MAS241 ANALYSIS 1 QUIZ 5

Problem 1. *(15 points) In this problem, you should use Theorem 6.5.1.*

Theorem 6.5.1) *If $\{f_k\}$ converges uniformly to f_0 on the compact set $[a, b]$ and if each f_k is integrable on $[a, b]$, then f_0 is also integrable on $[a, b]$. Furthermore,*

(1) *If $F_k(x) = \int_a^x f_k(t)dt$, then $\{F_k\}$ converges uniformly to the function $F_0(x) = \int_a^x f_0(t)dt$ on $[a, b]$.*
(2) *In particular, $\lim_{k\to\infty} \int_a^b f_k(x)dx = \int_a^b f_0(x)dx$.*

Now, define a function f_0 as

$$f_0 = \begin{cases} 0 & x \in \mathbb{R}/\mathbb{Q} \\ \frac{1}{m} & x = \frac{n}{m} \in \mathbb{Q},\ \gcd(m,n) = 1 \end{cases}.$$

Show that f_0 is integrable on $[a, b]$ and evaluate $F_0(x)$ for $x \in [a, b]$.

Solution 1. *Let $f_k = \frac{1}{m}$ when $x = \frac{n}{m} \in \mathbb{Q}$, $\gcd(m,n) = 1$, $m \le k$, and $f_k = 0$ otherwise. We only need to show that f_k converges uniformly to f_0 and each f_k is integrable on $[a, b]$.*

(1) Uniform convergence

$$f_0 - f_k = \begin{cases} 0 & x \in \mathbb{R}/\mathbb{Q} \\ \frac{1}{m} & x = \frac{n}{m} \in \mathbb{Q},\ \gcd(m,n) = 1,\ m > k \end{cases}.$$

So, for all $x \in [a, b]$, we have

$$|f_0 - f_k| < \frac{1}{k}$$

since $m > k$. And by Archimedes Principle, for any $\epsilon > 0$, we can find k such that $\frac{1}{k} < \epsilon$. This inequality holds for all $x \in [a, b]$, we can get uniform convergence.

(2) Integrable
Since every f_k are equal to zero except at finitely many points, let the nonzero points be $c_1, \cdots, c_n$ with $c_i < c_{i+1}$. By Lemma 1 of 6.2, f_k is integrable on $[c_i, c_{i+1}]$ and therefore, it is integrable on $[a, b]$. Now, we can use Theorem 6.2.10 and

$$\int_a^b f_k(x)dx = \int_a^b 0dx = 0.$$

Now, we may apply the Theorem 6.5.1. So, f_0 is integrable on $[a, b]$ and since $F_k(x) = \int_a^x f_k(t)dt = 0$, we have $F_0(x) = 0$ for all $x \in [a, b]$. □

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Problem 2. *(15 points) Let $\{f_k\}$ be a sequence of continuously differentiable functions on $[a, b]$ such that $\lim_{k\to\infty} f_k = f_0$ pointwisely on $[a, b]$ and $\lim_{k\to\infty} f_k' = g$ pointwisely on $[a, b]$. Prove or disprove that for $x \in [a, b]$,*

$$f_0(x) - f_0(a) = \int_a^x g(t)dt.$$

Solution 2. *Disprove.*
The counterexample is $f_k(x) = 2^{3k-2}x^3 - 3 \times 2^{2k-2}x^2 + 1$ on $[0, \frac{1}{2^{k-1}}]$ or $f_k(x) = \exp(-\frac{x^2}{(x-\frac{1}{k})^2})$. Let take f_k as first one.

$$f_k(x) = \begin{cases} 1 & x \in (-\infty, 0) \\ 2^{3k-2}x^3 - 3 \times 2^{2k-2}x^2 + 1 & x \in [0, \frac{1}{2^{k-1}}] \\ 0 & x \in (\frac{1}{2^{k-1}}, \infty) \end{cases}$$

