(1) For a given real number a we define $a^+ = \max(0, a) \geq 0$ and $a^- = \max(-a, 0) \geq 0$.
10 points For a real valued function $f \in BV(a, b)$, define

$$V(f; a, b) = \sup \left\{ \sum_{j=1}^{p} |\Delta f_j| \; : \; \pi \in \Pi[a, b] \right\}, \quad V^{\pm}(f; a, b) = \sup \left\{ \sum_{j=1}^{p} (\Delta f_j)^{\pm} \; : \; \pi \in \Pi[a, b] \right\}.$$

Let $V_f^{\pm}(x) = V^{\pm}(f; a, x)$ and $V_f(x) = V(f; a, x)$. Prove the following using definition.

(a) $V_f^+(x)$ and $V_f^-(x)$ are monotone on (a, b).

(b) $0 \leq V_f^{\pm}(x) \leq V_f(x)$ for all $x \in (a, b)$.

(c) If f is discontinuous at $c \in (a, b)$, then V_f is also discontinuous at c.

(d) $V_f(x) = V_f(x)^+ + V_f^-(x)$ for all $x \in (a, b)$.

Solution. (a) Choose any x and y satisfying $a < x < y < b$. We will show $V_f^{\pm}(x) \leq V_f^{\pm}(y)$. Choose any $\epsilon > 0$. By the property of sup, there exists $\pi_{\epsilon,x}^{\pm} \in \Pi[a, x]$ such that

$$V_f^{\pm}(x) \leq \epsilon + \sum_{j=1}^{n^{(\pm)}} \left(f(x_j^{(\pm)}) - f(x_{j-1}^{(\pm)}) \right)^{\pm}, \quad \text{where } \pi_{\epsilon,x}^{\pm} = \{a = x_0^{(\pm)}, x_1^{(\pm)}, \cdots, x_{n^{(\pm)}}^{(\pm)} = x\}.$$

In addition, because $\pi_{\epsilon,y}^{\pm} := \pi_{\epsilon,x}^{\pm} \cup \{y\} \in \Pi[a, y]$,

$$\sum_{j=1}^{n^{(\pm)}+1} \left(f(x_j^{(\pm)}) - f(x_{j-1}^{(\pm)}) \right)^{\pm} \leq V_f^{\pm}(y), \quad \text{where } x_{n^{(\pm)}+1}^{(\pm)} = y.$$

Combining the inequalities, we obtain $V_f^{\pm}(x) \leq \epsilon + V_f^{\pm}(y)$. Since ϵ can be any positive number, $V_f^{\pm}(x) \leq V_f^{\pm}(y)$.

(b) Fix $x \in (a, b)$. Because $\Pi[a, x]$ is nonempty, $V_f^{\pm}(x) \geq 0$. We will show $V_f^{\pm}(x) \leq V_f(x)$. Choose any $\epsilon > 0$ and set $\pi_{\epsilon,x}^{\pm}$ as in part (a). Because $c^{\pm} \leq |c|$ for any $c \in \mathbb{R}$,

$$V_f^{\pm}(x) \leq \epsilon + \sum_{j=1}^{n^{(\pm)}} \left(f(x_j^{(\pm)}) - f(x_{j-1}^{(\pm)}) \right)^{\pm} \leq \epsilon + \sum_{j=1}^{n^{(\pm)}} \left| f(x_j^{(\pm)}) - f(x_{j-1}^{(\pm)}) \right| \leq \epsilon + V_f(x).$$

Since ϵ can be any positive number, $V_f^{\pm}(x) \leq V_f(x)$.

(c) We will show the contrapositive statement. Suppose V_f is continuous at $c \in (a, b)$ and let $\epsilon > 0$ be given. Then,

$$\exists\, \delta > 0 \text{ such that } |V_f(x) - V_f(c)| < \epsilon \text{ for all } x \in N(c; \delta) \cup (a, b).$$

Thus, for any $x \in N(c;\delta) \cup (a,b)$, the trivial partition $\{x, c\}$ or $\{c, x\}$ gives

$$|f(x) - f(c)| \le \left\{ \begin{matrix} V(f;x,c) & \text{if } x < c, \\ V(f;c,x) & \text{if } x > c \end{matrix} \right\} \le |V_f(x) - V_f(c)| < \epsilon,$$

which completes the proof. (The proof is similar if we replace V_f by $V_f^{\pm}$ in the problem.)

