MAS242 ANALYSIS I QUIZ 1

Problem 1. *(15 points) Prove that*
a sequence $\{x_n\}$ converges in $\mathbb{R}$ if and only if each of its proper subsequences is Cauchy in $\mathbb{R}$.

Proof.

$$
\begin{aligned}
& \text{A sequence } \{x_n\} \text{ converges in } \mathbb{R}. \\
\Longleftrightarrow\; & \text{A sequence } \{x_n\} \text{ is Cauchy in } \mathbb{R}.(\boldsymbol{Theorem 1.4.4}) \\
\Longleftrightarrow\; & \text{For each } \epsilon > 0, \quad \exists N \text{ such that } \forall n, m > N, \quad |x_n - x_m| < \epsilon \\
\Longrightarrow\; & \text{For any subsequence} \{x_{n_k}\}, \quad \text{For each } \exists K \text{ such that } \forall k, l > K, \quad n_k, n_l > N \text{ and so } |x_{n_k} - x_{n_l}| < \epsilon \\
\Longleftrightarrow\; & \text{Each of its subsequences is Cauchy in } \mathbb{R}
\end{aligned}
$$

$$\therefore (\Longrightarrow)$$

$$
\begin{aligned}
& \text{A sequence } \{x_n\} \text{ does not converge in } \mathbb{R}. \\
\Longleftrightarrow\; & \text{A sequence } \{x_n\} \text{ is not Cauchy in } \mathbb{R}.(\boldsymbol{Theorem 1.4.4}) \\
\Longleftrightarrow\; & \text{There exists } \epsilon > 0, \quad \text{such that } \forall N, \quad \exists N_1, N_2 > N \text{ satisfying } |x_{N_1} - x_{N_2}| > \epsilon \\
\Longrightarrow\; & \text{There exists a proper subsequence } \{x_{n_k}\}, \quad \text{such that } \forall m > 0, \quad |x_{n_{2m}} - x_{n_{2m-1}}| > \epsilon \\
\Longleftrightarrow\; & \text{There exists a proper subsequence which is not Cauchy in } \mathbb{R}
\end{aligned}
$$

$$\therefore (\Longleftarrow)$$

□

Problem 2. *(15 points) Fix any $c > 0$. Let x_1 be any positive number and define $x_{k+1} = \sqrt{(x_k^2 + c/x_k^2)/2}$.*

(1) *Prove that $\{x_k\}$ converges.*

(2) *Use this sequence to calculate $\sqrt{2}$, accurate to two decimal places.*

Proof. (1) First, x_k is positive for all $k > 0$.

$$x_{k+1}^2 - \sqrt{c} = (x_k^2 - 2\sqrt{c} + c/x_k^2)/2 = (x_k^2 - \sqrt{c})^2/2x_k^2 \geq 0$$

Thus $x_k \geq c$ for all $k > 1$.

$$
\begin{aligned}
& x_{k+1}^2 - \sqrt{c} = (x_k^2 - \sqrt{c})^2/2x_k^2 < (x_k^2 - \sqrt{c})/2, \quad \forall k > 1 \\
\Longrightarrow\; & x_{k+1}^2 - \sqrt{c} < (x_2^2 - \sqrt{c})/2^{k-1} \to 0 \text{ as } k \to \infty \\
\Longrightarrow\; & x_k \to \sqrt[4]{c} \text{ as } k \to \infty
\end{aligned}
$$

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(2) Put $c = 2$ and let $y_k = x_k^2$. Then $y_{k+1} - \sqrt{2} < (y_k - \sqrt{2})/2$ for $k > 1$.

$$\begin{aligned} &\text{Let } y_1 = 1 \\ \implies &y_2 = \frac{3}{2} = 1.5 \\ &y_3 = \frac{17}{12} = 1.416.. \\ &y_4 = \frac{577}{408} = 1.414.. \end{aligned}$$

Therefore $\sqrt{2} = 1.41..$

$\square$