Group1 HW2

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Contribution Details:

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Anar Rzayev HW Group Exercises 20180788

75) That's not true; take $x_k = \sum_{j=1}^{k} \frac{1}{j} = 1 + \frac{1}{2} + \dots + \frac{1}{k}$

$$x_{k+1} - x_k = \left(1 + \frac{1}{2} + \dots + \frac{1}{k} + \frac{1}{k+1}\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right) = \frac{1}{k+1}$$ with

$\lim_{k\to\infty} (x_{k+1} - x_k) = \lim_{k\to\infty} \frac{1}{k+1} = 0$, because for $\forall \varepsilon > 0$,

we can find $k_0 \in \mathbb{N}$ such that $k_0 > \frac{1}{\varepsilon} - 1$, or just $\varepsilon(k_0 + 1) > 1$ and taking $k > k_0$, we find that

$\left| \frac{1}{k+1} - 0 \right| = \frac{1}{k+1} < \frac{1}{k_0 + 1} < \varepsilon$, meaning $\lim_{k\to\infty} \frac{1}{k+1} = 0$ ✓

However, if we assume $\{x_k\}$ is Cauchy, then from the famous theorem in Chapter 1.4 $\Rightarrow$ it should be bounded; meaning $x_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$ should be bounded

We'll prove it's impossible $\Rightarrow$ Take $k = 2^m$, with $m \geq 1$

$$x_{2^m} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^j+1} + \dots + \frac{1}{2^{j+1}}\right) + \dots + \frac{1}{2^m} \geq \sum_{j=0}^{m-1} \left(\frac{1}{2^j+1} + \dots + \frac{1}{2^j+2^j}\right) \geq \sum_{j=0}^{m-1} \frac{1}{2^j+2^j} \cdot 2^j = \sum_{j=0}^{m-1} \frac{1}{2} =$$

$$= \frac{1}{2} \cdot m$$

Hence, $\boxed{x_{2^m} \geq \frac{1}{2}m \text{ for each } m \geq 1}$ If $\{x_k\}$ was bounded $|x_k| \leq M$ would need to be true $\Rightarrow$ $M \geq x_k$ for each k

Choosing $m \in \mathbb{N}$ such that $m > 2M$, $M \geq x_{2^m} \geq \frac{1}{2}m >$ $> M$ $\boxed{\times}$ Hence, $\boxed{\{x_k\} \text{ is not Cauchy}}$ $\boxed{V}$.

8a) Take $x_k = \sum_{j=1}^{k} \frac{1}{j} = 1 + \frac{1}{2} + \dots + \frac{1}{k}$, the sequence in $\mathbb{R}$

Consider $c_k = 1 - \frac{1}{k+1}$ for $k > 0$, where c_k is a strictly monotone increasing sequence: If $a > b > 0$, $a+1 > b+1$ or $\frac{1}{a+1} < \frac{1}{b+1}$, $\frac{-1}{a+1} > \frac{-1}{b+1}$ and $c_a = 1 - \frac{1}{a+1} > 1 - \frac{1}{b+1} = c_b \Rightarrow$ For $a > b > 0 \Leftrightarrow c_a > c_b$ which concludes $\{c_k\}$ is strictly monotone increasing. Moreover, as in the previous task, $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N} \Rightarrow \frac{1}{k_0+1} < \varepsilon$ and then $k > k_0$ is chosen, $|c_k - 1| = \left|\frac{1}{k+1}\right| = \frac{1}{k+1} < \frac{1}{k_0+1} < \varepsilon$, hence

$\boxed{\lim_{k\to\infty} c_k = 1}$ As for $k \geq 2$, we can observe that

$$|x_{k+1}-x_k| = \left|\left(1+\frac{1}{2}+\dots+\frac{1}{k}+\frac{1}{k+1}\right)-\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right)\right| = \frac{1}{k+1} =$$

$$= \frac{k}{k+1}\cdot\frac{1}{k} = \left(1-\frac{1}{k+1}\right)\cdot\frac{1}{k} = C_k\cdot\left(\left(1+\frac{1}{2}+\dots+\frac{1}{k-1}+\frac{1}{k}\right) - \left(1+\frac{1}{2}+\dots+\frac{1}{k-1}\right)\right)$$