(d) Fix $x \in (a,b)$ and let $\epsilon > 0$. By the property of sup, $\exists\ \pi, \pi^{\pm} \in \Pi[a,x]$ such that

$$V_f^{\pm}(x) \le \frac{\epsilon}{2} + \sum_{j=1}^{n^{(\pm)}} \left(f(x_j^{(\pm)}) - f(x_{j-1}^{(\pm)}) \right)^{\pm}, \quad \pi^{\pm} = \{a = x_0^{(\pm)}, x_1^{(\pm)}, \cdots, x_{n^{(\pm)}}^{(\pm)} = x\}, \tag{1}$$

$$V_f(x) \le \epsilon + \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|, \quad \pi = \{a = x_0, x_1, \cdots, x_n = x\}. \tag{2}$$

In addition, we have $|c| = c^+ + c^-$ for all $c \in \mathbb{R}$. Therefore, the inequality (1) implies

$$V_f^+(x) + V_f^-(x) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} + V_f(x),$$

and the inequality (2) implies

$$V_f(x) \le \epsilon + V_f^+(x) + V_f^-(x).$$

Since ϵ can be any positive number, we conclude $V_f(x) = V_f^+(x) + V_f^-(x)$.

(2) Let f and g be continuous on $[a, b]$. Prove or disprove that (a) $f \in R(a, b)$, (b) $f \in RS(g; a, b)$.

10 points

Solution. (a) The statement is true. The statement and its proof are exactly the same as that of Theorem 6.2.7 in the textbook.

(b) The statement is not true. In fact, for any $g \notin BV[a, b]$, there exist some f continuous on $[a, b]$ such that $f \notin RS(g; a, b)$. In addition, if we define $g : [-1, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then g is continuous on $[-1, 1]$ but not in $BV[-1, 1]$.

(3) 10 points (True or False problem) Consider the following six statements:

p_1 : f is continuous on (a,b), p_2 : f is uniformly continuous on (a,b),

p_3 : f is differentiable on (a,b), p_4 : f has an antiderivative on (a,b),

p_5 : f is $R(a,b)$, p_6 : f is an indefinite integral of some $g \in R(a,b)$.

There are 30 possible statements in the form $p_i \implies p_j$. Find (a) true statements and (b) false statements among them. (Proof is not needed.)

Solution. (a) The true statements among $p_i \implies p_j$ are as follows. (11/30)

	$p_2 \implies p_1$	$p_3 \implies p_1$			$p_6 \implies p_1$
					$p_6 \implies p_2$
$p_1 \implies p_4$	$p_2 \implies p_4$	$p_3 \implies p_4$			
	$p_2 \implies p_5$				$p_6 \implies p_4$
	$p_2 \implies p_6$				$p_6 \implies p_5$

(b) The false statements among $p_i \implies p_j$ are as follows. (19/30)

$p_1 \not\Longrightarrow p_2$			$p_4 \not\Longrightarrow p_1$	$p_5 \not\Longrightarrow p_1$	
$p_1 \not\Longrightarrow p_3$	$p_2 \not\Longrightarrow p_3$	$p_3 \not\Longrightarrow p_2$	$p_4 \not\Longrightarrow p_2$	$p_5 \not\Longrightarrow p_2$	
			$p_4 \not\Longrightarrow p_3$	$p_5 \not\Longrightarrow p_3$	$p_6 \not\Longrightarrow p_3$
$p_1 \not\Longrightarrow p_5$		$p_3 \not\Longrightarrow p_5$	$p_4 \not\Longrightarrow p_5$	$p_5 \not\Longrightarrow p_4$	
$p_1 \not\Longrightarrow p_6$		$p_3 \not\Longrightarrow p_6$	$p_4 \not\Longrightarrow p_6$	$p_5 \not\Longrightarrow p_6$	

(4) 10 points

(a) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f'(x)| < 10$. Show that f is uniformly continuous.

