Analysis I. Midterm Exam. 13:20–15:20 April 20, 2021

name: ID#: Score: / 100

Guidelines for the exam:

(1) Make answers short and points clear. Otherwise, it will be considered incorrect.
(2) There are 10 problems for 10 points each. Each sub-problem has the same weight.
(3) You are allowed to use books and notes. Any direct help from people is not allowed.
(4) Zoom should be on all the time.
(5) Exam ends at 15:20. Scan your exam and upload it by 15:40 (if you have trouble with KLMS, submit your exam in e-mail, hykim0615@kaist.ac.kr).

Part A: Prove the problem using definitions but not theorems. You may use the completeness axiom for $\mathbb{R}$ in the book.

(1) The summation and subtraction of two nonempty sets $A, B \subset \mathbb{R}$ are defined as $A \pm B = \{a \pm b : a \in A, b \in B\}$. Let A and B be bounded.
  (a) Show that $\sup(A + B) = \sup A + \sup B$.
  (b) Show that $\inf B \le \inf A$ and $\sup A \le \sup B$ if $A \subset B$.
  (c) Let $C = \emptyset$, the empty set. What should be $\sup C$ and $\inf C$ to keep the above relation (b)? Explain your answer with one or two sentences. (This problem is related to the definition of the limsup and liminf).
(2) Prove or disprove. (Depending on the type of the statement, proving may mean finding an example and disproving may not.)
  (a) If $A \subset \mathbb{R}$ consists of infinitely many real numbers, there exists at least one limit point of A.
  (b) If $\{x_k\}$ is a bounded and monotone increasing sequence, the sequence converges.
  (c) Let $S = \{x \in \mathbb{R} : x = x_k \text{ for some } k \in \mathbf{N}\}$ for a given sequence $x_i \in \mathbb{R}$. Then, $y \in \mathbb{R}$ is a limit point of S if and only if y is a cluster point of the sequence x_i.

Part B: You may use any theorem or lemma in the book for the following problems if needed.

(3) The open set of the Euclidean space $\mathbb{R}^n$ is always with the L^2-norm. However, we may provide other norms. In the case we do not call it the Eucliean space anymore. Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and define $\|\mathbf{x}\|_1 = |x| + |y|$, $\|\mathbf{x}\|_2 = \sqrt{x^2 + y^2}$, and $\|\mathbf{x}\|_\infty = \max(|x|, |y|)$.
  (a) Show that $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_\infty$ are norms.
  (b) Sketch the unit balls with respect to these three norms, i.e., sketch $B_i = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_i < 1\}$ for $i = 1, 2$ and ∞.
(4) Let $C_1, C_2 \subset \mathbb{R}$ be compact and $S \subset \mathbb{R}$ be open.
  (a) Suppose that $S \ne \emptyset$ and $S \ne \mathbb{R}$. Show that S is not closed. (This means there is no other clopen set in $\mathbb{R}$ except $\mathbb{R}$ and $\emptyset$.)
  (b) Prove that $C_1 \cup C_2$ is compact. (Refer theorems you use clearly.)
  (c) If $C_1 \cap C_2 = \emptyset$, there exists two open sets U_1, U_2 such that $C_1 \subset U_1$, $C_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$.
(5) Every bounded subset $S \subset \mathbb{R}$ has a supremum in $\mathbb{R}$ if and only if $\mathbb{R}$ is Cauchy complete. (In other words, the Cauchy completeness is equivalent to Axiom 1.1.1.)
  (a) Prove the part for ($\Rightarrow$).
  (b) Pove the other part for ($\Leftarrow$).
(6) Prove or disprove.
  (a) The product $(0, 1) \times (0, 1) \subset \mathbb{R}^2$ is an open set.
  (b) If $\{C_k\}$ is nested closed nonempty subsets of $\mathbb{R}$, then $\cap_{k=1}^{\infty} C_k \ne \emptyset$.
  (c) For any set $S \subset \mathbb{R}^n$, its closure is same as the closure of its interior S^0, i.e., $\overline{S^0} = \overline{S}$.
(7) Prove or disprove.

(a) Let $S \subset \mathbb{R}^n$ be a nonempty domain, $C_\infty(S)$ be the continuous function space with the uniform norm, $F \subset C_\infty(S)$ is a dense subset, and $f_0 \in C_\infty(S)$. Show that there exists a Cauchy sequence $\{f_k\} \subset F$ that converges to f_0 uniformly.

(b) For a continuous function $f : [a, b] \to \mathbb{R}$, there exists a sequence of step functions $s_k : [a, b] \to \mathbb{R}$ that converges to f uniformly.

(c) For a step function $s : [a, b] \to \mathbb{R}$, there exists a sequence of continuous functions $f_k : [a, b] \to \mathbb{R}$ that converges to s uniformly.

(8) Prove the followings.

(a) Let $f : [a, b] \to \mathbb{R}$ be differentiable function and $|f'(x)| < 1$. Then, f is uniformly continuous.

(b) Use the mean value theorem to prove Bernoulli's inequality:

For every $x > -1$ and every $k \in \mathbf{N}$, $(1 + x)^k \geq 1 + kx$.

(9) Define a function $f : [0, 1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in the lowest terms,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Determine where f is continuous. Explain why.

(b) Determine where f is differentiable. Explain why.

Part C: Justification is not needed for true-false problems.

(10) (a) State if the followings are true or false.

(i) A boundary point of a set S is a limit point of S or an isolated point. There is no else.

(ii) If an isolated boundary point is deleted from S, it is not a boundary point anymore.

(iii) If a limit point is deleted from S, it is not a boundary point anymore.

(iv) A set S is closed if it contains all of its limit points, but miss some isolated points.

(v) A set S contains all of its boundary points if and only if it contains all of its limit points.

(b) The above questions tell us that the definition of the closed set in the textbook is bad. Give a better definition and explain why.