Group 1 HW 6 (6 May 2021)

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$\boxed{4.58}$ $\lim_{x\to0^+} x^x = \lim_{x\to0^+}(e^{\ln x})^x = e^{\lim_{x\to0^+} x\ln x}$ by thm

Consider $\lim_{x\to0^+} x\ln x = \lim_{(1/y)\to0^+} \frac{1}{y}\ln(\frac{1}{y})$

$= \lim_{y\to\infty} \frac{-\ln y}{y}$

$= -\lim_{y\to\infty} \frac{\frac{d}{dy}\ln y}{\frac{d}{dy}y}$ by L'hospital Form 2

$= -\lim_{y\to\infty}\frac{1}{y}$

$= 0$

$\therefore \lim_{x\to0^+} x^x = e^{\lim_{x\to0^+}x\ln x} = e^0 = \underline{1}$

$\boxed{4.85}$ Let $f(x)=\begin{cases} 0 & ; x=0 \\ e^{-1/x^2} & ; x\neq 0\end{cases}$

For $x\neq0$, $f(x) = e^{-1/x^2}$ is differentiable in all degrees in $(-\infty,0)\cup(0,\infty)$

We will prove that $\forall n\in\mathbb{N}\cup\{0\}$, $f^{(n)}(x) = (e^{-1/x^2})R(x)$, for some rational function $R(x)$; $x\neq0$

Basic step: $n=0 \to f^{(0)}(x) = (e^{-1/x^2})(1)$ true for 0

Induction step: Let $f^{(n-1)}(x) = (e^{-1/x^2})R(x)$ for some rational function $R(x)$

$\therefore f^{(n)}(x) = \frac{d}{dx}(e^{-1/x^2})\left(\frac{P(x)}{Q(x)}\right)$; for some polynomial $P(x), Q(x)$ such that $R(x)=\frac{P(x)}{Q(x)}$

$= (e^{-1/x^2})\left(\frac{2}{x^3}\right)\left(\frac{P(x)}{Q(x)}\right) + (e^{-1/x^2})\left(\frac{P'(x)Q(x)-P(x)Q'(x)}{(Q(x))^2}\right)$

$= (e^{-1/x^2})\left[\frac{2P(x)Q(x)+P'(x)Q(x)x^3-P(x)Q'(x)x^3}{(Q(x))^2x^3}\right]$

since $2P(x)Q(x)+P'(x)Q(x)x^3-P(x)Q'(x)x^3$ and $(Q(x))^2x^3$ are polynomials $\therefore$ True for n

We will prove that $\forall n\in\mathbb{N}\cup\{0\}$, $f^{(n)}(0)=0$

Basic step: $n=0 \to f^{(0)}(0) = f(0) = 0$ true for 0

Induction step: Let $f^{(n-1)}(x) = \begin{cases}(e^{-1/x^2})R(x) & ; x\neq0\\ 0 & ; x=0\end{cases}$ for some rational function $R(x)$

$\therefore f^{(n)}(0) = \lim_{h\to0}\frac{f^{(n-1)}(0+h)-f^{(n-1)}(0)}{h}$

$= \lim_{h\to0}\frac{1}{h}e^{-1/h^2}R(h)$

$= \lim_{h\to0}\frac{1}{h}e^{-1/h^2}\frac{P(h)}{Q(h)}$ for some polynomial $P(x), Q(x)$; $R(x)=\frac{P(x)}{Q(x)}$

Suppose $Q(x) = x^k Q_0(x)$, for some $k \in \mathbb{N} \cup \{0\}$, and some polynomial $Q_0(x)$; $Q_0(0) \neq 0$

$\therefore f^{(n)}(0) = \lim_{h \to 0} \frac{1}{h} e^{-1/h^2} \frac{P(h)}{Q(h)} = \left(\lim_{h \to 0} \frac{1}{h^{k+1}} e^{-1/h^2} \right) \left(\lim_{h \to 0} \frac{P(h)}{Q_0(h)} \right) = \frac{P(0)}{Q_0(0)} \left(\lim_{h \to 0} \frac{1}{h^{k+1}} e^{-1/h^2} \right)$

We will prove that $\forall k \in \mathbb{N} \cup \{0\}$; $\lim_{x \to \infty} \frac{x^{k+1}}{e^{x^2}} = 0$

Basic step: $k=0 \longrightarrow \lim_{x \to \infty} \frac{x}{e^{x^2}} = \lim_{x \to \infty} \frac{1}{2xe^{x^2}} = 0$ by L'hospital Form 2

