Group 1 HW 10

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$\boxed{6.60}$ Let $f \in R(0,1)$ $\therefore$ $f^2 \in R(0,1)$

Suppose $\left[\int_0^1 f(x)dx\right]^2 > \int_0^1 f^2(x)dx$

Let $\int_0^1 f(x)dx = I_1$, $\int_0^1 f^2(x)dx = I_2$ and $\Delta = I_1^2 - I_2 = \left[\int_0^1 f(x)dx\right]^2 - \int_0^1 f^2(x)dx > 0$

$$\left.\begin{array}{l} \text{for } \varepsilon_1 = \frac{\Delta}{4I_1} > 0,\ \exists \pi_1 \in \Pi[0,1], \forall \pi \in \Pi[0,1] : \pi \geq \pi_1 \longrightarrow |S(f;\pi) - I_1| < \varepsilon_1 \\ \text{for } \varepsilon_2 = \frac{\Delta}{2} > 0,\ \exists \pi_2 \in \Pi[0,1], \forall \pi \in \Pi[0,1] : \pi \geq \pi_2 \longrightarrow |S(f^2;\pi) - I_2| < \varepsilon_2 \end{array}\right\} \text{for every choice of } S_i$$

Let $\pi_0 = \pi_1 \vee \pi_2 \geq \pi_1, \pi_2$ $\therefore$ $|S(f;\pi_1) - I_1| < \varepsilon_1$, $|S(f^2;\pi_2) - I_2| < \varepsilon_2$

$\therefore$ $I_1 - \varepsilon_1 < S(f;\pi_0) < I_1 + \varepsilon_1$, $I_2 - \varepsilon_2 < S(f^2;\pi_0) < I_2 + \varepsilon_2$

$\therefore$ $S(f;\pi_0)^2 > (I_1 - \varepsilon_1)^2 = I_1^2 - 2I_1\left(\frac{\Delta}{4I_1}\right) + \left(\frac{\Delta}{4I_1}\right)^2 > I_1^2 - \frac{\Delta}{2} = \frac{I_1^2}{2} + \frac{I_2}{2}$

$\therefore$ $S(f^2;\pi_0) < I_2 + \varepsilon_2 = \frac{I_1^2}{2} + \frac{I_2}{2} < S(f;\pi_0)^2$

Let $\pi_0 = \{x_0, \ldots, x_p\}$ and $S(f;\pi) = \sum_{i=1}^{p} f(x_i)\Delta x_i$, $S(f^2;\pi) = \sum_{i=1}^{p} f^2(x_i)\Delta x_i$

Let $a_i = f(x_i)\sqrt{\Delta x_i}$, $b_i = \sqrt{\Delta x_i}$ $\therefore$ By Cauchy-Shwartz I.E, $\left[\sum_{i=1}^{p} a_i b_i\right]^2 \leq \left[\sum_{i=1}^{p} a_i^2\right]\left[\sum_{i=1}^{p} b_i^2\right]$

$\therefore$ $\left[\sum_{i=1}^{p} f(x_i)\Delta x_i\right]^2 \leq \left[\sum_{i=1}^{p} f^2(x_i)\Delta x_i\right]\left[\sum_{i=1}^{p} \Delta x_i\right]$

$\therefore$ $S(f;\pi_0)^2 \geq S(f^2;\pi_0)\ (1-0)$ contradicts

$\therefore$ $\left[\int_0^1 f(x)dx\right]^2 \leq \int_0^1 f^2(x)dx$

$\boxed{6.81}$ a) Let $\varepsilon > 0$, Let $k_0 = \left[\frac{1}{\varepsilon}\right] \geq \frac{1}{\varepsilon}$

$\therefore$ for $k > k_0$, $\forall x \in [0,1] : |f_k(x) - 0| = \left|\frac{x^k}{k}\right| \leq \frac{1^k}{k} < \frac{1}{k_0} \leq \varepsilon$

$\therefore$ $f_k(x) \longrightarrow h(x)$ uniformly as $k \to \infty$; $\forall x \in [0,1] : h(x) = 0$

b) Let $g(x) = \begin{cases} 0 & ; x \in [0,1) \\ 1 & ; x = 1 \end{cases}$, $f'(x) = x^{k-1}$

For $x \in \{0,1\}$, $\forall \varepsilon > 0, \exists k_0 = 0, \forall k > k_0, |f_k(x) - g(x)| = 0 < \varepsilon$

