Group1 HW1

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Contribution details:

Unurtuvshin Javkhlantugs → 1.12, 1.15

GADISA SHANKO FIRISA → 1.4, 1.26 and merging files

Anar Rzayev → 1.35, 1.39, and 1.41

Pasawat Viboonsunti → 1.23, 1.39

Murad Aghazada → 1.8, 1.27

$\boxed{1.23}$ Suppose that $S \subseteq \mathbb{R}$ is bounded and infinite; $\mu = \sup S$

Is μ neccesarily a limit point of S?

Consider $S = [0,1] \cup \{2\}$ $\therefore$ smallest upperbound of S is 2 $\therefore$ $\mu = 2$

However, $N'(2,1) = \{x \in S \mid 0 < |x-2| < 1\} = \emptyset$ $\therefore$ μ is not a limit point

$\boxed{1.39}$ Prove that $\lim_{n\to\infty} x_n = 0 \longleftrightarrow \lim_{n\to\infty} |x_n| = 0$

($\to$) let $\varepsilon > 0$ $\therefore$ $\exists k_0 \in \mathbb{N}, \forall k \geq k_0 : x_k \in N(0,\varepsilon) = \{x \mid |x| < \varepsilon\}$

$\therefore$ $\forall k \geq k_0 : |x_k| < \varepsilon$ $\therefore$ $\forall k \geq k_0 : |x_k| \in N(0,\varepsilon)$

$\therefore$ $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0 : |x_k| \in N(0,\varepsilon)$ $\therefore$ $\lim_{n\to\infty} |x_n| = 0$

($\leftarrow$) let $\varepsilon > 0$ $\therefore$ $\exists k_0 \in \mathbb{N}, \forall k \geq k_0 : |x_k| \in N(0,\varepsilon) = \{x \mid |x| < \varepsilon\}$

$\therefore$ $\forall k \geq k_0 : |x_k| < \varepsilon$ $\therefore$ $\forall k \geq k_0 : |x_k| \in N(0,\varepsilon)$

$\therefore$ $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0 : x_k \in N(0,\varepsilon)$ $\therefore$ $\lim_{n\to\infty} x_n = 0$ ■

$\boxed{1.4}$ suppose S is a nonempty set of real numbers that is bounded above. Let $\mu = \sup S$. prove that μ is unique.

<u>proof</u>

suppose μ and ν are distinct numbers such that $\mu = \sup S$ and $\nu = \sup S$. Then, $x \leq \mu$ for all $x \in S$ and $\mu \leq M$ for any upper bound M of S. similarly, $x \leq \nu$ for all $x \in S$ and $\nu \leq N$ for any upper bound N of S. Now, since ν is a supremum of S, it must be an upper bound. so taking $M = \nu$, we get $\mu \leq \nu$. Similarly, since μ is a supremum, it must be an upper bound of S. Then, setting $N = \mu$, we get $\nu \leq \mu$. Combining the two inequalities $\mu \leq \nu$ and $\nu \leq \mu$, we deduce that $\mu = \nu$. Thus, $\mu = \sup S$ is unique.

$\boxed{1.26}$ prove that a nonempty finite set has no limit points.

<u>proof</u>

suppose $S = \{x_1, x_2, ---, x_n\}$. Let $d_i = |x_i - x|$ for $i \in [1, n]$ and let $S_1 = \{d_1, d_2, ---, d_n\}$. Since S_1 is bounded, it has an infimum by the completeness axiom. So let $d = \inf S_1$.
Now, take $\varepsilon < d$. Then, $N'(x; \varepsilon) \cap S = \phi$. Thus, S doesn't have a limit point.

1.12 $T \subseteq S$, $\inf S \le \inf T \le \sup T \le \sup S$

~~Assume $x \in S$ but $x \notin T$~~

Assume that $l = \inf T$, ~~also~~ if $\exists x \in S$ s.t. $x < l$ and $x = \inf S$, then $\inf S < \inf T$.