(b) Show that $f(x) = e^x$ is <u>not</u> uniformly continuous on $\mathbb{R}$.

Solution. (a) Choose any $\epsilon > 0$. We let $\delta = \frac{1}{10}\epsilon$. Suppose that $x, y \in \mathbb{R}$ satisfy

$$|x - y| < \delta.$$

If $x = y$, of course we have $|f(x) - f(y)| = 0 < \epsilon$. Suppose $x \neq y$. By the mean value theorem, there exists some $z \in \mathbb{R}$ between x and y such that

$$|f(x) - f(y)| = |f'(z)||x - y| < 10\delta = \epsilon.$$

Therefore, f is uniformly continuous.

(b) We have to show that

$$\exists\ \epsilon > 0 \text{ such that } \forall\ \delta > 0,\ \exists\ x, y \in \mathbb{R} \text{ such that } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon.$$

Let $\epsilon = 1$ and choose any $\delta > 0$. We let $x = \ln \frac{2}{\delta}$ and $y = x + \frac{\delta}{2}$. First, we observe that

$$|x - y| = \frac{\delta}{2} < \delta.$$

Also, by the mean value theorem, there exists $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y).$$

Because $f'(z) = e^z > e^x = \frac{2}{\delta}$, we have

$$|f(x) - f(y)| \geq \frac{2}{\delta}|x - y| = \epsilon,$$

which completes the proof.

(5) Let f be a continuous, non-negative function on $[0,1]$. Show that

10 points

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \int_0^1 f^2(x)\,dx.$$

Solution. Because f is continuous on $[0,1]$, f^2 also is continuous on $[0,1]$, thus both are integrable. We set

$$p(\lambda) := \int_0^1 (f(x)+\lambda)^2\ dx = \lambda^2 + 2\lambda \int_0^1 f(x)\,dx + \int_0^1 f^2(x)\,dx.$$

Because $p(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$, particularly for $\lambda = \lambda_0 = -\int_0^1 f(x)\,dx$, we conclude

$$0 \leq p(\lambda_0) = \int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2.$$

(6) Let $f_k \in R(0,1)$, $f_k \to f_0$ uniformly on $[0,1]$ as $k \to \infty$. Show that f_0 is in $R(0,1)$.

10 points

Solution. Note that this is a part of Theorem 6.5.1 in textbook.

Solution 1. (Proof using Riemann condition) See the proof of Theorem 6.5.1.

Solution 2. (Proof using definition) We simplify the notation $f = f_0$. Choose any $\epsilon > 0$. Because $f_k \to f$ uniformly on $[0,1]$,

$$\exists N \in \mathbb{N} \text{ such that for all } k \geq N \text{ and } x \in [0,1], \quad |f_k(x) - f(x)| < \frac{\epsilon}{3}.$$

Therefore, for all $k \geq N$,

$$L\left(-\frac{\epsilon}{3} + f_k\right) \leq L\left(f\right) \leq U\left(f\right) \leq U\left(\frac{\epsilon}{3} + f_k\right). \tag{3}$$

Because $f_k \in R(0,1)$ for all k,

$$L\left(-\frac{\epsilon}{3} + f_k\right) = -\frac{\epsilon}{3} + \int_0^1 f_k(x)\,dx$$

and

$$U\left(\frac{\epsilon}{3} + f_k\right) = \frac{\epsilon}{3} + \int_0^1 f_k(x)\,dx.$$

Plugging these into the inequality (3), we obtain

$$U(f) - L(f) \leq U\left(\frac{\epsilon}{3} + f_k\right) - L\left(-\frac{\epsilon}{3} + f_k\right) = \frac{2}{3}\epsilon < \epsilon.$$

Because ϵ can be any positive number, we conclude $U(f) = L(f)$, so $f \in R(0,1)$.

