Group1 HW8

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Contribution Details

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11) Since f is continuous on $[a,b]$, Theorem 6.2.7 implies that f is integrable on $[a,b]$ $(f \in R(a,b))$ Since we found f is in $R[a,b]$, choosing $g(x)=f(x)$ yields that $\int_a^b f^2(x)dx = 0$ Obviously, continuity of f implies that $f(x)\cdot f(x)$ will also be continuous $\Rightarrow$ $\boxed{f^2(x) - \text{continuous on } [a,b]}$ (Proof is mentioned on Theorem 3.3.1)

Thus, f^2 - continuous and non-negative on $[a,b]$ where $\int_a^b f^2(x)dx = 0 \Rightarrow$ Then, according to Theorem 6.2.9, we get $f^2(x) = 0$ for $\forall x \in [a,b] \Rightarrow$ $\boxed{f(x)=0 \text{ for } \forall x \in [a,b]}$ $\boxed{\checkmark}$ ■

12) Since f,g - continuous functions on $[a,b]$, Theorem 6.2.7 implies that f,g - integrable on $[a,b]$ $(f,g \in R[a,b])$. Let $h(x) = f(x) - g(x)$, where Theorem 3.3.1 concludes that $\boxed{h(x) - \text{continuous}}$ So, from Theorem 6.2.7 we get that as $h(x)$ - continuous on $[a,b]$ $\Rightarrow \boxed{h(x) \in R[a,b]}$ and according to Theorem 6.1.1 $\Rightarrow$

$$1\cdot\int_a^b f(x)dx + (-1)\int_a^b g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx = 0 = \int_a^b (f(x)-g(x))dx$$

$$\Rightarrow \boxed{\int_a^b h(x)dx = 0}$$

Henceforth, $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 0 = \int_a^b (f(x)-g(x))\,dx = \int_a^b h(x)\,dx$

$\boxed{\int_a^b h(x)\,dx = 0 \text{ and } h(x)\text{-continuous on } [a,b]}$ $\Box$ If $\exists c \in [a,b]$ s.t. $h(c)=0$, then we get $f(c)-g(c)=0$ ✓ Assume $h(x)>0$ for all $x\in[a,b]$

Then, h-continuous and $h(x)>0$ for $\forall x\in[a,b]$ where $\int_a^b h(x)\,dx = 0 \Rightarrow$ from Theorem 6.2.8., we obtain $h(x)=0$ for $\forall x \in [a,b]$ which is $\boxed{\times}$ Hence, $\exists x_0 \in [a,b] \Rightarrow h(x_0) \leq 0$

If $h(x_0) = 0 \Rightarrow f(x_0) = g(x_0)$ for some $x_0 \in [a,b]$ and we're done ✓

So, assume $\boxed{h(x_0)<0}$ According to Theorem 6.1.1 $\Rightarrow$

$\int_a^b ((-1)f(x) + 0\cdot f(x))\,dx = \int_a^b (-f(x))\,dx = (-1)\int_a^b f(x)\,dx + 0\cdot\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx \Rightarrow \boxed{\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx \text{ for any function } f \text{ on } [a,b]}$

If $h(y)<0$ for $\forall y \in [a,b]$, then taking $t(y) = -h(y)$ yields $t(y) = -h(y) > 0$ and $\int_a^b -h(y)\,dy = -\int_a^b h(y)\,dy = 0 \Rightarrow \int_a^b t(y) = 0$ where $t(y)$-continuous on $[a,b]$ and $t(y)>0$ for $\forall y\in[a,b]$; However, in the above, we proved this can't be possible $\boxed{\times}$ $\exists y_0 \in [a,b] \Rightarrow h(y_0) \geq 0$ If $h(y_0) = f(y_0) - g(y_0) = 0$, then we're done ✓ $\Rightarrow$ So, assume $\boxed{h(y_0)>0}$

$h(x_0)<0<h(y_0)$ for some $x_0, y_0 \in [a,b]$ Since $h(x)$ is continuous on $[a,b]$, $h(x)$ will be continuous on $[x_0, y_0]$ as well. Then, applying Intermediate Value Theorem concludes that $\exists z_0 \in [x_0, y_0]$ such that $h(z_0)=0 \Rightarrow f(z_0)-g(z_0)=0$ or just $f(z_0)=g(z_0)$ for some $z_0 \in [a,b]$ ✓ In conclusion, for all cases, we proved $\boxed{\exists c \in [a,b] \text{ s.t. } f(c)=g(c)}$ ■

