Analysis I
Homework Assignment 2 & 3

Problem 2.4

We need to show that $\| . \|_\infty$ satisfies the 3 properties of any norm.

1) $\| . \|_\infty = \max \{|x_1|, |x_2|\} >= 0$ for any $\mathbf{x}$ in R^2, and if $\|\mathbf{x}\|_\infty = 0$, then $\max \{|x_1|, |x_2| \} = 0$ ==> $x_1 = 0 = x_2$ ==> $\mathbf{x} = \mathbf{0}$. So, $\| . \|_\infty$ is positive-definite.
2) For any c in R and any $\mathbf{x}$ in R^2, one has
$\|c\mathbf{x}\|_\infty = \max \{ |cx_1, |cx_2| \} = |c| \max \{|x_1|, |x_2| \} = |c| \|\mathbf{x}\|_\infty$ ==> $\| . \|_\infty$ satisfies the absolute homogeneity property.
3) For any $\mathbf{x}$, $\mathbf{y}$ in R^2, one has
$\| \mathbf{x} + \mathbf{y} \|_\infty = \max \{|x_1 + y_1|, |x_2 + y_2| \} <= \max \{|x_1| + |y_1|, |x_2| + |y_2| \} <= \max \{|x_1|, |x_2|\} + \max\{|y_1|, |y_2|\} = \|\mathbf{x}\|_\infty + \| \mathbf{y}\|_\infty$ ==> $\| . \|_\infty$ satisfies the subadditivity property.

Problem 2.12

Let $\{x_k\}$ be an arbitrary convergent sequence in R^n and let x_0 be its limit.
a) Suppose that there is another limit of $\{x_k\}$, say x'. Let $e > 0$ be any real number $< \|x_0 – x'\|/2$. There exists a natural number k_1 for which $\|x_k – x_0\| < e$ for all $k > k_1$. Likewise, there exists a natural number k_2 such that $\|x_k – x'\| < e$ for all $k > k_2$. Then for all $k > \max(k_1, k_2)$:
$$2e < \| x_0 – x'\| <= \| x_k – x_0\| + \| x_k – x'\| < 2e,$$
contradiction. So, the limit of a convergent sequence is unique.
b) Let $e > 0$ be any real number. There exists a k_1 for which $\| x_k – x_0\| < e$ for all $k > k_1$, where x_0 is the limit of the sequence $\{x_k\}$. So, $\|x_k\| <= \| x_k – x_0\| + \|x_0\| < e + \|x_0\|$ for all $k > k_1$. Let $M = \max (\|x_1\|, …, \|x_{k_1}\|, e + \|x_0\|)$. Then $\|x_k\| <= M$ for all k.
c) Let $e > 0$ be any real number. There exists a k_1 such that $\|x_k – x_0\| < e/2$ for all $k >= k_1$ (where x_0 is the limit of the sequence $\{x_k\}$). So, for all $m >= k >= k_1$,
$\|x_k – x_m\| <= \|x_k – x_0\| + \|x_m – x_0\| <= e$ ===>
$\{x_k\}$ is Cauchy.

Problem 2.15

For any given $e > 0$, there exists a k_1 such that
$$\| x_m – x_k| < e \text{ for all } m >= k >= k_1.$$
In particular, for $k = k_1$:
$\|x_m – x_{k_1}\| < e$ for all $m >= k_1$ ===>
$\|x_m \| < \|x_{k_1}\| + e$ for all $m >= k_1$.
Let $M = \max(\|x_1\|, …, \|x_{k_1 – 1}\|, \|x_{k_1}\| + e)$ ==>
$\|x_k\| <= M$ for every natural number k ==> $\{x_k\}$ is bounded.

Problem 2.17

Note that any deleted neighborhood $N'(x_0; e)$ of x_0 contains infinitely many points of S [If a deleted neighborhood $N'(x_0; e)$ contains only finitely many points, say $x_{i_1}, … , x_{i_k}$, let $e_1 = \min(\|x_{i_1} – x_0\|, …, \|x_{i_k} – x_0\|)$. Then $N'(x_0; e_1)$ contains no points of S, contradiction]. Let x_1 be a point of S in $N'(x_0; e)$, and let x_2 be a point of S in $N'(x_0; e/2)$, which is different from x_1. Having chosen the points x_1, x_2, …, x_k of S, let x_{k + 1} be a point of S in N'(x_0; e/{k + 1}), different from x_1, x_2, …, x_k. Then {x_k} is a sequence of distinct points in S. Observe that ||x_k – x_0|| < e/k for every natural number k. Let e' > 0 be any real number. Then there exists a k_1 for which e/k_1 < e' ==> for any k > k_1, ||x_k – x_0|| < e/k <= e/k_1 < e' ==> lim{k goes to infinity} x_k = x_0.

