ANALYSIS I. MIDTERM EXAM. 13:20–15:20 APRIL 20, 2021

NAME: ID#: SCORE: / 100

Guidelines for the exam:

(1) Make answers short and points clear. Otherwise, it will be considered incorrect.
(2) There are 10 problems for 10 points each. Each sub-problem has the same weight.
(3) You are allowed to use books and notes. Any direct help from people is not allowed.
(4) Zoom should be on all the time.
(5) Exam ends at 15:20. Scan your exam and upload it by 15:40 (if you have trouble with KLMS, submit your exam in e-mail, hykim0615@kaist.ac.kr).

Part A: Prove the problem using definitions but not theorems. You may use the completeness axiom for $\mathbb{R}$ in the book.

(1) The summation and subtraction of two nonempty sets $A, B \subset \mathbb{R}$ are defined as $A \pm B = \{a \pm b : a \in A, b \in B\}$. Let A and B be bounded.
  (a) Show that $\sup(A + B) = \sup A + \sup B$.
  (b) Show that $\inf B \le \inf A$ and $\sup A \le \sup B$ if $A \subset B$.
  (c) Let $C = \emptyset$, the empty set. What should be $\sup C$ and $\inf C$ to keep the above relation (b)? Explain your answer with one or two sentences. (This problem is related to the definition of the limsup and liminf).

(a) **(+4 pts)** Since $x = a + b \le \sup A + \sup B$, $\sup A + \sup B$ is an upper bound. So from the definition, we have $\sup(A + B) \le \sup A + \sup B$. And for all $x = a + b$, $x \le \sup(A+B)$ and we get $a \le \sup(A+B)-b$ for some fixed $b \in B$. Then, $\sup(A+B)-b$ is an upper bound for $a \in A$. From definition, we have $\sup(A) \le \sup(A + B) - b$ and equivalently, $b \le \sup(A + B) - \sup(A)$. Again, $\sup(A + B) - \sup(A)$ is an upper bound for $b \in B$, from the definition again, we get $\sup(A) + \sup(B) \le \sup(A + B)$.

(b) **(+4 pts)** Let $a \in A$. Then, $a \in B$ and hence $a \le \sup B$, i.e., $\sup B$ is an upper bound of A. Since $\sup A$ is the smallest upper bound, $\sup A \le \sup B$.

(c) **(+2 pts)** Since $C \subset B$ for any set, $\sup C \le \sup B$ for any $B \subset \mathbb{R}$ if the relation holds. The only way to keep it is to define $\sup C = -\infty$. Similarly, $\inf C$ should be $+\infty$.

(2) Prove or disprove. (Depending on the type of the statement, proving may mean finding an example and disproving may not.)
  (a) If $A \subset \mathbb{R}$ consists of infinitely many real numbers, there exists at least one limit point of A.
  (b) If $\{x_k\}$ is a bounded and monotone increasing sequence, the sequence converges.
  (c) Let $S = \{x \in \mathbb{R} : x = x_k \text{ for some } k \in \mathbb{N}\}$ for a given sequence $x_i \in \mathbb{R}$. Then, $y \in \mathbb{R}$ is a limit point of S if and only if y is a cluster point of the sequence x_i.

(a) **(+3 pts)** Counter example: $A = N$. For all point $x \in \mathbb{R}$, there is a deleted neighborhood $N'(x, \epsilon)$ which is an emptyset. $\epsilon = \frac{1}{2}$ for all $x \in N$ and $\epsilon = \frac{1}{2}\min(x - n, n + 1 - x)$ for all $n < x < n + 1$.

(b) **(+4 pts)** Prove: From the completeness axiom, we always have $\mu = \sup\{x_k\}$. If there is $k \in N$ such that $x_k = \mu$, then all $x_{k+a} = \mu$ also since x_k is monotone increasing and μ is a supremum. So from the definition, x_k converges to μ. Now, consider the case $x_k < \mu$. Assume that it doesn't converge to μ. Then, for some $\epsilon > 0$, we may find an infinite subsequence x_{k_i} with $k_i < k_{i+1}$ such that $x_{k_i} \le \mu - \epsilon$. For every natural number $n \in N$, we have $n < k_i$ and from monotonicity, $x_n \le \mu - \epsilon$ also. So, $\mu - \epsilon$ is an upper bound of sequence $\{x_k\}$ and it violates the definition of supremum μ.

