1 Let f be four times continuously differentiable.

10 points

1. (4pts) Use L'Hopital's rule to show that $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$.
2. (6pts) Use Taylor's theorem and find the convergence order of the above convergence. (Find a largest possible integer $\alpha > 0$ such that $\frac{f(x+h)+f(x-h)-2f(x)}{h^2} - f''(x) = O(h^\alpha)$ as $h \to 0$.)
   (Lesson: L'Hopital's rule gives convenience and Taylor's theorem gives detail.)

Solution.

1. (3 pts) Applying the L'Hospital's rule twice, we get

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = \lim_{h\to 0} \frac{f''(x+h)+f''(x-h)}{2} = f''(x).$$

   (1 pt) Since both numerators and denominators of each quotients tend to 0 as $h \to 0$, we may use the L'Hospital's rule twice.

2. The Taylor expansion gives

$$f(x \pm h) = f(x) \pm f'(x)h + \frac{1}{2}f''(x)h^2 \pm \frac{1}{3!}f'''(x)h^3 + \frac{1}{4!}f^{(4)}(s_\pm)h^4 \qquad (2pts)$$

   for some $s_\pm \in N(x,h)$. Thus we have

$$\begin{aligned}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} - f''(x) &= \frac{f''(x)h^2 + \frac{2}{4!}f^{(4)}(s_\pm)h^4}{h^2} - f''(x) \\ &= \frac{f^{(4)}(s_+)+f^{(4)}(s_-)}{12}h^2 = O(h^2) \qquad (4pts)\end{aligned}$$

2

10 points

1. (5pts) Show that the total variation $V(\sin x; 0, 2\pi) = 4$. (Use Definition 5.3.2.)
2. (5pts) Show that quotient $\dfrac{f}{g}$ is in $BV(a,b)$ if f and g are uniformly continuous and have no zero on $[a,b]$. (You may use theorems in Section 5.3.)

Solution.

1. Take a partition $\pi_0 = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ of $[0, 2\pi]$. Then $V(f, \pi_0) = \sum_{j=1}^{4} |\Delta f_j| = 4$ for $f(x) = \sin x$.
   (2 pts) Hence $V(\sin x; 0, 2\pi) = \sup_\pi V(f, \pi) \geq V(f, \pi_0) = 4$.
   Let π_1 be an arbitrary partition of $[0, 2\pi]$ and let $\pi_2 = \pi_0 \vee \pi_1$. Then $V(f; \pi_1) \leq V(f, \pi_2) = 4$ since $\sin x$ is monotone between each partition points of π_0.
   (3 pts) Thus $V(\sin x; 0, 2\pi) = \sup_\pi V(f, \pi) \leq 4$.

2. (2 pts) Actually, what we need is $f, g \in BV(a,b)$ and continuity of g on the compact interval $[a,b]$.
   (1 pt) If we let $m = \inf_{x \in [a,b]} |g(x)|$, then $m = |g(x_0)| > 0$ for some $x_0 \in [a,b]$ by theorem 3.2.4.
   (1 pt) Thus theorem 5.3.6 implies $1/g \in BV(a,b)$ and
   (1 pt) Theorem 5.3.5 implies $f/g \in BV(a,b)$.

3 Prove or disprove.

10 points

1. (5pts) If f is continuous on a compact set $[a, b]$, then $f \in BV(a, b)$ (of bounded variation).

2. (5pts) If f is continuous on a compact set $[a, b]$, then $f \in R[a, b]$ (of Riemann integrable).

Solution.

1. (False) Let $f(x) = x \sin(1/x)$ on $[0, 1]$ with $f(0) = 0$. Then f is continuous on $[0, 1]$. We may take a partition π_N with
$$\frac{1}{x_{2n}} = n\pi, \ \frac{1}{x_{2n+1}} = n\pi + \pi/2$$
for $n = 1, 2, 3, \cdots, 2N$, $x_0 = 1$, $x_{2N+1} = 0$. Then we can see that
$$V(x \sin(1/x), \pi_N) \geq \sum_{n=1}^{N} \frac{2}{n\pi + \pi/2} \to \infty$$
as $N \to \infty$.
(If your counterexample and justifying are correct, you will get 5 points. Otherwise, 0 point.)

2. (True) This is Theorem 6.2.7. In other words, you are asked to prove a theorem. If you said, it is a theorem and hence true, you will get 2 points. If you included a proof, 5 points.

