# Group 1 Homework 9

Contribution details:

Anar Rzayev – 21, 23, 35, 41

Murad Aghazada – 20, 45, 46

Pasawat Viboonsunti – 34, 39

MAS241

$\boxed{6.45}$ Consider $\int_a^b (tf+g)^2$ where $t$ is fixed real number. Since $(tf+g)^2 \geq 0 \Rightarrow \int_a^b (tf+g)^2 \geq 0$

Since $f, g \in R[a,b] \Rightarrow f^2, fg, g^2 \in R[a,b]$

Thus, we can write $\int_a^b (tf+g)^2 = t^2 \int_a^b f^2 + 2t \int_a^b fg + \int_a^b g^2 \geq 0$. Now, quadratic expression wrt $t$ is $\geq 0 \Rightarrow$ discriminant $\leq 0 \Rightarrow 4\left(\int_a^b fg\right)^2 \leq 4 \int_a^b f^2 \int_a^b g^2$

$\Rightarrow \left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$

$\boxed{6.20}$ Take $f$ as follows:

$$f(x) = \begin{cases} -1 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}^c \end{cases} \Rightarrow \begin{matrix} M_j = 1 \\ m_j = 1 \end{matrix} \Rightarrow \begin{matrix} L(f) = a-b \\ U(f) = b-a \end{matrix}$$

$L(f) \neq U(f) \Rightarrow f \notin R[a,b]$

$|f(x)| = 1 \Rightarrow S(|f|, \pi) = \sum \Delta x_j = b-a \Rightarrow I = b-a$

$\Rightarrow |f| \in R[a,b]$. Thus, contradiction

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$\boxed{6.46}$ i) $\|f\|_2 = \sqrt{\int_a^b f^2} \geq 0$

if $f=0 \Rightarrow S(f^2,\pi)=0 \Rightarrow I = \int_a^b f^2 = 0$

$\Rightarrow \|f\|_2 = 0$

if $\|f\|_2 = 0 \Rightarrow \int_a^b f^2 = 0$; $f^2 \in C[a,b]$ and $f^2 \geq 0$

By theorem 8 $f^2=0 \Rightarrow f=0$

ii) $\|cf\|_2 = \sqrt{\int_a^b c^2 f^2} = \sqrt{c^2 \int_a^b f^2} = |c| \sqrt{\int_a^b f^2} =$

$= |c| \, \|f\|_2$

iii) $\|f+g\|_2 = \sqrt{\int_a^b (f+g)^2} \overset{?}{\leq} \|f\|_2 + \|g\|_2 =$

$= \sqrt{\int_a^b f^2} + \sqrt{\int_a^b g^2}$ Since both sides $\geq 0$, take square

$\Leftrightarrow \int_a^b f^2 + 2\int_a^b fg + \int_a^b g^2 \leq \int_a^b f^2 + \int_a^b g^2 + 2\sqrt{\int_a^b f^2 \int_a^b g^2}$

$\Leftrightarrow$

$\int_a^b fg \leq \sqrt{\int_a^b f^2 \int_a^b g^2}$

$\int_a^b fg \leq \int_a^b |f| \cdot |g| \leq \sqrt{\int_a^b |f|^2 \int_a^b |g|^2} = \sqrt{\int_a^b f^2 \int_a^b g^2}$

(by 6.45)

$\boxed{6.34}$ Let $I = \int_0^1 f(x)\,dx$ Let $\varepsilon > 0$

$\therefore \exists \pi_0 \in \Pi[0,1], \forall \pi \succeq \pi_0 : U(f,\pi) - L(f,\pi) < \varepsilon/2$ (by Reimann Condition)

Let $\pi_0 = \{x_0, \dots, x_p\}$ ; $\alpha = \min\{\Delta x_i \mid i \in \{1,\dots,p\}\}$

Let $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ ; $B = \sum_{i=1}^{p} (M_i - m_i)$

Let $K_0 = \lceil \max(\frac{1}{\alpha}, \frac{2B}{\varepsilon}) \rceil$

For all $k > k_0$, $k > k_0 \geq \frac{1}{\alpha}$ $\therefore \forall i \in \{1,\dots,p\} : \frac{1}{k_0} \leq \alpha \leq \Delta x_i$

$\therefore \forall i \in \{1,\dots,p\}, \exists 0 < j \leq k : \frac{j-1}{k} \leq x_{i-1} < \frac{j}{k} \leq x_i$ $\therefore$ $m_i \leq f(\frac{j}{k}) \leq M_i$

For $0 < j \leq k$ such that $\nexists i \in \{1,\dots,p\} : \frac{j-1}{k} \leq x_{i-1} < \frac{j}{k} \leq x_i$ $\therefore \exists i \in \{1,\dots,p\} : x_{i-1} \leq \frac{j-1}{k} < \frac{j}{k} \leq x_i$

$\therefore$ $m_i \leq f(\frac{j}{k}) \leq M_i$

$\therefore$ $\frac{1}{k}\sum_{j=0}^{k} f(\frac{j}{k}) \leq \varepsilon$ ......?