For $x \in (0,1)$, $\forall \varepsilon > 0, \exists k_0 = \frac{\ln \varepsilon}{\ln x} + 1, \forall k > k_0, |f_k(x) - g(x)| = |x^{k-1}| < e^{\ln x (k_0 - 1)} = e^{\ln \varepsilon} = \varepsilon$

$\therefore$ $f_k(x) \longrightarrow g(x)$ pointwisely as $k \to \infty$; $g(x) = \begin{cases} 0 & ; x \in [0,1) \\ 1 & ; x = 1 \end{cases} \neq h'(x)$

$\boxed{7.5}$ a) Consider $1-\delta < x < 1+\delta \rightarrow 1-\delta < 2-x < 1+\delta \therefore 1-\delta < f(x) < 1+\delta$

$\forall \varepsilon > 0, \exists \delta = \varepsilon, \forall x \in (1-\delta, 1+\delta)$: $|f(x) - f(1)| = |f(x) - 1| < \delta = \varepsilon$

$\therefore \lim_{x \to 1} f(x) = f(1) \therefore f(x)$ is cont. at $x = 1$

By lemma in 7.3, $f \in RS(g; a, b)$ and $\int_0^2 f(x) dg(x) = f(1)(g(1^+) - g(1^-))$

$= 1(3-1)$

$= 2$

b) Let $\pi = \{x_0, \ldots, x_p\} \in \Pi[0,2] \therefore S(f, g, \pi) = \sum_{i=1}^{p} f(s_i)(g(x_i) - g(x_{i-1}))$

$\therefore \exists i \in \{1, \ldots, p\}: x_{i-1} < 1 \leq x_i$

$\therefore U(f, g, \pi) = f(M_i)(g(x_i) - g(x_{i-1})) = 2f(M_i)$, $L(f, g, \pi) = f(m_i)(g(x_i) - g(x_{i-1})) = 2f(m_i)$

$\therefore U(f, g) = \max(f(1), f(1^-))$, $L(f, g) = \min(f(1), f(1^-))$

$\therefore |U(f, g) - L(f, g)| = |f(1) - f(1^-)| > 0$ ($\because f$ discont at 1 to the left)

$\therefore f \notin RS(f, g)$

60 $\cancel{63}$ 65 70 71 72 80 81 88

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$$\sum\left(\sqrt{f^2(x)\,\Delta x}^{\,2}\right)\sum\left(\sqrt{\Delta x}^{\,2}\right)$$

$(x+y)^2 \leq x^2+y^2$

$x^2+2xy+y^2 \leq x^2+y^2$

$2x\sin(1/x) \qquad x^2\cos(1/x)\left(-\frac{1}{x^2}\right)$

$= 2x(s$

$\boxed{MA5241}$ Assume $F'(C)$ exists

$\boxed{6.65}$ Since there is jumpdiscontinuity at $c \Rightarrow$

$\Rightarrow f^+(c)$ and $f^-(c)$ exists, but are not equal.

Assume $f(x) \to A$ as $x \to c^+$ Let $\varepsilon > 0 \Rightarrow \exists \delta > 0$

s.t whenever $0 < t - c < \delta \Rightarrow A - \varepsilon < f(t) < A + \varepsilon$

Take $0 < h < \delta \Rightarrow h(A-\varepsilon) < \int_c^{c+h} f(t)dt < h(A+\varepsilon)$

$\Rightarrow A - \varepsilon < \frac{\int_c^{c+h} f(t)dt}{h} < A+\varepsilon \Rightarrow A - \varepsilon < \frac{F(c+h) - F(c)}{h} <$

$< A+\varepsilon$ $\Rightarrow \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = A \Rightarrow F'(c) = A$

Similarly if $f(x) \to B$ as $x \to c^- \Rightarrow F'(c) = B$

However, $A \neq B \Rightarrow$ contradiction.