(c) **(+3 pts)** Counter example: $x_k = C$, a constant. Then, $S = \{C\}$ is one point set and it is not a limit point. But it is a cluster point of the sequence x_i.

Part B: You may use any theorem or lemma in the book for the following problems if needed.

(3) The open set of the Euclidean space $\mathbb{R}^n$ is always with the L^2-norm. However, we may provide other norms. In the case we do not call it the Euclidean space anymore. Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and define $\|\mathbf{x}\|_1 = |x| + |y|$, $\|\mathbf{x}\|_2 = \sqrt{x^2 + y^2}$, and $\|\mathbf{x}\|_\infty = \max(|x|, |y|)$.

(a) Show that $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_\infty$ are norms.

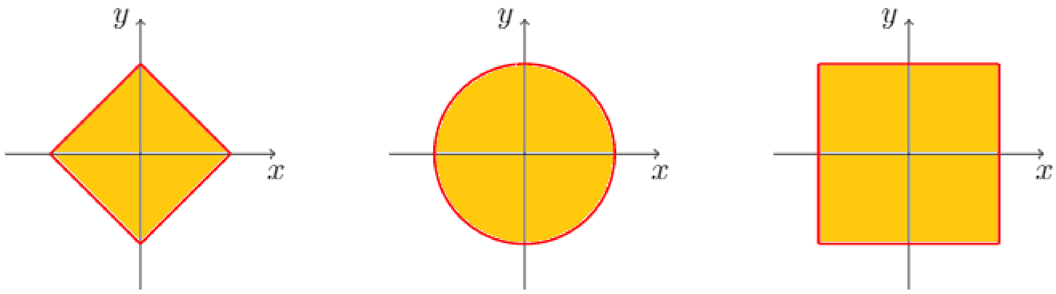
(b) Sketch the unit balls with respect to these three norms, i.e., sketch $B_i = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_i < 1\}$ for $i = 1, 2$ and ∞.

(a)

- (***Positive Definiteness***)
  $\|\mathbf{x}\|_1 = |x| + |y| \geq 0$ and $\|\mathbf{x}\|_1 = 0 \iff |x| + |y| = 0 \iff x = y = 0 \iff \mathbf{x} = 0$, $\|\mathbf{x}\|_\infty = \max(|x|, |y|) \geq 0$ and $\|\mathbf{x}\|_\infty = 0 \iff \max(|x|, |y|) = 0 \iff x = y = 0 \iff \mathbf{x} = 0$ **(+2 pts)**
- (***Absolute Homogeneity***)
  $\|c\mathbf{x}\|_1 = |cx| + |cy| = c(|x| + |y|) = c\|\mathbf{x}\|_1$, $\|c\mathbf{x}\|_\infty = \max(|cx|, |cy|) = c\max(|x|, |y|) = c\|\mathbf{x}\|_\infty$ **(+1 pts)**
- (***Subadditivity***)
  $\|\mathbf{x}_1 + \mathbf{x}_2\|_1 = |x_1 + x_2| + |y_1 + y_2| \leq |x_1| + |y_1| + |x_2| + |y_2| = \|\mathbf{x}_1\|_1 + \|\mathbf{x}_2\|_1$, $\|\mathbf{x}_1 + \mathbf{x}_2\|_\infty = \max(|x_1 + x_2|, |y_1 + y_2|) \leq \max(|x_1|, |y_1|) + \max(|x_2| + |y_2|) = \|\mathbf{x}_1\|_\infty + \|\mathbf{x}_2\|_\infty$ **(+2 pts)**

Therefore both are norms.

(b) **(+2 pts)** **(+1 pts)** **(+2 pts)**



$\|\mathbf{x}\|_1 < 1.$ $\|\mathbf{x}\|_2 < 1.$ $\|\mathbf{x}\|_\infty < 1.$

(4) Let $C_1, C_2 \subset \mathbb{R}$ be compact and $S \subset \mathbb{R}$ be open.
 (a) Suppose that $S \neq \emptyset$ and $S \neq \mathbb{R}$. Show that S is not closed. (This means there is no other clopen set in $\mathbb{R}$ except $\mathbb{R}$ and $\emptyset$.)
 (b) Prove that $C_1 \cup C_2$ is compact. (Refer theorems you use clearly.)
 (c) If $C_1 \cap C_2 = \emptyset$, there exists two open sets U_1, U_2 such that $C_1 \subset U_1$, $C_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$.