~~If there isn't such x exist~~

If there is no such x in S, then $\inf S = \inf T$ because $T \subseteq S$

Hence $\inf S \le \inf T$

Similarly assume $u = \sup T$, if $\exists x \in S$ s.t. $x > u$ and $x = \sup S$, then $\sup T < \sup S$. If there is no such x in S, then $\sup S = \sup T$ since $T \subseteq S$. ~~Either we~~ Therefore $\sup T \le \sup S$.

$\Rightarrow \inf S \le \inf T \le \sup T \le \sup S$

1.15 $S_1 + S_2 = \{x_1 + x_2 : x_i \text{ in } S_i, i = 1,2\}$, $\sup(S_1+S_2) = \sup S_1 + \sup S_2$, $\inf(S_1+S_2) = \inf S_1 + \inf S_2$

$x_1 \in S_1$, $x_2 \in S_2$, ~~by~~ by definition of infimum $\inf S \le x$ for all x in S. ~~$\exists x \in S$ s.t.~~

$\inf S \le x = x_1 + x_2$. ① ~~Assume that $\inf(S_1+S_2) \ne \inf S_1 + \inf S_2$~~.

~~$\inf S_1 + \inf S_2$~~ $m_1 = \inf S_1 \le x_1$ for all x_1 in S_1 $\rightarrow m_1 \in S_1$

$m_2 = \inf S_2 \le x_2$ for all x_2 in S_2 $\rightarrow m_2 \in S_2$

From ① $\inf S_1 + \inf S_2 \le x_1 + x_2$ for all x_1 in S_1 & x_2 in S_2 and $\inf S \le x = x_1 + x_2$. On one hand, the lowest value $x_1 + x_2$ can have is $\inf S$. On the other hand, the lowest value $x_1 + x_2$ can have is $\inf S_1 + \inf S_2$ for all x_1 in S_1 and for all x_2 in S_2. Therefore $\inf S_1 + \inf S_2 = \inf(S_1 + S_2)$. Similarly $\sup S_1 + \sup S_2 = \sup(S_1 + S_2)$

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$\boxed{1.8}$ Let's apply "Theorem 1.1.2" on real numbers $c+\sqrt{2}$ and $d+\sqrt{2}$.

So there is a rational number r satisfying $c+\sqrt{2} < r < d+\sqrt{2}$. Then, obviously $r-\sqrt{2}$ is irrational number between given real numbers c, d.

$\boxed{\text{Theorem 1.1.2:}}$ Between any two real numbers there is a rational number.

$\boxed{1.27}$ a) ~~Obviously, limit point x can't be equal t~~

Obviously, if x is limit point, $x>b$ and $x<a$ is impossible as we can take ε sufficiently small so that $S \cap N'(x,\varepsilon) = \emptyset$

For x, $a \le x \le b$, we already proved in $\boxed{1.8}$ that between any real numbers $c<d$, there is irrational number. So $N'(x,\varepsilon) \cap S \neq \emptyset$ for $\forall x \in [a,b]$

b) Answer is all real numbers. According to Theorem 1.1.2 there exists a rational number between any two distinct real numbers. So, for $\forall x \in R$, $N'(x,\varepsilon) \cap S \neq \emptyset$

$\boxed{1.27}$ c) Again applying $\boxed{1.8}$, we obtain answer is all real numbers

d) ~~[illegible]~~ I claim that if $c<d$ are real numbers there is $c<\frac{p}{2^k}<d$. (with $p\in\mathbb{Z}, k\in\mathbb{N}$)

Proof goes like this since $d-c>0$

by Archimedes principle $\exists k\in\mathbb{N}$ s.t $2^k(d-c)>1$

$$\Rightarrow 2^k\cdot d > 2^k\cdot c+1 \Rightarrow \exists p\in\mathbb{Z} \text{ s.t}$$

$$p\in(2^k\cdot c, 2^k\cdot d) \Rightarrow c<\frac{p}{2^k}<d.$$

It clearly means that answer is all real numbers.