$= C_k\left(x_k - x_{k-1}\right) = C_k\left|x_k - x_{k-1}\right|$, meaning that $k\geq 2$

$\boxed{|x_{k+1}-x_k| = C_k\,|x_k-x_{k-1}| \leq C_k|x_k-x_{k-1}|}$ However,

from previous task, we proved that $\boxed{\{x_k\} \text{ is not necessarily Cauchy}}$

$\boxed{\{x_k\}\to \text{not Cauchy}}$ counterexample $\boxed{V}$

g1) We'll prove the following inequality:

$||x|-|y|| \leq |x-y|$ for any reals x,y

Proof: From the triangle inequality, $|x| = |(x-y)+y| \leq$

$\leq |x-y|+|y| \Rightarrow \boxed{|x|-|y| \leq |x-y|}$ Similarly, $|x-y| = |y-x| \Rightarrow$

$|y| = |(y-x)+x| \leq |y-x|+|x| = |x-y|+|x| \Rightarrow -|y| \geq -|x-y|-|x|$

$\boxed{-|x-y| \leq |x|-|y|}$ Thus, $\boxed{-|x-y| \leq |x|-|y| \leq |x-y|}$ or just

$||x|-|y|| \leq |x-y|$ V

Let's first prove that if $\{x_k\}\to x_0$, then $\{|x_k|\}\to |x_0|$

$\{x_k\} \to x_0$ means for $\forall \varepsilon > 0, \exists k_0 \in N, \forall k > k_0 \Rightarrow |x_k - x_0| < \varepsilon$

Since $\big||x_k| - |x_0|\big| \le |x_k - x_0| < \varepsilon \Rightarrow$ we find that ✓

$\forall \varepsilon > 0, \exists k_0 \in N \Rightarrow \forall k > k_0, \big||x_k| - |x_0|\big| < \varepsilon$, or $\boxed{\{|x_k|\} \to |x_0|}$

About the converse, we claim it's not true. Take

$x_k = (-1)^k \Rightarrow |x_k| = 1$ for all $k \in N$, meaning that

$\big||x_k| - 1\big| = 0 < \varepsilon$ for each $\varepsilon > 0$; hence, $|x_k| \to 1$, but

since $x_n = (-1)^n$ has only the values $\{1, -1\}$ with getting

them iteratively, we can conclude that $\boxed{\{x_n\} \text{ does not even converge}}$

(it has 2 cluster points, so diverges). Thus,

the converse is false $\boxed{V}$ ∎

93) $x_1 = 1$, $x_{k+1} = \dfrac{2x_k + 3}{4}$ for $k \in N \Rightarrow x_{k+1} = \dfrac{x_k}{2} + \dfrac{3}{4}$

For $k \geq 2$, $x_k = \dfrac{x_{k-1}}{2} + \dfrac{3}{4}$ and $x_k - x_{k-1}$ should be found

$2x_k = x_{k-1} + \dfrac{3}{2}$, $\boxed{x_{k-1} = 2x_k - \dfrac{3}{2}}$ $x_k - x_{k-1} = x_k - 2x_k + \dfrac{3}{2} =$

$= \dfrac{3}{2} - x_k$

$\boxed{x_k - x_{k-1} = \dfrac{3}{2} - x_k}$ $x_{k+1} - x_k = \dfrac{3}{4} - \dfrac{x_k}{2} = \dfrac{1}{2}\left(\dfrac{3}{2} - x_k\right)$, so

$\boxed{x_{k+1}-x_k=\frac{1}{2}\left(\frac{3}{2}-x_k\right)}$ For $k\geq 2$, $|x_{k+1}-x_k|=\frac{1}{2}\left|\frac{3}{2}-x_k\right|=$

$=\frac{1}{2}|x_k-x_{k-1}| \Rightarrow \boxed{|x_{k+1}-x_k|=\frac{1}{2}|x_k-x_{k-1}|\leq C|x_k-x_{k-1}| \text{ for } k\geq 2}$ ✠

Choosing any real C with $\frac{1}{2}\leq C<1$, we found the above inequality for $\forall k\geq 2 \Rightarrow$ $\boxed{\{x_k\} \text{ is contractive}}$ ✠

