MAS242 ANALYSIS I QUIZ 3

Problem 1. *(15 points) For* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *in* $\mathbb{R}^2, define$

$$d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$$

(1) Show that d_1 is a metric on $\mathbb{R}^2$.(5pt)

(2) Define a $1-$neighborhood $N_1(\mathbf{x}; s)$ of $\mathbf{x} = (x_1, x_2)$ to be $N_1(\mathbf{x}; s) = \{\mathbf{y} \in \mathbb{R}^2 : d_1(\mathbf{x}, \mathbf{y}) < s\}$. Let $N(\mathbf{x}; r)$ be any (Euclidean) neighborhood of $\mathbf{x}$. Show that there exist positive r_1 and r_2 such that

$$N_1(\mathbf{x}; r_1) \subset N(\mathbf{x}; r) \subset N_1(\mathbf{x}; r_2)$$

(10pt)

Proof. (1) Let $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$.
Positive definiteness : $|x_1 - y_1| \geq 0, |x_2 - y_2| \geq 0 \implies d_1(\mathbf{x}, \mathbf{y}) \geq 0$
$d_1(\mathbf{x}, \mathbf{y}) = 0 \iff |x_1 - y_1| = |x_2 - y_2| = 0 \iff \mathbf{x} = \mathbf{y}$
Symmetry : $d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d_1(\mathbf{y}, \mathbf{x})$
The triangle inequality : $d_1(\mathbf{x}, \mathbf{z}) = |x_1 - z_1| + |x_2 - z_2| = |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$
$\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| = d_1(\mathbf{x}, \mathbf{y}) + d_1(\mathbf{y}, \mathbf{z})$

(2) Let $r_1 = r, r_2 = 2r$.
For $\mathbf{y} \in N_1(\mathbf{x}; r)$, $|y_1 - x_1| + |y_2 - x_2| \leq r$.
$(y_1 - x_1)^2 + (y_2 - x_2)^2 \leq (y_1 - x_1)^2 + (y_2 - x_2)^2 + 2|y_1 - x_1||y_2 - x_2| = \|\mathbf{x} - \mathbf{y}\|^2 \leq r^2$
$\therefore N_1(\mathbf{x}; r) \subset N(\mathbf{x}; r)$

For $\mathbf{y} \in N(\mathbf{x}; r), \|\mathbf{x} - \mathbf{y}\|^2 \leq r^2$.
$|y_1 - x_1| < r$ and $|y_2 - x_2| < r$. So, $|y_1 - x_1| + |y_2 - x_2| \leq 2r$
$\therefore N(\mathbf{x}; r) \subset N_1(\mathbf{x}; r_2)$.

□

Problem 2. *(15 points) Prove that, if* $\{\mathbf{x}_k\}$ *is a bounded sequence in* $\mathbb{R}^n$ *and* $\mathbf{y}_1, \ldots, \mathbf{y}_M$ *are cluster points of* $\{\mathbf{x}_k\}$*, then* $S = \{\mathbf{x}_k : k \in \mathbb{N}\} \cup \{\mathbf{y}_1, \ldots, \mathbf{y}_M\}$ *is closed*

Proof. Let $\mathbf{x}_0$ be a limit point of S.
For each $\epsilon_k \downarrow 0, \exists n_k \in \mathbb{N}$ such that $0 < \|\mathbf{x}_{n_k} - \mathbf{x}_0\| < \epsilon_k$.
For any $\epsilon > 0, \epsilon_m < \epsilon$ for some m.
Then $\{\mathbf{x}_{n_k}\}$ is a subsequence of $\{\mathbf{x}_k\}$ that converges to $\{\mathbf{x}_0\}$.
By theorem 2.1.10 in textbook, $\mathbf{x}_0$ is a cluster point of $\{\mathbf{x}_k\}$.
$\therefore$ A limit point of S is a cluster point of $\{\mathbf{x}_k\}$.
By theorem 2.2.4 in textbook, S is closed. □

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