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$\boxed{MAS241}$

$\boxed{6.7.1}$

a) if $x = 1$ or -1 $\Rightarrow f_k(x) = \frac{1}{2} \Rightarrow f_0(x) = \frac{1}{2}$

if $-1 < x < 1 \Rightarrow |x| < 1 \Rightarrow x^{2k} \to 0$ as $k \to \infty$

Thus $\frac{x^{2k}}{1+x^{2k}} \to \frac{0}{1+0} = 0$

$$f_0(x) = \begin{cases} \frac{1}{2} & \text{if } |x| = 1 \\ 0 & \text{otherwise } (|x| < 1) \end{cases}$$

b) $\|f_k - f_0\|_\infty = \sup(|g(x)|)$ where

$g(x) = 0$ at $x = 1; -1$ and $g(x) = \frac{x^{2k}}{1+x^{2k}}$ at $-1 < x < 1$

if $-1 < x < 1 \Rightarrow x^2 < 1$

$g(x) = \frac{1}{\left(\frac{1}{x^2}\right)^k + 1} < \frac{1}{2}$ Let $\frac{1}{2} > \varepsilon > 0$

$\frac{1}{2} - \varepsilon < \frac{x^{2k}}{1+x^{2k}} < \frac{1}{2} \Rightarrow \frac{\frac{1}{2} - \varepsilon}{\frac{1}{2} + \varepsilon} < x^{2k}$

take $x > \sqrt[2k]{\frac{\frac{1}{2} - \varepsilon}{\frac{1}{2} + \varepsilon}} \Rightarrow \lim_{k \to \infty} \|f_k - f_0\|_\infty = \frac{1}{2} \neq 0$

MAS241

6.71

c)

$$0 \leq f_k(x) \leq x^{2k}$$

$$\Rightarrow \int_{-1}^{1} 0 \leq \int_{-1}^{1} f_k(x)\,dx \leq \int_{-1}^{1} x^{2k}\,dx$$

$$\Rightarrow 0 \leq \int_{-1}^{1} f_k(x)\,dx \leq \left. \frac{x^{2k+1}}{2k+1} \right|_{-1}^{1} = \frac{2}{2k+1}$$

$$\lim_{k\to\infty} 0 = \lim_{k\to\infty} \frac{2}{2k+1} = 0$$

$$\Rightarrow \lim_{k\to\infty} \int_{-1}^{1} f_k(x)\,dx = 0$$

d) Consider $h(x) = 0$ for $\forall x \in [-1, 1]$

$f_0(x) \neq h(x)$ at only $x = 1$ and $x = -1$

~~By lemma~~ By Thm 10 from textbook

$$\Rightarrow \int_{-1}^{1} f_0 = \int_{-1}^{1} h = 0 \Rightarrow \int_{-1}^{1} f_0(x)\,dx = 0 =$$

$$= \lim_{k\to\infty} \int_{-1}^{1} f_k(x)\,dx$$

MAS241

6.80 Let's apply $\boxed{\text{Thm } 6.5.1}$ to f'_k

$\Rightarrow \int_a^x f'_k(t)dt$ converges uniformly to $\int_a^x g(t)dt$

Since $f'_k \in C[a,b] \Rightarrow f'_k \in R[a,b]$

$\Rightarrow \int_a^x f'_k(t)dt = f_k(x) - f_k(a)$

Thus $f_k(x) - f_k(a)$ converges uniformly to $\int_a^x g(t)dt$. Note that ~~...~~ uniform convergence implies also pointwise convergence.

(Because $|f_k(x) - f_0(x)| \leq \underbrace{\sup|f_k(x) - f_0(x)|}_{= \|f_k - f_0\|_\infty} < \varepsilon$)

Thus $\lim_{k\to\infty}(f_k(x) - f_k(a)) = f_0(x) - f_0(a) = \int_a^x g(t)dt$

On the other hand, since f'_k are all continuous

$\Rightarrow g$ is continuous as well $\Rightarrow (f_0(x) - f_0(a))' = g(x)$

$\Rightarrow f_0'(x) = g(x)$

MAS241

7.3 Let $\varepsilon > 0$

Assume that $\exists \Pi_0^1 \in \Pi(a,b), \Pi_0^2 \in \Pi(a,c)$

then take $\Pi_0 = \Pi_0^1 / \Pi_0^2 \cup \{c\}$

We know that whenever $\Pi^1 \geq \Pi_0^1$ and $\Pi^2 \geq \Pi_0^2$

$$\left| S(f,g,\Pi^1) - \int_a^b f\,dg \right| < \frac{\varepsilon}{2}$$

and

$$\left| S(f,g,\Pi^2) - \int_a^c f\,dg \right| < \frac{\varepsilon}{2}$$

Take $\Pi \geq \Pi_0 \in \Pi(c,b)$

$$\left| S(f,g,\Pi) - \int_a^b f\,dg + \int_a^c f\,dg \right| = \left| S(f,g,\Pi^1) - S(f,g,\Pi^2) - \right.$$

$$\left. - \int_a^b f\,dg + \int_a^c f\,dg \right| \leq \left| S(f,g,\Pi^1) - \int_a^b f\,dg \right| + \left| S(f,g,\Pi^2) - \right.$$

$$\left. - \int_a^c f\,dg \right| < \varepsilon \Rightarrow \int_c^b f\,dg = \int_a^b f\,dg - \int_a^c f\,dg$$

as desired.