From the famous theorem, we know ✠ $\boxed{\{x_k\} \text{ will be Cauchy sequence}}$

Considering that $x_1=1$, all the $\{x_k\}$ will be real numbers

As $\{x_k\}\subset\mathbb{R}$ is Cauchy $(x_k\in\mathbb{R})$, from the famous theorem, $\boxed{\{x_k\} \text{ will be convergent on } \mathbb{R}}$ ✠ Thus, it has

a limit $\Rightarrow$ knowing that $\lim_{k\to\infty} x_k = \lim_{k\to\infty}\frac{2x_{k-1}+3}{4}$ where

$\{x_k\}$ is convergent to some limit x_0. As $x_{k+1}=\frac{2x_k+3}{4}$

and rational functions preserve limits,

$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty}\frac{2x_k+3}{4}=\frac{2x_0+3}{4}$; thus, taking the limit

in recursive relation, we see $x_0=\frac{2x_0+3}{4}=\frac{3}{4}+\frac{x_0}{2}\Rightarrow$

$\frac{x_0}{2}=\frac{3}{4}$ or just $x_0=\frac{6}{4}=\frac{3}{2}$, $\boxed{x_0=\frac{3}{2}}$ ✠ Since $x_0>0$, we

conclude that $\boxed{\{x_k\} \text{ converges to } x_0=\frac{3}{2}}$ ✠ $\boxed{V}$ ▪

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$\boxed{1.47}$ Let $\{x_k\}$ be a sequence $\ni$ $\forall k$: $x_k>0$, $\lim_{k\to\infty}\frac{x_{k+1}}{x_k}=L>0$

$\therefore$ $\forall\varepsilon>0, \exists k_0\in\mathbb{N}, \forall k>k_0$: $\left|\frac{x_{k+1}}{x_k}-L\right|<\varepsilon$ $\therefore$ $L-\varepsilon<\frac{x_{k+1}}{x_k}<L+\varepsilon$

a) suppose $L<1$ $\therefore$ for $\varepsilon=\frac{1-L}{2}$, $\exists k_0\in\mathbb{N}, \forall k>k_0$: $\frac{x_{k+1}}{x_k}<L+\left(\frac{1-L}{2}\right)=\frac{1}{2}+\frac{L}{2}$

Let $M=\frac{1}{2}+\frac{L}{2}<1$ $\therefore$ $\forall k>k_0$: $x_{k+1}<Mx_k<x_k$

Claim $\lim_{k\to\infty}x_k=0$

let $\varepsilon>0$ be arbitary,

let $k_1=\max\{k_0+\lceil\log_M(\varepsilon/x_{k_0})\rceil, k_0\}$ $\therefore$ $k_1-k_0\geq\log_M(\varepsilon/x_{k_0})>0$

for $k>k_1$, $|x_k|=x_k<x_{k_1}<Mx_{k_1-1}<M^2x_{k_1-2}<\dots<M^{k_1-k_0}x_{k_0}$

Since $k_1-k_0\geq\log_M(\varepsilon/x_{k_0})$ and $0<M<1$ $\therefore|x_k|<M^{k_1-k_0}x_{k_0}\leq M^{\log_M(\varepsilon/x_{k_0})}x_{k_0}=\varepsilon$

$\therefore$ $\forall\varepsilon>0, \exists k_1, \forall k<k_1$: $|x_k|<\varepsilon$

b) suppose $L>1$ for $\varepsilon=\frac{L-1}{2}$, $\exists k_0\in\mathbb{N}, \forall k>k_0$: $\frac{x_{k+1}}{x_k}>L-\varepsilon=\frac{L}{2}+\frac{1}{2}>1$

Let $M=\frac{1}{2}+\frac{L}{2}<1$ $\therefore$ $\forall k>k_0$: $x_{k+1}>Mx_k>x_k$

Let $X>0$ be arbitary, $k_1=\max\{k_0, k_0+\lceil\log_M(X/x_{k_0})\rceil\}$ $\therefore$ $k_1-k_0\geq\log_M(X/x_{k_0})>0$