Induction step: Let $\lim_{x \to \infty} \frac{x^i}{e^{x^2}} = 0$ for $i = 0, 1, \dots, k$

$$\lim_{x \to \infty} \frac{x^{k+1}}{e^{x^2}} = \lim_{x \to \infty} \frac{(k+1)x^k}{2xe^{x^2}}$$ by L'hospital Form 2

$$= \frac{k+1}{2} \left(\lim_{x \to \infty} \frac{x^{k-1}}{e^{x^2}} \right)$$

$$= 0$$ by induction hypothesis

Consider $\lim_{h \to 0^+} \frac{1}{h^{k+1}} e^{-1/h^2} = \lim_{1/x \to 0^+} x^{k+1} e^{-x^2} = \lim_{x \to \infty} \frac{x^{k+1}}{e^{x^2}} = 0$

Consider $\lim_{h \to 0^-} \frac{1}{h^{k+1}} e^{-1/h^2} = \lim_{-1/x \to 0^-} (-x)^{k+1} e^{-x^2} = (-1)^{k+1} \lim_{x \to \infty} \frac{x^{k+1}}{e^{x^2}} = 0$

$\therefore \lim_{h \to 0^+} \frac{1}{h^{k+1}} e^{-1/h^2} = \lim_{h \to 0^-} \frac{1}{h^{k+1}} e^{-1/h^2} = 0 \qquad \therefore \lim_{h \to 0} \frac{1}{h^{k+1}} e^{-1/h^2} = 0$

$\therefore f^{(n)}(0) = \frac{P(0)}{Q_0(0)} \left(\lim_{h \to 0} \frac{1}{h^{k+1}} e^{-1/h^2} \right) = 0$

$\therefore k^{th}$ Taylor polynomial of f at $x_0 = 0$ is $P_k(x) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!}(x-0)^i = \sum_{i=0}^{k} 0 = 0$

$\therefore$ For $x = 0$, $\lim_{k \to \infty} P_k(x) = 0 = f(x)$

For $x \neq 0$, $\lim_{k \to \infty} P_k(x) = 0 \neq e^{-1/x^2} = f(x)$

Consider $\forall x \neq 0$, $\exists \varepsilon = \frac{1}{2} e^{-1/x^2} > 0$, $\forall k \in \mathbb{N}$, $|f(x) - P_k(x)| = |f(x) - 0| = e^{-1/x^2} > \varepsilon$

$\therefore P_k(x)$ converges to f if and only if $x = 0$

68) Since $f(c+0)+f(c-0)-2f(c)=2f(c)-2f(c)=0$ where $F(x)=f(c+x)+f(c-x)-2f(c)$, $G(x)=x^2 \Rightarrow F(0)=G(0)=0$

Then, applying l'Hopital's Rule for $\frac{F(x)}{G(x)} \Rightarrow$

$\lim_{h\to 0}\frac{F(h)}{G(h)}=\lim_{h\to 0}\frac{F'(h)}{G'(h)}$ where f is diff'ble on some nbd of c, $f''(c)$ exists and $G(x)$ does not vanish on the deleted nbd ($G(x)=x^2=0 \Leftrightarrow x=0$)

$F'(x)=f'(c+x)+f'(c-x)(-1)=f'(c+x)-f'(c-x)$ and

$G'(x)=2x \Rightarrow \lim_{h\to 0}\frac{F'(h)}{G'(h)}=\lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h}$ and

from exercise 4.7, we know $\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h}=$

$=g'(c)$ where g is diff'ble on nbd $N(c)$ of point c

Taking $g(x)=f'(x)$ where f is diff'ble on nbd of c and $f''(c)$-exists $\Rightarrow \lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h}=f''(c)$

Thus, $\boxed{\lim_{h\to 0}\frac{f(c+h)+f(c-h)-2f(c)}{h^2}=f''(c)}$ ✓

71) a) For $x>0$, $f(x)=e^{-1/x}$ and $\frac{df(x)}{dx}=f'(x)=e^{-1/x}\cdot\frac{d}{dx}\left(\frac{-1}{x}\right)$

$=e^{-\frac{1}{x}}\cdot\frac{1}{x^2}$, $\boxed{x>0 \Rightarrow f'(x)=\frac{e^{-\frac{1}{x}}}{x^2}}$ ⊠ $x\leq 0 \Rightarrow f(x)=0$, or just $\boxed{x<0 \Rightarrow f'(x)=0}$ ⊠ ✓

Thus, $x\neq 0 \Rightarrow f$ is diff'ble at x as shown above with desired $f'(x)$ values

b) Let's first evaluate $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}$ and $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}$ to prove that they are equal to 0 and this'd be $f'(0)$

$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^-}\frac{0-0}{x}=0$ since $f(x)=0$ for $x\leq 0$ (specifically, $f(0)=0$)

$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}-0}{x}=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{x}$ where $f(0)=0$ and

$f(x)=e^{-1/x}$ with $x>0 \Rightarrow$ Let $\frac{1}{x}=t\to\infty \Rightarrow \lim_{t\to\infty} te^{-t}=$

$=\lim_{t\to\infty}\frac{t}{e^t}$, since $\lim_{t\to\infty} t=\lim_{t\to\infty} e^t=\infty$, applying L'Hopital gives

