Group1 HW6

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Contribution Details

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$\boxed{\#3}$ Let $C = \{ S_\alpha : \alpha \in A \}$ and let for every finite subcollection $S_{\alpha_1}, ---, S_{\alpha_n}$ of C, we have $S_{\alpha_1} \cap S_{\alpha_2} \cap \cdots \cap S_{\alpha_n} \neq \phi$.

<u>claim</u>: $S \subset \mathbb{R}^n$ is compact if and only if every collection $C = \{ S_\alpha : S_\alpha \text{ closed}, S_\alpha \subseteq S \}$ of closed subsets of S with finite intersection property has a nonempty intersection.

<u>Proof</u>

$(\longrightarrow)$ suppose $S \subset \mathbb{R}^n$ is compact. Then, we will show that if $C = \{ S_\alpha : S_\alpha \text{ closed}, S_\alpha \subseteq S \}$ is a collection of closed subsets of S with finite intersection property, then $\bigcap_\alpha S_\alpha \neq \phi$.

We will prove the contrapositive (so an equivalent statement) of this claim. That is, if $\bigcap_\alpha S_\alpha = \phi$, then the collection $C = \{ S_\alpha : S_\alpha \text{ closed}, S_\alpha \subseteq S \}$ has some finite elements such that $S_{\alpha_1} \cap \cdots \cap S_{\alpha_n} = \phi$.

So, let $\bigcap_\alpha S_\alpha = \phi$. Then, $\left(\bigcap_\alpha S_\alpha \right)^c = S$.

$$\Rightarrow \quad S = \left(\bigcap_\alpha S_\alpha \right)^c = \bigcup_\alpha S_\alpha^c \quad \text{by De Morgan's laws.}$$

Since S_α is closed, S_α^c is open and an arbitrary union of open sets $\bigcup_\alpha S_\alpha^c$ is also open.

Then, the collection of open sets $\{S_\alpha^c\}$ forms an open cover of S. Since S is compact, there is a finite subcover of S. Let $S_{\alpha_1}, ---, S_{\alpha_n}$ is this finite subcover. That is, $S = S_{\alpha_1} \cup S_{\alpha_2} \cup \dots \cup S_{\alpha_n}$.

$\Rightarrow S = \bigcup_{i=1}^{n} S_{\alpha_i}$. $\Rightarrow S^c = \phi = \left(\bigcup_{i=1}^{n} S_{\alpha_i}\right)^c$.

$\Rightarrow \phi = \bigcap_{i=1}^{n} S_{\alpha_i}^c$ by de Morgan's laws.

$\Rightarrow S_{\alpha_1} \cap S_{\alpha_2} \cap --- \cap S_{\alpha_n} = \phi$. This proves the contrapositive of our statement. Thus, for every collection $C = \{S_\alpha ; S_\alpha \text{ closed}, S_\alpha \subseteq S\}$ with the finite intersection property has a nonempty intersection.

($\Leftarrow$) suppose conversely that every collection $C = \{S_\alpha : S_\alpha \text{ closed}, S_\alpha \subseteq S\}$ with the finite intersection property has a nonempty intersection. That is, by taking its contrapositive, if $\bigcap_\alpha S_\alpha = \phi$, then the collection $C = \{S_\alpha\}$ has some finite elements such that $S_{\alpha_1} \cap S_{\alpha_2} \cap \dots \cap S_{\alpha_n} = \phi$. We will then prove that S is compact. Let the collection open sets $\{V_\alpha\}$ be an open cover of S. $\Rightarrow S = \bigcup_\alpha V_\alpha$.

Then, $S^c = \phi = \left(\bigcup_\alpha V_\alpha\right)^c = \bigcap_\alpha V_\alpha^c$.

Since each V_α is open, then V_α^c is closed. By our assumption, if the intersection of the closed subsets S_α is empty (i.e., $\bigcap_\alpha S_\alpha = \phi$), then the collection $C = \{S_\alpha ; S_\alpha \text{ closed}, S_\alpha \subseteq S\}$ contain some finite elements $S_{\alpha_1}, ---, S_{\alpha_n}$ such that $S_{\alpha_1} \cap S_{\alpha_2} \cap \dots \cap S_{\alpha_n} = \phi$.

