ANALYSIS I. FINAL EXAM. 13:20–15:20 JUNE 15, 2021

NAME: ID#: SCORE: / 110

Guidelines for the exam:

(1) Make answers short and your point clear.
(2) You are allowed to use books and notes. However, discussion is not.
(3) Zoom should be on all the time.
(4) You may use any theorem except when you are asked to prove it. However, check conditions when you use a theorem.
(5) Exam ends at 15:20. Scan your exam and upload it by 15:30 (if you have trouble with KLMS, submit your exam in e-mail, hykim0615@kaist.ac.kr).

(1) Let f be infinite times differentiable.
 (a) (4pts) Use L'Hopital's rule to show that $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$.
 (b) (6pts) Use Taylor's theorem and find the convergence order of the above convergence. (Find a largest possible integer $\alpha > 0$ such that $\frac{f(x+h)+f(x-h)-2f(x)}{h^2} - f''(x) = O(h^\alpha)$ as $h \to 0$.)
 (Lesson: L'Hopital's rule gives convenience and Taylor's theorem gives detail.)

(2) (a) (5pts) Show that the total variation $V(\sin x; 0, 2\pi) = 4$. (Use Definition 5.3.2.)
 (b) (5pts) Show that quotient $\dfrac{f}{g}$ is in $BV(a,b)$ if f and g are uniformly continuous and have no zero on $[a,b]$. (You may use theorems in Section 5.3.)

(3) Prove or disprove.
 (a) If f is continuous on a compact set $[a,b]$, then $f \in BV(a,b)$ (of bounded variation).
 (b) If f is continuous on a compact set $[a,b]$, then $f \in R[a,b]$ (of Riemann integrable).

(4) Let $L(f)$ and $U(f)$ be the lower and upper Riemann integrals, respectively. Let f and g be bounded functions on $[a,b]$. Let

$$A := L(f+g), \quad B := L(f) + L(g), \quad C := U(f) + U(g), \quad D := U(f+g).$$

 (a) (4pts) Order them in size. (for example $A \le B \le C \le D$)
 (b) (2 pts each) Show the three inequalities in part (a).

(5) Prove or disprove.
 (a) Let $f_n \in R[a,b]$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$. Then, $f \in R[a,b]$.
 (b) Let π_1, π_2 are two partitions of an interval $[a,b]$. Then, for any bounded function f, $L(f,\pi_1) \le U(f,\pi_2)$ (L and U are lower and upper Riemann sums).

(6) Let $f_k = \frac{kx}{1+kx}$ for $x \in [0,1]$ and $k = 1, 2, \cdots$. Answer the followings and explain why.
 (a) (3pts) Find a function f_0 such that $f_k(x) \to f_0(x)$ for all $x \in [0,1]$ as $k \to \infty$.
 (b) (3pts) Determine whether the convergence is uniform.
 (c) (4pts) Determine whether $\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 \left(\lim_{k\to\infty} f_k(x)\right)dx$.

(7) Prove or disprove

(a) A function $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$ is integrable. Then, there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

(b) Let $f : [a, b] \to \mathbb{R}^+$ be a nonnegative continuous function. For $z \in [a, b]$, let $G(z)$ be the area bounded by the graph of $y = f(x)$, x-axis, $x = a$, and $x = z$. Then,

$$\int_a^b f(x)dx = G(b) - G(a).$$

(8) Suppose that $f : [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [a, b]$ for some $K > 0$.

(a) (2pts) Show that f is integrable.

(b) (8pts) Show that, for every natural number k,

$$\Big| \int_0^1 f(x)dx - \frac{1}{k}\sum_{j=1}^{k} f\Big(\frac{j}{k}\Big)\Big| \le \frac{K}{2k}.$$

(9) Let $g : [a, b] \to \mathbb{R}$ be monotone increasing and $f \in RS[g; a, b]$.

(a) (8pts) Show that $|f| \in RS[g; a, b]$ and

$$\Big| \int_a^b f(x)dg(x)\Big| \le \int_a^b |f(x)|dg(x).$$

(b) (2pts) What is the corresponding relation if $g : [a, b] \to \mathbb{R}$ is monotone decreasing.

(10) (a) (5pts) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f'(x)| < 10$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous (Find $\delta > 0$ for a given $\epsilon > 0$).

(b) (5pts) Prove or disprove that if f and g are bounded and $f + g$ is in $R(0, 1)$, the f and g are in $R(0, 1)$.

(11) Consider the following six statements:

p_1 : f is continuous on $[a, b]$, p_2 : f is uniformly continuous on $[a, b]$,

p_3 : f is differentiable on $[a, b]$, p_4 : f has an antiderivative on $[a, b]$,

p_5 : f is $R[a, b]$, p_6 : f is the indefinite integral of some $g \in R[a, b]$.

There are 30 possible statements in the form of $p_i \Rightarrow p_j$. Find true statements among them. (You don't need to explain why. A complete answer is for 10 points. -1 point for each missing true relation. -2 points for each false relation. The minimum score for this problem is 0. Hint: There are 16 true relations. You may simply write such as 1 $\Rightarrow$ 3,5,6 / 2 $\Rightarrow$ 1,3,6 / and so on.)