$\lim_{t\to\infty}\frac{t}{e^t}=\lim_{t\to\infty}\frac{1}{e^t}=\lim_{t\to\infty}\frac{1}{\infty}=0 \Rightarrow \boxed{\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=0}$

Hence, $\boxed{\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=0}$ ⊠ which means that

$\boxed{\lim_{x\to 0}\frac{f(x)-f(0)}{x}=f'(0)=0}$ ⊠ from the definition of derivative $\Rightarrow f'(0)$ exists ✓

c) As proved previously, $f'(x) = \begin{cases} \frac{e^{-1/x}}{x^2}, & x>0 \\ 0, & x \leq 0 \end{cases}$

where we also proved $f'(0)=0$

For $x>0$, $f'(x) = \frac{e^{-1/x}}{x^2}$ and $\frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{e^{-1/x}}{x^2}\right) =$

$$= \frac{\frac{d}{dx} e^{-1/x} \cdot x^2 - e^{-1/x} \cdot 2x}{x^4} = \frac{e^{-1/x} \cdot \frac{1}{x^2} \cdot x^2 - 2xe^{-1/x}}{x^4} =$$

$= \frac{e^{-1/x} - 2xe^{-1/x}}{x^4} \Rightarrow$ for $\boxed{x>0,\ f''(x) = \frac{e^{-1/x}(1-2x)}{x^4}}$ and

for $x<0$, since $f'(x)=0 \Rightarrow \frac{d}{dx} f'(x) = f''(x) = 0$, meaning

$\boxed{x<0 \Rightarrow f''(x)=0}$ Thus, for $x \neq 0$, f' is diff'ble at x and $f''(x)$ have been evaluated accordingly

d) $\lim\limits_{x \to 0^-} \frac{f'(x) - f'(0)}{x} = \lim\limits_{x \to 0^-} \frac{0-0}{x} = 0$ since $f'(x)=0$ for $x \leq 0$

$\lim\limits_{x \to 0^+} \frac{f'(x) - f'(0)}{x} = \lim\limits_{x \to 0^+} \frac{e^{-1/x}/x^2}{x} = \lim\limits_{x \to 0^+} \frac{e^{-1/x}}{x^3}$ where

$f'(0)=0$, $f'(x) = e^{-1/x}/x^2$ for $x>0 \Rightarrow$ Let $\frac{1}{x} = t \xrightarrow{\text{with } x \to 0^+} \infty$,

$\lim\limits_{t \to \infty} \frac{e^{-t}}{1/t^3} = \lim\limits_{t \to \infty} \frac{t^3}{e^t} = \lim\limits_{t \to \infty} \frac{3t^2}{e^t} = \lim\limits_{t \to \infty} \frac{6t}{e^t} = \lim\limits_{t \to \infty} \frac{6}{e^t} = 0$

since $\lim\limits_{t \to \infty} e^t = \infty$ and $\lim\limits_{t \to \infty} \frac{1}{\infty} = 0 \Rightarrow \boxed{\lim\limits_{x \to 0^+} \frac{f'(x) - f'(0)}{x} = 0}$

Combining previous result, we get

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$\lim\limits_{x\to 0^-} \dfrac{f'(x)-f'(0)}{x} = \lim\limits_{x\to 0^+} \dfrac{f'(x)-f'(0)}{x} = 0 \Rightarrow$ Hence, this means from the definition of derivatives (take $g = f'$) $f''(0)$ exists and $\boxed{f''(0)=0}$ ✓

e) From previous part, we can conclude that

$$f''(x) = \begin{cases} \dfrac{e^{-1/x}(1-2x)}{x^4}, & x>0 \\ 0, & x\le 0 \end{cases} \qquad f'(x) = \begin{cases} \dfrac{e^{-1/x}}{x^2}, & x>0 \\ 0, & x \le 0 \end{cases}$$

Claim: $\lim\limits_{t\to\infty} \dfrac{t^k}{e^t} = 0$, where $k\in\mathbb{N}$

Proof: Since $\lim\limits_{t\to\infty} e^t = \infty$, applying L'Hospital's Rule iteratively, we get $\lim\limits_{t\to\infty} \dfrac{t^k}{e^t} = \lim\limits_{t\to\infty} \dfrac{k t^{k-1}}{e^t} = \dots = \lim\limits_{t\to\infty} \dfrac{k(k-1)\dots 2t}{e^t}$

$= \lim\limits_{t\to\infty} \dfrac{k!}{e^t} = 0$ since $\lim\limits_{t\to\infty} \dfrac{1}{e^t} = \lim\limits_{t\to\infty} \dfrac{1}{\infty} = 0$ ✓

Claim: $f^{(n)}(x) = \begin{cases} \dfrac{e^{-1/x}\cdot P(x)}{x^{2n}}, & x>0 \\ 0, & x\le 0 \end{cases}$ for each $n\in\mathbb{N}$ where we've $P(x)$ - polynomial $\deg P < 2^n$