(a) Since $S \neq \emptyset$, there exists $a \in S$. Since $S \neq \mathbb{R}$, there exists $b \in \mathbb{R} \setminus S$. For convenience, let $a < b$ and consider an interval $[a, b]$. Since S is open and $a \in S$, it is an interior point. Hence, exists $\epsilon > 0$ such that $(a, a+\epsilon) \subset S$. Let I be the maximal such interval and $c = \sup I$. Then, c is a boundary point of S. If $c \in S$, c is an interior point of S and hence (a, c) is not maximal. Hence, $c \notin S$. Hence, S is not closed. **(+4 pts)**

*You must show $bd(S) \neq \emptyset$. If you don't prove it, **(-2 pts)**.

(b) Compact set C_1 and C_2 are bounded and closed. $C_1 \cup C_2$ is clearly bounded and closed. Hence, it is compact. **(+3 pts)**

(c) Since C_1 and C_2 are bounded and closed, the distance between the two sets is positive. Let $\epsilon = \frac{1}{2} dist(C_1, C_2)$. Consider an open covering $\{N(x, \epsilon) : x \in C_1\}$ of C_1. Since C_1 is compact, there exists a finite subcover $\{N(x_i, \epsilon) : i = 1, cdots, N_1\}$. Let $U_1 = \cup_{i=1}^{N_1} N(x_i, \epsilon) \supset C_1$. Similarly, construct $U_2 \supset C_2$. Then, U_1 and U_2 are open. You can check $U_1 \cap U_2 \neq \emptyset$ easily. **(+3 pts)**

(5) Every bounded subset $S \subset \mathbb{R}$ has a supremum in $\mathbb{R}$ if and only if $\mathbb{R}$ is Cauchy complete. (In other words, the Cauchy completeness is equivalent to Axiom 1.1.1.)
 (a) Prove the only if part for ($\Rightarrow$).
 (b) Prove the if part for ($\Leftarrow$).

(a) Let x_i be a Cauchy sequence and $A_k = \{x \in \mathbb{R} : x = x_i \text{ for some } i > k\}$. Since a Cauchy sequence is bounded, A_k are bounded. Let $\mu_k = \sup A_k$. Then, μ_k is a decreasing sequence and bounded below. Therefore, there exists $\mu_\infty \in \mathbb{R}$ the limit of μ_k. **(+3 pts)**
Now we show μ_∞ is the limit of x_i. Let $\epsilon > 0$. Then, there exists k_0 such that $|x_i - x_j| < \epsilon$ whenever $i, j > k_0$ and $|\mu_\infty - \mu_{k_0}| < \epsilon$. Therefore, there exists $j > k_0$ such that $|\mu_\infty - x_j| < 2\epsilon$. Hence, for any $i > k_0$, we have

$$|x_i - \mu_\infty| \leq |x_i - x_j| + |x_j - \mu_\infty| \leq 3\epsilon.$$

(+2 pts)

(b) Since S is bounded, there is a upper bound M. Choose $a_1 \in S$ and let $b_1 = M$. Let's define a_i, b_i inductively as

$$\begin{cases} a_{i+1} = a_i, \quad b_{i+1} = \frac{a_i+b_i}{2} & \text{if } [\frac{a_i+b_i}{2}, b_i] \cap S = \emptyset \\ a_{i+1} = \frac{a_i+b_i}{2}, \quad b_{i+1} = b_i & \text{if } [\frac{a_i+b_i}{2}, b_i] \cap S \neq \emptyset \end{cases}$$

