 \times \frac{\sqrt{3}}{2} \Rightarrow 2bh = \sqrt{3}a^2 = \frac{\sqrt{3}b^2}{3} = \frac{b^2}{\sqrt{3}} \Rightarrow h = \frac{b}{2\sqrt{3}}$

So, $c = a + 2h = a + \frac{b}{\sqrt{3}}$

$\Rightarrow (c-a)^2 = (\frac{b}{\sqrt{3}})^2 = c^2 + a^2 - 2ac = \frac{b^2}{3} \Rightarrow 3a^2 + 3c^2 - 6ac = b^2$



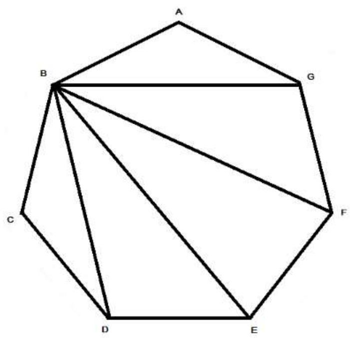
Heptagon

Here every angle is $\frac{5\pi}{7}$.

So, $b^2 = a^2 + a^2 - 2a^2 \cos \frac{5\pi}{7} = 2a^2(1 - \cos \frac{5\pi}{7})$

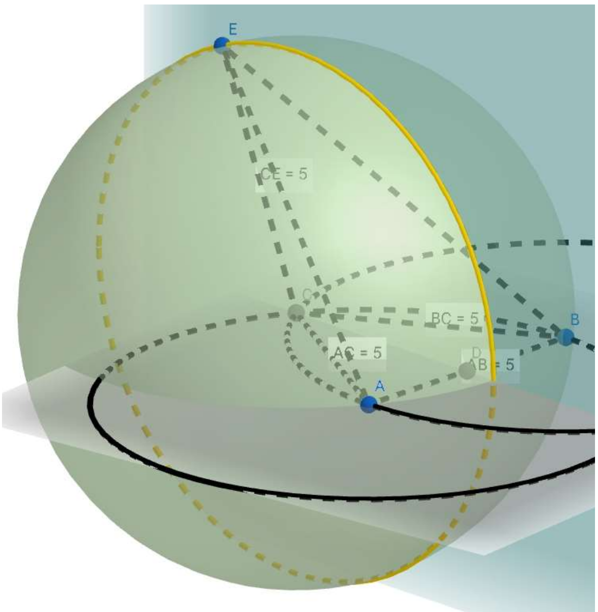
$c^2 = b^2 + a^2 - 2ab\cos(\frac{5\pi}{7} - \frac{1}{2}(\pi - \frac{5\pi}{7})) = a^2 + b^2 - 2ab\cos \frac{4\pi}{7}$

(I do not know how to evaluate cosine of these angles in fraction format so I can obtain a good figure)



I have not given much thought about arrangements of four points in space that realize two distances, and you have provided some nice examples. Are there further possibilities?

Yes, I found other possibilities.



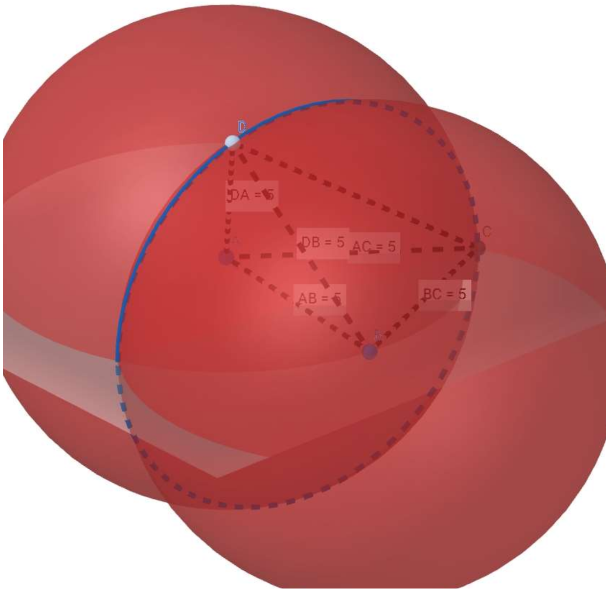
Case 1

Here the base triangle ABC is equlateral with side length a. Now we draw a sphere with radius a centered at C. Also, we draw the plane perpendicular to AB passing through its midpoint. The intersection circle (yellow in the picture) of the sphere and the perpendicular plane would be the fourth point, E's locus.

Because, now $CE=a$ and $AE=BE=b$.

(b changes as it moves along its locus)

<u>2-dimension fact:</u> When the E falls on the same plane as triangle ABC, it becomes the condition 1 and 3 respectively.



Case 2

And here, the base triangle ABC is equilateral with side length a as previous. But we take two spheres centered at A and B with radius a. These two spheres intersect at a circle (blue in the picture) which would be the locus of the fourth point, D.

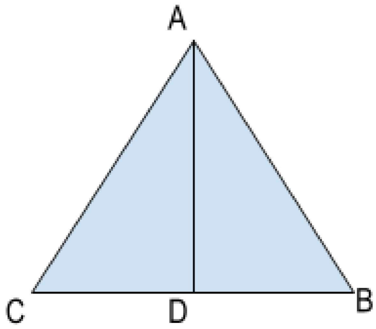
Now we get, $AD=BD=a$ and $CD=b$.

b varies as it moves along its locus.

