# Group1 HW7

## Date: 13 May 2021

Contribution Details

Anar Rzayev → Problem 6, 8, & 18

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$\boxed{5.9}$ For $f(x) = \begin{cases} -1 & ; x=0 \\ x^2 & ; 0<x<1 \\ 7/4 & ; x=1 \\ \sqrt{x+3} & ; 1<x<2 \\ 3 & ; x=2 \end{cases}$ ; $S=\{0,1,2\}$

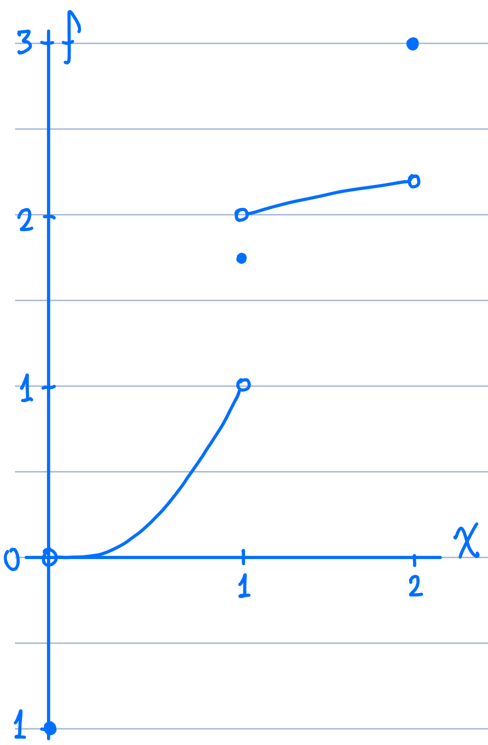
$$\therefore u(x) = \begin{cases} f(x)-f(x^-) & ; x\neq 0 \\ 0 & ; x=0 \end{cases} = \begin{cases} 0 & ; x=0 \\ x^2-x^2 & ; 0<x<1 \\ 7/4-1^2 & ; x=1 \\ \sqrt{x+3}-\sqrt{x+3} & ; 1<x<2 \\ 3-\sqrt{2+3} & ; x=2 \end{cases} = \begin{cases} 3/4 & ; x=1 \\ 3-\sqrt{5} & ; x=2 \\ 0 & ; x\in[0,1)\cup(1,2) \end{cases}$$

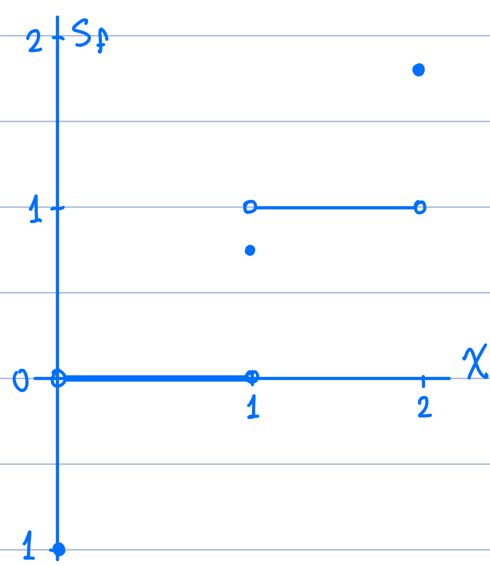
$$\therefore v(x) = \begin{cases} 0 & ; x=2 \\ f(x^+)-f(x) & ; x\neq 2 \end{cases} = \begin{cases} 0^2-(-1) & ; x=0 \\ x^2-x^2 & ; 0<x<1 \\ \sqrt{2+2}-7/4 & ; x=1 \\ \sqrt{x+3}-\sqrt{x+3} & ; 1<x<2 \\ 0 & ; x=2 \end{cases} = \begin{cases} 1 & ; x=0 \\ 1/4 & ; x=1 \\ 0 & ; x\in(0,1)\cup(1,2] \end{cases}$$

