(1) 10 points. Let $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{c} = (c_1, \cdots, c_n)$, and $g_1(t) = f(t, c_2, \cdots, c_n)$. Prove the following.

(a) If f is continuous at $\mathbf{x} = \mathbf{c}$, $g_1(t)$ is continuous at $t = c_1$.

(b) If $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$ for any sequence $\mathbf{x}_k$ that converges to $\mathbf{c}$, $f(\mathbf{x})$ is continuous at $\mathbf{c}$.

Solution. (a) Let $\epsilon > 0$ be given arbitrarily. Because f is continuous at $\mathbf{x} = \mathbf{c}$,

$$\exists\ \delta > 0 \quad \text{such that for all } \mathbf{x} \in N(\mathbf{c}; \delta), \quad |f(\mathbf{x}) - f(\mathbf{c})| < \epsilon.$$

For any $t \in N(c_1; \delta)$, we have $(t, c_2, \cdots, c_n) \in N(\mathbf{c}; \delta)$, and thus,

$$|g(t) - g(c_1)| = |f(t, c_2, \cdots, c_n) - f(\mathbf{c})| < \epsilon.$$

Therefore, $g_1(t)$ is continuous at $t = c_1$.

(b) Suppose to the contrary that f is discontinuous at $\mathbf{c}$, which means

$$\exists\ \epsilon_0 > 0 \quad \text{such that for all } \delta > 0,\ \exists\ \mathbf{x} \in N(\mathbf{c}; \delta) \text{ satisfying} \quad |f(\mathbf{x}) - f(\mathbf{c})| \geq \epsilon_0.$$

Fix such $\epsilon_0 > 0$. Then, for each $n \in \mathbb{N}$, there exists some $\mathbf{x}_n \in N(\mathbf{c}; \frac{1}{n})$ such that

$$|f(\mathbf{x}_n) - f(\mathbf{c})| \geq \epsilon_0.$$

We have $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{c}$ but $f(\mathbf{x}_n) \not\to f(\mathbf{c})$. Therefore, we proved the contrapositive statement of part (b).

(2) Let $f : [a, b] \to [a, b]$ be continuous. Prove or disprove the following.

10 points

(a) There exists at least one $x \in [a, b]$ such that $f(x) = x$. (Fixed point theorem)

(b) The image $f([a, b])$ is closed.

(c) The inverse image $f^{-1}([c, d])$ is connected for any $[c, d] \subset [a, b]$.

Solution. We can show that (a) and (b) are true but (c) is false as follows.

(a: True) If $f(a) = a$ or $f(b) = b$, then the statement holds. Suppose otherwise, that is, $f(a) > a$ and $f(b) < b$. The function $g(x) = f(x) - x$ is continuous on $[a, b]$ and satisfies $g(a) < 0$ and $g(b) > 0$. Thus, by intermediate value theorem,

$$\exists\, x_0 \in (a, b) \quad \text{such that} \quad g(x_0) = 0.$$

In other words, $f(x_0) = x_0$.

(b: True) Because $[a, b]$ is closed and bounded, $[a, b]$ is compact by Heine-Borel theorem. Since f is continuous and $[a, b]$ is compact, $f([a, b])$ is compact as well. Therefore, $f([a, b])$ is closed as well.

(c: False) We will present a counterexample. Define $f : [a, b] \to [a, b]$ by

$$f(x) = \begin{cases} x & \text{if } a \le x \le \frac{a+b}{2}, \\ b - x & \text{if } \frac{a+b}{2} < x \le b. \end{cases}$$

Let $c = a$ and $d = \frac{2a+b}{3}$. Then, one can show

$$f^{-1}([c, d]) = \left[a, \frac{2a+b}{3}\right] \cup \left[\frac{a+2b}{3}, b\right],$$

which is disconnected.

(3) 10 points Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove or disprove the following.

(a) For any $\epsilon > 0$, there exists a piecewise constant function $s : [a, b] \to \mathbb{R}$ such that $\|f - s\|_\infty < \epsilon$. (+5 points for the proof of corrected version.)

