# Group1 HW4

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Contribution Details:

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29) a) $S^\circ = (a,b)$

$S' = [a,b]$ $\Rightarrow \bar{S} = S' \cup S = [a,b]$

$bd(S) = \{a, b\}$

b) $S^\circ = \{(x,0) \subset \mathbb{R}^2 : a < x < b\}$

$S' = \{(x,0) \subset \mathbb{R}^2 : a \leq x \leq b\}$

$\bar{S} = S' \cup S = S' = \{(x,0) \subset \mathbb{R}^2 : a \leq x \leq b\}$

$bd(S) = \{(a,0); (b,0)\}$

c) any interval contains both rational and irrational numbers

$\Rightarrow S^\circ = \emptyset$ ; $bd(S) = \mathbb{R}$ $\Rightarrow \bar{S} = bd(S) \cup S^\circ =$
$= \mathbb{R}$ ; $S' = \mathbb{R}$

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d) $S^\circ = \emptyset$ ; $S' = R^n$ ; $bd(S) = R^n$

$\bar{S} = R^n$

e) $S^\circ = \emptyset$ ; $S' = S$ ; $bd(S) = S$

$\Rightarrow \bar{S} = S$

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48) Take $C_k = (0, \frac{1}{k})$

assume $\exists a$ s.t $a \in C_k$ for $\forall k$

By Archimedes principle

$\exists n \ \forall k \geq n \ \ a \cdot k > 1$

$\Rightarrow a > \frac{1}{k} \Rightarrow a \notin C_k$

Contradiction

55) a) Note that $\overline{N(x_0;\varepsilon_k)} = N(x_0;\varepsilon_k)^\circ \cup$

$\cup \, bd(N(x_0;\varepsilon_k)) = \{ y \subset \mathbb{R}^n / \| x_0 - y \| \leq \varepsilon_k \}$

Since $C_k$ is closure $\Rightarrow$ it's closed

$$\| y - x_0 \| + \| x_0 \| \geq \| y \|$$

$$\Rightarrow \| y \| \leq \| x_0 \| + \varepsilon_k$$

$\Rightarrow$ bounded

$x_0 \in C_k \Rightarrow$ nonempty.

b) Assume $\exists x_1 \neq x_0 \;\; x_1 \in C_k$

for any $k \in \mathbb{N} \Rightarrow \| x_0 - x_1 \| \leq \varepsilon_k$

Since $\varepsilon_k$ converges to $0 \Rightarrow x_0 = x_1$

58) a) $[-1,0) = (-2,0) \cap X$

$(0,1) = (0,1) \cap X$

Since $(-2,0)$ and $(0,1)$ are open in $R$ $\Rightarrow [-1,0]$ and $(0,1)$ are open in $X$

Similarly

$[-1,0) = [-1,0] \cap X$

$(0,1) = [0,1] \cap X$

b) $(-\frac{1}{2}, \frac{1}{2})$ is open in $R \Rightarrow$

$\Rightarrow S$ is open in $X$.

$S = (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$

$\frac{1}{2}$ is boundary point but not in $S$

$\Rightarrow S$ is not closed in $X$

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c) $x_{2k} = 1 - \frac{1}{2k+1}$

$x_{2k+1} = -1 + \frac{1}{2k+2}$

1 and -1 are limit points

$\{1, -1\} \cap X = \{-1\} = \{x_0\}$

$x_0 = -1$ is the only relative limit point in $X$.

e) $x_n = \frac{1}{n+1}$ for $n \geq 1$ is obviously Cauchy sequence

$\cancel{|x_n - x_m| < \max\left(\frac{1}{n+1}, \frac{1}{m+1}\right) < \frac{1}{2}}$

but $x_n \to 0$ $\quad 0 \notin X$

f) Since $[-1/k, 1/k]$ form a closed, bounded interval $C_k = [-1/k, 1/k] \cap X$ are nested, bounded, relatively closed sets.

$$\bigcap_{k=1}^{\infty} [-1/k, 1/k] = [\sup(-1/k), \inf(1/k)] =$$
$$= \{0\}$$

However $0 \cap X = \emptyset$.

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36) We will provide counterexample for which $S \subseteq \mathbb{R}^n$, and $\overline{(S^\circ)}$ is different from $\bar{S}$

Take $S=\{a\}$, where $a \in \mathbb{R}^n \Rightarrow$ Since there doesn't exist neighborhood $N(a)$ that is completely inside the set $S \Rightarrow$ There does not exist interior points of $S$, as $N(a) = (a-\varepsilon, a+\varepsilon)$ will never be contained in $S \Rightarrow \boxed{S^\circ = \emptyset}$

Claim: $\emptyset$ is closed

Prf: As it's true that for every element $x \in \mathbb{R}^n$, there is an open ball $B(x,r)$ with $x \in B(x,r) \subseteq \mathbb{R}^n$ (which is obvious) then, we conclude that $\mathbb{R}^n$ is open or its complement $\emptyset$ is closed ✓

From the definition, closure is the smallest closed set which contains the given set (original), and since we proved $\emptyset$-closed and $\emptyset$ contains itself (the set $\emptyset$)
$\Rightarrow \boxed{\bar{\emptyset} = \emptyset}$ $\boxed{\overline{(S^\circ)} = \emptyset}$ ☒ However, $\bar{S} = S \cup S' = \{a\} \cup S'$ where if $\exists x_0 \in \mathbb{R}^n$ exist in $S'$, then $x_0$ - limit point of $S \Rightarrow N'(x_0, \varepsilon) \cap S \neq \emptyset$ But we know if $S$ has finite number of elements, then there doesn't

exist a limit point since we talk about deleted neighbourhoods and choosing very small $\varepsilon > 0 \Rightarrow$ There does not exist $x_0$ or just $\boxed{S' = \emptyset}$ $\bar{S} = S \cup S' = \{a\}$, $\boxed{\bar{S} = \{a\}}$ ⊠ As we obtain,