$\boxed{6.39}$ consider $g : [a,b] \to \mathbb{R}$ ; $g(x) = \begin{cases} \lim_{t \to a^+} f(t) & ; x = a \\ \lim_{t \to b^-} f(t) & ; x = b \\ f(x) & ; \text{otherwise} \end{cases}$

$\therefore$ $g$ is continuous and bounded on $[a,b]$

By thm 7, $g \in R[a,b]$

Since $\forall x \in (a,b) : f(x) = g(x)$ $\therefore$ By lemma 2, $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ $\therefore f \in R[a,b]$

41) Given any subinterval $[x,y] \subset [a,b]$, one has $\int_x^y |f'(t)|dt \geq$
$\geq \left|\int_x^y f'(t)dt\right| = |f(y)-f(x)|$ due to Fundamental Theorem of Calculus

Therefore, we at once obtain for any partition $P$ of $[a,b]$, the estimate $S(P) := \sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| \leq \int_a^b |f'(t)|dt$

Since the RHS does not depend on the chosen $P$, we can conclude that $V_{[a,b]}(f) := \sup_P S(P) \leq \int_a^b |f'(t)|dt$

In reality, we have equality here: Looking at the graph, of a reasonable $f$, we immediately realize that $V_{[a,b]}(f)$ is the sum of all "infinitesimal changes" of $f$ added up with a positive sign. For the proof, however, we need the continuity of $f'$ over the whole interval $[a,b]$.

Given an $\varepsilon > 0$, there is a partition $P$ with $|f'(x)-f'(y)| \leq$
$\leq \varepsilon$ for any two points $x, y$ in the same subinterval of $P$ (choosing $\delta > \underbrace{\|P\|}_{\text{gauge}}$ will yield $|x-y| \leq \|P\| < \delta$). Using this estimate partition $P$, we have for each subinterval $[x_{k-1}, x_k]$, the

$$\int_{x_{k-1}}^{x_k} |f'(t)|dt \leq (|f'(x_k)|+\varepsilon)(x_k - x_{k-1}) = \left|\int_{x_{k-1}}^{x_k} f'(x_k)dt\right| +$$
$$+\varepsilon(x_k - x_{k-1}) \leq \left|\int_{x_{k-1}}^{x_k} f'(t)dt\right| + 2\varepsilon(x_k - x_{k-1}) = |f(x_k)-f(x_{k-1})|$$
$$+2\varepsilon(x_k - x_{k-1}),$$ because of the Fundamental Theorem of Calculus

Summing over $k$, we obtain

$$\int_a^b |f'(t)|dt = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'(t)|dt \leq S(P) + 2\varepsilon \cdot (b-a) \leq V_{[a,b]}(f) + 2\varepsilon(b-a)$$

choosing $\varepsilon \to \dfrac{\varepsilon'}{2(b-a)}$ for any choice of $\varepsilon' > 0$, we reveal that

$\int_a^b |f'(t)|dt \leq V_{[a,b]}(f) + \varepsilon'$ where $\varepsilon' > 0$ is arbitrary. Thus, combining it with initial result $V_{[a,b]}(f) \leq \int_a^b |f'(t)|dt$ proves that indeed $\boxed{V_{[a,b]}(f) = \int_a^b |f'(t)|dt}$ $\boxed{V}$ ■ $+$

a1) Notice that $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$

So, $f^+ - f^- = \frac{1}{2}(2f) = f \Rightarrow$ If both $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are in $R[a,b]$, then $f^+ - f^-$ will also be in $R[a,b]$, aka $f \in R[a,b]$ $\boxed{\text{If } f^+, f^- \in R[a,b] \Rightarrow f \in R[a,b]}$

Now, assume that exactly one of $\{f^+, f^-\}$ is in $R[a,b]$

- If $f^+ \in R[a,b]$, but $f^- \notin R[a,b]$, then it does not necessarily follow that $f \in R[a,b]$: Consider $\boxed{f(x) = \begin{cases} -1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}}$ on $[a,b]$

