Analysis I - Homework Assignment 5

Problem 5.5

Let $S = \{x_1, x_2, ...,\}$ be the set of all points in $[a, b]$ where f is discontinuous (since f is monotone, the set of discontinuities of f is countable). For any two sets T and T' such that $T \in T' \in [a, b]$,

$\sum_{x_j \in S\cap T'} u(x_j) =$ sup$\{\sum_{x_j\in F} u(x_j) : F \in S\cap T', F$ finite$\} \geq$ sup$\{\sum_{x_j\in F} u(x_j) : F \in S\cap T, F$ finite $\} = \sum_{x_j\in S\cap T}$ holds,

because if $F \subset S \cap T$, then $F \subset S \cap T'$. The same is true for the function v as well. So, for any x, y with $a \leq x < y \leq b$:

$s_f(y) = f(a)+\sum_{x_j\in S\cap(a,y]} u(x_j)+\sum_{x_j\in S\cap[a,y)} v(x_j) \geq f(a)+\sum_{x_j\in S\cap(a,x]} u(x_j)+\sum_{x_j\in S\cap[a,x)} v(x_j) = s_f(x)$. So, s_f is monotone increasing on $[a, b]$.

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Problem 5.7

Lemma. Let f be a monotone increasing function on $[a, b]$. Then for any two points x and y such that $a \leq x < y \leq b$, one has

$$s_f(y) - s_f(x) \leq f(y) - f(x).$$

Proof. Let $S = (x_1, x_2, \dots)$ be the set of points in $[a, b]$ where f is discontinuous (since f is monotone, the set of points where f is discontinuous is countable). The inequality above is equivalent to

$$f(a) + \sum_{x_j\in S\cap(a,y]} u(x_j) + \sum_{x_j\in S\cap[a,y)} v(x_j) - f(a) - \sum_{x_j\in S\cap(a,x]} u(x_j) - \sum_{x_j\in S\cap[a,x)} v(x_j) \leq f(y) - f(x)$$

$$\iff \sum_{x_j\in S\cap(x,y]} u(x_j) + \sum_{x_j\in S\cap[x,y)} v(x_j) \leq f(y) - f(x)$$

$$\iff \sum_{x_j\in S\cap(x,y)} u(x_j) + \sum_{x_j\in S\cap(x,y)} v(x_j) + \sum_{x_j\in S\cap[y,y]} u(x_j) + \sum_{x_j\in S\cap[x,x]} v(x_j) \leq f(y) - f(x).$$

Let $\pi = (x, y_1, y_2, \dots, y_p, y)$ be any partition of the interval $[x, y]$. Then

$$f(x^+) - f(x) + f(y_1^+) - f(y_1^-) + \cdots + f(y_p^+) - f(y_p^-) + f(y) - f(y^-) =$$

$$= -f(x) + (f(x^+) - f(y_1^-) + (f(y_1^+) - f(y_2^-) + \cdots + (f(y_p^+) - f(y^-)) + f(y) \leq f(y) - f(x),$$

since $f(z^+) - f(t^-) \leq 0$ for any z, t with $z < t$.

So,

$$f(x^+) - f(x) + (\sum_{j\in J(\pi)} f(y_j^+) - f(y_j^-)) + f(y) - f(y^-) \leq f(y) - f(x))$$

holds for every partition $\pi = (x, y_1, y_2, \ldots, y_p, y)$ of $[x, y]$.

Therefore

$$\sum_{x_j \in S \cap (x,y)} u(x_j) + \sum_{x_j \in S \cap (x,y)} v(x_j) = \sum_{x_j \in S \cap (x,y)} u(x_j) + v(x_j) = \sum_{x_j \in S \cap (x,y)} f(x_j^+) - f(x_j^-) \leq f(y^-) - f(x^+)$$

$$\implies \sum_{x_j \in S \cap (x,y)} u(x_j) + \sum_{x_j \in S \cap (x,y)} v(x_j) + \sum_{x_j \in S \cap [y,y]} u(x_j) + \sum_{x_j \in S \cap [x,x]} v(x_j) \leq$$

$$\leq f(y) - f(y^-) + f(x^+) - f(x) + f(y^-) - f(x^+) = f(y) - f(x),$$

as desired. The proof of the lemma is completed.