Problem 2.25

Let T = {x_k : k in N}. Then S = T ∪ {x_0}. Obviously, x_0 is a limit point of S. Suppose that S has another limit point, say x'. Then any deleted neighborhood N'(x'; e) contains infinitely many points of S ==> infinitely points of T ==> x' is a limit point of T ==> x' is a cluster point of the sequence {x_k} ==> there exists a subsequence {x_{k_j}} of {x_k} that converges to x'; contradiction, since all the subsequences of {x_k} converges to x_0 (as x_0 is the limit of the sequence {x_k}). So, x_0 is the only limit point of S and x_0 is in S ==> S is closed.

Problem 2.29

a) interior(S) = (a, b), closure(S) = [a, b], derived set(S) = [a, b], bd(S) = {a, b} (consists of points a and b) (every point in the interval (a,b) has a neighborhood which is completely contained in S, and if x' is any point that does not belong to (a,b), then any neighborhood of x' contains a point that belongs to S-complement ==> interior(S) = (a,b). It is easy observe that, the only points whose every neighborhood contains a point of S are those that are contained in [a,b], so closure(S) = [a,b].
If x' is a point that N(x'; e) contains both point of S and S-complement for every e > 0, then x' = a or x' = b ==> bd(S) = {a, b}. Finally, the only points whose every neighborhood contains a point of S are those that are contained in [a, b] ==> derived set(S) = [a,b])

b) interior(S) = { (x, 0) in R^2 : a < x < b }, closure(S) = { (x, 0) in R^2 : a <= x <= b }, derived set(S) = { (x, 0) in R^2 : a <= x <= b }, bd(S) = {(a,0), (b,0) } (consists of points (a,0) and (b,0) in R^2).
(the same reasoning as in (a))

c) interior(S) = empty set, closure(S) = R (the set of real numbers), derived set(S) = R, bd(S) = R.
(if x is any rational number, then every neighborhood of x will contain irrational numbers ==> interior(S) = empty set.
For any real number x and any positive real number e, the neighborhood N(x;e) contains both rational numbers and irrational numbers ==> derived set(S) = closure(S) = bd(S) = R.)

d) interior(S) = empty set, closure(S) = R^n, derived set(S) = R^n, bd(S) = R^n.
(the same reasoning as in (c))

e) interior(S) = empty set, closure (S) = S, derived set (S) = S, bd(S) = S.

Problem 2.43

For a point x, d(x, S) = 0 if and only if x is in the closure of S.
So, a set S in R^n is closed <==> S contains all of its limit points <==> S = closure(S) <==> d(x, S) = 0 if and only if x is a point of S <==> d(x, S) > 0 if and only if x is a point of S-complement.
(<==> means "if and only if")

Problem 2.52
We assume that the following is true:
If {C_k} is a nested sequence of closed, bounded, non-empty subsets of R^n (where n is fixed), then
∩(k : 1 to infinity) C_k is non-empty.
Furthermore, if lim{k goes infinity} d(C_k) = 0, then ∩(k : 1 to infinity) C_k = {x_0}, where x_0 is a point of R^n.

Let {a_k} and {b_k} be any two sequences (in R) such that a_k < b_k (for every natural number k), b_k <= b_{k – 1} (for every k > 1), and a_k >= a_{k – 1} (for every k > 1). Then C_k = {(x, 0, …, 0) in R^n : a_k <= x <= b_k } is a non-empty, bounded, closed subset of R^n and {C_k} is a nested sequence ==> (according to the assumption above)
∩(k : 1 to infinity) C_k is non-empty ==>
x_0 = (c, 0, …, 0) ∈ ∩(k : 1 to infinity) C_k for some real number c ==>
c ∈ ∩(k : 1 to infinity) [a_k, b_k] ==>
∩(k : 1 to infinity) [a_k, b_k] is not empty. Note that lim{k goes to infinity} d(C_k) = 0 <==> lim{k goes to infinity} sqrt{(b_k – a_k)^2} = 0 <==> lim{k goes to infinity} (b_k – a_k) = 0. So, if lim{k goes to infinity} (b_k – a_k) = 0, then ∩(k : 1 to infinity) C_k = {x_0 = (c, 0, …, 0)} (i.e., the intersection consists of a single point) for some x_0 in R^n (and c is a real number) ==> ∩(k : 1 to infinity) [a_k, b_k] = {c}.
So, Cantor's criterion holds in R.