Then, $\{a_k\}$ is an increasing Cauchy sequence and $\{b_k\}$ is a decreasing Cauchy sequence.
$\because |a_k - a_l|, |b_k - b_l| \leq (b_1 - a_1)/2^{\min(k,l)-1}$
(+2 pts)
So $\{a_k\}, \{b_k\}$ converges in $\mathbb{R}$. Let a_∞, b_∞ be the limits of $\{a_k\}, \{b_k\}$, respectively.
Indeed, $a_\infty = b_\infty$. $\because |a_\infty - b_\infty| \leq |a_k - b_k| \to 0$
By definition of b_i, $(b_\infty, \infty) \cap S = \emptyset$.($\iff$ b_∞ is a upper bound of S.)
If $b_\infty \in S$, then $b_\infty = \sup(S)$. Suppose $b_\infty \notin S$. Then, any $\epsilon > 0$, $\exists a_k \in S$ such that $b_\infty - a_k < \epsilon$. This implies $b_\infty = \sup(S)$. **(+3 pts)**

Therefore any bounded subset S has a supremum.

*If you wrote only as a list of theorems, you have got at most 5 points. .

(6) Prove or disprove.

(a) The product $(0,1)\times(0,1)\subset\mathbb{R}^2$ is an open set.

(b) If $\{C_k\}$ is nested closed nonempty subsets of $\mathbb{R}$, then $\cap_{k=1}^{\infty}C_k\neq\emptyset$.

(c) For any set $S\subset\mathbb{R}^n$, its closure is same as the closure of its interior S^0, i.e., $\overline{S^0}=\overline{S}$.

(a) Let $\mathbf{x}=(x,y)\in(0,1)\times(0,1)$. Let $r=\min(|x|,|y|,|1-x|,|1-y|)$. Then, $N(\mathbf{x},r)\subset(0,1)\times(0,1)$ and hence $\mathbf{x}$ is an interior point. Hence, $(0,1)\times(0,1)$ is open. **(+4 pts)**

(b) Counter example: Let $C_k=[k,\infty)$. Then, $\{C_k\}$ are nested closed nonempty subsets of $\mathbb{R}$ (unbounded). However, $\cap_{k=1}^{\infty}C_k=\emptyset$. **(+3 pts)**

(c) Counter example: For any set $S\subset\mathbb{R}^n$ with isolated points, $\overline{S^0}\neq\overline{S}$. **(+3 pts)**

(7) Prove or disprove.

(a) Let $S\subset\mathbb{R}^n$ be a nonempty domain, $C_\infty(S)$ be the continuous function space with the uniform norm, $F\subset C_\infty(S)$ is a dense subset, and $f_0\in C_\infty(S)$. Show that there exists a Cauchy sequence $\{f_k\}\subset F$ that converges to f_0 uniformly.

(b) For a continuous function $f:[a,b]\to\mathbb{R}$, there exists a sequence of step functions $s_k:[a,b]\to\mathbb{R}$ that converges to f uniformly.

(c) For a step function $s:[a,b]\to\mathbb{R}$, there exists a sequence of continuous functions $f_k:[a,b]\to\mathbb{R}$ that converges to s uniformly.

(a) **(+4 pts)** Let $\{\epsilon_k\}\downarrow 0$. For given $\epsilon>0$, $\exists k_0$ such that $\epsilon_k\le\epsilon$ for all $k\ge k_0$. Since F is dense, $\exists f_m\in F$ such that $\|f_m-f_0\|_\infty<\epsilon_m/2$ for all m. Then, for all $n,m\ge k_0$, $\|f_m-f_n\|_\infty=\|f_m-f_0+f_0-f_n\|_\infty\le\|f_m-f_0\|_\infty+\|f_0-f_n\|_\infty\le\epsilon_m/2+\epsilon_n/2\le\epsilon$.

(b) **(+3 pts)** For any $\epsilon>0$, choose $m\in\mathbb{N}$ such that $\frac{1}{2^m}<\epsilon$. Define a set $E_{n,m}:=f^{-1}((\frac{n-1}{2^m},\frac{n}{2^m}])$. Define functions s_m as $s_m(x)=\frac{n-1}{2^m}$ if $x\in E_{n,m}$. Then $\|f-s_m\|_\infty\le 2^{-m}$

(c) **(+3 pts)** False. WLOG, set $a=0,\ b=2$. Define a function $s:[0,2]\to\mathbb{R}$ as

$$s(x)=\begin{cases}2, & \text{if } x>1,\\ 0, & \text{if } x\le 1.\end{cases}$$

For $\epsilon<1$, suppose there exists a function f_k such that $\|s-f_k\|_\infty<\epsilon$. Then, for any $x\in[0,2]$, $f_k(x)\ge 1-\epsilon>1$ or $f_k(x)\le\epsilon<1$. Since f_k is continuous, $f_k(c)=1$ for some $c\in(0,2)$ by the intermediate value theorem. (contradiction)

(8) Prove the followings.