10) According to Theorem 6.1.3, we can obtain the proceedings $\rightarrow S(f,\Pi)=\sum_{j=1}^{p} f(s_j)\Delta x_j \le \sum_{j=1}^{p} g(s_j)\Delta x_j \le$
$\le \sum_{j=1}^{p} h(s_j)\Delta x_j$ since $f(x)\le g(x)\le h(x)$ for $\forall x \in [a,b]$
$\boxed{s_j \text{ is taken arbitrarily}}$ on interval $[x_{j-1}, x_j]$
and $\Delta x_j = x_j - x_{j-1} > 0$ with f,g,h - bounded functions on $[a,b]$
So, $S(f,\Pi)\le S(g,\Pi)\le S(h,\Pi)$ for $\forall \Pi \in \Pi[a,b]$
We also know that $\forall \varepsilon>0$, $\exists \Pi_0^1 \in \Pi[a,b]$ for which $|S(f,\Pi)-I|<\varepsilon$ w.e. $\Pi \ge \Pi_0^1$ Similarly, we deduce $\forall \varepsilon>0$, $\exists \Pi_0^2 \in \Pi[a,b]$ for which $|S(h,\tilde{\Pi})-I|<\varepsilon$ whenever $\tilde{\Pi}\ge \Pi_0^2$. Let $\Pi_0 = \Pi_0^1 \vee \Pi_0^2 \Rightarrow I-\varepsilon < S(f,\Pi)\le$
$\le S(g,\Pi) \le S(h,\Pi) < I+\varepsilon \Rightarrow \forall \varepsilon>0$, $\exists \Pi_0 \in \Pi[a,b] \Rightarrow$
$|S(g,\Pi)-I|<\varepsilon$ w.e. $\Pi \ge \Pi_0$ Thus, $\boxed{g \in R[a,b] \text{ and } \int_a^b g(x)dx = I}$ ■

MAS241

$\boxed{6.7}$ Note: value of f at 1 wasn't given, so I assume $f(y) = 1$ for $0 < y \leq 1$ rather than $0 < y < 1$.

Let's define $f \circ g$ on $[0,1]$ if $x = 0 \Rightarrow f(g(0)) = 1$

if $x \in \{\frac{p}{q} \mid (p,q) = 1\} \cap [0,1] \Rightarrow f(g(x)) = f(\frac{1}{q}) = 1$

since $0 < \frac{1}{q} \leq 1$

if $x \in \mathbb{Q}^c \cap [0,1] \Rightarrow f(g(x)) = f(0) = 0$

Now, considering the fact that between any two numbers there are both rational and irrational numbers we see that $m_j = \inf\limits_{x \in [x_{j-1}, x_j]} f(x) = 0$

$M_j = \sup\limits_{x \in [x_{j-1}, x_j]} f(x) = 1$

$\Rightarrow L(fg) = \sup(\sum m_j \Delta x_j) = \sup(0) = 0$

$U(fg) = \inf(\sum M_j \Delta x_j) = \inf(\sum \Delta x_j) = \inf(b-a) = b - a =$
$= 1 - 0 = 1 \Rightarrow L(fg) \neq U(fg) \Rightarrow f \circ g \notin R[0,1]$

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$\boxed{\text{MAS241}}$

$\boxed{6.19}$ Let's consider the intervals $[x_{j-1}, x_j]$
if we choose irrational number, then it won't be of the form $p/2^q$ obviously $\Rightarrow m_j = 0$

We claim that, it is possible to choose arbitrarily close numbers to x_j that are in the form $\frac{p}{2^q}$

Let $\varepsilon > 0$ choose q sufficiently large so that $2^q \cdot \varepsilon > 1$ $\Rightarrow$ the interval $[2^q(x_j - \varepsilon), 2^q \cdot x_j]$ will contain natural number since its length > 1

$\Rightarrow p \in [2^q(x_j - \varepsilon), 2^q x_j] \Rightarrow x_j - \varepsilon \leq \frac{p}{2^q} \leq x_j$