Proof: The cases $n=1,2$ hold true ✓ Assume it's true for $n=k$. We have to show for $n=k+1$? Since we've $f^{(k)}(x)=0$ for $x<0$ and $f^{(k)}(x) = \dfrac{e^{-1/x}P(x)}{x^{2k}}$ for $x>0$ $\deg P < 2^k$

Let $x>0 \Rightarrow f^{(k)}(x)=\dfrac{e^{-1/x}P(x)}{x^{2^k}}$ and $\dfrac{d}{dx}f^{(k)}(x)=f^{(k+1)}(x)=$

$$=\frac{\frac{d}{dx}e^{-1/x}P(x)\cdot x^{2^k}-e^{-1/x}P(x)\cdot\frac{d}{dx}x^{2^k}}{\left(x^{2^k}\right)^2}=$$

$$=\frac{x^{2^k}\left(\frac{d}{dx}e^{-1/x}\cdot P(x)+e^{-1/x}\cdot P'(x)\right)-e^{-1/x}P(x)2^k x^{2^k-1}}{x^{2^{k+1}}}=$$

$$=\frac{P(x)\cdot e^{-1/x}\cdot\frac{1}{x^2}\cdot x^{2^k}+e^{-1/x}P'(x)x^{2^k}-e^{-1/x}P(x)2^k x^{2^k-1}}{x^{2^{k+1}}}$$

$$=\frac{e^{-1/x}\left[P(x)x^{2^k-2}+P'(x)x^{2^k}-P(x)x^{2^k-1}\cdot 2^k\right]}{x^{2^{k+1}}}, \text{ so}$$

Let $\boxed{\deg P=m<2^k} \Rightarrow \boxed{f^{(k+1)}(x)=\dfrac{e^{-1/x}Q(x)}{x^{2^{k+1}}} \text{ for } x>0}$

where $\deg Q\le m+2^k-1<2^k+2^k-1=2^{k+1}-1<2^{k+1}$ since $\deg\left(P(x)x^{2^k-2}\right)=m+2^k-2$ and $\deg\left(P'(x)x^{2^k}\right)=m-1+2^k$, $\deg\left(P(x)x^{2^k-1}2^k\right)=m+2^k-1 \Rightarrow$ Therefore, we've

$$\boxed{f^{(k+1)}(x)=\frac{e^{-1/x}Q(x)}{x^{2^{k+1}}} \text{ for } x>0 \quad \begin{array}{l}\deg Q<2^{k+1}\\ Q-\text{polynomial}\end{array}}$$

Now, for $x<0 \Rightarrow f^{(k)}(x)=0$ and $f^{(k+1)}(x)=\dfrac{d}{dx}f^{(k)}(x)=0$

$x<0 \Rightarrow f^{(k)}(x)=0$ and $\frac{d}{dx} f^{(k)}(x) = f^{(k+1)}(x)=0$, meaning

$\boxed{f^{(k+1)}(x)=0 \text{ for } x<0}$ Thus, $x\neq 0 \Rightarrow f^{(k)}(x)$ is diff'ble at x with computed $f^{(k+1)}(x)$ values

Moreover, since $f^{(k)}(x)=0$ at $x=0 \Rightarrow f^{(k)}(0)=0$ or

$$\boxed{\lim_{x\to 0^-} \frac{f^{(k)}(x)-f^{(k)}(0)}{x} = \lim_{x\to 0^-} \frac{0-0}{x}=0}$$ because we have $f^{(k)}(x)=0$ for $x\leq 0$

$$\lim_{x\to 0^+} \frac{f^{(k)}(x)-f^{(k)}(0)}{x} = \lim_{x\to 0^+} \frac{e^{-1/x}P(x)/x^{2k}}{x}$$ where we had

$f^{(k)}(x) = \frac{e^{-1/x}P(x)}{x^{2k}}$ for $x>0 \Rightarrow \lim_{x\to 0^+} \frac{e^{-1/x}P(x)}{x^{2k+1}} = ?$

Let $P(x) = a_m x^m + \dots + a_1 x + a_0$ with $m<2^k<2^{k+1}$ and let $\frac{1}{x}=t \Rightarrow x\to 0^+$ means $t\to\infty$ and $\frac{P(x)}{x^{2^{k+1}}} = \frac{a_m x^m + \dots + a_0}{x^{2^{k+1}}} =$

$$= a_m\left(\frac{1}{x}\right)^{2^{k+1}-m} + \dots + a_1\left(\frac{1}{x}\right)^{2^{k+1}-1} + a_0\left(\frac{1}{x}\right)^{2^{k+1}} =$$

$= a_m t^{2^{k+1}-m} + \dots + a_1 t^{2^{k+1}-1} + a_0 t^{2^{k+1}}$, meaning that

$$\lim_{t\to\infty} \frac{1}{e^t}\left[a_m t^{2^{k+1}-m} + \dots + a_1 t^{2^{k+1}-1} + a_0 t^{2^{k+1}}\right] =$$

$$= \lim_{t\to\infty} a_m \cdot \frac{t^{2^{k+1}-m}}{e^t} + \dots + a_1 \cdot \frac{t^{2^{k+1}-1}}{e^t} + a_0 \cdot \frac{t^{2^{k+1}}}{e^t}$$ where