So, there exist some finite elements of $\{V_\alpha^c\}$ $V_{\alpha_1}^c, ---, V_{\alpha_n}^c$ such that

$$V_{\alpha_1}^c \cap V_{\alpha_2}^c \cap \dots \cap V_{\alpha_n}^c = \phi.$$

$$\Rightarrow \bigcap_{i=1}^{n} V_{\alpha_i}^c = \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right)^c = \phi$$

$$\Rightarrow \bigcup_{i=1}^{n} V_{\alpha_i} = \phi^c = S.$$

$$\Rightarrow S = V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}.$$

So S has a finite subcover for arbitrary open cover $\{V_\alpha\}$. Thus, S is compact.

HW 6 Suggested Exercises

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Exercise 2: Cantor's Criterion

Topological Statement Theorem: Let S be a topological space. A decreasing nested sequence of non-empty compact, closed subsets of S has a non-empty intersection. In other words, supposing $(C_k)_{k\geq 0}$ is a sequence of non-empty compact, closed subsets of S satisfying

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots,$$

it follows that

$$\bigcap_{k=0}^{\infty} C_k \neq \emptyset$$

The closedness condition may be omitted in situations where every compact subset of S is closed, for example when S is Hausdorff.

Proof. Assume, by way of contradiction, that $\bigcap_{k=0}^{\infty} C_k = \emptyset$. For each k, let $U_k = C_0 \backslash C_k$. Since $\bigcup_{k=0}^{\infty} U_k = C_0 \backslash \bigcap_{k=0}^{\infty} C_k$ and $\bigcap_{k=0}^{\infty} C_k = \emptyset$, we have $\bigcup_{k=0}^{\infty} U_k = C_0$. Since the C_k are closed relative to S and therefore, also closed relative to C_0, the U_k, their set complements in C_0, are open relative to C_0. Since $C_0 \subset S$ is compact and $\{U_k \mid k \geq 0\}$ is an open cover (on C_0) of C_0, a finite cover $\{U_{k_1}, U_{k_2}, \ldots, U_{k_m}\}$ can be extracted. Let $M = \max_{1\leq i\leq m} k_i$. Then $\bigcup_{i=1}^{m} U_{k_i} = U_M$ because $U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1} \cdots$, by the nesting hypothesis for the collection $(C_k)_{k\geq 0}$. Consequently, $C_0 = \bigcup_{i=1}^{m} U_{k_i} = U_M$. But then $C_M = C_0 \backslash U_M = \emptyset$, a contradiction.

Statement for Real Numbers The theorem in real analysis draws the same conclusion for closed and bounded subsets of the set of real numbers R. It states that a decreasing nested sequence $(C_k)_{k\geq 0}$ of non-empty, closed and bounded subsets of R has a non-empty intersection.

This version follows from the general topological statement in light of the Heine-Borel theorem, which states that sets of real numbers are compact if and only if they are closed and bounded. However, it is typically used as a lemma in proving said theorem, and therefore warrants a separate proof. As an example, if $C_k = [0, 1/k]$, the intersection over $(C_k)_{k\geq 0}$ is $\{0\}$. On the other hand, both the sequence of open bounded sets $C_k = (0, 1/k)$ and the sequence of unbounded closed sets $C_k = [k, \infty)$ have empty intersection. All these sequences are properly nested.

This version of the theorem generalizes to $\mathbf{R}^n$, the set of n -element vectors of real numbers, but does not generalize to arbitrary metric spaces. For example, in the space of rational numbers, the sets

$$C_k = [\sqrt{2}, \sqrt{2} + 1/k] = (\sqrt{2}, \sqrt{2} + 1/k)$$

are closed and bounded, but their intersection is empty. Note that this contradicts neither the topological statement, as the sets C_k are not compact, nor the variant below, as the rational numbers are not complete with respect to the usual metric. A simple corollary of the theorem is that the Cantor set is nonempty, since it is defined as the intersection of a decreasing nested sequence of sets, each of which is defined as the union of a finite number of closed intervals; hence each of these sets is non-empty, closed, and bounded. In fact, the Cantor set contains uncountably many points.

Theorem. Let $(C_k)_{k \geq 0}$ be a sequence of non-empty, closed, and bounded subsets of R satisfying

$$C_0 \supset C_1 \supset \cdots C_n \supset C_{n+1} \cdots$$

Then,

$$\bigcap_{k=0}^{\infty} C_k \neq \emptyset$$

Variant in complete metric spaces In a complete metric space, the following variant of Cantor's intersection theorem holds.

