Group1 HW3

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4) Since $||x||_\infty = \max(|x_1|, |x_2|)$, we can deduce that $\boxed{||x||_\infty \geq 0}$ as $|x_1|$ and $|x_2| \geq 0 \Rightarrow$ If we observe that $||x||_\infty = 0 \Leftrightarrow \max(|x_1|, |x_2|) = 0$, without loss of gener.

If $|x_1| = \max(|x_1|, |x_2|) = 0 \Leftrightarrow |x_2| \leq |x_1| = 0$ with $0 \leq |x_2| \leq |x_1| = 0 \Leftrightarrow x_1 = x_2 = 0 \Leftrightarrow x = (0,0) = 0$ ✓

If $|x_2| = \max(|x_1|, |x_2|) = 0 \Leftrightarrow 0 \leq |x_1| \leq |x_2| = 0 \Leftrightarrow x_1 = x_2 = 0 \Leftrightarrow x = (0,0) = 0$ ✓ Hence, $\boxed{||x||_\infty = 0 \Leftrightarrow x = 0}$

So, <u>positive definiteness</u> ✓

If $|x_1| \geq |x_2|$, $||x||_\infty = |x_1|$ and $||cx||_\infty = \max(|cx_1|, |cx_2|)$

Generally, $||cx||_\infty = \max(|cx_1|, |cx_2|) = |c| \max(|x_1|, |x_2|) = |c| \cdot ||x||_\infty \Rightarrow \boxed{||cx||_\infty = |c| \cdot ||x||_\infty}$ So, <u>absolute homogeneity</u> ✓

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \Rightarrow ||x+y||_\infty = \max(|x_1+y_1|, |x_2+y_2|)$, If $|x_1+y_1| \leq |x_2+y_2| \Rightarrow$
$||x+y||_\infty = |x_2+y_2| \leq |x_2| + |y_2| \leq \max(|x_1|, |x_2|) + \max(|x_2|, |y_2|) = ||x||_\infty + ||y||_\infty$, where we used the fact that

$|x_2| \leq \max(|x_1|, |x_2|)$ and $|y_2| \leq \max(|y_1|, |y_2|) \Rightarrow$ ✓

If $|x_1+y_1| > |x_2+y_2| \Rightarrow \|x+y\|_\infty = |x_1+y_1| \leq |x_1| + |y_1| \leq$
$\leq \max(|x_1|, |x_2|) + \max(|y_1|, |y_2|)$ where we used the triangle ineq. and $|x_1| \leq \max(|x_1|, |x_2|)$; $|y_1| \leq \max(|y_1|, |y_2|)$

Hence, $\boxed{\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty} \Rightarrow$ Subadditivity ✓

From these, we deduce $\boxed{\|\cdot\|_\infty \text{ is a norm in } \mathbb{R}^2}$ ☒

7) $(\|x+y\| + \|y\|)^2 = \|x+y\|^2 + \|y\|^2 + 2 \cdot \|x+y\| \cdot \|y\| =$
$= \langle x+y, x+y \rangle + \langle y, y \rangle + 2 \cdot \|x+y\| \cdot \|y\| = \langle x, x \rangle + \langle y, y \rangle +$
$+ 2\langle x, y \rangle + \langle y, y \rangle + 2 \cdot \|x+y\| \cdot \|y\| = \langle x, x \rangle + 2\langle x, y \rangle + 2\langle y, y \rangle$
$+ 2\|x+y\| \cdot \|y\| \overset{?}{\geq} (\|x\|)^2 = \langle x, x \rangle \Leftrightarrow \langle x, y \rangle + \langle y, y \rangle +$
$+ \|x+y\| \cdot \|y\| \overset{?}{\geq} 0 \Leftrightarrow \|x+y\| \cdot \|y\| \overset{?}{\geq} -(\langle x, y \rangle + \langle y, y \rangle)$
$\Leftrightarrow \|x+y\| \cdot \|y\| \overset{?}{\geq} -\langle x+y, y \rangle \Leftrightarrow \langle x+y, y \rangle \overset{?}{\geq} -\|x+y\| \cdot \|y\|$

This is true from the Cauchy-Schwartz, and since we used famous theorems, we eventually proved that $(\|x+y\| + \|y\|)^2 \geq (\|x\|)^2$, since expressions inside brackets are positive ($\|x\| \geq 0$)