$\boxed{1.27}$ c) Again applying $\boxed{1.8}$, we obtain answer is all real numbers.

d) Firstly, I claim that 0 is the only limit point of the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$. Obviously if $x \neq 0$ (and since $x \le 1$) $\exists m \in \mathbb{N}$ $\frac{1}{m+1} < x \le \frac{1}{m}$. Then choose $\varepsilon < \min(\frac{1}{m} - x, x - \frac{1}{m+1})$

$$\Rightarrow N'(x, \varepsilon) \cap \{\tfrac{1}{n} \mid n \in \mathbb{N}\} = \emptyset \Rightarrow \boxed{x = 0} \quad \square$$

Therefore taking $m = n$ in original set $S = \{\frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N}\}$. We get that 0 is limit point of original set. Now, fix n. let $\varepsilon > 0$, $\exists m > 0$

$m\varepsilon > 1 \Rightarrow \frac{1}{m} < \varepsilon \Rightarrow \frac{1}{n} + \frac{1}{m} \in N'(\frac{1}{n}, \varepsilon)$

So $\frac{1}{n}$ is also limit point of S.

Now, assume there is $x \neq \frac{1}{n}, 0 \to$ limit point of S. Obviously, $\exists \varepsilon > 0$ s.t $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cap (x - \varepsilon, x + \varepsilon) = \emptyset$ (since 0 is only limit point of $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $x \neq \frac{1}{n}$) $(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$ must have infinitely many common elements with S (since x is limit point)

$\boxed{1.27}$ d) (continued) let's pick any

let it be $\frac{1}{m}+\frac{1}{n}$ $\qquad x-\frac{\varepsilon}{2}<\frac{1}{m}+\frac{1}{n}$

Since there is no $\frac{1}{i}$ inside $(x-\varepsilon, x+\varepsilon)$

both $\frac{1}{n}$ and $\frac{1}{m}\leq x-\varepsilon$

Let $n\geq m$ $\Rightarrow \frac{1}{n}\leq\frac{1}{m}\leq x-\varepsilon=x-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}<$

$<\frac{1}{m}+\frac{1}{n}-\frac{\varepsilon}{2}$ $\quad\Rightarrow \frac{\varepsilon}{2}<\frac{1}{n}\Rightarrow m\leq n<\frac{2}{\varepsilon}$

Contradiction to $(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})\cap S$ having infinitely many elements.

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35) a) Since $\frac{1}{k}\ln k > 0$ for all $k \geq 2 \Rightarrow$

$e^{\frac{1}{k}\ln k} = (e^{\ln k})^{\frac{1}{k}} = k^{\frac{1}{k}} > e^0 = 1$, or equivalently

$\boxed{\forall k \geq 2 \Rightarrow k^{\frac{1}{k}} > 1}$ So, $\boxed{\exists y_k > 0,\ k^{\frac{1}{k}} = 1 + y_k}$ ✓

b) Since $(1+y_k)^k = (k^{\frac{1}{k}})^k = k$ and using binomial theorem

$k = 1 + \binom{k}{2}y_k^2 + \binom{k}{1}y_k + \ldots > 1 + \binom{k}{2}y_k^2 = 1 + \frac{k(k-1)}{2}y_k^2$, as

$\binom{k}{2} = \frac{k!}{2!(k-2)!} = \frac{(k-1)k}{2} \Rightarrow$ using the fact that $y_k > 0 \Rightarrow$

$\binom{k}{j}y_k^j > 0$ for $3 \leq j \leq k$ and $\boxed{k > 1 + \frac{1}{2}k(k-1)y_k^2 \text{ for } k \geq 2}$ ✓

c) $k-1 > \frac{1}{2}k(k-1)y_k^2$, and $k-1 \geq 1 \Rightarrow$ we can delete $(k-1)$ from both sides; $1 > \frac{1}{2}ky_k^2 \Rightarrow \frac{2}{k} > y_k^2$ and since $y_k > 0$ we find $\boxed{\sqrt{\frac{2}{k}} > y_k}$ ✓

d) Let $x_k = \sqrt{\frac{2}{k}}$, from the Archimedes' principle, we know $\exists t \in \mathbb{N},\ 2 < t\varepsilon^2$ for each $\varepsilon > 0$ ($M = 2$, $\varepsilon = \varepsilon^2$ in theorem)