$\boxed{\overline{(S^\circ)} = \emptyset \text{ and } \bar{S} = \{a\} \text{ are different}}$ ⊠ ⊠ $\Rightarrow$ ⊠ statement is not true for every $S$ in $R^n$

38) We will find a set $S$ in $\mathbb{R}^n$ for which $Bd(\bar{S})$ is different from $Bd(S) \Rightarrow$ Take $S = Q$ with the usual topology induced from $\mathbb{R}$. Let $I = Q^c$ - irrational numbers

Since from the Book, we know that $\overline{(\bar{S})} = \bar{S}$ and $\bar{S} \cap \overline{(S^c)} = Bd(S) \Rightarrow Bd(S) = Bd(Q) = \bar{Q} \cap \overline{(Q^c)} =$

$= \bar{Q} \cap \bar{I} = R \cap R = R$, as the closure of the set of rationals is all of $R$ because every real number is a limit of a sequence of rationals (i.e. $3, 3.1, 3.14, \ldots$ converges to $\pi$) Similarly, the closure of the set of irrationals is also $R \Rightarrow \boxed{\bar{Q} = \bar{I} = R}$ and $\boxed{Bd(S) = R}$ ⊠

On the other hand, $\bar{S} = \bar{Q} = R \Rightarrow Bd(\bar{S}) = Bd(R) =$

$= \bar{R} \cap \overline{(R^c)} = R \cap \bar{\emptyset} = R \cap \emptyset = \emptyset$, where we proved previously that

$\bar{\emptyset}=\emptyset$ and $\bar{R}=R\cup R'=R\cup R=R$ since all limit points of $R$ are situated on $R$ itself $\Rightarrow$ $\boxed{bd(\bar{S})=\emptyset}$ ☒ showing that $\boxed{bd(S)=R \text{ and } bd(\bar{S})=\emptyset \text{ are different}}$ ☒ Not true for every set $S$ in $\mathbb{R}^n$ ✓

42) $S$ - any bounded set in $\mathbb{R}^n \Rightarrow d(\bar{S})=d(S)$

<u>Claim:</u> If $\{a_n\}$ converges to $a$, and $\{b_n\}$ converges to $b$, then the sequence of real numbers $c_n=\|a_n-b_n\|$ converges to $\|a-b\|$

<u>Prf:</u> From triangle ineq, $\|a_n-b_n\|=\|a_n-a+a-b+b-b_n\| \leq \|a_n-a\|+\|a-b\|+\|b-b_n\| \Rightarrow \boxed{\|a_n-b_n\|-\|a-b\| \leq \leq \|a_n-a\|+\|b_n-b\|}$

Denote $d(x,y)=\|x-y\|$, then

$d(a_n,b_n)-d(a,b) \leq d(a_n,a)+d(b_n,b)$

$d(a,b)=\|a-a_n+a_n-b_n+b_n-b\| \leq d(a_n,a)+d(a_n,b_n)+d(b_n,b)$

$\Rightarrow -\left(d(a_n,a)+d(b_n,b)\right) \leq d(a_n,b_n)-d(a,b)$, therefore

$\boxed{|d(a_n,b_n)-d(a,b)| \leq d(a_n,a)+d(b_n,b)}$ ☒ From the conver- -gence, we know

$\forall \varepsilon>0, \exists N$ - large $\Rightarrow d(a_n,a)$ and $d(b_n,b)<\frac{\varepsilon}{2}$, so we get

$|d(a_n,b_n)-d(a,b)| \leq d(a_n,a)+d(b_n,b)<\varepsilon$ which means

$\boxed{d(a_n,b_n)\rightarrow d(a,b)}$ ☒ ✓

Now, let's return to set diameters. As $\bar{A} = A \cup A' \Rightarrow A \subset \bar{A}$ and $\sup\{d(a,b) \mid a,b \in A\} \leq \sup\{d(a,b) \mid a,b \in \bar{A}\}$ meaning that diameter $(A) \leq$ diameter $(\bar{A})$ Now suppose $d(\bar{A}) > d(A)$. Then, we can find some $a', b' \in \bar{A}$ such that $d(a', b') > d(A)$. Certainly, this means either $a' \notin A$ or $b' \notin A$ (otherwise, $d(a', b') \leq d(A)$ would be true whenever $a', b' \in A$), so either $a'$ or $b'$ is a limit point of $A$ (since $a', b' \in \bar{A} = A \cup A'$) Either way, we can construct sequences $\{a_n\} \subset A$, $\{b_n\} \subset A$ (not necessarily distinct elements), such that $a_n \to a'$ and $b_n \to b'$. Now, choose $\varepsilon = \frac{1}{2}\left(d(a', b') - d(A)\right)$ where $\varepsilon > 0 \Rightarrow \forall n \in \mathbb{N}$, $a_n$ and $b_n \in A$, we have $d(a_n, b_n) \leq d(A) < d(a', b') \Rightarrow |d(a_n, b_n) - d(a', b')| = d(a', b') - d(a_n, b_n) \geq d(a', b') - d(A) = 2\varepsilon > \varepsilon$ which contradicts what we showed in the claim that $d(a_n, b_n) \to d(a', b')$ should satisfy $\boxed{*}$ Hence $d(\bar{A}) > d(A)$ is not true $\Rightarrow d(\bar{A}) \leq d(A) \leq d(\bar{A})$

$\boxed{d(\bar{A}) = d(A)}$ ■