1 (5 points) For $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$, define

$$||\mathbf{x}||_\infty = \max\{|x_1|, |x_2|\}$$

and

$$d_\infty(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||_\infty.$$

Prove that $d_\infty(\mathbf{x}, \mathbf{y})$ is a metric on $\mathbb{R}^2$.

Solution. First, we will show that $||\cdot||_\infty$ is a norm on $\mathbb{R}^2$. **(+1 points)** It is clear that $||\cdot||_\infty$ is positive definite and absolutely homogeneous, so the only remainder is subadditivity. **(+1 points)** To prove it, with no loss of generality, assume that $|x_1| \geq |x_2|$, and $|y_1| \geq |y_2|$. Then $||\mathbf{x}||_\infty = |x_1|$, and $||\mathbf{y}||_\infty = |y_1|$. Thus, we have

$$||\mathbf{x}||_\infty + ||\mathbf{y}||_\infty = |x_1| + |y_1|.$$

However, since $|x_1| + |y_1| \geq |x_1 + y_1|$, and $|x_1| + |y_1| \geq |x_2| + |y_2| \geq |x_2 + y_2|$ by assumption,

$$|x_1| + |y_1| \geq \max\{|x_1 + y_1|, |x_2 + y_2|\}.$$

That implies the subadditivity, i.e.,

$$||\mathbf{x}||_\infty + ||\mathbf{y}||_\infty \geq ||\mathbf{x} + \mathbf{y}||_\infty. \quad \textbf{(+2 points)}$$

Therefore, $||\cdot||_\infty$ is a norm on $\mathbb{R}^2$. Then by **Theorem 2.1.4**, $d_\infty(\mathbf{x}, \mathbf{y})$ is a metric on $\mathbb{R}^2$. **(+1 points)**

2 Define the set D on $\mathbb{R}^2$ as

5 points

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y > 0\}.$$

Is D an open set or a closed set on $\mathbb{R}^2$? Explain your answer.

Solution. D is neither an open set nor a closed set. **(+1 points)**
First, consider a point in $\mathbb{R}^2$, $\mathbf{x} = (0, 1)$. Then $\mathbf{x} \in D$, but no neighborhood of $\mathbf{x}$ contained in D. Thus, $\mathbf{x}$ is contained in D but not a interior point of D. Therefore, D is not open. **(+2 points)**
Next, consider a point in $\mathbb{R}^2$, $\mathbf{y} = (0, 0)$. Then $\mathbf{y} \notin D$, but every neighborhood of $\mathbf{y}$ contains points in D and also points not in D. Thus, $\mathbf{y}$ is a boundary point of D but not contained in D. Therefore, D is not closed. **(+2 points)**