Let x_0 be any point in $[a, b]$. Since f is continuous at every point of $[a, b] \cap S^c$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if x is a point of $[a, b] \cap S^c$ and $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| < \epsilon \implies |s_f(x) - s_f(x_0)| < \epsilon$. Since x_0 was chosen arbitrarily, we conclude that s_f is continuous at every point of $[a, b] \cap S^c$.

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Problem 5.9

We first observe that f is continuous on $(0, 1)$ and $(1, 2)$. However since $f(0) = -1 \neq 0 = f(0^+)$, $f(1^-) = 1 \neq 7/4 \neq f(1^+) = 2$, and $f(2) = 3 \neq f(2^-) = \sqrt{5}$, f is not continuous at the points $x = 0, 1, 2$.

The saltus function associated with f is:

$s_f(0)$ = -1,

$s_f(x) = f(0) + f(0^+) - f(0) = 0$ for $0 < x < 1$,

$s_f(1) = f(0) + f(0^+) - f(0) + f(1) - f(1^-) = 3/4$,

$s_f(x) = 3/4 + f(1^+) - f(1) = 1$ for $1 < x < 2$,

$s_f(2) = 1 + 3 - \sqrt{5} = 4 - \sqrt{5}$.

The function $f_c = f - s_f$ is then defined by

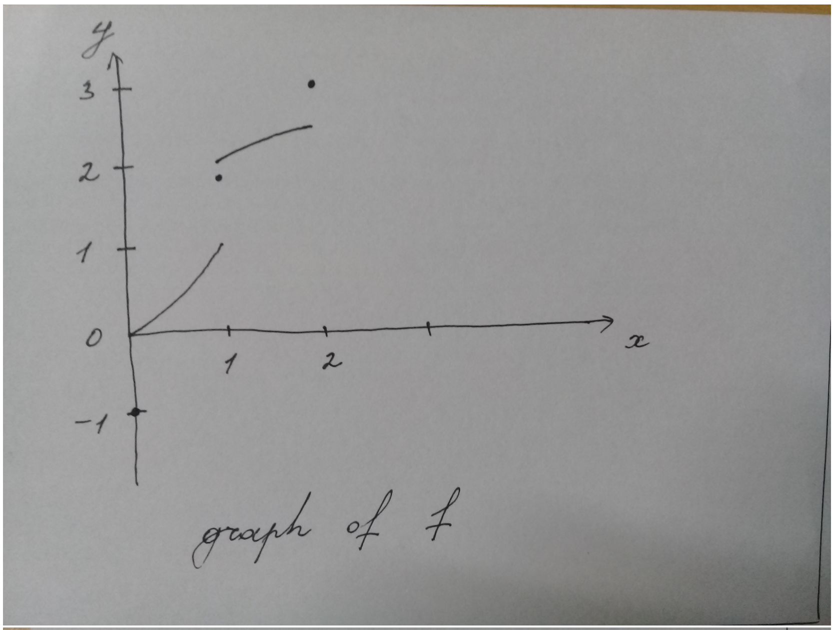
$f_c(0) = 0$,

$f_c(x) = x^2$ for $0 < x < 1$,

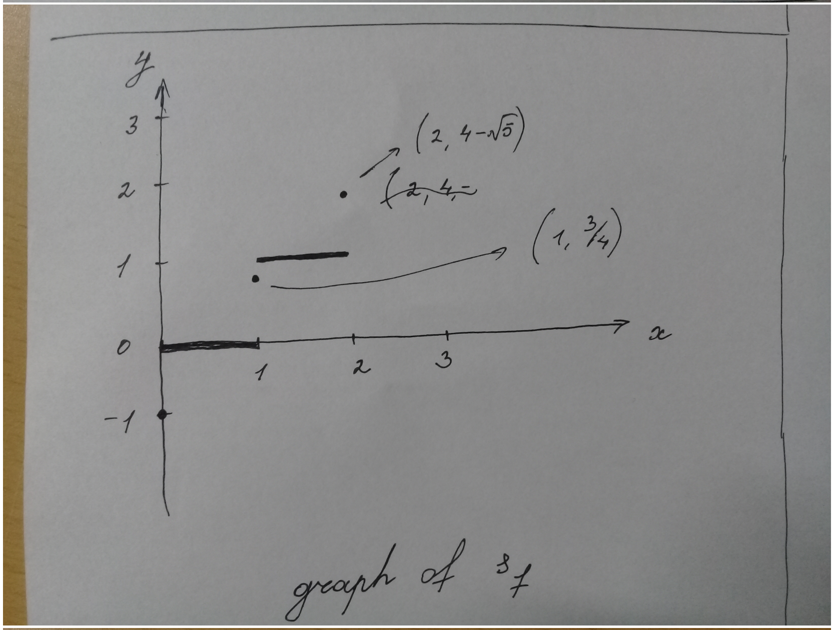
$f_c(1) = 1$,