(b) If f is uniformly continuous and strictly monotone, inverse function f^{-1} is also uniformly continuous.

Solution. We can show that (a) and (b) are true as follows.

(a: true) The function f is continuous on a compact interval, so f is uniformly continuous. Thus, for the given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$x, y \in [a, b],\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Let $I_n := [a + (n-1)\delta, a + n\delta)$ and $m_n := a + (n + 0.5)\delta$ for each $n \in \mathbb{N}$. Since $I_n \cap [a, b]$ partitions $[a, b]$, we can define $s : [a, b] \to \mathbb{R}$ by

$$s(x) = f(m_n) \quad \text{if } x \in I_n \cap [a, b].$$

Then s is piecewise constant and

$$|f(x) - s(x)| = |f(x) - f(m_n)| < \epsilon \quad \text{whenever } x \in I_n \cap [a, b].$$

(b) Since f is strictly monotone, f is one-to-one, so $f^{-1} : f([a, b]) \to [a, b]$ exists. Also, by the strict monotonicity, the inverse image of a relatively open interval by f^{-1} is again a relatively open interval, so f is continuous. Since f is continuous and $[a, b]$ is compact, $f([a, b])$ aiso is compact. Since f^{-1} is continuous, it is uniformly continuous on the compact set $f([a, b])$.

(4) 10 points

(a) Let $f_k : (a,b) \to \mathbb{R}$ be a Cauchy sequence in $C_\infty((a,b))$. Show that f_k converges to a continuous function (i.e., prove the theorem.)

(b) Prove or disprove that the limit is uniformly continuous if f_k are all uniformly continuous.

Solution. We can show that both of them are true as follows.

(a: True) For each $x \in (a,b)$, the sequence $\{f_k(x)\}_{k=1}^\infty$ converges since $\mathbb{R}$ is complete. Thus, we can define the pointwise limit function $f : (a,b) \to \mathbb{R}$ as

$$f(x) = \lim_{k\to\infty} f_k(x) \quad \text{for all } x \in (a,b).$$

We will show that f is continuous and $\|f_k - f\|_\infty \to 0$. Let $\epsilon > 0$ be given. Since $f_k : (a,b) \to \mathbb{R}$ is a Cauchy sequence in $C_\infty((a,b))$,

$$\exists\ N \in \mathbb{N} \quad \text{such that for all } m,n \geq N, \quad \|f_m - f_n\|_\infty < \frac{\epsilon}{2}.$$

Claim. $\|f_n - f\|_\infty < \epsilon$ for all $n \geq N$.

(Proof for the claim) For each $x \in (a,b)$, choose $K = K(x)$ that satisfies

$$|f_k(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } k \geq K.$$

Let $m = \max\{K, N\}$. Then, for any $n \geq N$, we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, f_k converges (uniformly) to f. Moreover, since a limit of a uniformly convergent sequence continuous functions is continuous, f is continuous.

(b: True) (Of course, "the limit" in the statement means the limit with respect to the norm $\|\cdot\|_\infty$.) Suppose that f_k are all uniformly continuous and $\|f_k - f\|_\infty \to 0$ as in part (a). We will show that f is uniformly continuous. Let $\epsilon > 0$ be given. Then, for some $n \in \mathbb{N}$,

$$\|f_n - f\|_\infty < \frac{\epsilon}{3}.$$

Since f_n is uniformly continuous,

$$\exists\ \delta > 0 \quad \text{such that for all } x,y \in (a,b) \text{ satisfying } |x-y| < \delta, \quad |f_n(x) - f_n(y)| < \frac{\epsilon}{3}.$$

Therefore, for all $x, y \in (a,b)$ satisfying $|x-y| < \delta$, we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq |f_n(x) - f_n(y)| + 2\|f - f_n\|_\infty \\
&< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.
\end{aligned}$$

(5) 10 points

(a) Prove or disprove that a set $A = \{f \in C_\infty(\mathbb{R}) : \|f\|_\infty \le 1\}$ is closed.