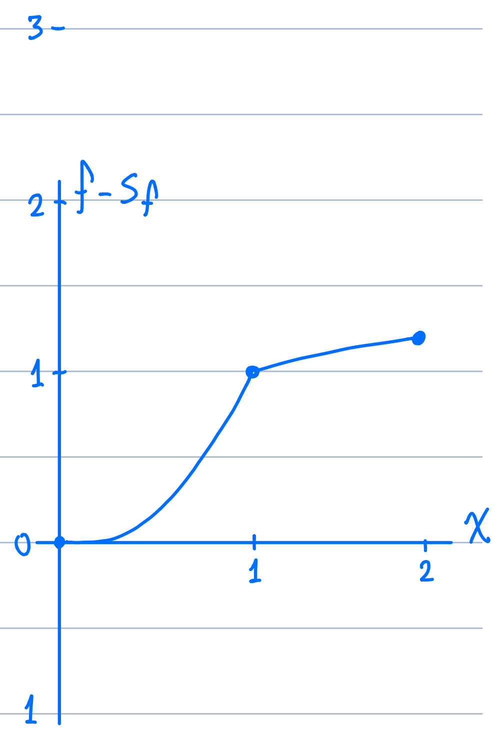
$$\therefore S_f(x) = f(0) + \sum_{y\in S\cap(0,x]} u(y) + \sum_{y\in S\cap[0,x)} v(y) = \begin{cases} f(0) & ; x=0 \\ f(0)+v(0) & ; 0<x<1 \\ f(0)+u(1)+v(0) & ; x=1 \\ f(0)+u(1)+v(0)+v(1) & ; 1<x<2 \\ f(0)+u(1)+u(2)+v(0)+v(1) & ; x=2 \end{cases}$$

$$= \begin{cases} -1 & ; x=0 \\ -1+1 & ; 0<x<1 \\ -1+3/4+1 & ; x=1 \\ -1+3/4+1+1/4 & ; 1<x<2 \\ -1+3/4+(3-\sqrt{5})+1/4 & ; x=2 \end{cases} = \begin{cases} -1 & ; x=0 \\ 0 & ; 0<x<1 \\ 3/4 & ; x=1 \\ 1 & ; 1<x<2 \\ 4-\sqrt{5} & ; x=2 \end{cases}$$

$$\therefore f-S_f(x) = \begin{cases} 0 & ; x=0 \\ x^2 & ; 0<x<1 \\ 1 & ; x=1 \\ \sqrt{x+3}-1 & ; 1<x<2 \\ \sqrt{5}-1 & ; x=2 \end{cases} = \begin{cases} x^2 & ; 0\leq x\leq 1 \\ \sqrt{x+3}-1 & ; 1<x\leq 2 \end{cases}$$



$\boxed{5.10}$ $f(x)=\begin{cases}0 & ; x=0\\ \sin(1/x) & ; x\neq 0\end{cases}$ ; Suppose $f\in BV(0,1)$

$\therefore \exists M>0, \forall \pi=\{x_0,\dots,x_p\}\in \Pi[0,1]: V(f;\pi)=\sum_{i=1}^{p}|\Delta f_i|\le M$

$\therefore$ For a fixed $M$, let $m=\lceil M\rceil+1$ ; $\pi=\{x_0,\dots,x_m\}$, $x_i=\begin{cases}0 & ; i=0\\ \frac{1}{(m+\frac{1}{2}-i)\pi} & ; 0<i<m\\ 1 & ; i=m\end{cases}$

$\therefore |\Delta f_1|=|f(x_1)-f(x_0)|=\left|\sin\left(\left(m-\frac{1}{2}\right)\pi\right)\right|=1 \quad (\because m\in\mathbb{N})$

$|\Delta f_m|=|f(x_m)-f(x_{m-1})|=|\sin 1-\sin(\frac{3}{2}\pi)|=\sin 1+1$

for $i=2,\dots,m-1$, $|\Delta f_i|=|f(x_i)-f(x_{i-1})|=\left|\sin\left(\left(m+\frac{1}{2}-i\right)\pi\right)-\sin\left(\left(m+\frac{3}{2}\right)\pi\right)\right|=2 \quad (\because m\in\mathbb{N})$

$\therefore V(f;\pi)=\sum_{i=1}^{m}|\Delta f_i| = 1+2(m-2)+\sin 1+1 = 2(m-1)+\sin 1 = 2\lceil M\rceil+\sin 1 > M \quad (\because M>0)$

Contradicts $\therefore$ Suppose $f\notin BV(0,1)$

$\boxed{5.19}$ Suppose $f$ is of bounded variation on $\mathbb{R}$ but is unbounded

Since $f$ is unbounded, $\forall M>0, \exists x\in\mathbb{R}; |f(x)|>M$

$\therefore$ for $M=V(f;\mathbb{R})+|f(0)|$, $\exists x_0\in\mathbb{R}; |f(x_0)|>V(f;\mathbb{R})+|f(0)|$