$2^{k+1}-m>0 \Rightarrow$ since previous claim proved that

$\lim\limits_{t\to\infty} \frac{t^k}{e^t} = 0$ for $\forall k \in \mathbb{N} \Rightarrow \lim\limits_{x\to 0^+} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} =$

$$= \lim_{t\to\infty} a_m \cdot \frac{t^{2^{k+1}-m}}{e^t} + \dots + a_1 \cdot \frac{t^{2^{k+1}-1}}{e^t} + a_0 \cdot \frac{t^{2^{k+1}}}{e^t} =$$

$= a_m \cdot 0 + \dots + a_1 \cdot 0 + a_0 \cdot 0 = 0$ where $2^{k+1} - m > 0 \Rightarrow$

$$\boxed{\lim_{x\to 0^+} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = 0 = \lim_{x\to 0^-} \frac{f^{(k)}(x) - f^{(k)}(0)}{x}}$$ So, this means

$f^{(k)}(x)$ is diff'ble at $x=0$: $\boxed{f^{(k+1)}(0) = 0}$ Thus, we get

$$f^{(k+1)}(x) = \begin{cases} \frac{e^{-1/x} Q(x)}{x^{2^{k+1}}}, & x > 0 \quad (Q\text{-polynomial, } \deg Q < 2^{k+1}) \\ 0, & x \le 0 \quad (\text{at the end, we found } f^{(k+1)}(0) = 0) \end{cases}$$

Hence, $n = k+1$ is true $\Rightarrow$ For all $n \geq 1$, this satisfies

From the claim, we get $\boxed{f^{(n)}(0) = 0 \text{ for } \forall n \in \mathbb{N}}$ ■

Note: Derivative of function is local property. When we're given function and point where to evaluate derivative, it suffices to look at function values in arbitrarily small nbd. That implies the derivative of func. known by analytical express, piecewise can be computed in the pieces using these analytical expressions, regardless the other pieces. Moreover, at junction points, we can find left and right derivatives to see if they match ■

3) $\Rightarrow \preceq$ is a partial order on $\Pi[a,b]$

From the definition of partial order, we have to prove

i) <u>reflexive:</u> Let $\pi = \{x_0, x_1, \dots, x_p\}$ be any partition of $\Pi[a,b]$. Since $\{x_0, x_1, \dots, x_p\} \subseteq \{x_0, x_1, \dots, x_p\}$, we can deduce that $\pi \preceq \pi$ where $\pi = \{x_0, x_1, \dots, x_p\}$. Hence $\Rightarrow$ $\pi \preceq \pi$ for all π on $\Pi[a,b]$ ✓

ii) <u>antisymmetric:</u> Let $\pi_1 = \{x_0, x_1, \dots, x_p\}$, $\pi_2 = \{y_0, y_1, \dots, y_q\}$ be any partitions on $\Pi[a,b]$. If $\pi_1 \preceq \pi_2$, then from the definition, as sets $\Rightarrow$ $\{x_0, x_1, \dots, x_p\} \subseteq \{y_0, y_1, \dots, y_q\}$

Similarly, since $\pi_2 \preceq \pi_1$ holds true, then from the def as sets $\Rightarrow$ $\{y_0, y_1, \dots, y_q\} \subseteq \{x_0, x_1, \dots, x_p\}$ If $A \subseteq B$ and $B \subseteq A$ then obviously $\Rightarrow A = B$, or just $\boxed{\{x_0, x_1, \dots, x_p\} = \{y_0, y_1, \dots, y_q\}}$

<u>This concludes the fact that</u> $\pi_1 = \pi_2$ from def of partition sets ✓

$\boxed{\text{If } \pi_1 \preceq \pi_2 \text{ and } \pi_2 \preceq \pi_1 \Rightarrow \pi_1 = \pi_2}$

iii) <u>transitive:</u> Let $\pi_1 = \{x_0, x_1, \dots, x_p\}$, $\pi_2 = \{y_0, y_1, \dots, y_q\}$, and $\pi_3 = \{z_0, z_1, \dots, z_r\}$ where $\pi_1 \preceq \pi_2$ and $\pi_2 \preceq \pi_3$. Then from the def of partition as sets $\Rightarrow$ $\{x_0, x_1, \dots, x_p\} \subseteq \{y_0, \dots, y_q\}$ and $\{y_0, y_1, \dots, y_q\} \subseteq \{z_0, z_1, \dots, z_r\}$. As we know from famous property