$\Rightarrow M_j = x_j \Rightarrow U(f;\Pi) = \sum_{j=1}^{p} M_j \Delta x_j = \sum_{j=1}^{p} (x_j^2 - x_j x_{j-1}) =$

$= \sum_{j=1}^{p} \frac{x_j^2 - x_{j-1}^2 + (x_j - x_{j-1})^2}{2} = \sum_{j=1}^{p} \frac{x_j^2 - x_{j-1}^2}{2} + \frac{(x_1 - x_0)^2 + \dots + (x_p - x_{p-1})^2}{2} \geq$

$\geq \frac{x_p^2 - x_0^2}{2} = \frac{1}{2} \Rightarrow U(f) = \inf(U(f;\Pi)) \geq \frac{1}{2}$

$L(f) = \sup(L(f;P)) = \sup(0) = 0$

$\Rightarrow L(f) \neq U(f) \Rightarrow f \notin R[0,1]$

MAS 2411

$\boxed{5.24}$ Let $c=a$ and assume f is continuous at c. We show that $V_f(c^+)=V_f(c)$ (it'll be enough since a is left-hand-side bound). Let $\varepsilon>0 \Rightarrow \exists \delta>0$

$|f(x)-f(c)|<\varepsilon$ whenever $c\le x<c+\delta$

Take partition $\Pi=\{x_0,\dots,x_p\}\in\Pi(a,b)$ s.t

$$V(f;a,b)-\varepsilon<\sum_{j=1}^{p}|\Delta f_j|\le V(f;a,b)$$

refine Π so that $\|\Pi\|<\delta \Rightarrow x_1\in N(a,\delta) \Rightarrow |f(x_1)-f(a)|<\varepsilon$

$$V(f;a,b)-\varepsilon<|f(x_1)-f(a)|+\sum_{j=2}^{p}|\Delta f_j|<\varepsilon+V(f;x_1,b)$$

$$\Rightarrow V(f;c,b)-V(f;x_1,b)<2\varepsilon$$

$$\Rightarrow 0\le V_f(x_1)-V_f(a)=V(f;a,x_1)-V(f;a,a)=V(f;a,x_1)=$$

$$=V(f;a,b)-V(f;x_1,b)<2\varepsilon \Rightarrow x_1 \text{ is arbitrary in } N(a,\delta)$$

$\Rightarrow V_f(a^+)=V_f(a)$. (Similar procedure for b)

Assume V_f is continuous at a. Let $\varepsilon>0$, $\exists\delta>0$

if $a\le x<a+\delta \Rightarrow |V_f(x)-V_f(c)|<\varepsilon$; $0\le|f(x)-f(c)|\le V(f;c,x)=$

$=V_f(x)-V_f(c)<\varepsilon \Rightarrow f(a^+)=f(a)$. (Similar procedure for b)

Thus, we are done!

Pasawat 20190886 HW8 MAS241

$\boxed{6.1}$ Claim: $\forall 0 \le a < b \le 1, \exists c \in (a,b) : c \in \mathbb{Q}$

Let $p = \lceil \frac{1}{b-a} \rceil \ge \frac{1}{b-a}$ $\therefore p \in \mathbb{Z}^+$; $2p(b-a) \ge 2$ $\therefore 2pb - 2 \ge 2pa$

$\therefore$ consider $n = \lfloor 2pb - 1 \rfloor \in \mathbb{Z}^+$; $2pa \le 2pb - 2 < \lfloor 2pb - 1 \rfloor \le 2pb - 1 < 2pb$

$\therefore 2pa < n < 2pb$ $\therefore \frac{n}{2p} \in (a,b)$; $\frac{n}{2p} \in \mathbb{Q}$

Claim: $\forall 0 \le a < b \le 1, \exists d \in (a,b) : d \in \mathbb{Q}^c$

<u>Case 1</u> $a, b \in \mathbb{Q}$, $\therefore a + \frac{1}{\sqrt{2}}(b-a) \in (a,b)$ is irrational

<u>Case 2</u> $a \in \mathbb{Q}, b \in \mathbb{Q}^c$ $\therefore \exists c \in (a,b) : c$ is rational

$\therefore \exists d \in (a,c) : d \in \mathbb{Q}^c$ from case 1 $\therefore d \in (a,b)$; $d \in \mathbb{Q}^c$