If $x_0>0$, let $\pi=\{0,x_0\}\in\Pi[0,x_0]$ $\therefore V(f;\pi)=|f(x_0)-f(0)|\le V(f;0,x_0)\le V(f;\mathbb{R})$

$\therefore |f(x_0)|\le|f(x_0)-f(0)|+|f(0)|\le V(f;\mathbb{R})+|f(0)|$

Contradicts with $|f(x_0)|>V(f;\mathbb{R})+|f(0)|$

$\therefore f$ is of bounded variation on $\mathbb{R}$, then $f$ is bounded

11/ Let $[a,b]$ be given interval containing $0$

$\Rightarrow a \leq 0 \leq b$. i) Assume $b \neq 0 \Rightarrow b > 0$

$\Rightarrow \exists m \ \forall n \geq m \quad \frac{2}{\pi(2n+1)} < b$

Let $x_n = \frac{2}{\pi(2n+1)} \Rightarrow f(x_n) = x_n \sin\left(\frac{1}{x_n}\right) =$

$= \frac{2}{\pi(2n+1)} \sin\left(\frac{\pi}{2}(2n+1)\right) = (-1)^n \cdot \frac{2}{\pi(2n+1)}$

$|f(x_n) - f(x_{n-1})| = \frac{2}{\pi(2n+1)} + \frac{2}{\pi(2n-1)} = \frac{2}{\pi} \cdot \frac{4n}{4n^2-1} \geq$

$\geq \frac{2}{\pi} \cdot \frac{1}{n} \quad (*)$

Let's consider following partition $\Pi = \{a, 0, x_n, \dots, x_m, b\}$

(we can take $n \to \infty$ as $x_i > 0$)

$\Rightarrow V(f;\Pi) \geq |f(x_n) - f(x_{n-1})| + \dots + |f(x_{m+1}) - f(x_m)| \geq$

$\geq \frac{2}{\pi}\left(\frac{1}{n} + \dots + \frac{1}{m+1}\right)$ by $(*)$. Since $m$ is fixed

and $\frac{1}{n} + \dots + \frac{1}{2} + \frac{1}{1}$ diverges $V(f;\Pi)$ is unbounded

ii) $b = 0$ $\exists m$ s.t $\forall n > m$ $a < -x_n$ (since $a < 0$)

$\Pi = \{a, -x_m, \dots, -x_n, 0\} \Rightarrow V(f;\Pi)$ is unbounded

similarly, since $x\sin\left(\frac{1}{x}\right) = (-x)\sin\left(\frac{-1}{x}\right)$

12

for $x \neq 0$ $\quad f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

$$\Rightarrow \|f'\|_\infty = \left|2x\sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right)\right| \le$$

$$\le \left|2x\sin\left(\tfrac{1}{x}\right)\right| + \left|\cos\left(\tfrac{1}{x}\right)\right| \le 2C+1$$

where $C = \max(|a|, |b|)$

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h\sin\left(\tfrac{1}{h}\right) = 0$$

since

$$\underset{\substack{\downarrow\\0}}{-|h|} \le \underset{\substack{\downarrow\\0}}{h\sin\left(\tfrac{1}{h}\right)} \le \underset{\substack{\downarrow\\0}}{|h|} \Leftarrow \text{Squeeze theorem}$$

Thus $\|f'\|_\infty$ is bounded for $\forall x \in [a,b]$

$\Rightarrow f \in BV(a,b)$

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6) Let $x, y \in [a,b]$ with $x<y$. Using the definitions of the saltus functions associated with the monotone increasing function $f$, $S_f(a) = f(a)$ and for $\forall \xi \in (a,b] \Rightarrow$

$$S_f(\xi) = f(a) + \sum_{\xi_j \in S(a,\xi]} u(\xi_j) + \sum_{\xi_j \in S[a,\xi)} v(\xi_j)$$

where: $S$ is the set of points in $[a,b]$ where $f$ is discontinuous

If $x=a$, then $a<y\leq b$ implies $S_f(x) = S_f(a) = f(a)$ and

$$S_f(y) = f(a) + \sum_{y_j \in S(a,y]} u(y_j) + \sum_{y_j \in S[a,y)} v(y_j) \geq f(a)$$ since

$u, v$ - are nonnegative $\Rightarrow S_f(y) - S_f(x) = S_f(y) - f(a) \geq 0$

$$0 \leq S_f(y) - S_f(x) = S_f(y) - f(a) = \sum_{y_j \in S(a,y]} u(y_j) + \sum_{y_j \in S[a,y)} v(y_j)$$

$$= \sum_{y_j \in S(x,y]} u(y_j) + \sum_{y_j \in S[x,y)} v(y_j),$$ hence we found that

$$\boxed{x=a,\quad x=a<y\leq b \Rightarrow S_f(y) - S_f(x) = \sum_{y_j \in S(x,y]} u(y_j) + \sum_{y_j \in S[x,y)} v(y_j)}$$