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If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ holds true. Therefore, $\boxed{\{x_0, x_1, \dots, x_p\} \subseteq \{z_0, z_1, \dots, z_r\}}$ becomes satisfied. From the definition of partition as sets, we conclude $\Pi_1 \preceq \Pi_3$ is true (where $\Pi_1 \preceq \Pi_2$ and $\Pi_2 \preceq \Pi_3$) $\boxed{V}$ ■

Since all these 3 properties are satisfied, we can conclude that $\boxed{\preceq \text{ is a partial order on } \Pi[a,b]}$ ✠ V ■

4) a) If $\Pi_1 \preceq \Pi_2$, then $C(\Pi_2) \subseteq C(\Pi_1)$

Let $\Pi \in C(\Pi_2)$ be any partition in $\Pi[a,b] \Rightarrow$ this means $\Pi_2 \preceq \Pi$ and as we had $\Pi_1 \preceq \Pi_2$, it is clear that using the fact "$\preceq$ is a partial order on $\Pi[a,b]$" implies it's transitive: If $\Pi_1 \preceq \Pi_2$ and $\Pi_2 \preceq \Pi_3$, then $\Pi_1 \preceq \Pi_3 \Rightarrow$ Since $\Pi_1 \preceq \Pi_2$ and $\Pi_2 \preceq \Pi$, we get $\boxed{\Pi_1 \preceq \Pi}$ $C(\Pi_1) = \{\Pi' \text{ in } \Pi[a,b] : \Pi_1 \preceq \Pi'\}$ where having $\Pi_1 \preceq \Pi$ means $\boxed{\Pi \in C(\Pi_1)}$ ✠ So, $\boxed{\Pi \in C(\Pi_2) \text{ is any partition in } \Pi[a,b] \Rightarrow \Pi \in C(\Pi_1)}$ ✠

which concludes $\boxed{C(\Pi_2) \subseteq C(\Pi_1)}$ ✠ $\boxed{V}$ ■

b) Notice that, if $\Pi_1 \preceq \Pi_2$, then $\|\Pi_2\| \leq \|\Pi_1\|$ where the gauge of a partition can only decrease as it is refined. More generally, the gauge can be same or decrease whenever it'll be refined

Let $\Pi_1 = \{x_0, x_1, \dots, x_p\}$ and $\Pi_2 = \{y_0, y_1, \dots, y_q\}$ where $\Pi_1 \leq \Pi_2$

From the def of partition sets, we get $\underline{\{x_0, x_1, \dots, x_p\} \subseteq \{y_0, y_1, \dots, y_q\}}$

Assume $\|\Pi_1\| = \Delta x_m$ and $\|\Pi_2\| = \Delta y_n$, where $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \dots, p$ and $\Delta y_j = y_j - y_{j-1}$ for $j = 1, \dots, q \Rightarrow \Delta x_m = \max\{\Delta x_i \mid i = 1, \dots, p\}$, and similarly, $\Delta y_n = \max\{\Delta y_j \mid j = 1, \dots, q\}$. Since $\{x_0, x_1, \dots, x_p\} \subseteq \{y_0, y_1, \dots, y_q\}$ all the points x_i will be present in $\{y_0, y_1, \dots, y_q\}$ where additional points could be feasible case. If there does not exist point y_t on the subinterval $[x_{m-1}, x_m]$, then $\Delta x_m = x_m - x_{m-1}$ will be still the maximum length and $\Delta y_n = \Delta x_m$ would hold true. Otherwise, if there would exist a y_t point on (x_{m-1}, x_m), then considering that $[x_0, x_1], [x_1, x_2], \dots, [x_{p-1}, x_p]$ subintervals can contain additional pts y_j where $[x_{m-1}, x_m]$ will have one such point at least and $\Delta x_m = x_m - x_{m-1}$ was the maximum length, we can obviously summarize that $\underline{\Delta y_n \leq \Delta x_m}$ will hold true because new additional pt y_t on $[x_i, x_{i+1}]$ subinterval reduces the previous length $\Delta x_{i+1} = x_{i+1} - x_i$ into smaller distances $y_t - x_i$ and $x_{i+1} - y_t$ for Π_2. Hence, We find $\Delta y_n = \|\Pi_2\| \le \Delta x_m = \|\Pi_1\| \Rightarrow \|\Pi_2\| \le \|\Pi_1\|$

Now, let $\pi \in C(\Pi_1)$ be any partition in $\Pi[a,b]$ and this means $\boxed{\Pi_1 \le \pi}$ from the statement. If we assume $\|\Pi_1\| < \varepsilon$, using the fact "if $\Pi_1 \le \Pi_2 \Rightarrow \|\Pi_2\| \le \|\Pi_1\|$"

we get $\|\pi\| \le \|\Pi_1\| < \varepsilon \Rightarrow \boxed{\|\pi\| < \varepsilon \text{ for all } \pi \in C(\Pi_1)}$

c) From the definition "least common refinement", it is immediate that $\Pi_1 \le \Pi_1 \vee \pi$ for any partition π in $\Pi[a,b] \Rightarrow$ from the previous rule, we get $\|\Pi_1 \vee \pi\| \le \|\Pi_1\|$ and since $\|\Pi_1\| < \varepsilon$, we conclude

$\|\Pi_1 \vee \pi\| \le \|\Pi_1\| < \varepsilon \Rightarrow \boxed{\|\Pi_1 \vee \pi\| < \varepsilon}$

$\boxed{4.84}$ Let $f(x) = (1+x)^{\alpha}$, where $x = a/b$ and $-1 < x < 1$.