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We have $\underline{\|x+y\|+\|y\| \geq \|x\|}$ or $\boxed{\|x+y\| \geq \|x\|-\|y\|}$ ⊠ ✓

20) We have to prove $\bigcup C_k = (0,3)$, where $C_k = \left[\frac{1}{k}, 3-\frac{1}{k}\right]$

Let $x \in (0,3)$ be an arbitrary point. Since $0 < x < 3 \Rightarrow 3-x > 0$

There exist large $k \in \mathbb{N}$ such that $k > \max\left(\frac{1}{x}, \frac{1}{3-x}\right)$

Thus, $\boxed{\frac{1}{x} < k}$ and $k > \frac{1}{3-x} \Rightarrow 3k - xk > 1$, $3 - x > \frac{1}{k}$ or

$\boxed{3-\frac{1}{k} > x}$ Hence, $\boxed{3-\frac{1}{k} > x > \frac{1}{k}}$ ($xk > 1$ implies $x > \frac{1}{k}$)

and this means $\boxed{x \in C_k}$ ⊠ So, we proved that if $x \in (0,3)$

is an arbitrary point, then $x \in C_k$ where $k > \max\left(\frac{1}{x}, \frac{1}{3-x}\right)$

and this implies $\boxed{x \in \bigcup C_k}$ ⊠ Now, let $\underline{x \in \bigcup C_k}$ be an

arbitrary point; then, there exist i such that $\underline{x \in C_i}$

So, $\frac{1}{i} \leq x \leq 3 - \frac{1}{i}$ and $0 < \frac{1}{i} \leq x \leq 3 - \frac{1}{i} < 3$; so,

$\boxed{x \in (0,3)}$ ⊠ For any $x \in \bigcup C_k$, we proved $x \in (0,3)$ ✓

Eventually, this concludes that $\boxed{\bigcup C_k = (0,3)}$ ⊠ ✓

21) Let's take the point $\underline{(0,0,0)}$, which is contained

on S as $\underline{r^2 > 0^2 + 0^2 = 0}$ ✓ If we take a neighborhood

centered at origin with fixed $\varepsilon > 0$, $N(0, \varepsilon)$ contains a

point which is not included on S. ($(0,0,c) \notin S$ as $c \neq 0$)
Specifically, take $(0,0,c)$ where $c < \varepsilon$. $\|(0,0,c)-(0,0,0)\|$

$= \sqrt{c^2} = c < \varepsilon$, with $c > 0$; hence, $(0,0,0)$ is not an interior point $\Rightarrow$ $\boxed{S \text{ is not open in } \mathbb{R}^3}$ ✓

24) Let $R = \|x_1 - x_2\| > 0$ be their distance. Consider the open neighbourhoods $U_i = \{x \in \mathbb{R}^n \mid \|x - x_i\| < \frac{R}{2}\}$ for $i = 1, 2$. Clearly, U_1 and U_2 are disjoint sets, in which they are open sets (easily understood from Definition)
As $x_1 \in U_1$ and $x_2 \in U_2$, we are done ✓

2.4 For $x=(x_1,x_2)\in\mathbb{R}^2$, let $\|x\|_\infty = \max(|x_1|,|x_2|)$

1) $\|x\|_\infty \geq |x_1|, |x_2|$ $\therefore \|x\|_\infty \geq 0 \quad \forall x\in\mathbb{R}^2$

2) $cx = (cx_1, cx_2)$ $\therefore \|cx\|_\infty = \max(|cx_1|,|cx_2|)$

$$= \begin{cases} |cx_1| ; & |x_1|\geq|x_2| \\ |cx_2| ; & |x_1|<|x_2| \end{cases}$$

$$= \begin{cases} c\max(|x_1|,|x_2|) ; & |x_1|\geq|x_2| \\ c\max(|x_1|,|x_2|) ; & |x_1|<|x_2| \end{cases}$$

$$= c\|x\|_\infty$$

3) Let $y=(y_1,y_2)$, $x=(x_1,x_2)$ $\therefore \|y\|_\infty \geq |y_1|, |y_2|$, $\|x\|_\infty \geq |x_1|, |x_2|$

$\therefore \|y\|_\infty + \|x\|_\infty \geq |y_1|+|x_1|, |y_2|+|x_2|$;