Now, let $a<x<y\leq b$, using the definition of saltus function

$$S_f(x) = f(a) + \sum_{x_j \in S(a,x]} u(x_j) + \sum_{x_j \in S[a,x)} v(x_j)$$ and similarly

$$S_f(y) = f(a) + \sum_{y_j \in S(a,y]} u(y_j) + \sum_{y_j \in S[a,y)} v(y_j)$$ from these two, we get

$$\boxed{\delta f(y)-\delta f(x)=\sum_{t_j\in S(x,y]} u(t_j)+\sum_{t_j\in S[x,y)} v(t_j) \text{ for } a<x<y\leq b}$$

Now, combining with previous result when $x=a$, we generalise

$$\boxed{\delta f(y)-\delta f(x)=\sum_{t_j\in S(x,y]} u(t_j)+\sum_{t_j\in S[x,y)} v(t_j) \text{ for } a\leq x<y\leq b}$$

Since both $u,v$ are nonnegative $\Rightarrow$ $\boxed{\delta f(y)-\delta f(x)\geq 0 \text{ for } x,y\in[a,b],\ x<y}$

Notice that, because $f$ is monotone increasing on $[a,b]$, $f$ will be monotone increasing on any closed interval $[m,n]$ which is subinterval of $[a,b]$

Let $S'$ be the set of points in $[m,n]$ where $f$'s discontinuous

Since $u(x')+v(x')=f(x'^+)-f(x'^-)$ for any $x'\in(m,n)$, we find $\sum_{x_j'\in S'} u(x_j')+\sum_{x_j'\in S'} v(x_j')\leq f(n)-f(m)$ holding true

Consider any partition of $[x,y]$ with finite number of points $F=\{x_0,x_1,\dots,x_p\}$ such that $x=x_0<x_1<\dots<x_p=y$ and $F\subseteq S\cap[x,y]$ $\Rightarrow$ Since $f$ is discontinuous at points of finite set $F$, using previous result gives $\sum_{x_j\in F} u(x_j)+\sum_{x_j\in F} v(x_j)\leq$ $\leq f(y)-f(x)$ (even if $f$ is continuous at some of points $x,y$ taking $x<x_i<y_i<y$ with boundaries $x_i, y_i$ gives $\sum_{x_i\in F} u(\cdot)+\sum_{x_i\in F} v(\cdot)\leq f(y_i)-f(x_i)$)

$\left( \sum_{t_j \in F} u(t_j) + \sum_{t_j \in F} v(t_j) \leq f(y_i) - f(x_i) \leq f(y) - f(x_i) \leq f(y) - f(x) \right)$

Therefore, regardless of $f$'s continuity at points $x, y \Rightarrow$ we get that

$\sum_{x_j \in F} u(x_j) + \sum_{x_j \in F} v(x_j) \leq f(y) - f(x)$ for sets containing finite number of pts $F \subseteq S \cap [x,y]$

Since $f(y) - f(x)$ is fixed wrt $x, y$ values and set $F$ is finite, the summations $\sum_{x_j \in F} u(x_j)$ and $\sum_{x_j \in F} v(x_j)$ will contain finitely many terms; therefore, taking supremum over all such finite sets $F$ to the LHS (taking supremum twice as finite set $F$ varies ($F \subseteq S \cap [x,y]$) along with $\sum_{x_j \in F} u(x_j)$, and $\sum_{x_j \in F} v(x_j)$ respectively (one-by-one, first to the left-sum, then to right-sum) (Initially, getting $\sum_{x_j \in F} u(x_j) \leq f(y) - f(x) - \sum_{x_j \in F} v(x_j)$ and taking supremum, then having $\sum_{x_j \in F} v(x_j) \leq f(y) - f(x) - \sum_{x_j \in S \cap [x,y]} u(x_j)$ would reveal) the result:

$$\sup\left\{ \sum_{x_j \in F} u(x_j) : \underset{F\text{-finite}}{F \subseteq S \cap [x,y]} \right\} + \sup\left\{ \sum_{x_j \in F} v(x_j) : \underset{F\text{-finite}}{F \subseteq S \cap [x,y]} \right\}$$

$$= \boxed{\sum_{x_j \in S \cap [x,y]} u(x_j) + \sum_{x_j \in S \cap [x,y]} v(x_j) \leq f(y) - f(x) \text{ for fixed values } x, y \in [a,b]}$$