$\therefore$ $x_{k_1}>Mx_{k_1-1}>\dots>M^{k_1-k_0}x_{k_0}\geq M^{\log_M(X/x_{k_0})}x_{k_0}=X$

$\therefore\forall X>0, \exists k_1$; $x_{k_1}>X$ $\therefore$ $\{x_k\}$ diverges to ∞

c) Let $\{y_k\},\{z_k\}$ be sequence $\ni$ $y_k=1$, $z_k=k$

$\forall\varepsilon>0, \forall k>0$, $\left|\frac{y_{k+1}}{y_k}-1\right|=0<\varepsilon$ $\therefore$ $\lim_{k\to\infty}\frac{y_{k+1}}{y_k}=1$

$\forall\varepsilon>0$, $\exists k_0=\lceil\frac{1}{\varepsilon}\rceil$, $\forall k>k_0$, $\left|\frac{z_{k+1}}{z_k}-1\right|=\frac{k+1}{k}-1=\frac{1}{k}<\frac{1}{k_0}\leq\varepsilon$ $\therefore\lim_{k\to\infty}\frac{z_{k+1}}{z_k}=1$

However y converges, but z diverges

d) $\forall\varepsilon>0$, $\exists k_0=\lceil\frac{1}{2\varepsilon}\rceil$, $\forall k>k_0$, $\left|\frac{(k+1)/2^{k+1}}{k/2^k}-\frac{1}{2}\right|=\frac{k+1}{2k}-\frac{1}{2}=\frac{1}{2k}<\frac{1}{2k_0}<\varepsilon$

$\therefore$ $\lim_{k\to\infty}\frac{(k+1)/2^{k+1}}{k/2^k}=\frac{1}{2}<1$ $\therefore$ $\{k/2^k\}$ converges to 0

1.95) Let $c>0, x_1>0$, Let $\{x_k\}$ be a sequence ; $\forall k>1$: $x_{k+1} = (x_k + c/x_k)/2$

a) Since $x_1>0$, let $x_k>0$ $\therefore$ $x_{k+1} = (x_k + c/x_k)/2 > 0$

$\therefore$ By induction $\forall k \in \mathbb{N}$; $x_k > 0$

for $k>1$: consider $0 < \left(\sqrt{x_{k-1}} - \sqrt{\frac{c}{x_{k-1}}}\right)^2 = x_{k-1} - 2\sqrt{c} + \frac{c}{x_{k-1}}$

$\therefore$ $x_k = \frac{1}{2}\left(x_{k-1} + \frac{c}{x_{k-1}}\right) > \sqrt{c}$

$x_k^2 > c$ $\therefore$ $x_k > \frac{c}{x_k}$ $\therefore$ $x_k > \frac{1}{2}\left(x_{k-1} + \frac{c}{x_k}\right) = x_{k+1}$

$\therefore \forall k>1 \therefore \sqrt{c} < x_{k+1} < x_k \leq x_2$

$\therefore$ $\{x_k\}$ for $k>1$ is bounded and monotonic

$\therefore$ $\{x_k\}$ converges $\therefore$ let $\lim\limits_{k\to\infty} x_k = x_0$

$\therefore x_0 = \lim\limits_{k\to\infty} x_k = \lim\limits_{k\to\infty} x_{k+1} = \lim\limits_{k\to\infty} \frac{1}{2}\left(x_k + \frac{c}{x_k}\right) = \frac{1}{2}\left[\lim\limits_{k\to\infty} x_k + c\Big/\left(\lim\limits_{k\to\infty} x_k\right)\right] = \frac{1}{2}\left(x_0 + \frac{c}{x_0}\right)$

$\therefore \frac{1}{2}x_0 = \frac{c}{2x_0}$ $\therefore$ $x_0 = \sqrt{c}$ $(\because \forall k ; x_k > 0)$

