Analysis I - Homework Assignment 6

Problem 6.3

Note that

$$h_1 = max(f,g) = max(f-g,0) + g = \frac{|f-g| + (f-g)}{2} + g = \frac{|f-g| + (f+g)}{2}$$

and

$$h_2 = min(f,g) = min(f-g,0) + g = \frac{(f-g) - |f-g|}{2} + g = \frac{(f+g) - |f-g|}{2},$$

because $max(a,0) = \frac{|a|+a}{2}$ and $min(a,0) = \frac{a-|a|}{2}$ for any function $a(x)$.

Since f and g are integrable on $[a,b]$, $f-g$ and $f+g$ are also integrable on $[a,b]$. Then, according to Theorem 6.2.5, $|f-g|$ is also integrable on $[a,b] \implies h_1$ and h_2 are integrable on $[a,b]$.

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Problem 6.6

Since f is integrable, Riemann's condition holds for f. Let ϵ be any positive real number. There exists a partition π_0 of $[a,b]$ such that, for every refinement $\pi = (a = x_0, x_1, \dots, x_{p-1}, x_p = b)$ of π_0, $U(f,\pi) - L(f,\pi) < \frac{\epsilon^2}{b-a} \implies$

$$\sum_{j=1}^{p} (M_j - m_j)\Delta x_j < \frac{\epsilon^2}{b-a} \implies ,$$

where $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$ and $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$, for $j = 1, \dots, p \implies$

$\sqrt{M_j} = \sup\{\sqrt{f(x)} : x \in [x_{j-1}, x_j]\}$ and $\sqrt{m_j} = \inf\{\sqrt{f(x)} : x \in [x_{j-1}, x_j]\} \implies$

$$U(\sqrt{f},\pi) - L(\sqrt{f},\pi) = \sum_{j=1}^{p} (\sqrt{M_j} - \sqrt{m_j})\Delta x_j \implies$$

$$U(\sqrt{f},\pi) - L(\sqrt{f},\pi) = \sum_{j=1}^{p} (\sqrt{M_j} - \sqrt{m_j})\Delta x_j \le$$

$$\le \sqrt{\sum_{j=1}^{p} \{M_j - m_j)\Delta x_j\} \{\sum_{j=1}^{p} (\Delta x_j\}} < \sqrt{\frac{\epsilon^2 (b-a)}{b-a}} = \epsilon,$$

using the Cauchy-Schwarz inequality and the fact that $\sum_{j=1}^{p} \Delta x_j = b-a$. So, for any refinement π of the partition π_0,

$$U(\sqrt{f},\pi) - L(\sqrt{f},\pi) < \epsilon$$

holds, which means that Riemann's condition is satisfied for $\sqrt{f} \implies \sqrt{f}$ is integrable on $[a,b]$.

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Problem 6.22

Clearly, $f^+(x) = max(f,0) = \frac{|f|+f}{2}$ and $f^-(x) = max(-f,0) = \frac{|f|-f}{2}$. As f is integrable on $[a,b]$, $|f|$ is also integrable $\implies f^+(x)$ and $f^-(x)$ are also integrable on $[a,b]$. Moreover, $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$. Using the linearity property of the Riemann integral, we deduce that

$$\int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx$$

and

$$\int_a^b |f(x)|dx = \int_a^b f^+(x)dx + \int_a^b f^-(x)dx.$$

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Problem 6.37

Since f is integrable on $[a,b]$, Riemann's condition is satisfied for f on $[a,b]$. There exists a partition $\pi = (x_0 = a, x_1, \ldots, x_{p-1}, x_p = b)$ such that $U(f,\pi) - L(f,\pi) = \sum_{i=1}^{p}(M_j - m_j)\Delta x_j < \epsilon$, where $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$ and $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$, for $j = 1, \ldots, p$. Let g and h be step functions on $[a,b]$ defined as follows:

$g(x) = \begin{cases} m_j, \text{if} x \in [x_{j-1}, x_j] \end{cases}$ and $h(x) = \begin{cases} M_j, \text{if} x \in [x_{j-1}, x_j] \end{cases}$.