The k^{th} Taylor polynomial of $f(x)$ is given by

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)\,(x - x_0)^n}{n!}$$

Since $f(x) = (1+x)^{\alpha}$, we have:

- $f'(x) = \alpha(1+x)^{\alpha-1}$. Then, $f'(x_0=0) = \alpha$.
- $f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$. $f''(0) = \alpha(\alpha-1)$.

$\vdots$

- $f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)\,(1+x)^{\alpha-k}$.

Then, $f^{(k)}(0) = \alpha(\alpha-1)\cdots(\alpha-k+1)$.

$$f^{(k)}(0) = \prod_{i=1}^{k}(\alpha - i + 1), \text{ where } k \geq 1.$$

$$\Rightarrow P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)\,x^n}{n!} = f(0) + \sum_{n=1}^{k} \frac{f^{(n)}(0)\,x^n}{n!}$$

$$P_k(x) = 1 + \sum_{n=1}^{k}\left(\frac{x^n}{n!} \cdot \prod_{i=1}^{n}(\alpha - i + 1)\right)$$

$\boxed{4.84}$

(b). Lagrange's remainder $R_k(0;x)$ is given by

$$R_k(0;x) = \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1} \quad \text{where}$$

$c \in (0,x)$.

However, $f^{(k+1)}(c) = \alpha(\alpha-1)\dots(\alpha-k)(1+c)^{\alpha-(k+1)}$

$$\Rightarrow R_k(0;x) = \frac{x^{k+1}}{(k+1)!} \cdot (1+c)^{\alpha-(k+1)} \prod_{i=1}^{k+1} (\alpha-i+1)$$

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$\boxed{5.4}$

Let $\pi_1 = \{x_0, x_1, ---, x_k\}$ and let

$$C(\pi_1) = \{\pi \mid \pi_1 \leq \pi\}$$

$$= \Big\{\{v_0, v_1, ---, v_m\} \;\Big|\; \{x_0, x_1, ---, x_k\} \subseteq \{v_0, ---, v_m\}\Big\}$$

(a). Let $\pi_2 = \{y_0, y_1, ---, y_n\}$ and $\pi_1 \leq \pi_2$.

Then, $\{x_0, x_1, ---, x_k\} \subseteq \{y_0, y_1, ---, y_n\}$.

Also, $C(\pi_2) = \{\pi \mid \pi_2 \leq \pi\}$

$$= \Big\{\{v_0, v_1, ---, v_s\} \;\Big/\; \{y_0, y_1, ---, y_n\} \subseteq \{v_0, ---, v_s\}\Big\}$$

- $\{x_0, x_1, ---, x_k\} \subseteq \{y_0, y_1, ---, y_n\}$

$\Rightarrow$ $\{x_0, x_1, ---, x_k\} \subseteq \{v_0, v_1, ---, v_s\}$ for each $\{v_0, v_1, ---, v_s\} \in C(\pi_2)$.

$\Rightarrow$ $\{x_0, x_1, ---, x_k\} \subseteq C(\pi_2)$ and therefore, $\pi_1 \leq \pi_i$ for each $\pi_i \in C(\pi_2)$.

$\Rightarrow$ $C(\pi_1) \subseteq \{\pi_i\} \subseteq C(\pi_2)$

$\Rightarrow$ $C(\pi_1) \subseteq C(\pi_2)$.

$\boxed{5.4}$

(b). Let $\|\pi_1\| < \varepsilon$.

$\Rightarrow \max\{\Delta x_i \mid 1 \le i \le k\} < \varepsilon$

Let $\pi = \{v_0, v_1, ---, v_m\} \in C(\pi_1)$.

Then, $\{x_0, x_1, ---, x_k\} \subseteq \{v_0, v_1, ---, v_m\}$.

$\Rightarrow \|\pi\| \le \|\pi_1\|$ for each $\pi \in C(\pi_1)$.

$\Rightarrow \|\pi\| < \varepsilon$

(c). Let $\|\pi_1\| < \varepsilon$ and let π be any partition of $[a, b]$.