(b) Prove or disprove that every infinite subset of A has a limit point (i.e., A is compact) or every sequence in A has a cluster point.

Solution. We can show that (a) is true but (b) is false, as follows.

(a: True) Let $g \in C_\infty(\mathbb{R})\backslash A$. Then, $K = \|g\|_\infty > 1$. For any $h \in C_\infty(\mathbb{R})$ satisfying $\|h - g\|_\infty < K - 1$, we have

$$\|h\|_\infty \ge \|g\|_\infty - \|g - h\|_\infty > K - (K-1) = 1,$$

which means

$$N(g; K-1) \subset C_\infty(\mathbb{R})\backslash A.$$

Therefore, A^c is open, which means A is closed.

(b: False) A is not compact, which can be shown as follows. Define

$$f(x) = \begin{cases} 1 + x & \text{if } -1 < x \le 0, \\ 1 - x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and set $f_n : \mathbb{R} \to \mathbb{R}$ to be the function defined as $f_n(x) = f(x - n)$ for each $n \in \mathbb{N}$. Then $B = \{f_n : n \in \mathbb{N}\}$ is an infinite subset of A.

Claim. B doesn't have a limit point in A.

(Proof for the claim) Suppose to the contrary that $g \in A$ is a limit point of B. Then, by the definition of the limit point, $f_n \in N'(g; 1)$ for some $n \in \mathbb{N}$. Because $\|\cdot\|_\infty$ is a norm, the condition $f_n \in N'(g; 1)$ is equivalent to

$$c = \|f_n - g\|_\infty \in (0, 1).$$

In particular, $|f_n(n) - g(n)| \le c$. For all $m \in \mathbb{N}$ such that $m \ne n$,

$$\|f_m - g\|_\infty \ge |f_m(n) - g(n)| \ge |f_m(n) - f_n(n)| - |f_n(n) - g(n)| \ge 1 - c \quad \text{whenever } m \ne n,$$

so we have $N'(g; \min\{c, 1 - c\}) \cap B = \emptyset$. This contradicts the assumption that g is a limit point.

(6) The following are false. Find counterexamples.

10 points

(a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is closed, $f(A)$ is closed.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is open, $f(A)$ is open.

(c) If A_i are open, $\cap_{i=1}^{\infty} A_i$ is open or the empty set.

(d) If $A_i \neq \emptyset$, are closed, and $A_1 \supset A_2 \supset \cdots$, then $\cap_{i=1}^{\infty} A_i \neq \emptyset$.

Solution. (a) Let $f(x) = e^x$ and $A = \mathbb{R}$. Then f is continuous and A is closed, but $f(A) = (0, \infty)$ is not closed.

(b) Let $f(x) = x^2$ and $A = (-1, 1)$. Then f is continuous and A is open, but $f(A) = [0, 1)$ is not open nor empty.

(c) Let $A_i = (-1/i, 1/i)$ for each $i \in \mathbb{N}$. Then A_i are open, but $\cap_{i=1}^{\infty} A_i = \{0\}$ is not open.

(d) Let $A_i = \cup_{j=i}^{\infty} \{j\}$ for each $i \in \mathbb{N}$. Then $A_i^c = (-\infty, i) \cup \bigcup_{j=1}^{\infty} (j, j+1)$ are open, which means A_i are closed. However, $\cap_{i=1}^{\infty} A_i = \emptyset$.

(7) Let $\liminf x_k = a$ and $\limsup x_k = b$. Show the following directly from definitions.

10 points

(a) Show that, if $a = b \in \mathbb{R}$, then x_k is a Cauchy sequence.

(b) Prove or disprove that, if $b < \infty$ and $\epsilon > 0$, there is $k_0 \in \mathbb{N}$ such that $x_k > b - \epsilon$ for all $k > k_0$.

(c) Prove or disprove that $a \leq b$.