Theorem. Suppose that X is a complete metric space, and $(C_k)_{k \geq 1}$ is a sequence of non-empty closed nested subsets of X whose diameters tend to zero:

$$\lim_{k \to \infty} diam\,(C_k) = 0$$

where $diam\,(C_k)$ is defined by

$$diam\,(C_k) = \sup\{d(x, y) \mid x, y \in C_k\}$$

Then the intersection of the C_k contains exactly one point:

$$\bigcap_{k=1}^{\infty} C_k = \{x\}$$

for some $x \in X$.

A converse to this theorem is also true: if X is a metric space with the property that the intersection of any nested family of non-empty closed subsets whose diameters tend to zero is non-empty, then X is a complete metric space. (To prove this, let $(x_k)_{k \geq 1}$ be a Cauchy sequence in X, and let C_k be the closure of the tail $(x_j)_{j \geq k}$ of this sequence.)

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1) For $S = (0,1) \subseteq \mathbb{R}$, and metric $d_0 : \mathbb{R} \to \mathbb{R}$; for $x,y \in \mathbb{R}$, $d_0(x,y) = \begin{cases} 0 ; & x=y \\ 1 ; & x \neq y \end{cases}$

Metric properties : 1) $d_0(x,y) \geq 0$ and $d_0(x,y) = 0 \longleftrightarrow x=y$ <u>true</u>

2) $d_0(x,y) = d_0(y,x)$ <u>true</u>

3) $d_0(x,z) \leq d(x,y) + d(y,z)$ <u>true</u>

$\begin{cases} \text{If } x=z \to d_0(x,z) = 0 \leq d(x,y)+d(y,z) & \text{true} \\ \text{If } x \neq z,y \to d_0(x,z) = 1 \leq d(x,y)+d(y,z) = 1 + d(y,z) & \text{true} \\ \text{If } x \neq z, x=y \therefore y \neq z \to d_0(x,z) = 1 \leq d(x,y)+d(y,z) = d(x,y)+1 & \text{true} \end{cases}$

With Euclidean norm, S is bounded

For $x \in \mathbb{R}, \varepsilon > 0$, Consider $N_0(x;\varepsilon) = \{y \in \mathbb{R} \mid d_0(x,y) < \varepsilon\} = \begin{cases} \{x\} ; & \varepsilon \leq 1 \\ \mathbb{R} ; & \varepsilon > 1 \end{cases}$

Consider any set A, for $x \notin A$, $N(x;\frac{1}{2}) \cap A = \emptyset$ $\therefore x$ is not boundary

for $x \in A$, $N(x;\frac{1}{2}) \cap A^c = \emptyset$, and $N(x;\frac{1}{2}) = \{x\} \subseteq A$ $\therefore x$ is interior and not boundary

$\therefore A$ has no boundary, and $\forall x \in A$: x is interior $\therefore A$ is open

$\therefore S$ is closed

Consider $A_k = (\frac{1}{k+1}, 1)$; $k \in \mathbb{N}$ $\therefore$ $S \subseteq \bigcup_{k=1}^{\infty} A_k = (0,1)$

But $\forall k \in \mathbb{N}, \exists x \in S ; x \notin A_k$ $\therefore$ none of the finite subcollection of $\{A_k\}$ is a cover

$\therefore S$ is not compact

2.75 Suppose that C is compact and that $\{S_\alpha\}$ with finite intersection property has empty intersection. Then union of their complements (which are all open) is exactly $C \Rightarrow$ open cover $\Rightarrow$ since C is compact $\exists$ finite subcover. Then take intersection of their complements, it should be $\neq \emptyset$ since $\{S_\alpha\}$ has finite intersection property.

Now, let $\{S'_\alpha\}$ be open cover of C. $\Rightarrow \bigcap_\alpha (S'_\alpha)^c = \emptyset$

$\Rightarrow \{(S'_\alpha)^c\}$ can't have finite intersection property.

$\Rightarrow$ we can find finite collection $\{\alpha_1, \ldots, \alpha_N\}$ s.t

$\bigcap_{i=1}^N (S'_{\alpha_i})^c = \emptyset \Rightarrow$ ~~$\bigcup_{i=1}^N S'_{\alpha_i}$~~ $\bigcup_{i=1}^N (S'_{\alpha_i}) = C$

Thus, $\exists$ finite subcover of $C \Rightarrow C$ is compact