Problem 2.53

a) Suppose that A is an empty set => B = R (the set of real numbers) => contradiction, because S is non-empty.
Let a ∈ A and b ∈ B ==> b is an upper bound for S, but a is not ==> there exists an x_0 ∈ S for which x_0 > a ==> a < x_0 <= b ==> x_0 ∈ S ∩ [a, b] and a < b.

Problem 2.56

b) C_k = closure(N(0; 1/k)) ∩ X ==> C_k = [-1/k, 1/k] ∩ X ==> C_k = (0, 1/k) ∩ X. Suppose that ∩ (k : 1 to infinity) C_k is non-empty; let x_0 ∈ ∩ (k : 1 to infinity) C_k. There exists a natural number k_0 for which x_0 > 1/k_0 ==> x_0 ∉ C_{k_0} ==> x_0 ∉ ∩ (k : 1 to infinity) C_k, contradiction. So ∩ (k : 1 to infinity) C_k is an empty set.

Problem 2.57

c) C_k = closure(N(x_0; e_k)) ∩ X = [x_0 – e_k, x_0 + e_k] ∩ X (because if S = (a, b), then closure(S) = [a, b]). Let x_1 be any real number different from x_0 ==> |x_0 – x_1| = e > 0 ==> there exists a natural number k_0 for which e > e_{k_0} ==> x_1 ∉ [x_0 – e_{k_0}, x_0 + e_{k_0}] = closure(N(x_0; e_{k_0})) ==> x_1 ∉ C_{k_0} ==> x_1 ∉ ∩(k : 1 to infinity) C_k. Note that x_0 ∉ ∩(k : 1 to infinity) C_k as well ==> ∩(k : 1 to infinity) C_k is empty.

It is obvious that the sets C_k are nonempty (since Q is dense in R), bounded (x_0 – e_k <= x <= x_0 + e_k for all x ∈ C_k), relatively closed in X (since the sets closure(N(x_0; e_k)) are closed), and C_k ∈ C_{k – 1} for all k > 1.

Problem 2.61

Suppose the claim is false; let U and V be two disjoint open sets for which
1) $R \in U \cup V$,
2) $R \cap U$ is non-empty and $R \cap V$ is non-empty.

Let $a \in R \cap U$ and $b \in R \cap V$. It follows that
1) $[a, b] \in U \cup V$,
2) $[a,b] \cap U$ is non-empty and $[a,b] \cap V$ is non-empty.
This means that the interval $[a,b]$ is disconnected; contradiction. So, R cannot be disconnected ==> R is connected.

Problem 2.68

a) $S = R^n$ is closed (S contains all of its limit points), but is not compact, because it is not bounded.
b) There is no such set; any compact subset S of R^n is closed.
c) $S = R^n$ is an open subset of R^n (for every point x in R^n, every neighborhood of x is contained R^n), and is not compact (since it is not bounded),
d) $S = \{x \in R^n : \|x\| <= 1 \}$ is compact, since it is closed and bounded. However, it is not open, as there is no neighborhood $N(x_0; e)$ for which $\|x_0\| = 1$ and $N(x_0; e) \in S$.

Problem 2.69

Let S be any compact set in R^n and let $A \in S$ be any closed subset of S. Then A is bounded, as S is bounded. Since A is also closed (in R^n), it follows that A is compact.