It is obvious that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a,b]$. So $0 \leq \int_a^b [f(x) - g(x)]dx$ and $0 \leq \int_a^b [h(x) - f(x)]dx$. Furthermore,

$$\int_a^b f(x) - g(x)]dx < \int_a^b h(x) - g(x)]dx = \sum_{j=1}^{p}(M_j - m_j)\Delta x_j < \epsilon$$

and

$$\int_a^b h(x) - f(x)]dx < \int_a^b h(x) - g(x)]dx = \sum_{j=1}^{p}(M_j - m_j)\Delta x_j < \epsilon.$$

Thus $g(x)$ and $h(x)$ satisfy the problem conditions.

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Problem 6.45

Let t be any real number, and define $m = \int_a^b f^2(x)dx$, $n = \int_a^b g^2(x)dx$, and $k = \int_a^b f(x)g(x)dx$. Let $a(t) = \int_a^b [tf(x) + g(x)]^2 dx$. Then

$$0 \leq a(t) = \int_a^b [t^2 f^2(x) + 2tf(x)g(x) + g^2(x)]dx = mt^2 + 2tk + n.$$

So, $mt^2 + 2tk + n \geq 0$ for every real number $t \implies$ the discriminant $\Delta = (2k)^2 - 4mn = 4k^2 - 4mn$ should be nonpositive $\implies k^2 \leq mn \implies$

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \left[\int_a^b f^2(x)dx\right]\left[\int_a^b g^2(x)dx\right],$$

as desired.

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Problem 6.65

For any $h > 0$ such that $c + h \leq b$, one has

$$\frac{F(c+h) - F(c)}{h} = \frac{\int_a^{c+h} f(t)dt - \int_a^c f(t)dt}{h} = \frac{\int_c^{c+h} f(t)dt}{h} = \frac{hf(c')}{h} = f(c'),$$

using the fact that $\int_c^{c+h} f(t)dt = hf(c')$ for some real number $c' \in [c, c+h]$.

Observe that, when h goes to 0, c', which is trapped between c and $c+h$, goes to c.

Similarly, for any $h < 0$ such that $c + h \geq a$, one has

$$\frac{F(c+h) - F(c)}{h} = \frac{\int_a^{c+h} f(t)dt - \int_a^c f(t)dt}{h} = \frac{\int_c^{c+h} f(t)dt}{h} = \frac{\int_{c+h}^{c} f(t)dt}{-h} = \frac{-hf(c")}{-h} = f(c"),$$

where the existence of real number $c" \in [c+h, c]$ such that $\int_{c+h}^{c} f(t)dt = -hf(c")$ has been taken into account. When h goes to 0, $c"$, which is trapped between $c+h$ and c, goes to c. Then

$$F'(c^+) = \lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = \lim_{c' \to c^+} f(c') = f^+(c)$$

and

$$F'(c^-) = \lim_{h \to 0^-} \frac{F(c+h) - F(c)}{h} = \lim_{c" \to c^-} f(c") = f^-(c),$$

since f is monotone increasing. We are given that $f^+(c) \neq f^-(c)$, hence $F'(c^+) \neq F'(c^-) \implies$ F is not differentiable at c.

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Problem 6.88

According to Taylor's theorem with integral remainder,

$$f(x) = p_k(x) + \frac{1}{k!} \int_{x_0}^{x} (x-t)^k f^{(k+1)}(t)dt,$$

where $p_k(x)$ is the kth Taylor polynomial of f about x_0. Let $f(t) = (x-t)^k f^{(k+1)}(t)$ and $g(t) = 1$. Since f is continuous (it is a product of continuous functions) and g is non-negative and integrable on $[a, b]$, we can apply Theorem 6.3.1 (we can assume, without loss of generality, that $x_0 < x$) :

$$\int_{x_0}^{x} f(t)g(t)dt = f(c) \int_{x_0}^{x} g(t)dt$$

for some number $c \in [x_0, x] \implies$

$$\int_{x_0}^{x} (x-t)^k f^{(k+1)}(t)dt = (x-c)^k f^{(k+1)}(c)(x-x_0) \implies$$

$$f(x) = p_k(x) + \frac{(x-c)^k f^{(k+1)}(c)(x-x_0)}{k!},$$

as required.