Since $|y_1|+|x_1| \geq |x_1+y_1|$ and $|y_2|+|x_2| \geq |x_2+y_2|$

$\therefore \|y\|_\infty + \|x\|_\infty \geq \max(|x_1+y_1|, |x_2+y_2|) = \|x+y\|_\infty$

$\therefore$ $(\|\ \|_\infty)$ is a norm

2.7 Since $(\|\ \|)$ is a norm, $\|x+y\|+\|y\| = \|x+y\|+\|-y\| \geq \|x+y-y\| = \|x\|$

(From subadditivity) $\therefore \|x+y\| \geq \|x\| - \|y\|$

$\boxed{2.19}$ a) Let $x_k = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$ be converging sequence in R^n. Assume it converges to

$$x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$$

By Theorem 2.1.7 for $\forall j \in [1,n]$ $x_j^{(k)}$ converges to $x_j^{(0)}$

Since converging sequences in R have unique limit $x_j^{(0)}$ is unique for $\forall j = 1, \dots, n$.

So x_0 is uniquely defined. ■

(b) let $x_k = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$ converge to $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ $\Rightarrow$ $x_j^{(k)}$ converges to $x_j^{(0)}$ for $\forall j = 1, \dots, n$. Since they are in R, they all are bounded $|x_j^{(k)}| \le M_j$

$$\|x_k\| = \sqrt{(x_1^{(k)})^2 + \dots + (x_n^{(k)})^2} \le |x_1^{(k)}| + \dots + |x_n^{(k)}| \le M_1 + \dots + M_n$$ ■

(c) Assume x_n converges to x_0 and let $\varepsilon > 0$.

$\exists k_0$ $\|x_k - x_0\| < \frac{\varepsilon}{2}$ $\Rightarrow$ $\|x_k - x_0 + x_0 - x_m\| \le \|x_k - x_0\| + \|x_m - x_0\|$

$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $\forall k, m > k_0$ ■

$\boxed{2.17}$ Since x_0 is limit point of set S

$$\exists x_{i_1} \in S \quad \|x_0 - x_{i_1}\| < 1 \leftarrow N'(x_0, 1) \Rightarrow x_{i_1} = y_1$$

$$\exists x_{i_2} \in S \quad \|x_0 - x_{i_2}\| < \frac{1}{2} \leftarrow N'(x_0, \tfrac{1}{2}) \Rightarrow x_{i_2} = y_2$$

$$N'(x_0, \tfrac{1}{3}) \Rightarrow x_{i_3} = y_3$$

$$\vdots$$

$$N'(x_0, \tfrac{1}{n}) \Rightarrow x_{i_n} = y_n$$

$$\vdots \qquad \vdots$$

We claim that ~~all~~ elements of the set $S' = \{y_1, y_2, y_3, \dots\}$ converge to x_0. Let $\varepsilon > 0$. By Archimedes' principle

$$\exists m \quad \forall i \geq m \quad i \cdot \varepsilon > 1 \Rightarrow \frac{1}{i} < \varepsilon \text{ for } \forall i \geq m$$

$$\Rightarrow \|x_0 - y_i\| < \frac{1}{i} < \varepsilon \text{ for } \forall i > m$$

So claim is proved and by 2.12(c) converging sequence is indeed Cauchy.

$\boxed{2.25}$ Let $T = \{x_k \mid k \in \mathbb{N}\}$ then $S = T \cup \{x_0\}$.

Obviously x_0 is limit point of S

Assume there is another one, let's say y_0

Then y_0 is limit point of T as well $\Rightarrow$

$\Rightarrow$ thus y_0 is cluster point of sequence $\{x_k\}$.

Then there is a subsequence converging to y_0, but since x_k converges to x_0, every subsequence should converge to $x_0 \Rightarrow x_0 = y_0$ $\therefore$ $\times$

Therefore it's closed.

Q.5

Let $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ and let

$$d_\infty(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_\infty .$$

(a). Show that d_∞ is a metric on $\mathbb{R}^2$.

(i). $d_\infty(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

since for $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $d(x_1, y_1) = |x_1 - y_1|$ and $d(x_2, y_2) = |x_2 - y_2|$ are metrics on $\mathbb{R}$, then

$|x_1 - y_1| \geq 0$ and $|x_2 - y_2| \geq 0$.

