Analysis I
Homework Assignment 1

Problem 1.26.

Let S be a non-empty finite set of size k and let $x_1 < x_2 < \ldots < x_k$ be its elements arranged in the increasing order.

i. If m is any real number $> x_k$, then the deleted neighborhood $N'(m; \varepsilon)$, with $\varepsilon = (m - x_k)/2$, contains no points of S => m is not a limit point.
ii. If m is any real number $< x_1$, then the deleted neighborhood $N'(n; \varepsilon)$, with $\varepsilon = (x_1 - n)/2$, contains no points of S => n is not a limit point.
iii. If $m = x_j$ for some $1 <= j <= k$, then let ε be any positive real number less than the distance between x_j and its closest "neighbor" (x_{j-1} or x_{j+1} if $1 < j < k$; x_2 if $j = 1$; x_{k-1} if $j = k$). Then, the deleted neighborhood $N'(m; \varepsilon)$ does not contain any points of S.
iv. If n is in the interval $[x_1, x_k]$, but is not an element of S, then there exists an index j such that $x_j < n < x_{j+1}$. Let $\varepsilon < \text{minimum}(x_{j+1} - n, n - x_j)$ be any positive real number. Obviously, $N'(n; \varepsilon)$ does not contain any elements of S. So, n is not a limit point of S.

Consequently, the set S has no limit points.

Problem 1.52

Let $\{x_k\}$ be a convergent sequence with limit x_0. Obviously, x_0 is a cluster point of $\{x_k\}$. Suppose that there exists another cluster point of $\{x_k\}$ – let's denote it by x'.
Let $\varepsilon < |x' - x_0|/2$ be any positive real number. Since x_0 is the limit, there is k_0 such that x_k is in $N(x_0; \varepsilon)$ for all $k > k_0$. Furthermore, since x' is a cluster point, there exists $k_1 > k_0$ for which $x_{k_1} \in N'(x'; \varepsilon) \subseteq N(x'; \varepsilon)$. So, x_{k_1} is in both $N(x_0; \varepsilon)$ and $N(x'; \varepsilon)$ =>

$|x_0 - x_{k_1}| < \varepsilon$ and $|x' - x_{k_1}| < \varepsilon$ =>

$|x_0 - x'| <= |x_0 - x_{k_1}| + |x' - x_{k_1}| < 2\varepsilon < |x' - x_0|$, contradiction.
So, there cannot exist another cluster point => a convergent sequence has exactly 1 cluster point => a sequence with at least 2 cluster points diverges.

Problem 1.53

Assume that a sequence $\{x_k\}$ converges to some value x_0. Let $\{y_k\} = \{x_{j_k}\}$ be any subsequence of $\{x_k\}$, and let $\varepsilon > 0$ be any real number. Since x_0 is the limit, there exists k_0 such that $x_k \in N(x_0; \varepsilon)$ for all $k > k_0$. Let m be the greatest integer such that $j_m <= k_0$. So, $j_{m+1} > k_0$ and $x_{j_k} \in N(x_0; \varepsilon)$ for all $k > m$ ==> $\{y_k\}$ converges to x_0 ==> every subsequence converges to x_0 (since $\{y_k\}$ was arbitrarily chosen).

Conversely, suppose that every subsequence of $\{x_k\}$ converges to some value x_0. Let $\{y_k\} = \{x_{2k-1}\}$ and $\{z_k\} = \{x_{2k}\}$, and let $\varepsilon > 0$ be any real number. There exist k_0 and k_1 such that $y_k \in N(x_0; \varepsilon)$ for all $k > k_1$, and $z_k \in N(x_0; \varepsilon)$ for all $k > k_2$. So, $y_k \in N(x_0; \varepsilon)$ and

$z_k \in N(x_0; \varepsilon)$ for all $k > \max(k_1, k_2) \implies x_k \in N(x_0; \varepsilon)$ for all $k > c$ (for some natural number c). In other words, $\{x_k\}$ converges to x_0.

Problem 1.72.