4 (10 points) Let $L(f)$ and $U(f)$ be the lower and upper Riemann integrals, respectively. Let f and g be bounded functions on $[a, b]$. Let

$$A := L(f + g), \quad B := L(f) + L(g), \quad C := U(f) + U(g), \quad D := U(f + g).$$

1. (4pts) Order them in size. (for example $A \leq B \leq C \leq D$)
2. (2 pts each) Show the three inequalities in part (a).

Solution.

1. $B \leq A \leq D \leq C$ (no partial points)
2. ($B \leq A$) Let $\epsilon > 0$. By definition of L, there are partitions π_1, π_2 such that $L(f) \leq L(f, \pi_1) + \epsilon$ and $L(g) \leq L(g, \pi_2) + \epsilon$ Thus we have

   $$L(f) + L(g) \leq L(f, \pi_1 \vee \pi_2) + L(g, \pi_1 \vee \pi_2) + 2\epsilon \leq L(f + g, \pi_1 \vee \pi_2) + 2\epsilon \leq L(f + g) + 2\epsilon.$$

   Since $\epsilon > 0$ is arbitrary, we get $L(f) + L(g) \leq L(f + g)$.

   ($A \leq D$) Theorem 6.2.3

   ($D \leq C$) Let $\epsilon > 0$. By definition of U, there are partitions π_1, π_2 such that $U(f, \pi_1) \leq U(f) + \epsilon$ and $U(g, \pi_2) \leq U(g) + \epsilon$ Thus we have

   $$U(f) + L(g) + 2\epsilon \geq U(f, \pi_1 \vee \pi_2) + U(g, \pi_1 \vee \pi_2) \geq U(f + g, \pi_1 \vee \pi_2) \geq U(f + g).$$

   Since $\epsilon > 0$ is arbitrary, we get $U(f) + U(g) \geq U(f + g)$.

- If you tried with $L(f, \pi) + L(g, \pi) \leq L(f + g, \pi)$, it is an incorrect method since $L(f)$ and $L(g)$ must be considered as supremums of the lower sums seperately, not considering the same partition π. Similar to $U(f + g) \leq U(f) + U(g)$. It does not give any points. (no partial points)

(5) Prove or disprove.

(a) (5 points) Let $f_n \in R[a,b]$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$. Then $f \in R[a,b]$.

(b) (5 points) Let π_1, π_2 be two partitions of an interval $[a,b]$. Then, for any bounded function f, $L(f,\pi_1) \le U(f,\pi_2)$. ($L$ and U are lower and upper Riemann sums.)

Solution. (a) This is false. Let $\mathbb{Q} \cap [a,b] = \{p_1, p_2, p_3, \ldots\}$. Consider

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{p_1, \ldots, p_n\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then $f_n \in R[a,b]$ because it has only finitely many discontinuities, and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$, but $f \notin R[a,b]$ as explained in Example 1 (p.249).

(b) This is true. See Theorem 6.2.2 (p.241). ◊

- There are no partial points. One gets either 0, 5, or 10 points.
- For (a), it is sufficient to give a valid counterexample.
- For (b), one has to give a proof. Stating, e.g., "See Theorem 6.2.2 (p.241)" gets no points.

(6) Let $f_k(x) = kx/(1+kx)$ for $x \in [0,1]$ and $k = 1, 2, \ldots$. Answer the followings and explain why.

(a) (3 points) Find a function f_0 such that $f_k(x) \to f_0(x)$ for all $x \in [0,1]$ as $k \to \infty$.

(b) (3 points) Determine whether the convergence is uniform.

(c) (4 points) Determine whether $\lim_{k\to\infty} \int_0^1 f_k(x)\,\mathrm{d}x = \int_0^1 (\lim_{k\to\infty} f_k(x))\,\mathrm{d}x$.

Solution. (a) Define

$$f_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

(b) The convergence is not uniform. Let $\varepsilon = 1/3$. For each $k_0 \ge 1$, take $k = k_0$. Then

$$\|f_k - f_0\|_\infty \ge \left| f_k\left(\frac{1}{k_0}\right) - f_0\left(\frac{1}{k_0}\right) \right| = \left| \frac{1}{2} - 1 \right| = \frac{1}{2} > \frac{1}{3} = \varepsilon.$$

(c) It is true.

$$\lim_{k\to\infty} \int_0^1 f_k(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_0^1 \frac{kx}{1+kx}\,\mathrm{d}x = \lim_{k\to\infty} \left(x - \frac{\log(1+kx)}{k} \right)\Bigg|_0^1 = \lim_{k\to\infty} \left(1 - \frac{\log(1+k)}{k} \right) = 1,$$

$$\int_0^1 \left(\lim_{k\to\infty} f_k(x) \right) \mathrm{d}x = \int_0^1 f_0(x)\,\mathrm{d}x = 1.$$ ◊

- There are no partial points.
- For (b) and (c), answers without complete proofs get no points.