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Problem 6.91

Taylor's theorem with integral remainder states that

$$f(x) = p_k(x) + \frac{1}{k!}\int_{x_0}^{x}(x-t)^k f^{(k+1)}(t)dt,$$

where $p_k(x)$ is the kth Taylor polynomial of f about x_0. Let $f(t) = (x-t)^{k-q}f^{(k+1)})(t)$ and $g(t) = (x-t)^q$, where q is any integer between 0 and k. Then $f(t)$ is continuous on $[a,b]$, and hence on $[x_0, x]$ (we are assuming without loss of generality that $x \geq x_0$), as it is a product of continuous functions. $g(t)$, on the other hand, is non-negative and continuous on $[x_0, x] \implies$ it is also integrable on $[x_0, x]$. Therefore Theorem 6.3.1 can be applied to f and g:

$$\int_{x_0}^{x} f(t)g(t)dt = f(c)\int_{x_0}^{x} g(t)dt$$

for some number $c \in [x_0, x] \implies$

$$\int_{x_0}^{x}(x-t)^k f^{(k+1)}(t) = (x-c)^{k-q}f^{(k+1)}(c)\int_{x_0}^{x}(x-t)^q dt =$$

$$= (x-c)^{k-q}f^{(k+1)}(c)\left[-\frac{(x-t)^{q+1}}{q+1}\right]\Bigg|_{x_0}^{x} = (x-c)^{k-q}f^{(k+1)}(c)\frac{(x-x_0)^{q+1}}{q+1}.$$

Therefore

$$f(x) = p_k(x) + \frac{(x-c)^{k-q}f^{(k+1)}(c)(x-x_0)^{q+1}}{(q+1)k!},$$

as required.

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Problem 6.59

Since $|f(a)| \leq K\int_a^a |f(t)|dt = 0$, $f(a) = 0$. Let $x \in (a, b]$ be any real number. For any $x' \in [a, x]$ the following inequality holds:

$$|f(x') \leq K\int_a^{x'} |f(t)|dt \leq K\int_a^x |f(t)|dt \implies$$

$$\max\{f(x') : x' \in [a,x]\} \leq K\int_a^x |f(t)|dt \implies .$$

Suppose that $\int_a^x |f(t)|dt > 0$. Then

$$\frac{\max\{f(x') : x' \in [a,x]\}}{\int_a^x |f(t)|dt} \leq K \implies$$

$$K \geq \frac{\max\{f(x') : x' \in [a,x]\}}{\int_a^x |f(t)|dt} \geq \frac{\max\{f(x') : x' \in [a,x]\}}{(x-a)\sup\{f(x') : x' \in [a,x]\}} \geq$$

$$\geq \frac{\max\{f(x') : x' \in [a,x]\}}{(x-a)\max\{f(x') : x' \in [a,x]\}} = \frac{1}{x-a}.$$

This is not possible for $x < \frac{1}{K} + a$. Therefore $\int_a^x |f(t)|dt = 0$ for all $x < \frac{1}{K} + a$. We deduce from theorem 6.2.9 that $|f(x)| = 0$ for every $x < \frac{1}{K} + a \implies f(x) = 0$ for every $x < \frac{1}{K} + a$.

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Problem 6.82

It is easy to check that $S_1(x) = x$, $C_2(x) = 1 + \int_0^x t dt = 1 + \frac{x^2}{2}$, $S_2(x) = \int_0^x C_2(t)dt = \int_0^x (1 + \frac{t^2}{2})dt = x + \frac{x^3}{6}$, $C_3(x) = 1 + \int_0^x S_2(t)dt = 1 + \int_0^x (t + \frac{t^3}{6})dt = 1 + \frac{x^2}{2} + \frac{x^4}{24}$.

We can prove by induction that

$$S_k(x) = \sum_{j=1}^{k} \frac{x^{2j-1}}{(2j-1)!}$$

and

$$C_k(x) = \sum_{j=1}^{k} \frac{x^{2(j-1)}}{(2(j-1))!}.$$