$\therefore \lim\limits_{k\to\infty} x_k = \sqrt{c}$

b) let $x_1 = 3$, $c = 5$ $\therefore$ $x_2 = 2.3333333$

$x_3 = 2.2380952$

$x_4 = 2.2360689$

$x_5 = 2.2360680$

$x_6 = 2.2360680$

$\therefore \sqrt{5} \approx 2.236068$

1.110) Let $X \subseteq \mathbb{R}$ be uncountable,

Case 1: X is bounded below

$\therefore$ Let $\{a_n\}$ be sequence such that $a_k = \begin{cases} \inf X & ; k=1 \\ \inf(X - \{a_1, \dots, a_{k-1}\}) & ; k>1 \end{cases}$

By definition, $\{a_n\}$ is monotone and increasing (not strictly)

Claim $\{a_n\}$ converges

Suppose $\{a_n\}$ diverges, Let $f: \mathbb{N} \to \mathbb{R}$; $f(k) = a_k$

$\therefore$ $\{a_n\}$ is unbounded above ($\because$ if bounded and monotone, it will converge)

$\therefore \forall x \in X, \exists k_0 \in \mathbb{N} :\ x < a_{k_0} = \inf(X - \{a_1, \dots, a_{k_0-1}\})$ $\therefore x \notin X - \{a_1, \dots, a_{k_0-1}\}$

$\therefore \exists k < k_0$; $x = a_k = f(k)$

$\therefore \exists f: \mathbb{N} \to X$; f is surjective

$\therefore$ X doesn't have larger cardinality than $\mathbb{N}$

$\therefore$ X is countable, contradiction

Let $\lim_{n\to\infty} a_n = a_0$ $\therefore$ $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k > k_0 : |a_k - a_0| < \varepsilon$

Claim: a_0 is limit point of X

let $\varepsilon > 0$ $\therefore$ $\exists k \in \mathbb{N} :\ a_0 - \varepsilon < a_k < a_0 + \varepsilon$

$\therefore$ $a_0 - \varepsilon < \inf(X - \{a_1, \dots, a_{k_0-1}\}) < a_0 + \varepsilon$

$\therefore \exists x \in X :\ a_0 - \varepsilon < x < a_0 + \varepsilon$ (else, $\inf(X - \{a_1, \dots, a_{k_0-1}\}) \geq a_0 + \varepsilon$)

Case 2: X is bounded above

$\therefore$ Let $\{a_n\}$ be sequence such that $a_k = \begin{cases} \sup X & ; k=1 \\ \sup(X - \{a_1, \dots, a_{k-1}\}) & ; k>1 \end{cases}$

By definition, $\{a_n\}$ is monotone and decreasing (not strictly)

Claim $\{a_n\}$ converges

Suppose $\{a_n\}$ diverges, Let $f: \mathbb{N} \to \mathbb{R}$; $f(k) = a_k$

$\therefore$ $\{a_n\}$ is unbounded below ($\because$ if bounded and monotone, it will converge)

$\therefore \forall x \in X, \exists k_0 \in \mathbb{N} :\ x < a_{k_0} = \sup(X - \{a_1, \dots, a_{k_0-1}\})$ $\therefore x \notin X - \{a_1, \dots, a_{k_0-1}\}$

$\therefore \exists k < k_0$; $x = a_k = f(k)$

$\therefore \exists f: \mathbb{N} \to X$; f is surjective

$\therefore$ X doesn't have larger cardinality than $\mathbb{N}$

$\therefore$ X is countable, contradiction

Let $\lim_{n\to\infty} a_n = a_0$ $\therefore \forall \varepsilon>0, \exists k_0 \in \mathbb{N}, \forall k>k_0 : |a_k - a_0| < \varepsilon$

Claim: a_0 is limit point of X

let $\varepsilon > 0$ $\therefore$ $\exists k \in \mathbb{N} : a_0 - \varepsilon < a_k < a_0 + \varepsilon$

$\therefore a_0 - \varepsilon < \sup(X - \{a_1, \dots, a_{k_0 - 1}\}) < a_0 + \varepsilon$

$\therefore \exists x \in X : a_0 - \varepsilon < x < a_0 + \varepsilon$ (else, $\sup(X - \{a_1, \dots, a_{k_0-1}\}) \leq a_0 - \varepsilon$)