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88) From the Theorem 6.3.1, we know that if f is continuous on $[a,b]$ and $g \in R[a,b]$; $g \geq 0 \Rightarrow \exists c \in [a,b]$ s.t.

$f(c)\int_a^b g = \int_a^b fg$ Now, take $g = 1 \Rightarrow f(c)(b-a) = \int_a^b f$

and for the f, we put $(x-t)^k f^{(k+1)}(t)$ with $a = x_0, b = x$

$\int_{x_0}^{x} (x-t)^k f^{(k+1)}(t)\,dt = (x-x_0)(x-c)^k f^{(k+1)}(c)$ $(\star)$

From Theorem 6.4.3 $\Rightarrow f(x) = P_k(x) + \frac{1}{k!}\int_{x_0}^{x}(x-t)^k f^{(k+1)}(t)\,dt$

Thus, by $(\star) \Rightarrow$ $\boxed{f(x) = P_k(x) + \frac{f^{(k+1)}(c)(x-c)^k(x-x_0)}{k!} \text{ for some } c \in [x_0, x]}$

72) a) if $x = 0 \Rightarrow f_k(x) = 0 \Rightarrow f_0(x) = 0$ for $x = 0$ ✓

if $x > 0$, since whenever $k \to \infty \Rightarrow kx,\ kx+1 \to \infty$ and from L'Hospital's Rule, $\lim_{k\to\infty} \frac{kx}{1+kx} = \lim_{k\to\infty} \frac{x}{x} = 1$, Thus $f_0(x) = 1$ for $0 < x \leq 1$ ✓

b) $\|f_k - f_0\|_\infty = \sup\left(\left|\frac{kx}{1+kx} - 1\right|\right) = \sup\left(\left|\frac{-1}{1+kx}\right|\right)$

$= \sup\left(\frac{1}{1+kx}\right)$ for $0 < x \leq 1$ (we don't need to consider $x = 0$ since $f_k - f_0$ gives zero which can't be supremum for positive function) It's clear that

$\underbrace{\frac{1}{1+kx}}_{>1} < \frac{1}{1} = 1$ for $0 < x \leq 1$. Let $1 > \varepsilon > 0$, choose $x < \frac{\varepsilon}{k(1-\varepsilon)}$

$\Rightarrow kx(1-\varepsilon) < \varepsilon$ and $(1-\varepsilon)(1+kx) < 1 \Rightarrow 1-\varepsilon < \frac{1}{1+kx} < 1$

meaning that $\sup\left(\frac{1}{1+kx}\right) = 1$; Thus, $\boxed{\|f_k - f_0\|_\infty \xrightarrow[k \to \infty]{} 1 \text{ as}}$

Since $f_0(x) = 0$ at $x = 0$, this concludes $\boxed{\{f_k\} \text{ does not converge uniformly to } f_0}$ $\boxed{V}$ ■ ⊠

c) $\int_0^1 f_k(x)\,dx = \int_0^1 \frac{kx}{1+kx}\,dx = \int_0^1 \left(1 - \frac{1}{1+kx}\right)dx = \int_0^1 1\,dx -$

$- \int_0^1 \frac{1}{1+kx}\,dx = x\Big]_0^1 - \frac{\ln(kx+1)}{k}\Big]_0^1 = 1 - \frac{\ln(k+1)}{k}$

From L'Hospital's Rule, $\lim_{k \to \infty}\left(1 - \frac{\ln(k+1)}{k}\right) = 1 - \lim_{k\to\infty} \frac{\ln(k+1)}{k}$ ($\to \infty$, $\to \infty$)