Then $\boxed{f^+ \equiv 0}$ and since $\forall c \in \mathbb{R}$ we had $\int_a^b c\,dx = c(b-a)$ for

$\Rightarrow \int_a^b 0\,dx = 0$ and $\boxed{f^+ \in R[a,b]}$ However, taking into account the $f^-$ function

$f^- = \begin{cases} 1, & \text{if } x \in Q \\ 0, & \text{otherwise} \end{cases}$ However, according to Example 1, we know $\boxed{f^- \text{ is not Riemann integrable on } [a,b]}$ ⊠

since $L(f^-,\Pi) = 0$ and $U(f^-,\Pi) = b-a$ where $\Pi$ is any partition of $[a,b]$. Consequently $L(f^-) = 0$ and $U(f^-) = b-a$ and $\boxed{f^- \text{ is not integrable}}$ ✓ (Note that between any reals, there always exist rational and irrational number)

Similarly, $L(f,\Pi) = (-1)(b-a) = a-b$ and $U(f,\Pi) = 0$ where $\Pi$ is any partition of $[a,b]$. Thus, $L(f) = a-b$ and $U(f) = 0$, wherein $\boxed{f \text{ is not integrable}}$ ⊠ So, constructing such $f$ has yielded

$\boxed{f^+ \in R[a,b],\ f^- \text{ is not in } R[a,b], \text{ but } f \text{ is not in } R[a,b]}$ ⊠

So, it does not necessarily follow that $f \in R[a,b]$ in this case

☑ - If $f^- \in R[a,b]$, but $f^+ \notin R[a,b]$, then it doesn't necessarily follow that $f \in R[a,b]$: Take $\boxed{f(x) = \begin{cases} 0, & \text{if } x \in Q \\ 1, & \text{if } x \notin Q \end{cases} \text{ on } [a,b]}$

$\boxed{f^- \equiv 0}$ ⊠ and $\boxed{f^- \in R[a,b]}$ ⊠ ⊠

But, $f^+ = \begin{cases} 0, & \text{if } x \in Q \\ 1, & \text{if } x \notin Q \end{cases}$ where using the fact that there exist rational and irrational numbers between any 2 values, $L(f^+,\Pi) = 0$ and $U(f^+,\Pi) = b-a$ where $\Pi \in \Pi[a,b] \Rightarrow L(f^+) = 0$ and $U(f^+) = b-a$ where we get $\boxed{f^+ \text{ is not integrable}}$ ⊠ Since $f^+ = f$, it is also derived that $\boxed{f \text{ is not integrable}}$. In conclusion, for such function $f$, $f^- \in R[a,b]$, $f^+ \notin R[a,b]$, and $f$ is not integrable in $R[a,b]$

So, $\boxed{\text{it does not necessarily follow that } f \text{ is integrable in this case}}$.

Anar Rzayev MAS241: HW8 Submission ID: 20190182

35) The given statement has insufficient details. In fact, statement becomes false whenever $f$ is decreasing. For instance, take $f(x) = 1 - x^2$ on $[0,1]$: This is continuous and maps $[0,1]$ one-to-one onto $[0,1]$, where $f^{-1}(x) = \sqrt{1-x}$, but $\int_0^1 (1-x^2)\,dx + \int_0^1 \sqrt{1-x}\,dx = x - \frac{x^3}{3}\Big]_0^1 + \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = 1 - \frac{1}{3} + \frac{2}{3} = \frac{4}{3} \neq 1$ $\boxed{\times}$ Therefore, we have to consider new assumptions. Specifically, assume $\boxed{f\text{-increasing (strictly)}}$. Then, by surjectivity (onto) of $f$, there exist $\exists t \in [0,1]$ s.t. $f(t) = 0 \Rightarrow f(0) \leq f(t) = 0$ and $f(0) \in [0,1]$ $\Rightarrow \boxed{f(0) = 0}$ Similarly, $\exists h \in [0,1]$ s.t. $f(h) = 1$ and $1 \geq f(1) \geq f(h) = 1 \Rightarrow \boxed{f(1) = 1}$ Let $\{x_0, x_1, \dots, x_N\}$ be a partition of $[0,1]$. Then, due to being strictly increasing, and mapping $[0,1]$ one-to-one onto $[0,1]$, we can deduce $\{f(x_0), f(x_1), \dots, f(x_N)\}$ is a partition of $[f(0), f(1)] = [0,1]$ (Conversely, if $P'_1 = \{f(x_0 = 0), f(x_1), \dots, f(x_n = 1)\}$ is a partition of $f^{-1}$ on $[f(0) = 0, f(1) = 1]$, then we know that

