1 (4 points) For a partition $\pi = \{x_0, x_1, \dots, x_p\}$ of the interval $[a, b]$, define $V(f, \pi; a, b)$ as

$$V(f, \pi; a, b) = \sum_{j=1}^{p} |\Delta f_j|.$$

Prove that if $\pi_2 \preceq \pi_1$, then $V(f, \pi_2; a, b) \leq V(f, \pi_1; a, b)$.

Solution. From the point of view of the mathematical induction, it is sufficient to prove the case that $\pi_2 = \{y_0, y_1, \dots, y_q\}$ and $\pi_1 = \{y_0, y_1, \dots, y_i, y, y_{i+1}, \dots, y_q\}$, where π_1 has only one additional point y, compares with π_2. However, in this case,

$$V(f, \pi_2; a, b) \leq V(f, \pi_1; a, b) \iff |f(y_{i+1}) - f(y_i)| \leq |f(y_{i+1}) - f(y)| + |f(y) - f(y_i)|,$$

which is trivial by triangle inequality.

2 Let f be function on $[a, b]$. f is called *absolutely continuous* on $[a, b]$ if for every positive number ϵ, there exist a positive number δ such that if a finite sequence of sub-intervals $\{(x_k, y_k)\}_{k=0}^{p}$ of $[a, b]$ satisfies

6 points

$$a \leq x_0 < y_0 \leq x_1 < y_1 \leq \cdots \leq x_p < y_p \leq b, \quad \text{and} \quad \sum_{k=0}^{p} (y_k - x_k) < \delta,$$

then

$$\sum_{k=0}^{p} |f(y_k) - f(x_k)| < \epsilon.$$

Prove that if f is absolutely continuous on $[a, b]$, then $f \in BV(a, b)$.
(*Hint: Use the result of problem 1.*)

Solution. Fix ϵ and suppose $\delta < b - a$. Choose $N \in \mathbb{N}$ such that

$$a + (N - 1)\delta < b \leq a + N\delta.$$

Define a partition of $[a, b]$, $\pi_0 = \{a = x_0, x_1, \ldots, x_N = b\}$, where

$$x_i = a + i\delta \quad \text{for} \quad 1 \leq i \leq N - 1.$$

Let π be a any partition of $[a, b]$ and $\pi_1 = \pi \vee \pi_0$. Then by problem 1,

$$V(f, \pi; a, b) \leq V(f, \pi_1; a, b).$$

However, if $\{x_i = y_0, y_1, \ldots, y_q = x_{i+1}\}$ is a set of points in π_1 between x_i and x_{i+1}, then

$$\sum_{k=1}^{q} |f(y_i) - f(y_{i-1})| < \epsilon$$

by the definition of absolutely continuous. Thus, $V(f, \pi_1; a, b) \leq N\epsilon$. Since N and ϵ are fixed values, we can conclude $f \in BV(a, b)$.