Solution. (a) Let $\epsilon > 0$ be given arbitrarily. Because $a, b \in \mathbb{R}$, the sequence is bounded as $|x_k| \leq M$ for some constant $M < \infty$. (Since there are no cluster points in the bounded sets $\{x \in [-M, M] : x \leq a - \frac{\epsilon}{2}\}$ and $\{x \in [-M, M] : x \geq b + \frac{\epsilon}{2}\}$, there are only finitely many terms of x_k in those sets. See Theorem 1.3.11.) Hence, there exists some $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} x_k &> a - \frac{\epsilon}{2} \quad \text{for all } k \geq N_1, \\ x_k &< b + \frac{\epsilon}{2} \quad \text{for all } k \geq N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then, for all $m, n \geq N$,

$$|x_m - x_n| < \left(b + \frac{\epsilon}{2}\right) - \left(a - \frac{\epsilon}{2}\right) = \epsilon.$$

(b: False) We will present a counterexample. Define a sequence x_k as

$$\begin{cases} x_{2k-1} = 0 & \text{for all } k \in \mathbb{N}, \\ x_{2k} = 1 & \text{for all } k \in \mathbb{N}. \end{cases}$$

Then the set of cluster points is $\{0, 1\}$, so $b = \sup\{0, 1\} = 1 < \infty$. If we set $\epsilon = 1$, however, $x_{2k-1} \leq 0 = b - \epsilon$ for all $k \in \mathbb{N}$, so there is no k_0 satisfying the statement.

(c: True) We will prove $a \leq b$. If $a = -\infty$ or $b = \infty$, it naturally holds that $a \leq b$, so we only have to consider the case when $a, b \in \mathbb{R}$, i.e., the case x_k is bounded. Then, for the set C of cluster points of x_k,

$$a = \inf C \leq \sup C = b.$$

(8) 10 points Let $C_1, C_2 \subset \mathbb{R}^n$ be two disjoint and closed sets.

(a) Give the definition for the distance between the two sets.

(b) Find an example that the distance of the two disjoint and closed sets is zero.

(c) Show that, if C_1 is bounded, there exist two open sets U_1 and U_2 such that $C_1 \subset U_1$, $C_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$.

Solution. (a) We define

$$\operatorname{dist}(C_1, C_2) = \inf\{\|\mathbf{x} - \mathbf{y}\| \,:\, \mathbf{x} \in C_1,\ \mathbf{y} \in C_2\},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

(b) For example, let $C_1 = \mathbb{N}$ and $C_2 = \{n + \frac{1}{n} \,:\, n \in \mathbb{N},\ n \geq 2\}$. Then,

$$C_1^c = (-\infty, 1) \cup \bigcup_{j=1}^{\infty} (j, j+1) \quad \text{and} \quad C_2^c = (-\infty, 2.5) \cup \bigcup_{j=2}^{\infty} \left(j + \frac{1}{j}, j + 1 + \frac{1}{j+1}\right)$$

are open, so C_1 and C_2 are closed. However,

$$0 \leq \operatorname{dist}(C_1, C_2) \leq \inf\left\{\left|n - \left(n + \frac{1}{n}\right)\right| \,:\, n \in \mathbb{N},\ n \geq 2\right\} = 0.$$

(c) Because C_2^c is open and $C_1 \subset C_2^c$, we have

$$\forall\ \mathbf{x} \in C_1,\ \exists\ r_{\mathbf{x}} > 0 \quad \text{such that} \quad N(\mathbf{x}; 2r_{\mathbf{x}}) \subset C_2^c.$$

By Heine-Borel theorem, C_1 is compact. Therefore, for the open cover $\{N(\mathbf{x}; r_{\mathbf{x}})\}_{x \in C_1}$ of C_1, there exists a finite subcover $\{N(\mathbf{x}_j; r_{\mathbf{x}_j})\}_{j=1}^{k}$. For $r := \min\left\{r_{\mathbf{x}_j} \,:\, j = 1, \cdots, k\right\}$, we set

$$U_1 = \cup_{j=1}^{k} N(\mathbf{x}_j; r_{\mathbf{x}_j}) \quad \text{and} \quad U_2 = \{\mathbf{x} \in \mathbb{R}^n \,:\, \operatorname{dist}(\{\mathbf{x}\}, C_2) < r\}.$$

