MAS242 ANALYSIS I QUIZ 1

Problem 1. *(15 points) Let S be a bounded infinite subset of $\mathbb{R}$.*
Prove that there exists a sequence of distinct points of S that converges to some point in $\mathbb{R}$

Proof. Bolzano-Wierestrass theorem guarantees that there exists at least one limit point of S in $\mathbb{R}$.

Let x be a limit point of S.
$\implies$ Every $\delta > 0$, there exists $y \in S$ such that $0 < |x - y| < \delta$.

Choose $x_1 \in S$ such as $0 < |x - x_1| < 1$ and inductively, choose $x_{i+1} \in S$ such as $0 < |x - x_{i+1}| < \frac{|x-x_i|}{2}$.
Then $\{x_n\}$ is a sequence of distinct points of S.

Given any $\epsilon > 0$, choose $N > \log \frac{1}{\epsilon}$.
$\implies$ For all $n > N$, $0 < |x - x_n| < \frac{1}{2^N} < \epsilon$.
$\implies$ $\{x_n\}$ converges to x. □

Problem 2. *(15 points) Prove or disprove following statements.*

(1) *Any bounded sequence which has unique limit point converges in the domain $\mathbb{R}$.*

(2) *There exists bounded convergent sequence which has two limit points in the domain $\mathbb{R}$.*

Proof. (1) Consider a sequence such as $x_{2n} = 1$ and $x_{2n-1} = \frac{1}{2n-1}$ for all $0 < n$. Then $\{x_n\}$ has unique limit point 0 as a set.(1 is a cluster point of a sequence but not a limit point of a set.)

However the sequence $\{x_n\}$ does not converge.
False.

(Solution for cluster point instead of limit point)

Let x be the unique cluster point of a sequence.
Suppose $\{x_n\}$ does not converge to x.
Then for some $\epsilon > 0$, there is infinite subset S of $\{x_n\}$ such that $\forall y \in S$, $|x - y| > \epsilon$. Since S is bounded infinite set in $\mathbb{R}$, there is a cluster point of S different from x. Contradiction.
True.

(2) In $\mathbb{R}$, any convergent sequence has only one cluster point.
False.

□

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