$f_c(x) = \sqrt{x + 3} - 1$ for $1 < x < 2$,

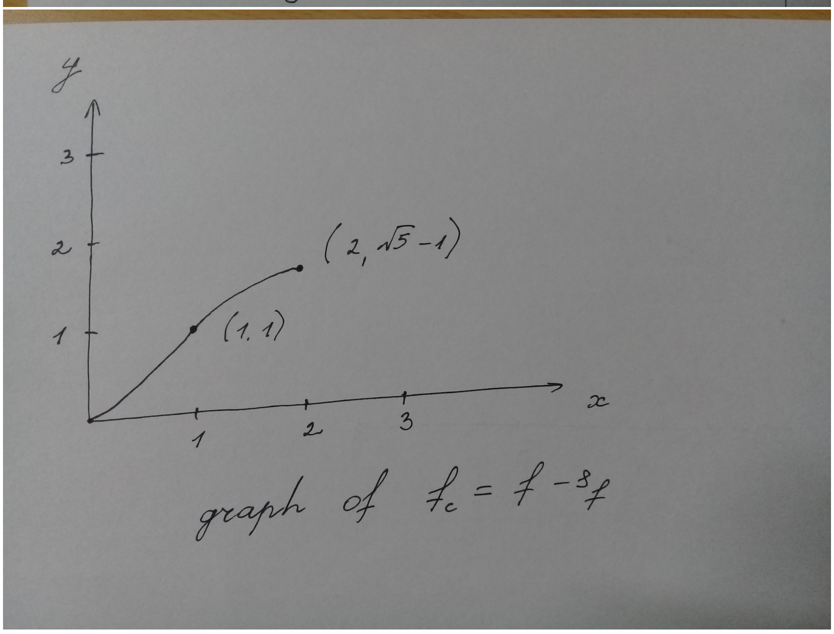
$f_c(2) = \sqrt{5} - 1$.



y
3
2
1
0
-1
1
2
x
graph of f



y
3
2
1
0
-1
1
2
3
x
(2, 4-√5)
(1, 3/4)
graph of ³f



y
3
2
1
(2, √5 - 1)
(1, 1)
1
2
3
x
graph of f_c = f - ³f

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Problem 5.15

Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be any polynomial. $f_k(x) = a_kx^k$ is continuous on $[a, b]$ and differentiable on (a, b). So, a_kx^k is of bounded variation on $[a, b]$, for any k. The function $p(x)$, being the sum of functions which are of bounded variation on $[a, b]$, is also of bounded variation on $[a, b]$. Since the choices of the interval $[a, b]$ and the polynomial $p(x)$ were arbitrary, we conclude that every polynomial is of bounded variation on every interval $[a, b]$.

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Problem 5.18

Let $x \in [a, b]$ be any real number and let $\pi = (a = x_0, x_1, \ldots, x_p = x)$ be any partition of $[a, x]$.

(a)

(1) $\sum_{j=1}^{p} |\Delta f_j| = \sum_{j\in J^+(\pi)} \Delta f_j + \sum_{j\in J^-(\pi)} |\Delta f_j| \leq V_j^+(x) + V_j^-(x) \implies V_j(x) \leq V_j^+(x) + V_j^-(x)$.