We are given that $x_1 = 0$, $x_2 = 1$, and $x_{k+1} = (x_{k-1} + x_k)/2$ for $k \geq 2$. Applying induction:

i. Base case : $|x_{j+1} - x_j| = (1/2)^{j-1}$ holds for $j = 1$.
ii. Inductive step : assume that $|x_{n+1} - x_n| = (1/2)^{n-1}$ for some natural number n. Then $|x_{n+2} - x_{n+1}| = |(x_n + x_{n+1})/2 - x_{n+1}| = |x_n - x_{n+1}|/2 = (1/2)^n$.

So, $|x_{j+1} - x_j| = (1/2)^{j-1}$ holds for all $j \in N$.

Problem 1.75.

Let $\{x_k\}$ be the sequence whose elements are defined as $x_k = 1 + ½ + \ldots + 1/k$.
Then $x_{k+1} - x_k = 1/(k+1)$ and thus $\lim_{k \to \infty} (x_{k+1} - x_k) = 0$. However, $\{x_k\}$ is not Cauchy, since $\{x_k\}$ is divergent. To prove this assertion, let's consider the element x_{2^m}, where $m \geq 3$ is a natural number :

$$x_{2^m} = 1 + ½ + \ldots 1/(2^m) = 1 + ½ + (1/3 + ¼) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots + 1/(2^m) > 1 + ½ + 2 * ¼ + 4 * 1/8 + \ldots + 2^{m-1} * 1/(2^m) = 1 + m/2.$$

This means that the sequence $\{x_k\}$ is unbounded => diverges => is not Cauchy.

Problem 1.82.

Let the sequences $\{C_k\}$ and $\{x_k\}$ be defined as $C_k = 1 - 1/(k+1)$ and $x_k = 1 + ½ + \ldots + 1/k$ for all $k \geq 1$. So, $\lim_{k \to \infty} (C_k) = 1$, $\{C_k\}$ is strictly monotone increasing, and $\{x_k\}$ diverges (proved in the problem 1.75). However, the property $|x_{k+1} - x_k| \leq C_k|x_k - x_{k+1}|$ is satisfied, because

$$|x_{k+1} - x_k| = 1/(k+1) = (1 - 1/(k+1)) * 1/k = C_k\,|x_k - x_{k+1}|.$$

So, a sequence $\{x_k\}$ satisfying the given properties is not necessarily convergent => is not necessarily Cauchy.

Problem 1.27.
d) $S = \{p/2^k : p \in Z, k \in N\}$.
Let $x = p_1/2^{k_1}$ for some $p_1 \in Z$, $k_1 \in N$, and $\varepsilon > 0$ be any real number. There exists $m \in N$ such that $1/2^m < \varepsilon$. Then

$$x - \varepsilon < p_1/2^{k_1} - 1/2^m = (p_1*2^m - 2^{k_1})/2^{k_1 + m}$$

and

$$x + \varepsilon > p_1/2^{k_1} + 1/2^m = (p_1*2^m + 2^{k_1})/2^{k_1 + m}.$$

So,

$$x + \varepsilon > (p_1 \cdot 2^m + 1) / 2^{k_1 + m} > x - \varepsilon \implies$$

$$(p_1 \cdot 2^m + 1) / 2^{k_1 + m} \in N'(x; \varepsilon).$$

So, any deleted neighborhood $N'(x; \varepsilon)$ contains a point of S. Therefore x is a limit point of S. We conclude that any element of S is a limit of S.

Now, let $x \notin S$ be a real number, and let $\varepsilon > 0$ be a real number such that $1/2^{m + 1} < \varepsilon$ for some natural number m. Obviously, $x \in (a, a + 1)$, where a is the integer part of x. So, $x \in (a, a + ½)$ or $x \in (a + ½, a + 1)$ (note that $x = a + ½$ is impossible, since $x \notin S$). Assume, without loss of generality that, $x \in (a, a + ½) \implies x \in (a, a + ¼)$ or $x \in (a + ¼, a + ½)$ (again, x cannot be equal to $a + ¼$) $\implies \ldots \implies x \in (a + A/2^m, a + (A + 1)/2^m)$ for some integer $A >= 0$. So,