(7) Prove or disprove.

(a) (5 points) A function $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$ is integrable. Then there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x$.

(b) (5 points) Let $f : [a, b] \to \mathbb{R}$ be a nonnegative continuous function. For $z \in [a, b]$, let $G(z)$ be the area bounded by the graph of $y = f(x)$, the x-axis, $x = a$, and $x = z$. Then

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a).$$

Solution. (a) This is false. Consider $f(x) = g(x) = x$ with $[a, b] = [-1, +1]$. Then

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \int_{-1}^{+1} x^2\,\mathrm{d}x = \frac{x^3}{3}\bigg|_{-1}^{+1} = \frac{2}{3}, \quad \text{but} \quad f(c)\int_a^b g(x)\,\mathrm{d}x = c\int_{-1}^{+1} x\,\mathrm{d}x = 0$$

for all $c \in [a, b]$.

(b) This is true. We have

$$G(z) = \int_a^z f(x)\,\mathrm{d}x,$$

since G is nonnegative and $a \le z$. Therefore,

$$G(b) - G(a) = G(b) = \int_a^b f(x)\,\mathrm{d}x. \qquad \Diamond$$

- There are no partial points. One gets either 0, 5, or 10 points.
- For (a), it is sufficient to give a valid counterexample.

(8) Suppose that $f : [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le K\,|x - y|$ for all $x, y \in [a, b]$ for some $K > 0$.

(a) (2 points) Show that f is integrable.

(b) (8 points) Show that, for every natural number k,

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right)\right| \le \frac{K}{2k}.$$

Solution. (a) Since continuous functions are integrable (Theorem 6.2.7, p.248), it suffices to show that f is continuous on $[a, b]$. Fix $y \in [a, b]$. Let $\varepsilon > 0$. Take $\delta = \varepsilon/K$. Then for all $x \in [a, b]$ with $|x - y| < \delta$ we have

$$|f(x) - f(y)| \le K\,|x - y| < K\delta = \varepsilon.$$

This means that f is continuous on $[a, b]$.

((b) is on the next page.)

(b) Observe that

$$\begin{aligned}
\left|\int_{(j-1)/k}^{j/k} f(x)\,\mathrm{d}x - \frac{1}{k} f\left(\frac{j}{k}\right)\right| &= \left|\int_{(j-1)/k}^{j/k} \left(f(x) - f\left(\frac{j}{k}\right)\right) \mathrm{d}x\right| \\
&\leq \int_{(j-1)/k}^{j/k} \left|f(x) - f\left(\frac{j}{k}\right)\right| \mathrm{d}x \\
&\leq \int_{(j-1)/k}^{j/k} K \left|x - \frac{j}{k}\right| \mathrm{d}x \\
&= \int_{(j-1)/k}^{j/k} -K \left(x - \frac{j}{k}\right) \mathrm{d}x \\
&= \left(-\frac{K}{2}\left(x - \frac{j}{k}\right)^2\right)\Bigg|_{(j-1)/k}^{j/k} \\
&= \frac{K}{2k^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right)\right| &= \left|\sum_{j=1}^{k}\int_{(j-1)/k}^{j/k} f(x)\,\mathrm{d}x - \sum_{j=1}^{k}\frac{1}{k} f\left(\frac{j}{k}\right)\right| \\
&= \left|\sum_{j=1}^{k}\left(\int_{(j-1)/k}^{j/k} f(x)\,\mathrm{d}x - \frac{1}{k} f\left(\frac{j}{k}\right)\right)\right| \\
&\leq \sum_{j=1}^{k}\left|\int_{(j-1)/k}^{j/k} f(x)\,\mathrm{d}x - \frac{1}{k} f\left(\frac{j}{k}\right)\right| \\
&\leq \sum_{j=1}^{k}\frac{K}{2k^2} \\
&= \frac{K}{2k}.
\end{aligned}$$

This completes the proof. ◊

- There are no partial points for (a).
- For (a), it is sufficient to state "Lipschitz functions are continuous" without proof.
- For (b), showing $|\ldots| \leq K/k$ instead of $|\ldots| \leq K/2k$ gets 4 points.

NAME: ID#: SCORE: / 110

Guidelines for the exam:

(1) Make answers short and your point clear.
(2) You are allowed to use books and notes. However, discussion is not.
(3) Zoom should be on all the time.
(4) You may use any theorem except when you are asked to prove it. However, check conditions when you use a theorem.
(5) Exam ends at 15:20. Scan your exam and upload it by 15:30 (if you have trouble with KLMS, submit your exam in e-mail, hykim0615@kaist.ac.kr).

