MAS241 Analysis 1 Quiz 8 Solution

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Problem 1. (18 points) Let g be a nonnegative continuous function on $[a, b]$. Let f be the function on $[a, b]$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(1) (6 points) Find $L(f)$ and $U(f)$ in terms of g.

(2) (12 points) Prove that f is Riemann integrable on $[a, b]$ if and only if g is identically zero on $[a, b]$.

Solution. (1) Observe that

$$L(f, \pi) = \sum_{j=1}^{p} m_j \Delta x_j = \sum_{j=1}^{p} 0 \cdot \Delta x_j = 0 \quad \text{and} \quad U(f, \pi) = \sum_{j=1}^{p} M_j \Delta x_j = \sum_{j=1}^{p} M_j^* \Delta x_j = U(g, \pi)$$

for all $\pi \in \Pi[a, b]$, where $\pi = \{x_0, x_1, x_2, \dots, x_p\}$,

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x), \quad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x), \quad M_j^* = \sup_{x \in [x_{j-1}, x_j]} g(x), \quad \text{and} \quad \Delta x_j = x_j - x_{j-1}.$$

Therefore,

$$L(f) = \sup_{\pi \in \Pi[a,b]} L(f, \pi) = \sup_{\pi \in \Pi[a,b]} 0 = 0 \quad \text{and} \quad U(f) = \inf_{\pi \in \Pi[a,b]} U(f, \pi) = \inf_{\pi \in \Pi[a,b]} U(g, \pi) = U(g).$$

In short, $L(f) = 0$ and $U(f) = U(g)$.

(2) The "if" direction is obvious. Let us show the "only if" direction. Suppose that f is Riemann integrable on $[a, b]$. Then $L(f) = U(f)$ by Theorem 6.2.4. That is, $U(g) = 0$. Towards contradiction, suppose that $g(x_0) > 0$ for some $x_0 \in [a, b]$. Then there exist $m > 0$ and a neighborhood N of x_0 such that

$$g(x) \geq m > 0 \quad \text{for all } x \in N \cap [a, b]$$

by Theorem 3.3.3. Let c and d be the endpoints of $N \cap [a, b]$ with $c < d$. Then

$$U(g) \geq m(d - c) > 0.$$

This is a contradiction. Therefore, g is identically zero on $[a, b]$. ◊

- This problem is a slight modification of Exercise 6.18.
- In (1), concluding $U(f) = \int_a^b g$ instead of $U(f) = U(g)$ is also a correct answer.
- In (2), although it is not necessary, one can use Theorem 6.2.7 and 6.2.9.

Problem 2. (12 points) Let f be a nonnegative Riemann integrable function on $[a, b]$. Prove that f^α is Riemann integrable on $[a, b]$ for all $\alpha > 1$. (Here, f^α is the function defined by $x \mapsto f(x)^\alpha$.)

Solution. Fix $\alpha > 1$. If $\|f\|_\infty = 0$, then $f = 0$ identically, so the statement is trivial. We assume $\|f\|_\infty > 0$. We will prove that f^α satisfies Riemann's condition on $[a, b]$. Let $\varepsilon > 0$ be given. Since f is Riemann integrable on $[a, b]$, it satisfies Riemann's condition: There exists $\pi_0 \in \Pi[a, b]$ such that every refinement π of π_0 satisfies

$$U(f, \pi) - L(f, \pi) < \frac{\varepsilon}{\alpha \|f\|_\infty^{\alpha-1}}.$$

Let π be a refinement of π_0. Say $\pi = \{x_0, \ldots, x_p\}$. Let

$$M_j = \sup_{[x_{j-1}, x_j]} f, \quad m_j = \inf_{[x_{j-1}, x_j]} f, \quad M_j' = \sup_{[x_{j-1}, x_j]} f^\alpha, \quad \text{and} \quad m_j' = \inf_{[x_{j-1}, x_j]} f^\alpha.$$

Note that $M_j' = M_j^\alpha$ and $m_j' = m_j^\alpha$, since $x \mapsto x^\alpha$ is monotone increasing. Observe that

$$\begin{aligned}
U(f^\alpha, \pi) - L(f^\alpha, \pi) &= \sum_{j=1}^{p} \left(M_j' - m_j'\right) \Delta x_j \\
&= \sum_{j=1}^{p} \left(M_j^\alpha - m_j^\alpha\right) \Delta x_j \\
&= \sum_{j=1}^{p} \alpha c_j^{\alpha-1} \left(M_j - m_j\right) \Delta x_j && \text{by the mean value theorem} \\
&\leq \sum_{j=1}^{p} \alpha \|f\|_\infty^{\alpha-1} \left(M_j - m_j\right) \Delta x_j \\
&= \alpha \|f\|_\infty^{\alpha-1} \sum_{j=1}^{p} \left(M_j - m_j\right) \Delta x_j \\
&= \alpha \|f\|_\infty^{\alpha-1} \left(U(f, \pi) - L(f, \pi)\right) \\
&< \varepsilon.
\end{aligned}$$

This completes the proof. $\Diamond$

- This problem is a slight modification of the proof of Theorem 6.2.5(iii).
- If one assumes α is an integer, then there is a deduction of 8 points.