$= 1 - \lim_{k \to \infty} \frac{\frac{1}{k+1}}{1} = \underset{\sim}{1}$ ✓

Consider $h(x) = 1$ for $\forall x \in [0,1] \Rightarrow h(x) = f_0(x)$ for $\forall x \neq 0$

on $[0,1]$. From Theorem 10, $\int_0^1 h(x)\,dx = \int_0^1 f_0(x)\,dx =$

$= \int_0^1 1\,dx = \int_0^1 x'\,dx = x\Big]_0^1 = 1 \Rightarrow$ $\boxed{\lim_{k \to \infty} \int_0^1 f_k(x)\,dx = \int_0^1 f_0 = 1}$ (+) $\boxed{V}$ ■ ⊠

10) a) If $x=0 \Rightarrow f_k(0)=0$ and $f_0(0)=0$ ✓

If $x=1 \Rightarrow f_k(1)=0$ and $f_0(1)=0$ ✓

If $0<x<1 \Rightarrow (1-x)^k \to 0$ as $k\to\infty$ but since $k\to\infty$, we need to use L'Hospital's rule as in the following:

$$\lim_{k\to\infty} \frac{kx}{\left(\frac{1}{1-x}\right)^k} = x \lim_{k\to\infty} \frac{1}{\left(\frac{1}{1-x}\right)^k \cdot \ln\left(\frac{1}{1-x}\right)} = x \lim_{k\to\infty} \frac{(1-x)^k}{\ln\left(\frac{1}{1-x}\right)} =$$

$\underline{0<1-x<1}$

$=0$ (we took derivative wrt k, not x) $\Rightarrow f_0(x)=0$ for $\forall x$

b) Let's find $\|f_k-f_0\|_\infty = \|f_k\|_\infty = \sup(|f_k|) =$
$=\sup(f_k)$ where $f_k(x)\geq 0$

(Arithmetic Mean-Geometric Mean inequality)

$x=0$ or $1 \Rightarrow f_k=0$

if $0<x<1 \Rightarrow$ from A.M.-G.M. $kx+\underbrace{(1-x)+\dots+(1-x)}_{k} \geq$
$\geq (k+1)\sqrt[k+1]{kx(1-x)^k}$ where $\left(\frac{k}{k+1}\right)^{k+1} \geq f_k(x)$ and since equality case exists $\left(x=\frac{1}{k+1}\right) \Rightarrow \boxed{\sup(f_k) = \left(\frac{k}{k+1}\right)^{k+1}}$

So, we have to identify whether $\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^{k+1}=0$

$$\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^{k+1} = c \Rightarrow \ln c = \ln \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^{k+1} = \lim_{k\to\infty}(k+1)\ln\left(\frac{k}{k+1}\right)$$

$$= \lim_{k\to\infty}\frac{\ln(k)-\ln(k+1)}{\frac{1}{k+1}} = \lim_{k\to\infty}\frac{\frac{1}{k}-\frac{1}{k+1}}{\frac{-1}{(k+1)^2}} = \lim_{k\to\infty} -\frac{(k+1)}{k} = -1 \Rightarrow$$

$\boxed{V}$

$c=e^{-1}\neq 0$, therefore $\boxed{\{f_k\} \text{ does not converge to } f_0 \text{ uniformly}}$

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c) For computing $\int_0^1 f_k(x)dx = \int_0^1 kx(1-x)^k dx$, we will use integration by parts with $f(x)=x$, $g(x)=\frac{-(1-x)^{k+1}}{(k+1)}$

$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \Rightarrow \int x\, g'(x)dx =$

$= \int x \cdot \frac{-\cancel{(k+1)}(1-x)^k \cdot (-1)}{\cancel{k+1}} dx = \int x(1-x)^k dx = -\frac{x(1-x)^{k+1}}{k+1} +$

$+ \int \frac{(1-x)^{k+1}}{k+1} dx = -\frac{x(1-x)^{k+1}}{k+1} + \frac{1}{k+1}\int (1-x)^{k+1} dx = \int x(1-x)^k dx$

$\int (1-x)^{k+1} dx = \frac{(1-x)^{k+2}}{k+2}(-1) + C \Rightarrow \int_0^1 x(1-x)^k dx = -\frac{x(1-x)^{k+1}}{k+1}\Big]_0^1$

$-\frac{1}{k+1} \cdot \frac{(1-x)^{k+2}}{k+2}\Big]_0^1 = \frac{-1}{k+1} \cdot \frac{0}{k+2} + \frac{1}{k+1} \cdot \frac{1}{k+2} = \frac{1}{(k+1)(k+2)}$