(7) Prove that, for any two bounded functions f and g on $[a, b]$,

10 points

$$L(f) + L(g) \leq L(f + g) \leq U(f + g) \leq U(f) + U(g).$$

Solution. • Note: In the original problem, the inequalities in red ink are in the opposite direction. However, if so, then there is a counterexample as follows. Let f and g be the real valued functions on $[0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}, \quad g(x) = 1 - f(x).$$

One can easily check that $U(f) = U(g) = 1$, $L(f) = L(g) = 0$, $L(f+g) = U(f+g) = 1$, so the original inequalities are false.

- (Proof of the corrected inequality) $L(f+g) \leq U(f+g)$ by Theorem 6.2.3. We wll prove the inequalities in red ink. Let $\epsilon > 0$ be given. There exists some $\pi_f, \pi_g \in \Pi[0, 1]$ such that

$$L(f) \leq \epsilon + L(f, \pi_f) \quad \text{and} \quad L(g) \leq \epsilon + L(g, \pi_g).$$

Let $\pi = \pi_f \vee \pi_g$. From Theorem 6.2.1, we have

$$L(f, \pi_f) \leq L(f, \pi) \quad \text{and} \quad L(g, \pi_g) \leq L(g, \pi).$$

Combining all the above inequalities, we have

$$L(f) + L(g) \leq 2\epsilon + L(f, \pi) + L(g, \pi).$$

Because $\inf_I f + \inf_I g \leq \inf_I (f + g)$ for any closed subinterval I of π, this implies

$$L(f) + L(g) \leq 2\epsilon + L(f + g, \pi).$$

Since ϵ can be any positive number, we conclude $L(f) + L(g) \leq L(f + g, \pi)$. Following the same procedure (changing sup and inf in each argument and adjusting the direction of inequalities) we can prove $U(f + g) \leq U(f) + U(g)$.

(8) 10 points Let f be continuously differentiable on $[a, b]$. Prove that $V(f; a, b) = \int_a^b |f'(x)|\, dx$.

Solution. Let $\pi = \{a = c_0, c_1, \cdots, c_n = b\} \in \Pi[a, b]$ be given. From mean value theorem, for each $j = 1, 2, \cdots, n$, we have

$$\exists\ d_j \in (c_{j-1}, c_j) \quad \text{such that} \quad f(c_j) - f(c_{j-1}) = f'(d_j)(c_j - c_{j-1}).$$

Therefore,

$$\sum_{j=1}^{n} |f(c_j) - f(c_{j-1})| = \sum_{j=1}^{n} |f'(d_j)|(c_j - c_{j-1}).$$

Using the inequality

$$\inf\{|f'(x)| \,:\, x \in [c_j, c_{j-1}]\} \le |f'(d_j)| \le \sup\{|f'(x)| \,:\, x \in [c_j, c_{j-1}]\},$$

we get

$$L(|f'|, \pi) \le \sum_{j=1}^{n} |f(c_j) - f(c_{j-1})| \le U(|f'|, \pi).$$

For each given $\epsilon > 0$, we choose π as follows:

$$\pi = \pi_1 \cup \pi_2 \cup \pi_3,$$

where

$$L(|f'|) \le \epsilon + L(|f'|, \pi_1),$$

$$U(|f'|) \ge \epsilon + U(|f'|, \pi_2),$$

$$V(f; a, b) \le \epsilon + \sum_{\pi_3} |f_j|.$$

Then we have

$$L(|f'|) - \epsilon \le V(f; a, b) \le 2\epsilon + U(|f'|).$$

Since f' is continuous on $[a, b]$, $L(|f'|) = U(|f'|) = \int_a^b |f'(x)|\, dx$. We let $\epsilon \to 0$ and complete the proof.

(9) 10 points Use the Cauchy form of the remainder for the function $f(x) = \ln(x+1)$ on $(-1, 1]$ to show that $\lim_{k\to\infty} R_k(0; x) = 0$ [uniformly] on $[-r, 1]$, where $0 < r < 1$.