Case 3: X is unbounded $\therefore$ Let $X_1 = \{x \in X \mid x \geq 0\}$, $X_2 = \{x \in X \mid x < 0\}$

if both X_1 and X_2 are countable, then X is countable

$\therefore$ either X_1 or X_2 is uncountable

$\therefore$ By case 1 & 2, X_1 or X_2 has a limit point

$\therefore$ X has a limit point

1.50 $\quad [a_{k+1}, b_{k+1}] \subseteq [a_k, b_k] \Rightarrow a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$

1) $b_1 \geq a_{k+1} \geq a_k \Rightarrow$ by Thm 1.3.7 a_k converges to a_0

2) $a_1 \leq b_{k+1} \leq b_k \Rightarrow$ by Thm 1.3.7 b_k converges to b_0

$$a_k \leq b_k \Rightarrow a_0 \leq b_0$$

$b_k \geq b_0 \geq a_0 \geq a_k \Rightarrow [a_0, b_0]$ is contained

in $\bigcap_{k=1}^{\infty} I_k \Rightarrow \neq \emptyset$

if $a_0 = b_0 = x_0 \Rightarrow$ for $\forall k$ $\; b_k \geq x_0 \geq a_k$

$$\Rightarrow [x_0] = \bigcap_{k=1}^{\infty} I_k$$

$\boxed{1.59}$ a) $v = \liminf x_k$ (finite) $\varepsilon > 0$

We claim that there are finitely many term to the left $v - \varepsilon$. Otherwise, we will have cluster point less than v. But, v is inf of cluster points.

b) Since it is already proved that v is in C (C = set of cluster point(s)), for any $\varepsilon, k > 0$

$\exists k_1 > k$ s.t $x_{k_1} \in N(v, \varepsilon) \Rightarrow x_{k_1} < v + \varepsilon$.

$\boxed{1.60}$ i) if $\liminf x_k = \limsup x_k = x_0 \Rightarrow$ there is only one cluster point and since liminf and lim sup are well-defined, x_k is bounded $\Rightarrow x_k$ converges to some x_1. Now, we show that $x_1 = x_0$. Assume $x_0 \neq x_1$ and let $\varepsilon = \frac{|x_0 - x_1|}{2} > 0$

By definition, there are finitely many terms outside of $N(x_1, \varepsilon)$ and there exist infinitely many terms inside $N(x_0, \varepsilon)$

Contradiction $\Rightarrow x_0 = x_1 \Rightarrow x_k$ converges to x_0

ii) Since x_k converges $\Rightarrow$ it is bounded $\Rightarrow$ has exactly one clusterpoint and it is $x_0 \Rightarrow \limsup x_k = \liminf x_k = x_0$

Bolzano-Weirstrass

1.105 Let $\{A_1, A_2, ---, A_n\}$ is a finite collection of countably infinite sets. We will prove that the set $A = \bigcup_{i=1}^{n} A_i$ is countably infinite.

Proof (Induction):

Let $n=2$. Then, $A = A_1 \cup A_2$. To prove that A countable, let's define the map $f: A_1 \cup A_2 \longrightarrow \mathbb{Z}$ as

$$f(x) = \begin{cases} f_1(x), & x \in A_1 \\ -f_2(x), & x \notin A_1 \end{cases}$$

To prove that $f(x)$ is a one-to-one mapping, we show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. So let $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$. Now, if $f(x_1) \leq 0$, $f(x_1) = f(x_2) = -f_2(x_1) = -f_2(x_2)$.
$\Rightarrow f_2(x_1) = -f(x_1) = -f(x_2) = f_2(x_2)$. However, $f_2(x)$ is an injective map and thus, $f_2(x_1) = f_2(x_2)$ implies $x_1 = x_2$. So, $f(x)$ is one-to-one. On the other hand, if $f(x_1) > 0$, then $f(x_1) = f(x_2) = f_1(x_1) = f_1(x_2)$. Since $f_1(x)$ is injective, $f_1(x_1) = f_1(x_2)$ implies $x_1 = x_2$. So, $f(x)$ is injective.

Now, let's define another function from the set of integers $\mathbb{Z}$ to the set of natural numbers as:

$g(x): \mathbb{Z} \longrightarrow \mathbb{N}$. Since $\mathbb{Z}$ is countable, then $g(x)$ is an injective map.