Fix $\varepsilon > 0$, any real $\Rightarrow$ Since $\exists t \in \mathbb{N}$, $\frac{2}{t} < \varepsilon^2$ or just $\sqrt{\frac{2}{t}} < \varepsilon \Rightarrow$ Take $k_0 = t$, then for all $k > k_0 = t$

$|x_k - 0| = \left|\sqrt{\frac{2}{k}}\right| = \sqrt{\frac{2}{k}} < \sqrt{\frac{2}{t}} < \varepsilon$; meaning that

from the def of convergence, $\boxed{\lim_{k\to\infty} x_k = 0} \Rightarrow$

$\boxed{\lim_{k\to\infty} \sqrt{\frac{2}{k}} = 0}$ ✓

e) We have $1 < 1 + y_k = k^{\frac{1}{k}} < 1 + \sqrt{\frac{2}{k}}$, equivalently

$\underbrace{\lim_{k\to\infty}\left(1 + \sqrt{\frac{2}{k}}\right) = 1 + 0 = 1}$ with $\underbrace{\lim_{k\to\infty} 1 = 1}$. From the

squeeze play theorem, $1 < k^{\frac{1}{k}} < 1 + \sqrt{\frac{2}{k}} \Rightarrow \boxed{\lim_{k\to\infty} k^{\frac{1}{k}} = 1}$

39) If $\{x_k\}$ converges to zero, then $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N} \Rightarrow$

for $\forall k > k_0$, $|x_k - 0| = |x_k| < \varepsilon$. Since $|x_k| = ||x_k||$

$\Rightarrow \underbrace{\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k > k_0 \Rightarrow ||x_k|| < \varepsilon}$, or from def, it

means $\lim_{k\to\infty} |x_k| = 0$; $\boxed{\{|x_k|\} \text{ converges to zero}}$ when

$||x_k| - 0| = |x_k - 0| = |x_k| = ||x_k||$ ✓ Conversely, if we

had $\{|x_k|\}$ converging to zero, then $\underline{\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}}$

for $\underline{\forall k > k_0} \Rightarrow ||x_k| - 0| = \underline{||x_k||} < \varepsilon$. As $||x_k|| = \underline{|x_k| < \varepsilon}$

this is becoming

$|X_k| = |X_k - 0| < \varepsilon$ for $\forall \varepsilon > 0$, $\exists k_0 \in N$, meaning Anar Rzayev

(X_k) converges to zero ✓ Note: $a > 0$, $|a| = a = ||a|| = |a| = \cdots$

$a < 0 \Rightarrow |a| = -a \Rightarrow ||a|| = |-a| = -a = |a|$ ✓

41) From the def, $\forall \varepsilon > 0$, $N'(X_0; \varepsilon) \cap S \neq \emptyset$; Thus, for $n \in N$, set $\varepsilon_n = \frac{1}{n}$ and conclude $N'(X_0; \frac{1}{n}) \cap S \neq \emptyset$

From the axiom of choice, $\forall n \in N \Rightarrow$ we can select $s_n \in N'(X_0; \frac{1}{n}) \cap S \Rightarrow$ I claim (s_n) is the required sequence

(1) As $s_n \in N'(X_0; \frac{1}{n}) \cap S$ for all $n \in N$, we get that $\boxed{s_n \neq X_0}$ and $\boxed{s_n \in S}$ ✓

(2) Let $\varepsilon > 0$ be given. We have to show that there exist $k_0 \in N$ s.t. for all $n > k_0 \Rightarrow |s_n - X_0| < \varepsilon$

By construction, $s_n \in N'(X_0; \frac{1}{n})$ or implying $|s_n - X_0| < \frac{1}{n}$ for $\forall n \Rightarrow$ Choosing $k_0 \in N$ for which (Archimedes' principle) $1 < k_0 \varepsilon \Rightarrow \frac{1}{k_0} < \varepsilon$ and $n > k_0 \Rightarrow |s_n - X_0| < \frac{1}{n} < \frac{1}{k_0} < \varepsilon$

$\Rightarrow \boxed{|s_n - X_0| < \varepsilon}$ ✓ Hence, (s_n) satisfies ✓