$$x < a + (A + 1) / 2^m \quad \text{and } x > a + A / 2^m.$$

Thus

$$x + \varepsilon > a + A/2^m + 1/2^{m + 1}$$

and

$$x - \varepsilon < a + (A + 1)/2^m - 1/2^{m + 1} = a + A/2^m + 1/2^{m + 1} \implies$$

$$a + A/2^m + 1/2^{m + 1} \in N'(x; \varepsilon)$$

(again x is not equal to $a + A/2^m + 1/2^{m + 1}$ since $x \notin S$) $\implies$
any deleted neighborhood $N'(x; \varepsilon)$ of x contains a point of S $\implies$
Any real number is a limit point of S.

e)

i. Let x be a real number equal to $1/m_0$ for some natural number m_0, and let $\varepsilon > 0$ be a real number with $1/n_0 < \varepsilon$ for some natural number n_0. Then $x < 1/m_0 + 1/n_0 < x + \varepsilon \implies 1/m_0 + 1/n_0 \in N'(x; \varepsilon) \implies$ x is a limit point of S. So, any number of the form $1/m$ for some natural number m is a limit point of S.
ii. Let $x > 2$ be any real number. Then $N'(x; x – 2)$ does not contain any points of S $\implies$ x is not a limit point of S. 2 is not a limit point either; $N'(2; ½)$ contains no points of S.
iii. Let x be a number such that $x = 1 + 1/m$ for some positive integer $m > 1$, and let $\varepsilon > 0$ be any real number such that $\varepsilon < \text{minimum}(1/(m – 1) – 1/m, 1/m – 1/(m + 1))$. Then $N'(x; \varepsilon)$ contains no points of S.
iv. Let x be a number such that $1 < x < 2$ and $x \neq 1 + 1/m$ for all $m \in \mathbb{N}$. Then $x \in (1 + 1/(m_0 + 1), 1 + 1/m_0)$ for some $m_0 \in \mathbb{N}$. Let $\varepsilon > 0$ be any number such that $\varepsilon < \text{minimum } (x – (1 + 1/(m_0 + 1)), 1 + 1/m_0 – x)$. The deleted neighborhood $N'(x; \varepsilon)$ contains no points of X. So, if $x > 1$, then x is not a limit point of S.
v. Finally, let x be a real number such that $0 < x < 1$ and $x \neq 1/m$ for all $m \in \mathbb{N}$. Then $x \in (1/(m_0 + 1), 1/m_0)$ for some $m_0 \in \mathbb{N}$. Let $\varepsilon > 0$ be any real number such that $\varepsilon < \text{minimum } (x – 1/(m_0 + 1), 1/m_0 – x)$. Then $N'(x; \varepsilon)$ contains no points of S.

In conclusion, the limit points of S are all the numbers of the form $1/m$ for some natural number m.

Problem 1.48.
a) Let ε be any positive real number such that ε < (1 – L)/2, and let k_0 be a natural number such that |(x_k)^{1/k} – L| < ε(1 – L)/2 for all k >= k_0. Then for all k >= k_0,

$$|x_k - L^k| = |(x_k)^{1/k} - L| \times |x_k^{(k-1)/k} L^0 + \ldots + x_k^{1/k} \times L^{k-2} + L^{k-1}| < \frac{\epsilon(1-L)}{2} |x_k^{(k-1)/k} \times L^0 + \ldots$$

$$+ x_k^{1/k} \times L^{k-2} + L^{k-1}| < \frac{\epsilon(1-L)}{2} |(L + \frac{\epsilon(1-L)}{2})^{k-1} \times L^0 + \ldots (L + \frac{\epsilon(1-L)}{2}) \times L^{k-2} + L^{k-1}| <$$

$$= \frac{\epsilon(1-L)}{2} |(L + \frac{\epsilon(1-L)}{2})^{k-1} + \ldots (L + \frac{\epsilon(1-L)}{2}) + 1| < \frac{\epsilon(1-L)}{2} \frac{1}{1 - (L + \frac{\epsilon(1-L)}{2})} = \frac{\epsilon}{2-\epsilon} < \epsilon,$$

since $0 < L + \frac{\epsilon(1-L)}{2} < L + \frac{(1-L)^2}{4} = \frac{1+2L+L^2}{4}$.

and (1 + 2L + L^2)/4 < 1 (also note that ε < 1 and one of the factors is a sum of the form 1 + a + … a^m, where a < 1, so there is an upper bound for this sum, for all the values of m).

