MAS241 ANALYSIS 1 QUIZ 5

Problem 1. *(21 points) Prove or disprove the following statements. You should write the proof or counterexample. If your answer is wrong, there will be -3 points deduction. Note that we always assume the Euclidean space with Euxlidean metric.*

(1) *If $\{C_k\}$ is a sequence of compact, nonempty subsets of $\mathbb{R}^n$ and satisfies $C_k \supseteq C_{k+1}$ for each k, then $\bigcap_{n=1}^{\infty} C_k = \{x_0\}$ for some point $x_0 \in \mathbb{R}^n$.*

(2) *Let f be continuous on $[a,b] \subset \mathbb{R}$. Define $g(x)$ on $[a,b]$ as follows: $g(a) = f(a)$ and $g(x) = \inf\{f(y) : y \in [a,x]\}$ for $x \in (a,b]$. Then, g is monotone decreasing and continuous on $[a,b]$.*

(3) *Let S be a compact subset of $\mathbb{R}^n$ and $\{C_k\}$ be a sequence of closed subsets of $\mathbb{R}^n$ which satisfies $\bigcap_{n=1}^{\infty} C_k = \emptyset$. Then, there exists a finite index set $A \subset \mathbb{N}$ which satisfies $S \cap \bigcap_{\alpha \in A} C_\alpha = \emptyset$.*

Solution 1. (1) *(Disprove) The counter example is $C_k = [-\frac{1}{k}, 1 + \frac{1}{k}]$, which is a nested sequence and $\bigcap_{n=1}^{\infty} C_k = [0,1]$. The correct statement is $\bigcap_{n=1}^{\infty} C_k \neq \emptyset$, which is Cantor's Criterion, Theorem 2.3.3.*

(2) *(Prove) If $x_1 \le x_2$, then $[a,x_1] \subseteq [a,x_2]$ and from the definition, we can get $g(x_2) \le g(x_1)$. So, it is monotone decreasing. Also, note that $g(x) \le f(x)$ for all $x \in [a,b]$.*

To show the continuity, let x_0 be any point on $[a,b]$. Since f is continuous, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Take same δ for continuity of function $g(x)$.

If $x_0 < x < x_0 + \delta$, then

$$g(x) = \inf\{f(y) : y \in [a,x]\} = \min(g(x_0), \inf\{f(y) : y \in [x_0,x]\}).$$

From monotone decreasing property, we have $g(x) \le g(x_0)$. For any $y \in [x_0,x]$, we already know that $|f(y) - f(x_0)| < \epsilon$ and we have

$$\inf\{f(y) : y \in [x_0,x]\} \ge f(x_0) - \epsilon \ge g(x_0) - \epsilon.$$

Finally, we get $g(x_0) - \epsilon \le g(x) \le g(x_0)$ or $|g(x) - g(x_0)| < \epsilon$.

If $x_0 - \delta < x < x_0$, we may apply the same argument for $g(x_0)$, and we get same results, $g(x_0) \le g(x) \le g(x_0) + \epsilon$ or $|g(x) - g(x_0)| < \epsilon$. Thus, $g(x)$ is continuous on arbitrary point $x_0 \in [a,b]$ and g is continuous.

(3) *(Prove) Let define the sequence of open sets $D_k = C_k^c$. Since $\bigcap_{n=1}^{\infty} C_k = \emptyset$, we have $\bigcup_{n=1}^{\infty} D_k = \mathbb{R}^n$. Thus, $\{D_k\}$ is a countable open cover of compact set S, we can find a finite index set $A \subset \mathbb{N}$ such that $S \subset \bigcup_{\alpha \in A} D_\alpha$. So, $S \cap (\bigcup_{\alpha \in A} D_\alpha)^c = \emptyset$ and finally, we have $S \cap \bigcap_{\alpha \in A} C_\alpha = \emptyset$.*

$\square$

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Problem 2. *(9 points) For $x \in \mathbb{R}$, let define function $f(x) = x$ if x is rational and $f(x) = -x$ if x is irrational. Show that f is continuous at only 1 point and discontinuous at others points.*

Solution 2. *The continuous point is $x = 0$. For any $\epsilon > 0$, take $\delta = \epsilon$. Then, for all x with $|x - 0| < \delta$, we have*

$$|f(x) - f(0)| = |f(x)| = |x| < \delta = \epsilon.$$

So, it completes the proof of continuity of $x = 0$.

If $x \neq 0$, let $|x| = a > 0$ and take $\epsilon = \frac{a}{2}$. But for any neighborhood $N'(x)$, there exist at least one rational point and irrational point. If x is rational, take the point $x' \in N'(x)$ as an irrational point and we have

$$|f(x) - f(x')| = 2a > \epsilon.$$

It means, there is no neighborhood $N'(x)$ which satisfies $N'(x) \subset f^{-1}(N(f(x), \epsilon))$.

If x is irrational, take the point $x' \in N'(x)$ as a rational point and from the same argument, we get $f(x)$ is discontinuous at all $x \neq 0$. $\square$