(a) Let $f:[a,b]\to\mathbb{R}$ be differentiable function and $|f'(x)|<1$. Then, f is uniformly continuous.

(b) Use the mean value theorem to prove Bernoulli's inequality:

For every $x>-1$ and every $k\in\mathbb{N}$, $(1+x)^k\ge 1+kx$.

(a) **(+5 pts)** For $x,y\in[a,b]$, $\exists c\in(x,y)$ such that $\frac{|f(x)-f(y)|}{|x-y|}=|f'(c)|<1$ by the mean value theorem. For any $\epsilon>0$, let $\delta=\epsilon$. Then, if $|x-y|<\delta$, $|f(x)-f(y)|<|x-y|<\epsilon$.

(b) **(+5 pts)** By the mean value theorem, $\frac{(1+x)^k-1}{(1+x)-1}=kc^{k-1}$ for some c between $1+x$ and 1. So, $(1+x)^k-1=kxc^{k-1}$. If $-1<x<0, 0<c<1\implies c^{k-1}\le 1\implies kxc^{k-1}\ge kx$. If $0<x, c>1\implies c^{k-1}\ge 1\implies kxc^{k-1}\ge kx$. If $x=0$, $1=1$ is clear. Therefore, $(1+x)^k\ge 1+kx$.

(9) Define a function $f : [0,1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in the lowest terms,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Determine where f is continuous. Explain why.
(b) Determine where f is differentiable. Explain why.

(a) **(+5 pts)** f is continuous only at irrational numbers. If x is a rational number, $f(x) = \frac{1}{q}$ for some $q > 0$. Let $\epsilon = \frac{1}{2q}$. Then, for any $\delta > 0$, there exists an irrational number $y \in N(x, \delta)$ and $|f(y) - f(x)| > \epsilon$. Hence, f is discontinuous at x. If x is an irrational number, for any $\epsilon > 0$, choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Let $a_{n,m} = |x - \frac{n}{m}|$ and $\delta < a_{n,m}$ for all $m < k$ and $n = 1, 2, \ldots, m$. Then, for all $y \in N(x, \delta)$, $f(y) < \frac{1}{k} < \epsilon$

(b) **(+5 pts)** f is nowhere differentiable. First, f is not differentiable at rational numbers since f is not continuous at those points. Let x be an irrational number. Then, for any $q \in \mathbb{N}$, there exists at least one rational number $r \in N(x, \frac{1}{q})$ such that $f(r) \geq \frac{1}{q}$. Therefore, $|\frac{f(r)-f(x)}{r-x}| \geq 1$ at those points and $|\frac{f(y)-f(x)}{y-x}| = 0$ for irrational numbers $y \in (0,1)$. Therefore, the limit does not exsits and hence not differentiable at the irrational number x.

Part C: Justification is not needed for true-false problems.

(10) (a) State if the followings are true or false.

(i) A boundary point of a set S is a limit point of S or an isolated point. There is no else.
(ii) If an isolated boundary point is deleted from S, it is not a boundary point anymore.
(iii) If a limit point is deleted from S, it is not a boundary point anymore.
(iv) A set S is closed if it contains all of its limit points, but miss some isolated points.
(v) A set S contains all of its boundary points if and only if it contains all of its limit points.

(b) The above questions tell us that the definition of the closed set in the textbook is bad. Give a better definition and explain why.

(a) **(+5 pts)** True statements: (i), (ii), (iv), (v). False statments: (iii)

(b) **(+5 pts)** A subset $A \subset \mathbb{R}^n$ is called closed if it contains all of its limit points. The reason why this is a better definition is that only the limit point matters to be a closed sets, but not isolated points.