So, for all ε < (1 – L)/2 there exists k_0 such that |x_k – L^k| < ε for all k >= k_0. Thus, for all ε_0 >= (1 – L)/2 there exists k_1 such that |x_k – L^k| < ε_0 for all k >= k_1. In conclusion, given any ε > 0, there exists k_0 such that |x_k – L^k| < ε for all k >= k_0 ==> lim_{k goes to infinity} x_k = L^k.

b) Let ε > 0 be any number such that L - ε > 1. So, there exists k_0 such that |(x_k)^{1/k} – L| < ε for all k >= k_0 ==> (x_k)^{1/k} > L - ε for all k >= k_0 ==> x_k > (L - ε)^k for all k >= k_0. Let M > 0 be any number, and L - ε = 1 + x, where x > 0. By the Archimedes' principle, there exists k_1 for which 1 + x*k_1 > M. On the other hand, appealing to Bernoulli's inequality one has (1 + x)^k_1 >= 1 + x*k_1 ==> (L - ε)^k_1 = (1 + x)^k_1 >= 1 + x*k_1 > M. So, (L - ε)^k > M for all k >= k_1 ==> x_k > (L - ε)^k > M for all k >= max(k_0, k_1) ==> {x_k} diverges to positive infinity.

c) L = 1 => for any ε > 0, there exists k_0 such that |(x_k)^{1/k} – 1| < ε for all k >= k_0 =>

(x_k)^{1/k} < 1 + ε => x_k < (1 + ε)^k for all k >= k_0.

Since the limit is order-preserving on convergent sequences and since lim_{k goes to infinity} (1 + ε)^k = positive infinity (because by Bernoulli's inequality one has (1 + ε)^k >= 1 + ε*k, and 1 + ε*k can be greater than any real number, for all sufficiently large values of k), the above inequality just implies that

lim_{k goes to infinity} <= positive infinity,

which was already known to us and does not even suffice to determine whether {x_k} converges or not.
Also (x_k)^{1/k} > 1 – ε, however this inequality also does not provide any useful information about the convergence of {x_k}. Because if ε < 1, then x_k > (1 – ε)^k and the only result that can be drawn from the last inequality is

lim_{k goes to infinity} (x_k) >= 0,

as lim_{k goes to infinity} (1 – ε)^k = 0 and limit is order-preserving on convergent sequences.
But this result already follows from the fact that all the elements of the sequence are positive.

If $\varepsilon >= 1$, then $(1 – \varepsilon)^k$ alternates between positive and negative numbers (as k increases) and comparing x_k with $(1 – \varepsilon)^k$ does not provide any useful information about the convergence of $\{x_k\}$. Consequently, if $L = 1$, no conclusion can be drawn about the convergence of the sequence $\{x_k\}$.

Problem 1.76.

Note that the sum

$$1 + 2/3 + (2/3)^2 + \ldots + (2/3)^k$$

approaches

$$1/(1 – 2/3) = 3$$

as k goes to infinity.
Let $\varepsilon > 0$ be any real number.
There exists k_0 for which $(2/3)^{k_0} < \varepsilon/6$.
Then for all $m > k > k_0$:

$|x_k – x_{k_0}| <= |x_k – x_{k – 1}| + \ldots |x_{k_0 + 1} – x_{k_0}| < (2/3)^{k – 1} + \ldots + (2/3)^{k_0} = (2/3)^{k_0} * ((2/3)^{k – 1 – k_0} + \ldots + 1) < (2/3)^{k_0} * 3 < \varepsilon/2$ and

$|x_m – x_k| <= |x_m – x_{k_0}| + |x_k – x_{k_0}| < 2 * \varepsilon/2 = \varepsilon$.