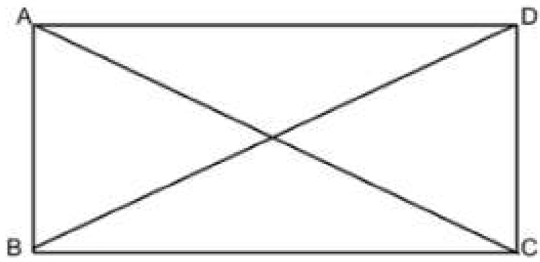
And these are all I get.

I think it would be useful for you to think about other possible ways of getting four points who distances are all integers. Can you arrange that all six integers are distinct?

Yes.

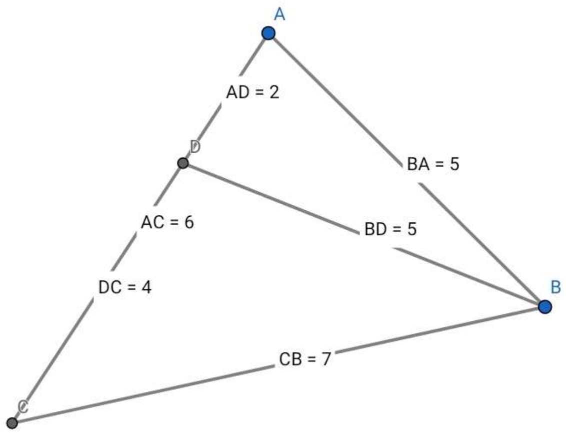


At first we take three points A, B, D who form a right angled triangle with integer side lengths. Now we reflect this triangle along any of the sides except the hypotenuse and get another triangle.
These four points have six distinct integer distances from each other.



Again, if we would reflect the same triangle along the perpendicular bisector of AB (or BC) we get a rectangle. Here the their distances all are integers (as we took the right triangle in such way). Here the number of distinct pair distances is 3.

As I was trying to form such conditions, I found one. (In the picturebelow)



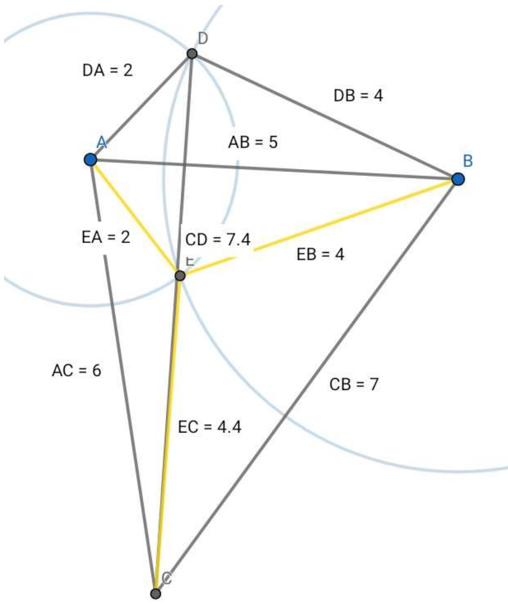
I do not know how they would be related if D is not on the line AC. However, I developed some calculation when it lies on AC.
Let, AD = a, AB = c, BD = b, BC = e, DC = d, and AC = $a + d$
Assume, $\angle ADB = \alpha$ and $\angle CDB = \pi - \alpha$
Applying cosine law in triangle ABD,
$c^2 = a^2 + b^2 - 2ab\cos\alpha$
$\Rightarrow \cos\alpha = \frac{a^2+b^2-c^2}{2ab}$
Now in triangle CBD,
$e^2 = d^2 + b^2 - 2bd\cos(\pi - \alpha) = d^2 + b^2 + 2bd\cos\alpha = d^2 + b^2 + \frac{2bd}{2ab}(a^2 + b^2 - c^2)$
So, $e^2 = d^2 + b^2 + \frac{d}{a}(a^2 + b^2 - c^2)$
If right hand side is a perfect square e can be an integer.

Finally, you have given a set of necessary conditions for six distances to be realized by four points in the form of twelve triangle inequalities. Suppose that you have six numbers a, b, c, d, e, f for which the sum of any two is greater than or equal to any one of the remaining; are there four points in space that realize this set of distances?

Yes.
In my previous note, for 3-D shapes, case-1 would follow this rule as long as $2a \geq b$ (where a is the side lengths of the base triangle, and b is the other three distances). And the triangle inequality holds for each of the four triangles. And for case 2 there, we can choose b, so that $2b \geq a$ (a be the green side lengths, and b be the red ones). It makes the above condition true.Also, there, in case 3, this condition always holds. However, as I mentioned in this note, in both cases, the above condition is always true.
So, it was not sufficient for six distances to be pair distances of four points on a plane. So, my assumption was not right.

Your necessary condition that any five of the six numbers is at least as great as the remaining one is a pretty trivial necessary condition, which is certainly not sufficient. Consider the set of numbers (1, 2, 4, 5, 6, 7).

This set of numberscertainly do not work. We can choose five distances randomly, but the sixth must belong to a certain set of numbers, depending on the other five we have chose. For the above set, if we consider {2, 4, 5, 6, 7} first, we definately do not get 1 as the sixth distance in either way. At first, from this set of 5 distances we can choose a triangle in ${}^5C_3$=10 ways.
Suppose we chose a triangle with lengths 5, 6, 7. Now we want to take the fourth point, D, to form another triangle with side lengths 2, 4, 5.



There we get four possible ways to choose D. Hence, four possible values for the sixth pair distance.
I could not exactly measure the sixth distance using my length chasing and trigonometry knowledge, but it is measurable somehow.
(Here, we could not take 2, 4, 7 as they cannot form any triangle. But we could take D on AC, then we would get two possible values for the six pair distance).
Here we could take first triangle with other three side lengths as well. That makes some certain possible values for the sixth one.