1 Define a sequence of real numbers (x_n) by

4 points

$$x_0 = 1, \qquad x_{n+1} = \frac{1}{2+x_n} \quad \text{for} \quad n \geq 0.$$

Show that (x_n) converges, and evaluate its limit.

Solution.

$$|x_{n+2} - x_{n+1}| \leq |\frac{1}{2+x_{n+1}} - \frac{1}{2+x_n}| = |\frac{x_n - x_{n+1}}{(2+x_{n+1})(2+x_n)}| \leq \frac{1}{4}|x_{n+1} - x_n|$$

So, it is a contractive sequence and therefore Cauchy sequence. **(+3 points)**
Thus it has a limit. We can get that limit is sqrt2-1 by a simple calculation. **(+1 points)**

□

2 Let (a_n) and (ϵ_n) be sequences of positive numbers. Assume that $\lim_{n\to\infty} \epsilon_n = 0$ and that there is a number k in $(0,1)$ such that $a_{n+1} \leq ka_n + \epsilon_n$ for every n. Prove that $\lim_{n\to\infty} a_n = 0$

6 points

(Hint : If you are stuck, start like this. Fix $\delta > 0$, and choose n_0 such that $\epsilon_n < \delta$ for all $n \geq n_0$. Then $a_{n_0+1} \leq ka_{n_0} + \epsilon_{n_0} < ka_{n_0} + \delta$)

Solution. Fix $\delta > 0$, and choose n_0 such that $\epsilon_n < \delta$ for all $n \geq n_0$. Then

$$\begin{aligned}
&a_{n_0+1} \leq ka_{n_0} + \epsilon_{n_0} < ka_{n_0} + \delta \\
&a_{n_0+2} \leq k^2 a_{n_0} + k\delta + \epsilon_{n_0+1} < k^2 a_{n_0} + (1+k)\delta \\
&a_{n_0+3} \leq k^3 a_{n_0} + (k+k^2)\delta + \epsilon_{n_0+2} < k^3 a_{n_0} + (1+k+k^2)\delta
\end{aligned}$$

and, by the induction Principle

$$a_{n_0+m} < k^m a_{n_0} + (1 + k + \cdots + k^{m-1}\delta) < k^m a_{n_0} + \frac{\delta}{1-k}$$

Letting $m \to \infty$, we find that

$$\limsup_{n\to\infty} a_n \leq \frac{\delta}{1-k} \qquad \textbf{(+4 points)}$$

Since δ is arbitraty, we have $\limsup_{n\to\infty} a_n \leq 0$, and thus $\lim_{n\to\infty} a_n = 0$ **(+2 points)**

$\square$

Remark.

A lot of mistakes.

1. $|a_{n+1} - a_n| < \delta$ does not implies that the sequence $\{a_n\}$ is Cauchy or contractive sequence.
2. When you don't know if there exists a limit of the sequence, be careful about taking a limit.
3. If you fix $\delta > 0$ then it is not true that $\frac{\delta}{1-k} < \epsilon$ for all ϵ

Notice about claim.

1. If you have any claim on your score, mail dhcho2440@kaist.ac.kr until 6, April, Monday.
2. Please read the attached remark in the solution before sending the mail.

Thanks.