MAS241 ANALYSIS I QUIZ 4

Problem 1. *(15 points) Prove or disprove. If the statement is wrong, give a counterexample.*

(1) *For every set S in $\mathbb{R}^n$, the complement of S^0 is the closure of S^c*

(2) *For every set S in $\mathbb{R}$, the closure of S and S have the same interiors.*

(3) *For every set S in $\mathbb{R}$, the interior of S and S have the same closures.*

Proof. (1) S^0 is the union of all open sets contained in S. By De Morgan's laws, the complement of S^0 is the intersection of all closed containing S^c, that is the closure of S^c

(2) Let $S = \mathbb{Q}$, then $\overline{S}$ is $\mathbb{R}$. $S^0 = \emptyset, \overline{S}^0 = \mathbb{R}$

(3) Let $S = \mathbb{Q}$, then S^0 is $\emptyset$. $\overline{S^0} = \emptyset, \overline{S} = \mathbb{R}$

□

Problem 2. *(15 points) Prove that $\mathbb{R}$ is connected and the only clopen(both close and open) subsets of $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$*

Proof. Assume $\mathbb{R}$ is disconnected.
There exist two nonempty, open sets U and V such that $\mathbb{R} \subseteq U \cup V$ and $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$
U $\cup$ V $\subseteq \mathbb{R} \implies V = U^c$. U is open, so V is closed.

Claim : The only clopen set of $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.
Let S $\neq \emptyset, \mathbb{R}$ be a clopen set. Because S is closed, S contains all boundary point of S.
Let $x \in S$ be a boundary point of S. Then $N(x) \cap S^c$ is nonempty.
But S is open, $\forall x \in S$ are interior points, $\exists \epsilon > 0$ such that $N(x; \epsilon) \subset S$ (contradiction)

Since V is clopen, V is either $\emptyset$ or $\mathbb{R}$. If V is $\emptyset$, it contradicts the assumption. If V is $\mathbb{R}$, then U is $\emptyset$. Then it contradicts the assumption again. Therefore $\mathbb{R}$ is not disconnected, i.e. $\mathbb{R}$ is connected. □

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