NAME: ID#: PIN(4 DIGIT): SCORE: / 110

Guidelines for the exam:

(1) Choose four digit PIN. Your score and grade will be announed with this PIN.
(2) Short answers are preferred.
(3) There are 11 problems for 10 points each.
(4) You are allowed to use lecture videos, books, and notes.
(5) Use any theorem in the book unless you are asked to prove it.
(6) Discussion is not allowed. If you get helped from a person, you will get F grade for the course.
(7) It is online exame. If there is typo in the exam, point it out and fix it by yourself.
(8) Exam ends at 11:20am. Scan your exam and upload (or email) it by 11:40am.

(1) For a given real number a we define $a^+ = \max(0, a) \geq 0$ and $a^- = \max(-a, 0) \geq 0$. For a real valued function $f \in BV(a, b)$, define

$$V(f; a, b) = \sup\left\{\sum_{j=1}^{p} |\Delta f_j| : \pi \in \Pi[a, b]\right\}, \quad V^{\pm}(f; a, b) = \sup\left\{\sum_{j=1}^{p} (\Delta f_j)^{\pm} : \pi \in \Pi[a, b]\right\}.$$

Let $V_f^{\pm}(x) = V^{\pm}(f; a, x)$ and $V_f(x) = V(f; a, x)$. Prove the followings using definition.
 (a) $V_f^+(x)$ and $V_f^-(x)$ are monotone on (a, b).
 (b) $0 \leq V_f^{\pm}(x) \leq V_f(x)$ for all $x \in (a, b)$.
 (c) If f is discontinuous at $c \in (a, b)$, then V_f is also discontinuous at c.
 (d) $V_f(x) = V_f^+(x) + V_f^-(x)$ for all $x \in (a, b)$.

(2) Let f and g be continuous on $[a, b]$. Prove or disprove that (a) $f \in R(a, b)$, (b) $f \in RS(g; a, b)$.

(3) (True or False problem) Consider the following six statements:

$p_1 : f$ is continuous on (a, b), $p_2 : f$ is uniformly continuous on (a, b),
$p_3 : f$ is differentiable on (a, b), $p_4 : f$ has an antiderivative on (a, b),
$p_5 : f$ is $R(a, b)$, $p_6 : f$ is an indefinite integral of some $g \in R(a, b)$.

There are 30 possible statements in the form of $p_i \Rightarrow p_j$.
 (a) Find true statements among them. (b) Find false statements. (Proof is not needed.)

(4) (a) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f'(x)| < 10$. Show that f is uniformly continuous.
 (b) Show that $f(x) = e^x$ is <u>not</u> uniformly continuous on $\mathbb{R}$.

(5) Let f be continuous, nonnegataive function on $[0, 1]$. Show that

$$\Big(\int_0^1 f(x)dx\Big)^2 \leq \int_0^1 f^2(x)dx.$$

(6) Let $f_k \in R(0, 1)$, $f_k \to f_0$ uniformly on $[0, 1]$ as $k \to \infty$. Show that f_0 is in $R(0, 1)$.

(7) Prove that, for any two bounded functions f and g on $[a, b]$,

$$L(f + g) \leq L(f) + L(g) \leq U(f) + U(g) \leq U(f + g).$$

(8) Let f be continuously differentiable on $[a, b]$. Prove that $V(f, a, b) = \int_a^b |f'(x)|dx$.

(9) Use the Cauchy form of the remainder for the function $f(x) = \ln(x + 1)$ on $(-1, 1]$ to show that $\lim_{k\to\infty} R_k(0; x) = 0$ [uniformly] on $[-r, 1]$, where $0 < r < 1$.

(10) Suppose that f is bounded and that g is incrasing on $[a, b]$. Let π' be obtained from the partition π by inserting one point x' in the partition interval (x_{k-1}, x_k). Prove that $L(f, g, \pi) \leq L(f, g, \pi')$ and $U(f, g, \pi') \leq U(f, g, \pi)$.

(11) Let $f \in C([a, b] \times [c, d])$, $h \in R(a, b)$, and $F(y) = \int_a^b f(x, y)h(x)dx$. Show that $F \in C([c, d])$.