Obviously, $C_1 \subset U_1$, $C_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$. U_1 is a union of open sets, so is an open set. To prove that U_2 is open, for each $\mathbf{x} \in U_2$, observe that

$$N(\mathbf{x}; r - \operatorname{dist}(\{\mathbf{x}\}, C_2)) \subset U_2.$$

(9) 10 points

(a) Prove or disprove that $f(x,y) = \frac{x^2y}{x^2+y^2}$ is continuous at $(x,y) = (0,0)$ if we set $f(0,0) = 0$.

(b) Show that $f : [0,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous.

Solution. (a) For $(x,y) \neq (0,0)$, using $|xy| \leq \frac{1}{2}(x^2+y^2)$, we have

$$|f(x,y) - f(0,0)| = \frac{x^2|y|}{x^2+y^2} \leq \frac{|x|(x^2+y^2)}{2(x^2+y^2)} = \frac{|x|}{2}.$$

Therefore, setting $\delta = \epsilon$ for given $\epsilon > 0$ shows the continuity at 0 as follows:

$$|(x,y) - (0,0)| < \delta \implies |f(x,y) - f(0,0)| \leq \frac{|x|}{2} \leq \frac{|(x,y)|}{2} < \frac{\delta}{2} < \epsilon.$$

(b) Let $\epsilon > 0$ be given. The restricted function $f : [0,2] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is continuous on a compact set, thus is uniformly continuous. So,

$$\exists\, \delta_1 > 0 \quad \text{such that for all } x,y \in [0,2] \text{ satisfying } |x-y| < \delta_1, \quad |f(x) - f(y)| < \epsilon.$$

On the other hand, for the restricted function $f : [1,\infty) \to \mathbb{R}$, we have

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2} \quad \text{for all } x,y \in [1,\infty).$$

Therefore, $\delta_2 = \epsilon$ satisfies that for all $x,y \in [1,\infty)$ satisfying $|x-y| < \delta_2$,

$$|f(x) - f(y)| < \frac{\delta_2}{2} = \frac{\epsilon}{2} < \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2, 1\}$. If $x,y \in [0,\infty)$ satisfy $|x-y| < \delta$, then $x,y \in [0,2]$ or $x,y \in [1,\infty)$, so the above argument proves that

$$|f(x) - f(y)| < \epsilon.$$

(10)
10 points

(a) Let $X = \{f : \mathbb{N} \to \{0, 1\}\}$ be the collection of all functions defined on natural numbers which have values of 0 or 1. Prove or disprove that the set is uncountable.

(b) Prove that a sequence x_k has a cluster point if $x_k \in (0, 1)$ for all k. (Do not use Heine-Borel.) (Note: You should consider the space $\mathbb{R}$, because cluster points can be 0 or 1.)

Solution. (a) The set X is uncountable, which can be proved as follows. Suppose to the contrary that X is countable, and enumerate the elements as $X = \{f_1, f_2, \cdots, f_k, \cdots\}$. Define a function $f : \mathbb{N} \to \{0, 1\}$ by

$$f(n) = \begin{cases} 0 & \text{if } f_n(n) = 1, \\ 1 & \text{if } f_n(n) = 0. \end{cases} \quad \text{for each } n \in \mathbb{N}.$$

Then, $f \neq f_n$ for all $n \in \mathbb{N}$, so $f \notin X$, which contradicts the definition of X. Therefore, X is uncountable.

(b) (The proof is the same as that for Theorem 1.3.11.) The sequence x_k is bounded. Consider the image $Y = \{x_k : k \in \mathbb{N}\}$. If Y is infinite, by Bolzano-Weierstrass theorem, Y has a limit point, which is a cluster point of the sequence x_k. Otherwise, if Y is finite, at least one of the values of the sequence repeats infinitely often, which becomes a cluster point of the sequence x_k.