Thus, $d_\infty(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq 0$.

(ii). To show that $d_\infty(\vec{x}, \vec{y}) = d_\infty(\vec{y}, \vec{x})$, we have

$d_\infty(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ and

$d_\infty(\vec{y}, \vec{x}) = \max\{|y_1 - x_1|, |y_2 - x_2|\}$.

However, since $|x_1 - y_1|$ and $|x_2 - y_2|$ are metrics, we have

$d(x_1, y_1) = d(y_1, x_1) \Rightarrow |x_1 - y_1| = |y_1 - x_1|$ and

$d(x_2, y_2) = d(y_2, x_2) \Rightarrow |x_2 - y_2| = |y_2 - x_2|$.

$$\Rightarrow d_\infty(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

$$= \max\{|y_1 - x_1|, |y_2 - x_2|\}$$

$$d_\infty(\vec{x}, \vec{y}) = d_\infty(\vec{y}, \vec{x})$$

2.5

(a). continued.

Let $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, and $\vec{z} = (z_1, z_2)$.

Then, we will show that

$$d_\infty(\vec{x}, \vec{z}) \leq d_\infty(\vec{x}, \vec{y}) + d_\infty(\vec{y}, \vec{z}).$$

We know

$$d_\infty(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

$$d_\infty(\vec{y}, \vec{z}) = \max\{|y_1 - z_1|, |y_2 - z_2|\}$$

$$d_\infty(\vec{x}, \vec{z}) = \max\{|x_1 - z_1|, |x_2 - z_2|\}$$

$$\Rightarrow d_\infty(\vec{x}, \vec{y}) + d_\infty(\vec{y}, \vec{z}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\}$$

$$d_\infty(\vec{x}, \vec{y}) + d_\infty(\vec{y}, \vec{z}) = \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\}$$

since $d(\vec{x}, \vec{y})$, $d(\vec{y}, \vec{z})$, $d(\vec{x}, \vec{z})$ are metrics, we have

$|x_1 - y_1| + |y_1 - z_1| \geq |x_1 - z_1|$ and

$|x_2 - y_2| + |y_2 - z_2| \geq |x_2 - z_2|$. Then,

$$\begin{aligned} d_\infty(\vec{x}, \vec{y}) + d_\infty(\vec{y}, \vec{z}) &= \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\} \\ &\geq \max\{|x_1 - z_1|, |x_2 - z_2|\} \\ &\geq d_\infty(\vec{x}, \vec{z}) \end{aligned}$$

$$\Rightarrow d_\infty(\vec{x}, \vec{z}) \leq d_\infty(\vec{x}, \vec{y}) + d_\infty(\vec{y}, \vec{z})$$

$\boxed{2.12}$

(a). Let $\{\vec{x}_k\}$ be a sequence in $\mathbb{R}^n$. suppose $\{\vec{x}_k\}$ converges to $\vec{x}_1$ and $\vec{x}_2$ such that $\vec{x}_1 \neq \vec{x}_2$.

Then, there is k_1 such that whenever $k \geq k_1$, $\vec{x}_k \in N(\vec{x}_1; \frac{\varepsilon}{2})$ and there is k_2 such that $\vec{x}_k \in N(\vec{x}_2; \frac{\varepsilon}{2})$ for $k \geq k_2$.

Let $k_0 = \max\{k_1, k_2\}$. Then, for any $k \geq k_0$, $\vec{x}_k \in N(\vec{x}_1; \frac{\varepsilon}{2})$ and $\vec{x}_k \in N(\vec{x}_2; \frac{\varepsilon}{2})$.

So, for $k \geq k_0$:

$$\|\vec{x}_2 - \vec{x}_1\| = \|\vec{x}_2 - \vec{x}_k + \vec{x}_k - \vec{x}_1\|$$

$$\leq \|\vec{x}_2 - \vec{x}_k\| + |\vec{x}_k - \vec{x}_1\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$$

$$\Rightarrow \|\vec{x}_2 - \vec{x}_1| < \varepsilon.$$

$$\Rightarrow \|\vec{x}_2 - \vec{x}_1\| = 0$$

$\Rightarrow \vec{x}_1 = \vec{x}_2$ — a contradiction.

So, $\lim_{k \to \infty} \vec{x}_k$ is unique.