Problem 2.70

$C_1, C_2, \ldots, C_k$ – compact subsets of R^n.
a) $\cup(j : 1 \text{ to } k)\ C_j$ is compact; $\cup(j : 1 \text{ to } k)\ C_j$ is closed, because each C_j is closed, and $\cup(j : 1 \text{ to } k)$ is bounded, as each C_j is bounded (let m_j be an upper bound for C_j; then $\max(m_1, m_2, \ldots, m_k)$ is an upper bound for $\cup(j : 1 \text{ to } k)\ C_j$) ==> $\cup(j : 1 \text{ to } k)\ C_j$ is compact.

b) $\cap(j : 1 \text{ to } k)\ C_j$ is compact; $\cap(j : 1 \text{ to } k)\ C_j$ is closed, since each C_j is closed,
and $\cap(j : 1 \text{ to } k)\ C_j \in C_1$ is bounded ==> $\cap(j : 1 \text{ to } k)\ C_j$ is compact.

Problem 3.17
If $x_0 \neq ½$ is a rational number in $[0,1]$, let $\{e_k\}$, where $e_1 < 1 – x_0$, be a sequence of positive irrational numbers that converges monotonically to 0 So, $\{x_0 + e_k\}$ converges to x_0, and $x_0 + e_k$ is irrational for all k. Then lim{k goes to infinity} $f(x_0 + e_k)$ = lim{k goes to infinity} $m(1 – x_0 – e_k) = m(1 – x_0) \neq f(x_0)$ ==> f is not continuous at x_0 (because f is continuous at c if and only if every Cauchy sequence $\{x_k\}$ such that lim{k goes to infinity} $x_k = c$, it follows that lim{k goes to infinity} = f(c) (where c is a limit point of the domain of f; in this problem, the domain of f is the interval $[0,1]$, and each point of this interval is a limit point).

If $x_0 \neq ½$ is an irrational number in [0,1], let $\{e_k\}$ be a sequence of positive irrational numbers that converges monotonically to 0, and satisfies $e_1 < 1 – x_0$ and $(x_0 + e_k) \in Q$ (the set of rational numbers) for every natural number k. Then $\{x_0 + e_k\}$ converges to x_0. So, lim{k goes to infinity} $f(x_0 + e_k)$ = lim{k goes to infinity} $m(x_0 + e_k) = mx_0 \neq f(x_0)$ ==> f is not continuous at x_0.

Combining the two results obtained above, we conclude that f is not continuous at any point $\neq ½$ in the interval [0,1]. It remains to show that f is continuous at $x = ½$. Let $e > 0$ be any real number, and let $a = e/m$.
If x is a rational number in the neighborhood N(1/2; a) : $|f(x) – f(1/2)| = |mx – m/2| = m|x – 1/2| < ma = e$;
if x is an irrational number in the neighborhood N(1/2; a) : $|f(x) – f(1/2)| = |m(1 – x) – m/2| = m|1/2 – x| < ma = e$.
So, if x in the neighborhood N(1/2; a), then f(x) is in the neighborhood N(f(1/2); e) ==> f is continuous at $x = ½$.

Problem 3.20

a) $x \to x/2$, $y \to x/2$: $f(x) = [f(x/2)]^2 >= 0$ ===> $f(x) >= 0$ for every real number x.

b) $f(x_0) = 0$ for some real number x_0 ==> $x \to x – x_0$, $y \to x_0$: $f(x) = f(x – x_0)f(x_0) = 0$ ==> $f(x) = 0$ for every real number x.

c) $f(x_0) \neq 0$ for some real number x_0 ==> $x \to x_0$, $y \to 0$: $f(x_0) = f(x_0)f(0)$ ==> $f(0) = 1$.

d) Suppose that lim{x goes to 0} f(x) = f(0). If $f(x_0) = 0$ some x_0, then f is identically 0 (according to (b)), so lim{x goes to c} f(x) = 0 = f(c) for every real number c ==> f is continuous on all of R. Consider the case $f(x) \neq 0$ for all x in R. Then lim{x goes to 0} f(x) = f(0) = 1. So,

lim{x goes to c} f(x) = lim{y goes to 0}f(y + c) = lim{y goes to 0} f(y)f(c) = lim{y goes to 0} f(y) × lim{y goes to 0} f(c) = f(c) ==> f is continuous at every point c of R.

