1 (4 points) Let $f \in \mathbf{C}^\infty$ from $\mathbb{R} \to \mathbb{R}$. Suppose that, for some positive integer n,

$$f(1) = f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = f^{(n)}(0) = 0$$

Prove that $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$.

Proof. By Rolle's Theorem, $f'(x_1) = 0$ for some $x_1 \in (0, 1)$. Then since $f'(0) = 0$, $f''(x_2) = 0$ for some $x_2 \in (0, x_1)$. Repeated applications of Rolle's theorem give $f^{(n)}(x_n) = 0$ for some $x_n \in (0, x_{n-1})$ and thus $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$. □

2 (6 points) Let $f(x) = x \log(1 + \frac{1}{x})$, where $x \in (0, \infty)$

1. Show that f is strictly monotonically increasing.
   (*Hint:* Consider some appropriate composition of function.)
2. Compute $\lim_{x \to 0} f(x)$ and $\lim_{x \to \infty} f(x)$.

Proof.

1. Consider $exp(f(x)) = \big(1 + \frac{1}{x}\big)^x$, Which is increasing function. As exponential is also increasing, so is f.
2. First one is
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\log(x+1) - \log(x)}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x+1} - \frac{1}{x}}{-\frac{1}{x^2}} = 0$$
   and second one is 1 since $\lim_{x \to \infty} (1 + \frac{1}{x})^x = e$

□