<u>Case 3</u> $a \in \mathbb{Q}^c, b \in \mathbb{Q}$ $\therefore \exists c \in (a,b) : c$ is rational

$\therefore \exists d \in (c,b) : d \in \mathbb{Q}^c$ from case 1 $\therefore d \in (a,b)$; $d \in \mathbb{Q}^c$

<u>Case 4</u> $a \in \mathbb{Q}^c, b \in \mathbb{Q}$ $\therefore \exists c \in (a,b) : c$ is rational

$\therefore \exists d \in (a,c) : d \in \mathbb{Q}^c$ from case 3 $\therefore d \in (a,b)$; $d \in \mathbb{Q}^c$

Let $f(x) = \begin{cases} 1 ; & x \in \mathbb{Q} \\ 0 ; & x \in \mathbb{Q}^c \end{cases}$, $g(x) = \begin{cases} -1 ; & x \in \mathbb{Q} \\ 0 ; & x \in \mathbb{Q}^c \end{cases}$

$\therefore (f+g)(x) = \begin{cases} 0 ; & x \in \mathbb{Q} \\ 0 ; & x \in \mathbb{Q}^c \end{cases}$ $\therefore (f+g)(x) = 0$; $\forall x \in \mathbb{R}$

Let $\pi = \{x_0, \dots, x_p\} \in \Pi[0,b]$

for $i \in \{1, \dots, p\}$: $\exists x \in [x_{i-1}, x_i] : x \in \mathbb{Q}^c$ $\therefore m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = 0$

$\exists x \in [x_{i-1}, x_i] : x \in \mathbb{Q}$ $\therefore M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = 1$

$$\left.\begin{aligned} \therefore L(f,\pi) &= \sum_{i=1}^{p} f(m_i)\Delta x_i = \sum_{i=1}^{p} 0(\Delta x_i) = 0 \\ U(f,\pi) &= \sum_{i=1}^{p} f(M_i)\Delta x_i = \sum_{i=1}^{p} 1(\Delta x_i) = (x_p - x_0) = 1 \end{aligned}\right\} L(f) \neq U(f) \therefore f \notin R(a,b)$$

However $(f+g)(x) \in R(a,b)$

$\therefore f+g \in R(a,b) \not\longrightarrow f, g \in R(a,b)$

$\boxed{6.18}$ Claim: $\forall 0 \le a < b \le 1, \exists c \in (a,b) : c \in \mathbb{Q}$

Let $p = \left\lceil \frac{1}{b-a} \right\rceil \ge \frac{1}{b-a}$ $\therefore p \in \mathbb{Z}^+$; $2p(b-a) \ge 2$ $\therefore 2pb - 2 \ge 2pa$

$\therefore$ Consider $n = \lfloor 2pb - 1 \rfloor \in \mathbb{Z}^+$; $2pa \le 2pb - 2 < \lfloor 2pb - 1 \rfloor \le 2pb - 1 < 2pb$

$\therefore$ $2pa < n < 2pb$ $\therefore$ $\frac{n}{2p} \in (a,b)$; $\frac{n}{2p} \in \mathbb{Q}$

Claim: $\forall 0 \le a < b \le 1, \exists d \in (a,b) : d \in \mathbb{Q}^c$

Case 1 $a, b \in \mathbb{Q}$, $\therefore$ $a + \frac{1}{\sqrt{2}}(b-a) \in (a,b)$ is irrational

Case 2 $a \in \mathbb{Q}, b \in \mathbb{Q}^c$ $\therefore \exists c \in (a,b) : c$ is rational

$\therefore \exists d \in (a,c) : d \in \mathbb{Q}^c$ from case 1 $\therefore$ $d \in (a,b)$; $d \in \mathbb{Q}^c$

Case 3 $a \in \mathbb{Q}^c, b \in \mathbb{Q}$ $\therefore$ $\exists c \in (a,b) : c$ is rational

$\therefore \exists d \in (c,b) : d \in \mathbb{Q}^c$ from case 1 $\therefore$ $d \in (a,b)$; $d \in \mathbb{Q}^c$

Case 4 $a \in \mathbb{Q}^c, b \in \mathbb{Q}$ $\therefore$ $\exists c \in (a,b) : c$ is rational

$\therefore \exists d \in (a,c) : d \in \mathbb{Q}^c$ from case 3 $\therefore$ $d \in (a,b)$; $d \in \mathbb{Q}^c$