In the end, combining with previous result of $S_f(y) - S_f(x)$ where $u, v$ - nonnegative, we conclude

$$Sf(y)-Sf(x)=\sum_{t_j\in S(x,y]} u(t_j)+\sum_{t_j\in S[x,y)} v(t_j)\le \sum_{t_j\in S[x,y]} u(t_j)+$$

$$+\sum_{t_j\in S[x,y]} v(t_j)\le f(y)-f(x)\Rightarrow \boxed{Sf(y)-Sf(x)\le f(y)-f(x) \text{ for } x,y\in[a,b],\ x<y}$$

This concludes problem $\boxed{7}$ ■

8) To see that $f_c$ is continuous from the left, first fix any $x$ in $(a,b]$ and any $\varepsilon>0$. Since $f(x^-)=\lim_{y\to x^-} f(y)$ exists, we can choose a $\delta>0$ such that, for all $y$ in $(x-\delta,x)$, $0\le f(x^-)-f(y)<\varepsilon$. Choosing any $y$ in $(x-\delta,x)$ and using previous computations from last problem

$$f_c(x)-f_c(y)=\big(f(x)-Sf(x)\big)-\big(f(y)-Sf(y)\big)=f(x)-f(y)-$$

$$-\big(Sf(x)-Sf(y)\big)=f(x)-f(y)-\Big(\sum_{t_j\in S(y,x]} u(t_j)+\sum_{t_j\in S[y,x)} v(t_j)\Big)$$

where we had discussed extensively on how to find $Sf(x)-Sf(y)$ with $x>y \Rightarrow f_c(x)-f_c(y)=f(x)-f(y)-u(x)-v(y)-$

$$-\Big(\sum_{t_j\in S(y,x)} u(t_j)+\sum_{t_j\in S(y,x)} v(t_j)\Big)=[f(x)-u(x)]-[f(y)+v(y)]-$$

$$-\sum_{t_j\in S(y,x)} u(t_j)+v(t_j)=f(x^-)-f(y^+)-\sum_{t_j\in S(y,x)} u(t_j)+v(t_j)\le$$

$f(y^+)\ge f(y)$ $\qquad$ $u,v$ - nonnegative

$\le f(x^-)-f(y^+)\le f(x^-)-f(y)<\varepsilon$. Thus, $\boxed{f_c \text{ is continuous from the left at } x\in(a,b]}$ $\boxed{8}$ ■

18) a) $\Rightarrow V_f(x) = V_f^+(x) + V_f^-(x)$ for $\forall x \in [a,b]$

Let $P = \{a = x_0, x_1, \dots, x_n = b\} \in \Pi[a,b]$. Then $\Delta f_k = f(x_k) - f(x_{k-1})$ is either positive, negative, or zero for $\forall k \in \{0,1,\dots,n\}$

If $\Delta f_k \geq 0$, then $k \in J^+(P)$. If $\Delta f_k < 0$, then $k \in J^-(P)$

Therefore, $V_f(P) = \sum_{j \in J^+(P)} |f(x_j) - f(x_{j-1})| + \sum_{j \in J^-(P)} |f(x_j) - f(x_{j-1})|$

$= \sum_{j \in J^+(P)} \Delta f_j + \sum_{j \in J^-(P)} |\Delta f_j| = p_f(P) + n_f(P)$ where the positive variation of $f$ associated with $P$ is denoted by $p_f(P)$ and is defined to be $p_f(P) = \sum_{k \in J^+(P)} \Delta f_k$. Negative variation of $f$ associated with $P$ is denoted as $n_f(P)$ and defined to be $n_f(P) = \sum_{k \in J^-(P)} |\Delta f_k| \Rightarrow$ Total Positive variation of $f$ on $[a,b]$ is denoted by $V^+(f;a,b)$ and $V^+(f;a,b) = \sup\{p_f(P) : P \in \Pi[a,b]\}$. Total Negative Variation of $f$ on $[a,b]$ is denoted by $V^-(f;a,b) = \sup\{n_f(P) : P \in \Pi[a,b]\}$. We also define total positive/negative variation function of $f$ to be given by $V_f^+(x) = V^+(f;a,x)$ and $V_f^-(x) = V^-(f;a,x)$