Then, the composite map $(g \circ f)(x): A_1 \cup A_2 \longrightarrow \mathbb{N}$ is one-to-one and the set $A = A_1 \cup A_2$ is countable.

1.105 continued

Now, suppose the set $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ is countable. We will show that the set $A = A_1 \cup A_2 \cup \cdots A_n = \bigcup_{i=1}^{n} A_i$ is countable.

Let $B = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$.

Then, $A = B \cup A_n$. Now, define a map $f : B \cup A_n \longrightarrow \mathbb{Z}$ such that

$$f(x) = \begin{cases} h(x), & x \in B \\ -g(x), & x \notin B \end{cases}$$

Following exactly similar argument as in the $n=2$ case, we deduce that $f(x)$ is one-to-one. Now, if $k(x) : \mathbb{Z} \to \mathbb{N}$, then $(k \circ f)(x) : B \cup A_n \longrightarrow \mathbb{N}$ is one-to-one.

so, $A = B \cup A_n = A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n$ is countable.

1.109

(a). Let $S = \{s_1, s_2, s_3, ---\}$ is countably infinite.

Then, for any subset A of S define a function $f_A : S \to \{0,1\}$ such that

$$f_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Now, let $g : P(S) \to 2^S = \{0,1\}^S$ such that

$$A \mapsto f_A$$

Then, the function g is a bijection between $P(S)$ and infinite binary sequences $\{0,1\}^S$. so $|P(S)| = |B|$, where $B = \{(b_1, b_2, b_3, ---) \mid b_i = 0 \text{ or } 1\}$ is the set of all infinite binary sequences.

(b). Let $B = \{(b_1, b_2, b_3, ---) : b_i = 0 \text{ or } 1\}$ be the set of all infinite binary sequences.

Then, each b_i is an infinite sequence of 0_s and $1s$.

That is, $b_i = b_{i1}\, b_{i2}\, b_{i3} ---$, where $b_{ij} = 0$ or 1.

Then, we can put B in an array of b_i's as follows:

$$\begin{array}{llll} b_{11} & b_{12} & b_{13} & ----- \\ b_{21} & b_{22} & b_{23} & ----- \\ \vdots & & & \\ b_{n1} & b_{n2} & b_{n3} & ----- \\ \vdots & & & \end{array}$$

Now, let's define another infinite sequence of 0_s and 1_s as:

$a_i = 1 - b_{ii}$. That's, $a_i = \begin{cases} 0, & \text{if } b_{ii} = 1 \\ 1, & \text{if } b_{ii} = 0 \end{cases}$

Then, a sequence $a = a_1 a_2 a_3 --- a_n ---$ belongs to B.

$\boxed{1.109}$

(b). continued

Since a belongs to B, let $a = b_j$ for some b_j in B.
Then, $a_j = b_{jj}$. However, from our definition of a_j, we have
$a_j = 1 - b_{jj}$ and thus, $a_j \neq b_{jj}$. So, the sequence $a = a_1 a_2 a_3 \cdots$
differs from b_i at the i^{th} place, for every i. Thus, the
sequence $a = a_1 a_2 a_3 \cdots$ doesn't belong to B — a contradiction.
Therefore, B is uncountable.

1.10

Let X be an arbitrary uncountable subset of the set of real numbers $\mathbb{R}$. Let $A = \{[i-1, i] \mid i \in \mathbb{Z}\}$ denote a collection of all closed intervals $[i-1, i] = A_i$.

Then, let $X_i = X \cap A_i$, where $A_i = [i-1, i]$ for some $i \in \mathbb{Z}$.

$\Rightarrow X = \bigcup_{i \in \mathbb{Z}} X_i$. Now, if all X_i's are countable, then X will the infinite union countable sets and so, X is countable too. So, not every X_i can be finite. Then, let X_j is infinite, for some $j \in \mathbb{Z}$. Now, the set $X_j \subseteq [j-1, j]$ has a limit point in the interval $[j-1, j]$. Let y be a limit point of X_j. Then, for any neighbourhood U of y, $U \cap [j-1, j]$ is also a limit point of y. So, y is a limit point of $X = \bigcup_{i \in \mathbb{Z}} X_i$ as well.