So, $|x_k – x_{k_0}| < \varepsilon/2$ for all $k >= k_0$ and $|x_m – x_k| < \varepsilon$ for all $m > k >= k_0$ ==>
$\{x_k\}$ is Cauchy.

Problem 1.110.

Let S be any uncountable set of real numbers.

Consider the sets $S_j = [j, j + 1)$ for all integers j. The number of these sets is countably infinite. It follows that there exists an integer k for which S_k contains infinitely many elements of S. To prove this assertion, let's proceed by the method of contradiction. Suppose that this is not true; that is, for each integer j, the set S_j contains only finitely many elements of S. Let M be a natural number such that each S_j contains at most M elements of S.
However, in such a situation the set S can written as a sequence of distinct points in the following way:

$\{s_{0, 1}, \ldots, s_{0, i_0}$ = [the elements of S in S_0; $i_0 <= M$],
$s_{1,1}, \ldots, s_{1, i_1}$ = [the elements of S in S_1; $i_1 <= M$],
$s_{-1, 1}, \ldots, s_{-1, i_{-1}}$ = [the elements of S in S_{-1}; $i_{-1} <= M$],
.
.
.
$s_{k, 1}, \ldots, s_{k, i_k}$ = [the elements of S in S_k; $i_k <= M$],
$s_{-k, 1}, \ldots, s_{-k, i_{-k}}$ = [the elements of S in S_{-k}; $i_{-k} <= M$],
.

$\vdots$

This means that S is a countable set, which contradicts the given information. So, there is at least one integer k for which the set S_k contains infinitely many of the elements of S $\Longrightarrow$ S has infinitely many elements in $[k, k + 1)$. Let S' denote the subset of S which contains all the elements of S in the interval $[k, k + 1)$. Then S' is a bounded, infinite set $\Longrightarrow$ has a limit point $\Longrightarrow$ S has a limit point.

Problem 1.77

Observe that $|x_{k+1} - x_k| = 1/(2k + 2)!$ and $|x_{k} - x_{k-1}| = 1/(2k)!$ for all $k \geq 2$. So,

$$|x_{k+1} - x_k| = 1/(2k + 2)! =$$
$$= \frac{1}{(2k + 1)(2k + 2)} \cdot |x_{k} - x_{k-1}| \leq 1/30 \cdot |x_{k} - x_{k-1}|$$

for all $k \geq 2$. Therefore $\{x_k\}$ is a contractive sequence $\Rightarrow$ is Cauchy.

The above inequality also implies that

$$|x_{k+1} - x_k| \leq (1/30)^{k-1} \cdot |x_2 - x_1|$$

for all $k \geq 1$.

Let $m > k \geq 2$ be natural numbers. Then

$$|x_{m} - x_{k}| \leq |x_{m} - x_{m-1}| + \ldots + |x_{k+1} - x_{k}| \leq$$
$$\leq (1/30)^{m-2} \cdot |x_2 - x_1| + \ldots + (1/30)^{k-1} \cdot |x_2 - x_1| <$$
$$< (1/30)^{k-1} \cdot |x_2 - x_1| \cdot (30/29) = (1/30)^{k-1} \cdot 1/24 \cdot 30/29, \qquad (***)$$

taking into account the facts

1) $1 + (1/30) + \ldots + (1/30)^n$ approaches $1/(1 - 1/30) = 30/29$ as n goes to infinity;
2) $x_2 - x_1 = 1/(4!) = 1/24$.

Note that $\{x_k\}$ is convergent since it is Cauchy. Let x_0 be its limit. Then one has

$$|x_0 - x_k| \leq (1/30)^{k-1} \cdot 1/24 \cdot 30/29,$$

letting m go to infinity in the expression (***).
(we can do this since $\lim_{m \to \infty} |x_m - x_k| = |x_0 - x_k|$, for a fixed k).

For $k = 4$,

$$|x_0 - x_4| \leq (1/30)^3 \cdot 1/24 \cdot 30/29 < (0.1)^6$$

and $x_4 = 1 - 1/(2!) + 1/(4!) - 1/(6!) + 1/(8!) = 0.5403025\ldots$
On the other hand, $\cos(1 \text{ rad}) = 0.5403023\ldots$