Then, $\pi_1 \le \pi_1 \vee \pi$

$\Rightarrow \|\pi_1 \vee \pi\| \le \|\pi_1\| < \varepsilon$

$\Rightarrow \|\pi_1 \vee \pi\| < \varepsilon$

$\boxed{4.72}$ a) $x \neq 0$ $\quad f'(x) = \left(e^{-\frac{1}{x^2}}\right)' = \dfrac{e^{-\frac{1}{x^2}} \cdot 2}{x^3}$

b) $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x} = 2\lim\limits_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x^4} = 2\lim\limits_{x\to\infty} \dfrac{x^4}{e^{x^2}} =$

$= 2\lim\limits_{x\to\infty} \dfrac{4x^3}{e^{x^2}\cdot 2x} = 4\lim\limits_{x\to\infty} \dfrac{x^2}{e^{x^2}} = 4\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 4\lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$

c) for $x \neq 0$ $\quad f''(x) = \left(\dfrac{2e^{-\frac{1}{x^2}}}{x^3}\right)' = \dfrac{4e^{-\frac{1}{x^2}} - 6e^{-\frac{1}{x^2}}\cdot x^2}{x^6} =$

$= \dfrac{e^{-\frac{1}{x^2}}\cdot(4-6x^2)}{x^6}$

---

a) for $x \neq 0$ $\quad f'(x) = \left(e^{-\frac{1}{x^2}}\right)' = \dfrac{2e^{-\frac{1}{x^2}}}{x^3}$

b) $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x} = \lim\limits_{x\to\infty} \dfrac{x}{e^{x^2}} =$

$\lim\limits_{x\to\infty} \dfrac{1}{2xe^{x^2}} = 0$

c) for $x \neq 0$ $\quad f''(x) = \left(\dfrac{2e^{-\frac{1}{x^2}}}{x^3}\right)' = \dfrac{e^{-\frac{1}{x^2}}(4-6x^2)}{x^6}$

d) $f''(0) = \lim\limits_{x\to 0} \dfrac{f'(x)-f'(0)}{x} = 2\lim\limits_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x^4} = 2\lim\limits_{x\to\infty} \dfrac{x^4}{e^{x^2}} =$

$= 2\lim\limits_{x\to\infty} \dfrac{4x^3}{2xe^{x^2}} = 4\lim\limits_{x\to\infty} \dfrac{x^2}{e^{x^2}} = 4\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 4\lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$

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$\boxed{4.72}$ (continued)

e) We will make similar claims to $\boxed{4.71}$

Firstly, if $p(x), q(x)$ are polynomials $\Rightarrow \lim_{x\to 0} \frac{p(x)}{q(x)} e^{\frac{-1}{x^2}} = 0$

Let m be smallest power of x in $q(x)$

Divide both top and bottom by $x^m \Rightarrow \lim_{x\to 0} \frac{\frac{p(x)}{x^m}}{\frac{q(x)}{x^m}} e^{\frac{-1}{x^2}}$

Note that $\lim_{x\to 0} \frac{q(x)}{x^m}$ doesn't vanish

and we can write $\lim_{x\to 0} \frac{p(x)}{x^m} e^{\frac{-1}{x^2}}$ as $\lim_{x\to\infty} \left(\sum_{n\in A} x^n e^{-x^2}\right)$,

but $\lim_{x\to\infty} \frac{x^{n} {\scriptstyle\to\infty}}{e^{x^2} {\scriptstyle\to\infty}} = n \lim_{x\to\infty} \frac{x^{n-1}}{2x e^{x^2}} = \frac{n}{2} \lim \frac{x^{n-2}}{e^{x^2}}$

Thus, we decrease power of x using L'Hopital's, nevertheless e^{x^2} always appears in bottom $\Rightarrow \boxed{\lim = 0}$

Next claim is $f^{(n)}(x) = \frac{p(x)}{q(x)} e^{\frac{-1}{x^2}}$ for $x \neq 0$ (p, q are polynomials)

We proceed by induction $f'(x) = \frac{2}{x^3} e^{\frac{-1}{x^2}}$

Assume $f^{(n)}(x) = \frac{p(x)}{q(x)} e^{\frac{-1}{x^2}} \Rightarrow f^{(n+1)}(x) = \left(\frac{p}{q} e^{\frac{-1}{x^2}}\right)' = \frac{q\left(p' e^{\frac{-1}{x^2}} + p\frac{2e^{\frac{-1}{x^2}}}{x^3}\right) - p e^{\frac{-1}{x^2}} q'}{q^2}$

$= e^{\frac{-1}{x^2}} \cdot \frac{(p'(x)q(x)x^3 + 2p(x)q(x) - p(x)q'(x))}{x^3 q(x)^2} = e^{\frac{-1}{x^2}} \cdot \frac{p_1(x)}{q_1(x)}$

Back to problem: Again we proceed by induction to prove $f^{(n)}(0) = 0$

for $n=1$ $f'(0) = 0$ ✓ Assume $f^{(n)}(0) = 0$

$\Rightarrow f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{p(x)}{q(x)} e^{-\frac{1}{x^2}} = 0$ done!

$\boxed{5.2}$ Assume that there is x_j in Π_1 but not in Π_2.

then $x_j \in \Pi_1 \vee \Pi_2 = \Pi_2$ contradiction