Since $V_f(a) = 0$, and for $x \in (a,b]$, $V_f(x) = V(f;a,x) = \sup\{\sum_{j=1}^{r} |\Delta f_j| : \pi \in \Pi[a,x]\}$

Applying ⊛ result from the previous page, we get the following:

$V_f(x) = V(f; a, x) = \sup\{V_f(P) : P \in \Pi[a,x]\} =$ 

we applied ⊛ result here

$= \sup\{p_f(P) + n_f(P) : P \in \Pi[a,x]\} = \sup\{p_f(P) : P \in \Pi[a,x]\}$

$+ \sup\{n_f(P) : P \in \Pi[a,x]\} = V^+(f; a, x) + V^-(f; a, x)$

$= V_f^+(x) + V_f^-(x)$ for $\forall x \in (a, b]$. Since $V_f(a) = 0$, we also find

$V_f(a) = 0 = V_f^+(a) + V_f^-(a) = 0 + 0$ ✓ Therefore, $x = a$ is also considered

$\Rightarrow \boxed{V_f(x) = V_f^+(x) + V_f^-(x) \text{ for } \forall x \in [a, b]}$ ⊠ (since we know that the sums

($p_f(P)$ and $n_f(P)$ are bounded summations for $P \in \Pi[a,x]$)
(and non-empty)

(Therefore, we could split the supremum summation $p_f(P) + n_f(P)$ into summation of supremums: $\sup\{p_f(P)\} + \sup\{n_f(P)\}$

(In fact, that step is also clear from previous problems/considerations as well

Since $x = a + b \le \sup A + \sup B$ where $a \in A, b \in B$, and $A, B$ - bounded

$\Rightarrow \sup A + \sup B \rightsquigarrow$ upper-bound. So, $\underline{\sup(A+B) \le \sup A + \sup B}$

$\forall x = a + b$, $x \le \sup(A+B) \Rightarrow a \le \sup(A+B) - b$ for some fixed

$b \in B \Rightarrow \sup(A+B) - b \rightsquigarrow$ upper bound for $a \in A \Rightarrow \sup A \le \sup(A+B)$

$- b$ and $b \le \sup(A+B) - \sup A \rightsquigarrow$ upper bound for $b \in B$ and so,

$\sup B \le \sup(A+B) - \sup A \Rightarrow \underline{\sup(A+B) \ge \sup A + \sup B} \Rightarrow \underline{\sup(A+B) =}$

$\underline{= \sup A + \sup B}$

In conclusion, verifying the supremum property, and taking into account the boundedness of $pf(P)$ and $nf(P)$ by the difference of right-end point's $f$ value and left-end point's $f$ value for each of their defined intervals (in fact, they are bounded by $V_f(x) = V(f;a,x) \leq V_f(b) \leq f(b) - f(a)$ (bounded above)) we can finalize that $\boxed{V_f(x) = V_f^+(x) + V_f^-(x) \text{ for } \forall x \in [a,b]}$ $\boxed{\text{V}}$ ■

b) Since $V_f^+(x) = V^+(f;a,x) = \sup\left\{ \sum_{j \in J^+(\pi)} \Delta f_j : \pi \in \Pi[a,x] \right\}$

where $\Delta f_j \geq 0$ from definition's statement, this concludes $\boxed{V_f^+(x) \geq 0}$ Similarly, $V_f^-(x) = V^-(f;a,x) =$

$= \sup\left\{ \sum_{j \in J^-(\pi)} |\Delta f_j| : \pi \in \Pi[a,x] \right\} \geq 0$ which is obvious to see $\Rightarrow \boxed{V_f^-(x) \geq 0}$

where $x \in (a,b]$

Hence, $V_f^+(a) = V_f^-(a) = 0$ guarantees that $x = a$ should also be included:

$\boxed{V_f^+(x), V_f^-(x) \geq 0 \text{ for } \forall x \in [a,b]}$ Using the previous part a) above, its relation gives us

$V_f(x) = V_f^+(x) + \underbrace{V_f^-(x)}_{\geq 0} \geq V_f^+(x)$ and $V_f(x) = \underbrace{V_f^+(x)}_{\geq 0} + V_f^-(x)$

$\geq V_f^-(x)$ for $x \in [a,b] \Rightarrow$ $\boxed{\begin{matrix} 0 \leq V_f^+(x) \leq V_f(x) \\ 0 \leq V_f^-(x) \leq V_f(x) \end{matrix} \text{ for } \forall x \in [a,b]}$

$\oplus$ $\boxed{\text{V}}$ ■