e) Observe that $f(2x) = f(x)^2$ ==> $f(3x) = f(x)f(2x) = f(x)^3$ ==> … $f(nx) = f(x)^n$ for every natural number n. So, setting x = 1 ==> $f(n) = f(1)^n$ for every natural number n. Also, setting $y \to x$: $f(-x) = 1/f(x)$ ==> $f(-n) = 1/f(1)^n = f(1)^{-n}$ for every natural number n ==> $f(x) = f(1)^x$ for every integer x. Then $x \to m/n$ (where m is any integer) in the equation $f(nx) = f(x)^n$ (where n is any natural number) ==> $f(m) = f(m/n)^n$ ==> $f(m/n) = f(m)^{1/n} = f(1)^{m/n}$ ==> $f(x) = f(1)^x$ holds for every rational number x. Let c be an irrational number, and let $\{c_k\}$ be a sequence of rational numbers that converges to c. Note that c is a limit point of the set of real numbers. So, as f is continuous on all of R,

f(c) = lim{x goes to c}f(x) = lim{k goes to infinity} $f(c_k)$ = lim{k goes to infinity} $f(1)^{c_k} = f(1)^c$ ==> $f(x) = f(1)^x$ for every irrational number x ==> $f(x) = f(1)^x$ for every real number x.

Problem 3.48

a) for every $y \in [c,d]$, there exists at least one $x \in [a,b]$ such that $f(x) = y$, since f maps [a,b] onto [c,d]. Furthermore, if there existed two different values $x = x_1$ and $x = x_2$ for which $f(x) = y$, then f would not be strictly monotone. This implies that for every $y \in [c,d]$, there exists a unique value $x \in [a,b]$ such that $f(x) = y$ ==> f^{-1} maps [c,d] one-to-one onto [a,b].

b) let y_1 < y_2 < y_3 be any three numbers in [c,d], and let x_1, x_2, and x_3 be the unique values for which f(x_1) = y_1, f(x_2) = y_2, and f(x_3) = y_3 (so f^{-1}(y_i) = x_i for i = 1,2,3). Since f is strictly monotone on [a,b], it follows that either x_1 < x_2 < x_3 or x_1 > x_2 > x_3 should hold ==> f^{-1} is strictly monotone on [c,d].

Problem 3.51

Note that tan(x) = sin(x)/cos(x), and since sin(x) and cos(x) are continuous on all of R, tan(x) is also continuous at every point it is defined; that is, tan(x) is continuous at every point x for which cos(x) ≠ 0. Furthermore, tan(x) maps (-π/2, π/2) one-to-one onto R. To prove this assertion, first notice that tan(x_1) = tan(x_2) for x_1, x_2 ∈ (- π/2, π/2) implies x_1 = x_2 ==> tan(x) is one-to-one on (- π/2, π/2). Secondly,

y = tan(x) = sin(x)/cos(x) ==>
sin(x) = cos(x) y ===>

1 = sin(x)^2 + cos(x)^2 = cos(x)^2 (y^2 + 1) ==>

which is solvable for every real number y, since cos(x)^2 can take any real value not exceeding 1. So, the claim is justified. Consequently, $\tan^{-1}$(x) maps R one-to-one onto (- π/2, π/2).

Another observation is that tan(x) is strictly monotone increasing on (- π/2, π/2). This, in turn, implies that $\tan^{-1}$(x) is strictly monotone increasing on R.

Let 0 < x_0 < π/2 be any real number. Consider the compact set [-x_0, x_0].
tan(x) is continuous and strictly increasing on [-x_0, x_0], and maps [-x_0, x_0] one-to-one onto [-tan(-x_0), tan(x_0)]. Therefore, $\tan^{-1}$(x) is continuous and strictly increasing on [-tan(-x_0), tan(x_0)], and maps [-tan(-x_0), tan(x_0)] one-to-one onto [-x_0, x_0]. Choosing x_0 close enough to π/2, one can make the interval [-tan(-x_0), tan(x_0)] as large as one wishes. So, $\tan^{-1}$(x) is continuous on all of R.

a) the function $\tan^{-1}$(x) has domain R (the set of real numbers), so f(x) has domain R^2 (there is no restriction on the values of x_1 and x_2). As noted above, $\tan^{-1}$(x) is strictly increasing on R. Since $\tan^{-1}$(0) = 0, and since the range of $\tan^{-1}$(x) is the interval (- π/2, π/2), it follows that the range of f(x) is the interval [0, π/2].

b) g(x) = $\tan^{-1}$(x) is continuous on all of R. h(**x**) = $x_1^2 + x_2^2$ is also continuous on all of R^2. Consequently, g(h(**x**)) = $\tan^{-1}(x_1^2 + x_2^2)$ is continuous at every point (x_1,x_2) of R^2.

c) As noted above, $\tan^{-1}$(x) is strictly increasing on R, and maps the real line one-to-one onto the interval (-π/2, π/2). The immediate consequence is that lim{x goes to infinity} $\tan^{-1}$(x) = π/2. Then

lim{||**x**|| goes to infinity} $\tan^{-1}(x_1^2 + x_2^2)$ = lim{x goes to infinity} $\tan^{-1}$(x) = π/2.

