MAS241 Analysis 1 Quiz 9

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Problem 1. (18 points) For each of the following sentences, we assume f is Riemann integrable on $[a, b]$ and F is the function on $[a, b]$ defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Mark **True** or **False**. Then **justify your answers in one or two sentences**. (There are no extra minus points if your answer is opposite to the correct one.)

(a) (6 points) The function $x \mapsto f(x)F(x)$ is Riemann integrable on $[a, b]$.

(b) (6 points) If F is differentiable at $c \in (a, b)$, then f is continuous at c.

(c) (6 points) There exists $c \in [a, b]$ such that

$$\frac{F(b) - F(a)}{b - a} = f(c).$$

Solution. (a) **True**. By the fundamental theorem of calculus (p.253), F is continuous on $[a, b]$, so it is Riemann integrable on $[a, b]$ by Theorem 6.2.7 (p.248). By Theorem 6.2.5 (p.246), $x \mapsto f(x)F(x)$ is Riemann integrable on $[a, b]$.

(b) **False**. Consider, e.g., $[a, b] = [0, 1]$ and

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2. \end{cases}$$

Then $F(x) = x$, so F is differentiable at $1/2 \in (0, 1)$, but f is not continuous at $1/2$.

(c) **False**. Consider, e.g., $[a, b] = [0, 1]$ and

$$f(x) = \begin{cases} x & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2. \end{cases}$$

Then

$$\frac{F(1) - F(0)}{1 - 0} = \frac{1}{2} \neq f(c) \quad \text{for all } c \in [0, 1]. \qquad \diamond$$

- This problem is to see how well you understand the fundamental theorem of calculus (p.253).
- For each problem, 2 points for a correct answer and 4 points for a correct justification.
- There are no extra minus points if one answers opposite to the correct one.

Problem 2. (12 points) Let f be a bounded function on $\mathbb{R}$, which is continuous on $\mathbb{R} \setminus S$ for some finite set S. Prove that f is Riemann integrable on every closed interval. (Hint: Consider a small neighborhood near each point in S.)

Solution. Fix $[a, b]$. Let $S \cap [a, b] = \{c_1, \ldots, c_n\}$. We claim that f satisfies Riemann's condition on $[a, b]$. Let $\varepsilon > 0$ be given. Take sufficiently small $0 < \delta < \varepsilon$. Let $N_i = (c_i - \delta, c_i + \delta)$, and let J_j be the disjoint closed intervals such that $[a, b] \setminus (N_1 \cup \cdots \cup N_n) = J_1 \cup \cdots \cup J_m$. For each j there is $\pi_j \in \Pi J_j$ such that $U\big(f, \pi_j'\big) - L\big(f, \pi_j'\big) < \varepsilon$ for any refinement π_j' of π_j, since f is continuous on J_j (Theorem 6.2.7, p.248). Let $\pi = (\pi_1 \cup \cdots \cup \pi_m) \cup (\{c_1 \pm \delta\} \cup \cdots \cup \{c_n \pm \delta\}) \in \Pi[a, b]$. Then for any refinement π' of π we have

$$
\begin{aligned}
U(f, \pi') - L(f, \pi') &= \left(\sum_{j=1}^{m} U\big(f, \pi_j'\big) + \sum_{i=1}^{n} U(f, \pi_i'') \right) - \left(\sum_{j=1}^{m} L\big(f, \pi_j'\big) + \sum_{i=1}^{n} L(f, \pi_i'') \right) \\
&= \sum_{j=1}^{m} \big(U\big(f, \pi_j'\big) - L\big(f, \pi_j'\big)\big) + \sum_{i=1}^{n} (U(f, \pi_i'') - L(f, \pi_i'')) \\
&< \sum_{j=1}^{m} \varepsilon + \sum_{i=1}^{n} 4\delta \, \|f\|_\infty \\
&= m\varepsilon + 4n\delta \, \|f\|_\infty < (m + 4n \, \|f\|_\infty) \, \varepsilon,
\end{aligned}
$$

where $\pi' = (\pi_1' \cup \cdots \cup \pi_m') \cup (\pi_1'' \cup \cdots \cup \pi_n'')$. $\Diamond$

Another Solution. Fix $[a, b]$. Let $S \cap [a, b] = \{c_1, \ldots, c_n\}$. Let $\varepsilon > 0$ be given. Take $0 < \delta < \varepsilon$ and $\pi \in \Pi[a, b]$ as in the above solution. Then we have

$$
\begin{aligned}
U(f) - L(f) &\le U(f, \pi) - L(f, \pi) \\
&= \left(\sum_{j=1}^{m} U(f, \pi_j) + \sum_{i=1}^{n} U(f, \{c_i \pm \delta\}) \right) - \left(\sum_{j=1}^{m} L(f, \pi_j) + \sum_{i=1}^{n} L(f, \{c_i \pm \delta\}) \right) \\
&= \sum_{j=1}^{m} (U(f, \pi_j) - L(f, \pi_j)) + \sum_{i=1}^{n} (U(f, \{c_i \pm \delta\}) - L(f, \{c_2 \pm \delta\})) \\
&< \sum_{j=1}^{m} \varepsilon + \sum_{i=1}^{n} 4\delta \, \|f\|_\infty \\
&= m\varepsilon + 4n\delta \, \|f\|_\infty < (m + 4n \, \|f\|_\infty) \, \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we obtain $U(f) - L(f) = 0$. Theorem 6.2.4 (p.243) completes the proof. $\Diamond$

- This problem is a slight modification of Exercise 6.39.
- A solution assuming the existence of a continuous function on $\mathbb{R}$ that equals f on $\mathbb{R} \setminus S$ gets no points. Also, a solution assuming the existence of $f(c+)$ and $f(c-)$ at $c \in S$ gets no points. That is because there are functions such as $f(x) = \sin(1/x)$ with $f(0) = 0$.
- If one cites Exercise 6.39 without any proof, then there is a deduction of at least 8 points.