For $f(x) = \begin{cases} x ; & x \in \mathbb{Q} \\ 0 ; & x \in \mathbb{Q}^c \end{cases}$, and $\pi = \{x_0, \dots, x_p\} \in \Pi[0,b]$

for $i \in \{1, \dots, p\}$: $\exists x \in [x_{i-1}, x_i] : x \in \mathbb{Q}^c$ $\therefore$ $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = 0$

since $\forall \varepsilon > 0, \exists x \in (x_i - \varepsilon, x_i) ; x \in \mathbb{Q}$ $(f(x) = x)$ $\therefore$ $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = x_i$

$L(f, \pi) = \sum_{i=1}^{p} f(m_i) \Delta x_i = \sum_{i=1}^{p} 0 (\Delta x_i) = 0$

$U(f, \pi) = \sum_{i=1}^{p} f(M_i) \Delta x_i = \sum_{i=1}^{p} (x_i)(x_i - x_{i-1})$

$= \sum_{i=1}^{p} \left(\frac{x_i + x_{i-1}}{2} + \frac{x_i - x_{i-1}}{2} \right) (x_i - x_{i-1})$

$= \frac{1}{2} \sum_{i=1}^{p} (x_i^2 - x_{i-1}^2) + \frac{1}{2} \sum_{i=1}^{p} (x_i - x_{i-1})^2$

$= \frac{1}{2} (x_p^2 - x_0^2) + \frac{1}{2} \sum_{i=1}^{p} (x_i - x_{i-1})^2$

$= \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{p} (x_i - x_{i-1})^2$

$\therefore L(f) = 0$, $U(f) = \frac{1}{2} + \frac{1}{2} \inf_{\pi \in \Pi[a,b]} \sum_{i=1}^{p} (x_i - x_{i-1})^2 = \frac{1}{2}$

$L(f) \ne U(f)$ $\therefore$ $f \notin R(a,b)$

$\boxed{6.10}$

Fix $\varepsilon > 0$. Since f and h are in $R[a,b]$, then there exists a partition π_0 such that for any refinement $\pi = \{x_0, ---, x_p\}$ of π_0, we have

$$\left| S(f,\pi) - \int_a^b f(x)dx \right| < \varepsilon \quad \text{and}$$

$$\left| S(h,\pi) - \int_a^b h(x)dx \right| < \varepsilon$$

Also, since $f(x) \leq g(x) \leq h(x)$, for all $x \in [a,b]$, then $S(f,\pi) \leq S(g,\pi) \leq S(h,\pi)$.

$$\Rightarrow -\varepsilon + \int_a^b f(x)dx < S(f,\pi) \leq S(g,\pi)$$

$$\leq S(h,\pi) < \varepsilon + \int_a^b h(x)\,dx$$

$$\Rightarrow$$

$$-\varepsilon + \int_a^b f(x)dx < S(g,\pi) < \varepsilon + \int_a^b h(x)dx$$

since $\int_a^b f(x)dx = \int_a^b h(x)dx = I$, then

$$-\varepsilon + I < S(g,\pi) < \varepsilon + I$$

$$\Rightarrow \left| S(g,\pi) - I \right| < \varepsilon$$

so, g is in $\mathcal{R}[a,b]$, and

$$I = \int_a^b g(x)dx = \int_a^b f(x)dx = \int_a^b h(x)dx$$

$\boxed{6.18}$

Let $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

Suppose $\pi = \{x_0, x_1, ---, x_p\}$ be a partition of $[0,1]$.

Then, $M_j' = \sup\{x : x \in [x_{j-1}, x_j]\}$

$m_j' = \inf\{x : x \in [x_{j-1}, x_j]\}$ when x is rational

and $M_j'' = 0$

$m_j'' = 0$ when x is irrational.

Thus, $M_j = \max\{M_j', M_j''\} = x_j$

$m_j = \min\{m_j', m_j''\} = 0$

$$\Rightarrow L(f, \pi) = \sum_{j=1}^{p} m_j \Delta x_j = 0 \text{ and}$$

$$U(f, \pi) = \sum_{j=1}^{p} x_j \Delta x_j$$

$\Rightarrow L(f) = 0$ and $U(f) = 1$

since $L(f) \neq U(f)$, $f \notin R[0,1]$.