(2) $V_f(x) \geq \sum_{j=1}^{p} |\Delta f_j| = \sum_{j\in J^+(\pi)} \Delta f_j + \sum_{j\in J^-(\pi)} |\Delta f_j| \implies V_f(x) \geq V_j^+(x) + \sum_{j\in J^-(\pi)} |\Delta f_j| \implies V_f(x) \geq V_j^+(x) + V_j^-(x)$.

Combining the results (1) and (2), the equality $V_f(x) = V_j^+(x) + V_j^-(x)$ is obtained.

(b) Obviously, $V_f(x) = V_f^+(x) + V_f-(x) \geq V_f^+(x) \geq 0$

and $V_f(x) = V_f^+(x) + V_f-(x) \geq V_f^-(x) \geq 0$.

(c) Let x_1' and x_2' be any two real numbers with $a \leq x_1' < x_2' \leq b$ and let $\pi_1 = (a = x_0, x_1, \ldots, x_p = x_1')$ be any partition of $[a, x_1']$. Then $\pi_2 = (a = x_0, x_1, \ldots, x_p = x_1', x_{p+1} = x_2')$ is a partition of $[a, x_2']$. So,

$\sum_{j\in J^+(\pi_2)} \Delta f_j = \sum_{j\in J^+(\pi_1)} \Delta f_j +$ max$(0, f(x_2') - f(x_1')) \implies$

$V_f^+(x_2') \geq \sum_{j\in J^+(\pi_1)} \Delta f_j +$ max$(0, f(x_2') - f(x_1')) \implies$

$V_f^+(x_2') \geq V_f^+(x_1') +$ max$(0, f(x_2') - f(x_1')) \geq V_f^+(x_1')$.

Similarly,

$\sum_{j\in J^-(\pi_2)} |\Delta f_j| = \sum_{j\in J^-(\pi_1)} |\Delta f_j| +$ max$(0, f(x_1') - f(x_2')) \implies$

$V_f^-(x_2') \geq \sum_{j\in J^-(\pi_1)} |\Delta f_j| +$ max$(0, f(x_1') - f(x_2')) \implies$

$V_f^-(x_2') \geq V_f^-(x_1') +$ max$(0, f(x_1') - f(x_2')) \geq V_f^-(x_1')$.

So, V_f^+ and V_f^- are both monotone increasing on $[a, b]$.

d)

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Problem 5.19

Let x be any real number and let $I = [0, x]$ if $x \geq 0$, $I = [x, 0]$ if $x < 0$. Then f is of bounded variation on $I \implies$

$\sum_{i=1}^{p} |\Delta f_j| \leq$ sup S for any partition π of $I \implies$

$|f(x) - f(0)| \leq$ sup $S \implies$

$|f(x)| \leq |f(0)| +$ sup $S \implies$

f is bounded on R.

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Problem 5.22

First note that for any $[a, b] \subset [c, \infty]$, $[c, b]$ is also in $[c, \infty]$, and $V(f; a, b) \leq V(f; c, b)$. So, $V(f; c, \infty) = \sup \{V(f; c, b) : b \geq c\}$.

Similarly, $V(f; -\infty, c) = \sup \{V(f; a, c) : a \leq c\}$ and $V(f; R) = \sup \{V(f; -x, x) : x > 0\}$.

Let $S = \{V(f; a, b) : [a, b]$ a compact interval $\}$. Then $V(f; R) = \sup S$ is finite. For any point $c \in R$,

$V(f; -\infty, c) = \sup \{V(f; a, b) : [a, b] \subset (-\infty, c]\} \leq \sup \{V(f; a, b) : [a, b]\} = V(f; R)$

and

$V(f; c, \infty) = \sup \{V(f; a, b) : [a, b] \subset (c, \infty]\} \leq \sup \{V(f; a, b) : [a, b]\} = V(f; R)$,

so $V(f; -\infty, c)$ and $V(f; c, \infty$ are finite.

Let $x > c$ be any real number. Then

$V(f; -x, x) = V(f; -x, c) + V(f; c, x) \implies$

$V(f; R) \geq V(f; -x, c) + V(f; c, x) \implies$

$V(f; R) \geq V(f; -\infty, c) + V(f; c, x) \implies$

$V(f; R) \geq V(f; -\infty, c) + V(f; c, \infty)$.