Solution. The Cauchy form of the remainder is

$$R_k(0; x) = \frac{f^{(k+1)}(c)}{k!} x(x-c)^k = \frac{(-1)^k x(x-c)^k}{(c+1)^{k+1}},$$

where c is strictly between 0 and x. We cannot show the uniform convergence with this alone unless we extract more information on c. Fix $0 < r < 1$. First, let $x \in [-r, 0)$ and $c \in (x, 0)$. We observe that

$$|R_k(0; x)| = \frac{|x|}{|c+1|}\left(\frac{c-x}{c+1}\right)^k \leq \frac{r}{1-r}\left(\frac{0-(-r)}{0+1}\right)^k = \frac{r^{k+1}}{1-r}.$$

Since $\frac{r^{k+1}}{1-r}$ is independent of x and $|r| = r < 1$, we have $\lim_{k\to\infty} R_k(0; x) = 0$ uniformly on $[-r, 0)$. Similarly, for $x \in (0, r]$ and $c \in (0, x)$,

$$|R_k(0; x)| = \frac{x}{1+c}\left(\frac{x-c}{1+c}\right)^k \leq \frac{r}{1+0}\left(\frac{r-0}{1+0}\right)^k = r^{k+1},$$

which shows $\lim_{k\to\infty} R_k(0; x) = 0$ uniformly on $(0, r]$. The pointwise convergence for each $x \in [r, 1]$ can be shown as

$$|R_k(0; x)| = \frac{x}{1+c}\left(\frac{x-c}{1+c}\right)^k \to 0.$$

However, if c gets close to 0 and x gets close to 1, $|R_k(0; x)|$ does not converge to 0 uniformly. (However, in fact, we know that the convergence is uniform on $[-r, 1]$ by Abel's theorem.)

(10) 10 points Suppose that f is bounded and that g is increasing on $[a, b]$. Let π' be obtained from the partition π by inserting one point x' in the partition interval (x_{k-1}, x_k). Prove that $L(f, g, \pi) \leq L(f, g, \pi')$ and $U(f, g, \pi') \leq U(f, g, \pi)$.

Solution. Because g is monotone increasing, we have

$$\begin{aligned}
&U(f, g, \pi') - U(f, g, \pi) \\
&= (g(x') - g(x_{k-1}))(\sup_{[x_{k-1}, x']} f - \sup_{[x_{k-1}, x_k]} f) + (g(x_k) - g(x'))(\sup_{[x', x_k]} f - \sup_{[x_{k-1}, x_k]} f) \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
&L(f, g, \pi') - L(f, g, \pi) \\
&= (g(x') - g(x_{k-1}))(\inf_{[x_{k-1}, x']} f - \inf_{[x_{k-1}, x_k]} f) + (g(x_k) - g(x'))(\inf_{[x', x_k]} f - \inf_{[x_{k-1}, x_k]} f) \geq 0.
\end{aligned}$$

(11) 10 points Let $f \in C([a,b] \times [c,d])$, $h \in R(a,b)$, and $F(y) = \int_a^b f(x,y)h(x)\,dx$. Show that $F \in C([c,d])$.

Solution. Let $\epsilon > 0$ be given. Since $f \in C([a,b] \times [c,d])$ and $[a,b] \times [c,d]$ is compact, there exists $\delta > 0$ such that

$$|f(x_1,y_1) - f(x_2,y_2)| < \epsilon \quad \text{whenever } |(x_1,y_1) - (x_2,y_2)| < \delta.$$

Choose any $s, t \in [c,d]$ satisfying $|t - s| < \delta$. Then,

$$\begin{aligned} |F(t) - F(s)| &= \left| \int_a^b (f(x,t) - f(x,s))\, h(x)\,dx \right| \\ &\le \int_a^b |f(x,t) - f(x,s)|\,|h(x)|\,dx. \end{aligned}$$

Because $|(x,t)-(x,s)| = |t-s| < \delta$, we have $|f(x,t)-f(x,s)| < \epsilon$ for all $x \in [a,b]$. Therefore,

$$|F(t) - F(s)| \le \epsilon \int_a^b |h(x)|\,dx.$$

(Since $h \in R(a,b)$, $|h| \in R(a,b)$, so we complete the proof.)