MAS241 ANALYSIS 1 QUIZ 10

When you disprove some statements, you should give us a counter example and explain why it violates the original statement. There are 3 problems and each of them is 15 points. But, you should choose 2 problems and solve them. Your final score is determined by 2 highest score. For example, if you get 10pts, 5pts, 5pts, the score is 15 pts, not 20 pts.

Problem 1. *(15 points) In this problem, you should use Theorem 6.5.1.*

Theorem 6.5.1) *If $\{f_k\}$ converges uniformly to f_0 on the compact set $[a,b]$ and if each f_k is integrable on $[a,b]$, then f_0 is also integrable on $[a,b]$. Furthermore,*

(1) *If $F_k(x) = \int_a^x f_k(t)dt$, then $\{F_k\}$ converges uniformly to the function $F_0(x) = \int_a^x f_0(t)dt$ on $[a,b]$.*
(2) *In particular, $\lim_{k\to\infty} \int_a^b f_k(x)dx = \int_a^b f_0(x)dx$.*

Now, define a function f_0 as

$$f_0 = \begin{cases} 0 & x \in \mathbb{R}/\mathbb{Q} \\ \frac{1}{m} & x = \frac{n}{m} \in \mathbb{Q},\ \gcd(m,n) = 1 \end{cases}.$$

Show that f_0 is integrable on $[a,b]$ and evaluate $F_0(x)$ for $x \in [a,b]$.

Problem 2. *(15 points) Let $\{f_k\}$ be a sequence of continuously differentiable functions on $[a,b]$ such that $\lim_{k\to\infty} f_k = f_0$ pointwisely on $[a,b]$ and $\lim_{k\to\infty} f_k' = g$ pointwisely on $[a,b]$. Prove or disprove that for $x \in [a,b]$,*

$$f_0(x) - f_0(a) = \int_a^x g(t)dt.$$

Problem 3. *(15 points) Suppose that g is defined by*

$$g(x) = \begin{cases} a, & \text{for } 0 \le x < 1 \\ b, & \text{for } 1 \le x \le 2. \end{cases}$$

Here, $a \neq b$ and $c \neq d$, and they are constants.

(1) *Let f be a function defined on $[0,2]$ by*

$$f(x) = \begin{cases} c, & \text{for } 0 \le x < 1 \\ d, & \text{for } 1 \le x \le 2. \end{cases}$$

Prove or disprove that f is in $RS[g;0,2]$. Evaluate $\int_0^2 f(x)dg(x)$ if it exists.

(2) *Let f be a function defined on $[0,2]$ by*

$$f(x) = \begin{cases} c, & \text{for } 0 \le x \le 1 \\ d, & \text{for } 1 < x \le 2. \end{cases}$$

Prove or disprove that f is in $RS[g;0,2]$. Evaluate $\int_0^2 f(x)dg(x)$ if it exists.

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