On the other hand,

$V(f; -x, x) = V(f; -x, c) + V(f; c, x) \implies$

$V(f; -x, x) \leq V(f; -\infty, c) + V(f; c, \infty) \implies$

$V(f; R) \leq V(f; -\infty, c) + V(f; c, \infty)$.

Thus $V(f; R) = V(f; -\infty, c) + V(f; c, \infty)$.

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Problem 5.23

As we established in problem 5.22, $V(f; c, \infty) = \sup \{V(f; c, b) : b \geq c\}$ and

$V(f; -\infty, c) = \sup \{V(f; a, c) : a \leq c\}$.

Claim. For any two real numbers $x_1 < x_2$, $V_f(x_1) + V(f; x_1, x_2) = V_f(x_2)$.

Proof. Let $x < x_1$ be any real number. Then $V(f; x, x_2) = V(f; x, x_1) + V(f; x_1, x_2)$. $(**)$

(1) $(**) \implies$

$V(f; -\infty, x_2) \geq V(f; x, x_1) + V(f; x_1, x_2) \implies$

$V(f; -\infty, x_2) \geq V(f; -\infty, x_1) + V(f; x_1, x_2) \implies$

$V_f(x_2) \geq V_f(x_1) + V(f; x_1, x_2)$.

(2) $(**) \implies$

$V(f; x, x_2) \leq V(f; -\infty, x_1) + V(f; x_1, x_2) \implies$

$V(f; -\infty, x_2) \leq V(f; -\infty, x_1) + V(f; x_1, x_2) \implies$

$V_f(x_2) \leq V_f(x_1) + V(f; x_1, x_2)$.

Combining (1) and (2), we conclude that $V_f(x_2) = V_f(x_1) + V(f; x_1, x_2)$. The claim is proven.

So, for any two real numbers $x_1 < x_2$,

$V_f(x_2) = V_f(x_1) + V(f; x_1, x_2) \geq V_f(x_1) \implies$

V_f is monotone increasing on R.

Moreover, $V_f(x_2) - V_f(x_1) = V(f; x_1, x_2) \geq f(x_2) - f(x_1) \implies$

$V_f(x_2) - f(x_2) \geq V_f(x_1) - f(x_1) \implies$

$V_f - f$ is monotone increasing on R.

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Problem 5.25

We know from problem 5.23 that

$$V_f(x_1) + V(f; x_1, x_2) = V_f(x_2),$$

for any two real numbers $x_1 < x_2$.

Since f is of bounded variation on any interval $[a, b]$, it is of bounded variation on $[c, x]$, for any $x \geq c$, and on $[x, c]$, for any $x \leq c$. So, the function $V_f(x)$ is defined by

$V_f(x) = V_f(c) + V(f; c, x)$, if $x \geq c$,

$V_f(x) = V_f(c) - V(f; x, c)$, if $x < c$.

Suppose that f is continuous at c. Since f is of bounded variation on $[c, x]$ for any $x \geq c$ and on $[x, c]$ for any $x < c$, we deduce from Theorem 5.5.2 that the functions $V(f; c, x)$, where $x \geq c$, and $V(f; x, c)$, where $x < c$, both are continuous at $x = c$. Consequently, $V_f(c^+)$ and $V_f(c^-)$ exist for the function $V_f(x)$. They can be shown to be equal:

$$V_f(c^+) = V_f(c) + V(f; c, c^+) = V_f(c) = V_f(c) - V(f; c^-, c) = V_f(c^-),$$

thereby implying that V_f is continuous at c.

Conversely, if we suppose that $V_f(x)$ is continuous at $x = c$, then $V_f(c^+)$ and $V_f(c^-)$ exist and are equal to each other $\implies$

$V_f(c) + V(f; c, c^+)$ and $V_f(c) + V(f; c^-, c)$ exist and are equal $\implies$ the function $V_f(f; c, x)$, where $x > c$, is continuous at c (also note that f is of bounded variation on every compact interval) $\implies$

f is continuous at c.