We can easily show that $f_k(x)$ converges pointwisely to f_0 which is

$$f_0 = \begin{cases} 1 & x \in (-\infty, 0] \\ 0 & x \in (0, \infty) \end{cases}.$$

Also, f_k is differentiable for all $x \in \mathbb{R}$, and we have

$$f_k' = \begin{cases} 3 \times 2^{3k-2}x^2 - 3 \times 2^{2k-1}x & x \in (0, \frac{1}{2^{k-1}}) \\ 0 & \text{otherwise} \end{cases}.$$

Note that it is continuous and it converges pointwisely to $g = 0$. Now, take $[a, b] = [0, 1]$. Then, for any $x \in (0, 1]$, we have $f_0(x) - f_0(a) = -1$, but $\int_a^x g(t)dt = 0$ always. So, the given statement is not true. □

Problem 3. *(15 points) Suppose that g is defined by*

$$g(x) = \begin{cases} g_1, & \text{for } 0 \le x < 1 \\ g_2, & \text{for } 1 \le x \le 2. \end{cases}$$

Here, $g_1 \neq g_2$ and $f_1 \neq f_2$.

(1) *Let f be a the function defined on $[0, 2]$ by*

$$f(x) = \begin{cases} f_1, & \text{for } 0 \le x < 1 \\ f_2, & \text{for } 1 \le x \le 2. \end{cases}$$

Prove or disprove that f is in $RS[g; 0, 2]$. Evaluate $\int_0^2 f(x)dg(x)$ if it exists.

(2) *Let f be a the function defined on $[0, 2]$ by*

$$f(x) = \begin{cases} f_1, & \text{for } 0 \le x \le 1 \\ f_2, & \text{for } 1 < x \le 2. \end{cases}$$

Prove or disprove that f is in $RS[g; 0, 2]$. Evaluate $\int_0^2 f(x)dg(x)$ if it exists.

Solution 3. (1) *Disprove.*

Consider a partition π of $[0, 2]$ which is $\pi = \{x_0, \cdots, x_p\}$. For every interval $[x_{j-1}, x_j]$ which doesn't contain 1, the increment $\Delta g_j = g(x_j) - g(x_{j-1})$ is zero. So, we only need to consider 2 cases, $1 \in (x_{j-1}, x_j)$ or $x_j = 1$ for some $0 < j < p$.

If $1 \in (x_{j-1}, x_j)$, then $\Delta g_j = g_2 - g_1$. But $f(s_j) = f_1$ if $s_j < 1$ and $f(s_j) = f_2$ if $s_j \ge 1$. So, a Riemann-Stieltjes sum $S(f, g, \pi)$ can be $f_1(g_2 - g_1)$ or $f_2(g_2 - g_1)$, we cannot find a number I of definition 7.1.1.

If $x_j = 1$, then we should consider two intervals of π, which are $[x_{j-1}, 1]$ and $[1, x_{j+1}]$. On the interval $[1, x_{j+1}]$, $g(x) = g_2$ is a constant and we have $\Delta g_{j+1} = 0$. On the interval $[x_{j-1}, 1]$, $\Delta g_j = g_2 - g_1$ and

$$f(x) = \begin{cases} f_1 & s_j < 1 \\ f_2 & s_j = 1. \end{cases}$$

So, we cannot find I again and f is not (Riemann-Stieltjes) integrable.

(2) *Prove.*

Again, we consider the partition in (1) and consider two cases. For $1 \in (x_{j-1}, x_j)$, we cannot find I with same reason as above.

If $x_j = 1$, then we should consider two intervals of π, which are $[x_{j-1}, 1]$ and $[1, x_{j+1}]$. On the interval $[1, x_{j+1}]$, $g(x) = g_2$ is a constant and we have $\Delta g_{j+1} = 0$. On the interval $[x_{j-1}, 1]$, $\Delta g_j = g_2 - g_1$ and $f(x) = f_1$ is a constant. So, we can find I which is $I = f_1(g_2 - g_1)$ and f is (Riemann-Stieltjes) integrable.

$\square$