1 Prove that (a) $\iff$ (b).

4 points

(a) Archimedes' principle holds.

(b) For any $c > 0$, there exists some k in $\mathbb{N}$ such that $k - 1 \leq c < k$.

Solution. ($\Rightarrow$)
Let $c > 0$ be given arbitrarily. Set

$$A = \{k \in \mathbb{N} \,|\, c < k\}.$$

We apply Archimedes' principle to $M = c$ and $\epsilon = 1$, which implies $A \neq \emptyset$. The last inequality holds for $k = \min A$. **(+2 points)**

($\Leftarrow$)
Let ϵ and M be any two positive real numbers. Then, there exists some k in $\mathbb{N}$ such that $\frac{M}{\epsilon} < k$. In other words, $M < k\epsilon$. **(+2 points)** $\square$

2 Let $S = \left\{ \sum_{k=1}^{n} \frac{1}{(k!)^2} \,\middle|\, n \in \mathbb{N} \right\}$

6 points

(a) Explain why $\sup S$ exists in $\mathbb{R}$.

(b) Prove that $\sup S$ is a limit point of S.

Solution. (a) For any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{(k!)^2} \leq \sum_{k=1}^{n} \frac{1}{(2^{k-1})^2} \quad (\because\ 2^{k-1} \leq k! \text{ for any } k = 1, 2, \cdots)$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{(2^{k-1})^2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

Therefore, $\frac{4}{3}$ is an upper bound of S. **(+2 points)** Because $\mathbb{R}$ is complete, $\sup S$ exists in $\mathbb{R}$. □

(b) Let $m = \sup S$. Choose any $\epsilon > 0$. We need to show that

$$S \cap N'(m; \epsilon) \neq \emptyset.$$

Because $m = \sup S$,

$$\exists\, x \in S \quad \text{such that} \quad m - \epsilon < x \leq m. \ \textbf{(+2 points)}$$

Fix any such x. Since $x \in S$, x is of the form

$$\sum_{k=1}^{n} \frac{1}{(k!)^2}.$$

We prove $x \neq m$ by contradiction. Suppose $x = m$ and let $y = x + \frac{1}{((n+1)!)^2}$. Then,

$$y \in S \quad \text{and} \quad y > x = m,$$

which contradicts that $m = \sup S$. Therefore, $x \in S \cap N'(m; \epsilon)$. **(+2 points)** □

Remark.

A lot of mistakes.

1. $\sup S - \epsilon < x_{k_0} \le \sup S$. Not $\sup S - \epsilon < x_{k_0} < \sup S$. It is sometimes a big deal.

2. It can happen $N(\sup S, \epsilon) \cap S \neq \emptyset$ and $N'(\sup S, \epsilon) \cap S = \emptyset$.

3. You need to show that $\sup S \notin S$.

4. That is not true if $\sup S$ is not a limit point implies that $\sup S - \epsilon$ is an upper bound. Consider that the case $\sup S \in S$ and $N'(\sup S, \epsilon) \cap S = \emptyset$ then $\sup S - \epsilon$ is not an upper bound