Problem 3.59

Let $g(x) = f(x) - x$ for all $x \in [a,b]$.
If $g(x) >= 0$ for all the values of x, then $g(b) >= 0 \implies f(b) >= b \implies f(b) = b \implies f(x) = x$ holds for $x = b$.
If $g(x) <= 0$ for all the values of x, then $g(a) <= 0 \implies f(a) <= a \implies f(a) = a \implies f(x) = x$ holds for $x = a$.
Suppose that there exist real numbers x_1 and x_2 in $[a,b]$ for which $g(x_1) < 0$ and $g(x_2) > 0$. By the intermediate value theorem, there should exist a value x_0 such that $x_1 < x_0 < x_2$ and $g(x_0) = 0 \implies f(x_0) = x_0 \implies f(x) = x$ holds for $x = x_0$.

Problem 3.76

a) Suppose that f is uniformly continuous on S. Let $\varepsilon > 0$ be any positive real number. Then there exists a real number $\delta > 0$ such that if x, y are any two positive real numbers with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$ holds. Observe that

$$|f(x) - f(y)| < \varepsilon \iff |\ln(x) - \ln(y)| < \varepsilon \iff e^{|\ln(x) - \ln(y)|} < e^\varepsilon \iff x/y < e^\varepsilon \text{ and } y/x < e^\varepsilon .$$

Let x be an arbitrary positive real number, and let $\delta' < \delta$ be any positive real number. According to the assumption above, $|f(x) - f(x + \delta')| < \varepsilon$ holds. So,

$$|f(x) - f(x + \delta')| < \varepsilon \implies (x + \delta')/x < e^\varepsilon \implies 1 + \delta'/x < e^\varepsilon .$$

Note that x was arbitrarily chosen, that is, it can be any positive real number. Let $x = s$ for some s with $0 < s < \delta'/(e^\varepsilon - 1)$. Then

$$1 + \delta'/x > 1 + \delta'/[\delta'/(e^\varepsilon - 1)] = 1 + (e^\varepsilon - 1) = e^\varepsilon,$$

contradiction. This means that $f(x) = \ln x$ is not uniformly continuous on $S = (0, \infty)$.

b) Let $\varepsilon > 0$ be any real number, and let $\delta = a(e^\varepsilon - 1)$. Then, if $|x - y| < \delta$ for any two real numbers x, y in $[a, \infty)$, it follows that (assume without loss of generality that $x >= y$ and let $\delta' = x - y$)

$$x/y = (y + \delta')/y = 1 + \delta'/y < 1 + \delta/a = 1 + (e^\varepsilon - 1) = e^\varepsilon \quad \text{and} \quad y/x <= 1 <= e^\varepsilon \implies$$

$$e^{|\ln x - \ln y|} < e^\varepsilon \implies |\ln x - \ln y| < \varepsilon \implies |f(x) - f(y)| < \varepsilon.$$

Consequently, $f(x) = \ln x$ is uniformly continuous on $[a, \infty)$ for any $a > 0$.

Problem 3.83

a) Let f be any continuous, periodic function on R. Then f is also continuous on the compact set $[0, p] \implies f([0,p]) = [m, M]$, where $m = \{\min(f(x)) : x \text{ in } [0,p]\}$ and $M = \{\max(f(x)) : x \text{ in } [0,p]\}$. Since f is periodic with period p, it follows that the range of f is $[m, M] \implies$ f is bounded on R.
Observe that, since f is continuous, it is uniformly continuous on any closed interval $[-kp, kp]$, where k is any natural number. Since there is no restriction on the values that k can take, we conclude that f is uniformly continuous on R.

b) Note that $\sin(x)$ and $\cos(x)$ are continuous on all of R, and $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$, that is, $\sin(x)$ and $\cos(x)$ both are periodic functions. According to (a), $\sin(x)$ and $\cos(x)$ are uniformly continuous on R.