(9) Let $g : [a, b] \to \mathbb{R}$ be monotone increasing and $f \in RS[g; a, b]$.
  (a) (8pts) Show that $|f| \in RS[g; a, b]$ and

$$\Big|\int_a^b f(x)dg(x)\Big| \le \int_a^b |f(x)|dg(x).$$

  (b) (2pts) What is the corresponding relation if $g : [a, b] \to \mathbb{R}$ is monotone decreasing.

(a) Consider the partition $\pi = \{x_0, \cdots, x_p\}$. For any interval $[x_{j-1}, x_j]$, let M_j and m_j denote the supremum and the infimum of f and M_j', m_j' of $|f|$. Then,

$$M_j - m_j = \sup\{f(x) - f(y) : x, y \in [x_{j-1}, x_j]\},$$
$$M_j' - m_j' = \sup\{|f(x)| - |f(y)| : x, y \in [x_{j-1}, x_j]\}.$$

We can easily find that $M_j' - m_j' \le M_j - m_j$. Since g is monotone increasing, by definition,

$$U(|f|, g, \pi) - L(|f|, g, \pi) \le U(f, g, \pi) - L(f, g, \pi) < \epsilon.$$

By Riemann condition, $|f| \in RS[g; a, b]$.
Now, since $f, -f \le |f|$, we have $\Big|\int_a^b f(x)dg(x)\Big| \le \int_a^b |f(x)|dg(x)$ from the Theorem 7.3.5 - (ii).
(b) If g is monotone decreasing, $-g$ is nomotone increasing. Hence, after replaceing g with $-g$, the relation will be $\Big|\int_a^b f(x)dg(x)\Big| \le -\int_a^b |f(x)|dg(x)$.

(10) (a) (5pts) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f'(x)| < 10$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous (Find $\delta > 0$ for a given $\epsilon > 0$).
  (b) (5pts) Prove or disprove that if f and g are bounded and $f + g$ is in $R(0, 1)$, the f and g are in $R(0, 1)$.

(a) Let $\epsilon > 0$ be given. Take $\delta = \frac{\epsilon}{10}$. Then, whenever $y \in N(x, \delta)$, we have

$$|f(y) - f(x)| \le 10|x - y| < \epsilon.$$

Since δ is indepent of x, f is uniformly continuous.
(b) It is a false statement. We may take f as a bounded function which is not Riemann integrable and $g = -f$. Then, $f + g = 0$ is Riemann integrable.

(11) Consider the following six statements:

p_1 : f is continuous on $[a, b]$, p_2 : f is uniformly continuous on $[a, b]$,
p_3 : f is differentiable on $[a, b]$, p_4 : f has an antiderivative on $[a, b]$,
p_5 : f is $R[a, b]$, p_6 : f is the indefinite integral of some $g \in R[a, b]$.

There are 30 possible statements in the form of $p_i \Rightarrow p_j$. Find true statements among them. (You don't need to explain why. A complete answer is for 10 points. -1 point for each missing true relation. -2 points for each false relation. The minimum score for this problem is 0. Hint:

There are 16 true relations. You may simply write such as 1 $\Rightarrow$ 3,5,6 / 2 $\Rightarrow$ 1,3,6 / and so on.)

Trues: $1 \Rightarrow 2, 4, 5$ / $2 \Rightarrow 1, 4, 5$ / $3 \Rightarrow 1, 2, 4, 5, 6$ / $4 \Rightarrow 5$ / $5 \Rightarrow$ *none* / $6 \Rightarrow 1, 2, 4, 5$
Falses: $1 \Rightarrow 3, 6$ / $2 \Rightarrow 3, 6$ / $3 \Rightarrow$ *none* / $4 \Rightarrow 1, 2, 3, 6$ / $5 \Rightarrow 1, 2, 3, 4, 6$ / $6 \Rightarrow 3$

Or, $3 \Rightarrow 6 \Rightarrow (1 \Leftrightarrow 2) \Rightarrow 4 \Rightarrow 5$.

Reason for the answer.
1. Since p_1 and p_2 are equivalent, there are basically 20 possibilities.
2. Even if $f(x) = F'(x)$, f is not necessarily continuous. Hence, we keep talking "contuously differentiable".
3. $f(x) = \int_a^x g(y)dy$ is continuous, but not necessarily differentiable. Hence, f is not an anti-derivative of g in general.
4. $1 \Rightarrow 6$ is false. $f(x) = x\sin(\frac{1}{x})$ on $[0, 1]$ is an example.