$P=\{x_0=0,\dots,x_n=1\}$ is a partition of $f$ on $[0,1]$.) Observe that

$$L(f,P)+U(f^{-1},P')=\sum_{k=0}^{n-1} f(x_k)(x_{k+1}-x_k)+\sum_{k=0}^{n-1} x_{k+1}\left(f(x_{k+1})-f(x_k)\right)$$

$$=\sum_{k=0}^{n-1} x_{k+1}f(x_{k+1})-x_k f(x_k)=x_n f(x_n)-x_0 f(x_0)=f(1)-f(0)=1$$

Similarly, we have $U(f,P)+L(f^{-1},P')=f(1)-f(0)=1$

Thus, for all $P'$, we get $1-U(f^{-1},P')=L(f,P)\leq\int_0^1 f(x)dx\leq$

$\leq U(f,P)=1-L(f^{-1},P')\Rightarrow$ Thus, $\int_0^1 f(x)dx=1-\int_0^1 f^{-1}(y)dy$

and $\boxed{\int_0^1 f(x)dx+\int_0^1 f^{-1}(y)dy=1}$ Alternatively, just notice that the following equality holds:

$$\sum_{i=0}^{n-1} f(x_i)(x_{i+1}-x_i)+\sum_{i=0}^{n-1} x_i\left(f(x_{i+1})-f(x_i)\right)+\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left(f(x_{i+1})-f(x_i)\right)$$

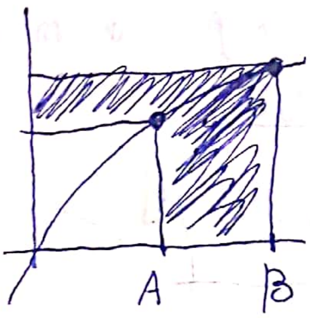
$=x_n f(x_n)-x_0 f(x_0)=f(1)-f(0)=1$, where the first two sums are Riemann sums for $\int_0^1 f$ and $\int_0^1 f^{-1}$, respectively. The $3^{rd}$ sum converges $\to 0$ as the size of partition $\to 0$ $\Big(\sum_{j=1}^{n} f(x_{j-1})(x_j-x_{j-1})$

$$+\sum_{j=1}^{n} f^{-1}(f(x_j))\left(f(x_j)-f(x_{j-1})\right)=\sum_{j=1}^{n} f(x_{j-1})(x_j-x_{j-1})+\sum_{j=1}^{n} x_j\left(f(x_j)-f(x_{j-1})\right)$$

$=1\cdot f(1)-0\cdot f(0)=1$, where the LHS $\xrightarrow{\text{converges}}\int_0^1 f+\int_0^1 f^{-1}$ as the partition norm $\to 0$ where RHS stays same$\Big)$

$\int_0^1 f(x)dx\to$ area under the curve between $x=0$, $x=1$, and $x$-axis

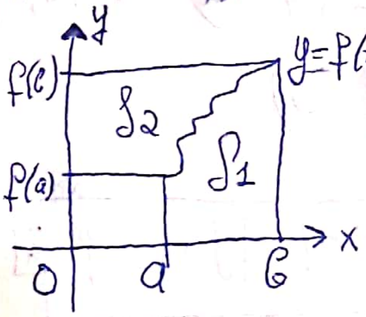
$\int_{f(0)}^{f(1)} f^{-1}(y)dy\to$ area left of the curve between $y=f(0)$, $y=f(1)$, and $y$-axis




$\int_a^b f(x)dx \to$ area under the curve between $x=a, x=b$, and the x-axis

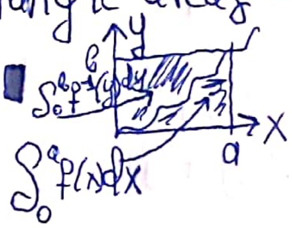
$\int_{f(a)}^{f(b)} f^{-1}(y)dy \to$ area left of the curve between $y=f(a)$, $y=f(b)$, and y-axis (Looking at the same graph "from" the y-axis) ⇝ it'll be graph of $f^{-1}$

Adding these areas together gives inverted L-shape. That's, larger rectangle with smaller removed: $1\times f(1) - 0\times f(0) = f(1) = 1$ (since $b=1$ and $a=0$ in the original problem statement)




$S_1 = \int_a^b f(x)dx$, $S_2 = \int_{f(a)}^{f(b)} f^{-1}(x)dx \Rightarrow$

Geometrically, $S_1 + S_2 = S_{big} - S_{small}$ where $S_{big}, S_{small}$ denotes big rectangle and small rectangle areas $\Rightarrow$ addition of 2 areas, making up a big rectangle substracting small one