(b). Let $\{\vec{x}_k\}$ be a sequence in $\mathbb{R}^n$ such that $\lim_{k \to \infty} \vec{x}_k = \vec{x}_0$.

Then, for every $\varepsilon > 0$, there exists k_0 such that whenever $k \geq k_0$, $\vec{x}_k \in N(\vec{x}_0; \varepsilon)$. Let $\varepsilon = 1$. Then, for $k \geq k_0$,

$$\|\vec{x}_k\| = \|\vec{x}_k - \vec{x}_0 + \vec{x}_0\| \leq \|\vec{x}_k - \vec{x}_0\| + \|\vec{x}_0\|$$

$$\|\vec{x}_k\| < \varepsilon + \|\vec{x}_0\|$$

$$\Rightarrow \|\vec{x}_k\| < 1 + \|\vec{x}_0\| \text{ for all } k \geq k_0.$$

2.12

(b). continued

Now, consider $\|\vec{x}_1\|, \|\vec{x}_2\|, ---, \|\vec{x}_{k_0-1}\|, \|\vec{x}_0\|+1.$

Let $M = \max\{ \|\vec{x}_1\|, \|\vec{x}_2\|, ---, \|\vec{x}_{k_0-1}\|, 1+\|\vec{x}_0\| \}.$

Then, $\|\vec{x}_k\| \leq M$ for all $k \in \mathbb{N}$.

(c). Let $\{\vec{x}_k\}$ be a sequence in $\mathbb{R}^n$ such that $\lim_{k\to\infty} \vec{x}_k = \vec{x}_0$.

Then, for every $\varepsilon > 0$, there exists k_0 such that whenever $k \geq k_0$, $\|\vec{x}_k - \vec{x}_0\| < \varepsilon/2$

so, for $k, m \geq k_0$,

$$\|\vec{x}_k - \vec{x}_m\| = \|\vec{x}_k - \vec{x}_0 + \vec{x}_0 - \vec{x}_m\|$$

$$\leq \|\vec{x}_k - \vec{x}_0\| + |\vec{x}_0 - \vec{x}_m\|$$

$$\|\vec{x}_k - \vec{x}_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$$

$$\Rightarrow \|\vec{x}_k - \vec{x}_m\| < \varepsilon \text{ as required.}$$

2.24

Let $\vec{x}_1$ and $\vec{x}_2$ be two distinct points in $\mathbb{R}^n$ and let $\varepsilon = \|\vec{x}_1 - \vec{x}_2\|$.
Then, suppose $U_1 = N(\vec{x}_1 ; \frac{\varepsilon}{2})$ and $U_2 = N(\vec{x}_2 ; \varepsilon/2)$ be neighborhoods of $\vec{x}_1$ and $\vec{x}_2$ with radius $\varepsilon/2$, respectively.
Now, we claim that U_1 and U_2 are disjoint open sets such that $\vec{x}_1 \in U_1$ and $\vec{x}_2 \in U_2$.
Clearly, $\vec{x}_1 \in N(\vec{x}_1 ; \varepsilon/2) = U_1$ and $\vec{x}_2 \in N(\vec{x}_2 ; \varepsilon/2) = U_2$.
To show that $U_1 \cap U_2 = \phi$, let $\vec{x}_3 \in U_1 \cap U_2$.
Then, $\vec{x}_3 \in N(\vec{x}_1 ; \varepsilon/2)$ and $\vec{x}_3 \in N(\vec{x}_2 ; \varepsilon/2)$.
That is, $\|\vec{x}_1 - \vec{x}_3\| < \frac{\varepsilon}{2}$ and $\|\vec{x}_3 - \vec{x}_2\| < \varepsilon/2$.
Since $\varepsilon = \|\vec{x}_1 - \vec{x}_2\|$,

$$\varepsilon = \|\vec{x}_1 - \vec{x}_2\| = \|\vec{x}_1 - \vec{x}_3 + \vec{x}_3 - \vec{x}_2\|$$

$$\Rightarrow \quad \varepsilon \leq \|\vec{x}_1 - \vec{x}_3\| + \|\vec{x}_3 - \vec{x}_2\|$$

$$\Rightarrow \quad \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\Rightarrow \quad \varepsilon < \varepsilon$ – a contradiction.

So, $U_1 \cap U_2 = \phi$.