**23)** For any two bounded functions $f$ and $g$ on $[a,b]$

$$L(f+g) \le L(f) + L(g) \le U(f) + U(g) \le U(f+g)$$

Let $\Pi$ be any partition of $[a,b] \Rightarrow$ Let's show that

$$L(f+g,\Pi) \geq L(f,\Pi) + L(g,\Pi)$$

<u>Proof:</u> Since $\inf\limits_{x\in[x_{j-1},x_j]} \{f(x)+g(x)\} \geq \inf\limits_{x\in[x_{j-1},x_j]} f(x) + \inf\limits_{x\in[x_{j-1},x_j]} g(x)$

we get $L(f+g,\Pi) = \sum_{j=1}^{p} m_{f+g,j}\Delta x_j \geq \sum_{j=1}^{p} m_{f,j}\Delta x_j + \sum_{j=1}^{p} m_{g,j}\Delta x_j$

$= L(f,\Pi) + L(g,\Pi)$

where $m_{f+g,j} = \inf_{x \in [x_{j-1}, x_j]} \{f(x)+g(x)\}$, $m_f = \inf_{x\in[x_{j-1},x_j]} \{f(x)\}$, and similarly

$m_{g,j} = \inf_{x\in[x_{j-1},x_j]} \{g(x)\} \Rightarrow \boxed{L(f+g,\pi) \geq L(f,\pi) + L(g,\pi)}$ ⊠ ✓

Using Theorem 6.2.1, if $\pi'$ – refinement of $\pi$, then $L(f,\pi) \leq L(f,\pi') \Rightarrow$ For any $\pi_1, \pi_2 \in \Pi[a,b]$, $L(f+g, \pi_1 \vee \pi_2) \geq$
$\geq L(f, \pi_1 \vee \pi_2) + L(g, \pi_1 \vee \pi_2) \geq L(f,\pi_1) + L(g,\pi_2)$. Thus,

$L(f+g) = \sup_{\pi\in\Pi[a,b]} L(f+g,\pi) \geq L(f,\pi_1) + L(g,\pi_2)$

Taking supremum among $\pi_1, \pi_2 \Rightarrow \boxed{L(f+g) \geq L(f) + L(g)}$ ⊠

Notice that we could take supremum of the first resultant inequality, but we knew $\sup(A+B) \leq \sup A + \sup B$ would be revealed $\left( \sup(L(f,\pi) + L(g,\pi)) \leq \sup L(f,\pi) + \sup L(g,\pi) \right)$

In the same manner, $U(f+g,\pi) \leq U(f,\pi) + U(g,\pi)$ for any $\pi \in \Pi[a,b]$ (Just note $\sup_{x\in[x_{j-1},x_j]} \{f(x)+g(x)\} \leq \sup_{x\in[x_{j-1},x_j]} \{f(x)\} +$
$+ \sup_{x\in[x_{j-1},x_j]} \{g(x)\}$) Otherwise, $U(f+g,\pi) = \sum_{j=1}^{p} M_{f+g,j} \Delta x_j \leq \sum_{j=1}^{p} M_{f,j} \Delta x_j$
$+ \sum_{j=1}^{p} M_{g,j} \Delta x_j = U(f,\pi) + U(g,\pi)$. Similarly, for $\forall \pi_1, \pi_2 \in \Pi[a,b]$
we know $U(f,\pi) \geq U(f,\pi')$ for any $\pi' \geq \pi$. Since $\pi_1 \vee \pi_2 \geq \pi_1, \pi_2$
$U(f+g, \pi_1 \vee \pi_2) \leq U(f, \pi_1 \vee \pi_2) + U(g, \pi_1 \vee \pi_2) \leq U(f,\pi_1) + U(g,\pi_2)$

Thus, $U(f+g, \Pi_1 \vee \Pi_2) \leq U(f, \Pi_1) + U(g, \Pi_2)$ and $U(f+g) =$

$= \inf_{\Pi \in \Pi[a,b]} U(f+g, \Pi) \leq U(f, \Pi_1) + U(g, \Pi_2)$. Taking infimum

among $\Pi_1$ and $\Pi_2$, we obtain $\boxed{U(f+g) \leq U(f) + U(g)}$

In conclusion, $\boxed{U(f) + U(g) \geq U(f+g) \geq L(f+g) \geq L(f) + L(g)}$

where we used Theorem 6.2.3 that $L(f) \leq U(f)$ for every bounded function $f$ on $[a,b]$ $\boxed{V}$ ■ $+$