$\boxed{\int_0^1 f_k(x)dx = \int_0^1 kx(1-x)^k dx = \frac{k}{(k+1)(k+2)}}$ Moreover, we get $\lim_{k\to\infty} \int_0^1 f_k(x)dx =$

$= \lim_{k\to\infty} \frac{k}{(k+1)(k+2)} = \lim_{k\to\infty} \frac{k}{k^2+3k+2} = \lim_{k\to\infty} \frac{1}{2k+3} = 0$ where we used L'Hospital's Rule where $k\to\infty$, $(k+1)(k+2)\to\infty \Rightarrow \boxed{\lim_{k\to\infty}\int_0^1 f_k(x)dx = 0}$

Since $f_0(x)=0$ for $\forall x \Rightarrow \int_0^1 f_0(x)dx = \int_0^1 0\,dx = C\Big]_0^1 = C - C = 0$

$\boxed{\int_0^1 f_0(x)dx = 0} \Rightarrow \boxed{\lim_{k\to\infty}\int_0^1 f_k(x)dx = \int_0^1 f_0(x)dx = 0}$ $\boxed{V}$

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63) The problem had some erratas: If we define function $F(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \in (0,1]$ and $F(0)=0$, then it is easy to show that $F'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\cdot\frac{(-1)}{x^2} = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ where $x \in (0,1]$. In the similar manner,

$F'(0) = \lim_{h\to 0}\frac{F(h)-F(0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right)$ where Squeeze Theorem gives $0 \le \left|h\sin\left(\frac{1}{h}\right)\right| \le |h|$ and as $|h| \to 0 \Rightarrow \left|h\sin\left(\frac{1}{h}\right)\right| \to 0$ revealing the result $F'(0)=0$. Moreover, $|F'(x)| = \left|2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \le 2|x| + |1| = 2|x| + 1 \le 3$, $x \in (0,1]$

$F'(0) = 0 \Rightarrow |F'(x)| \le 3$ i.e. $x \in [0,1]$ Thus, $\boxed{F'(x) \text{ is bounded}}$

Moreover, it is also clear that $F'(x)$ has a single (at most one) discontinuity point. Therefore, it is well-known that F will be Riemann integrable (this would still be true if it had a countable set of discontinuity points) (General Rule: a bounded func is Riemann-Integrable $\Leftrightarrow$ set of discontinuity pts has zero measure) Therefore, given problem is not correct as it mentioned that F' is not integrable on $[0,1]$. In order to resolve this issue, we will redefine the function $F(x)$:

$F(x) = x^2 \sin(1/x^2)$ for $x \in (0,1]$ and $F(0)=0$

Taking derivative, $F'(x) = 2x\sin\left(\frac{1}{x^2}\right) + x^2\cos\left(\frac{1}{x^2}\right)\cdot\left(x^{-2}\right)' =$

$= 2x\sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right)$ for $x \in (0,1]$ where we obtain

$F'(0) = \lim\limits_{x\to 0}\frac{F(x)-F(0)}{x} = \lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$. Now, $-x \leq x\sin\left(\frac{1}{x^2}\right) \leq$

$\leq x$ with squeeze giving $\lim\limits_{x\to 0}(-x) \leq \lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right) \leq$

$\leq \lim\limits_{x\to 0} x \Rightarrow \lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = 0$ and $\boxed{F'(0)=0}$ ✓

$F'(x) = 2x\sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) \Rightarrow$ So, $F'(x)$ is not bounded on $[0,1]$

Take $a_n = \sqrt{\frac{2}{(2n-1)\pi}} \Rightarrow \cos\left(\frac{1}{a_n^2}\right) = 1$ for all n as $a_n \to 0$

$\left|2x\sin\left(\frac{1}{x^2}\right)\right| \leq 2$ for all $x \in [0,1]$

and $\frac{2\cos\left(\frac{1}{a_n^2}\right)}{a_n} = \frac{2}{a_n} = \sqrt{2(2n-1)\pi} \to \infty$ as $n \to \infty$

Hence, F' is unbounded on $[0,1]$ and therefore, F' is unbounded $\Rightarrow$ As a result, we see that F' is not Riemann integrable on $[0,1]$ (since Riemann integrable functions are considered only if F' is bounded on given domain $[0,1]$, which is not the case for our function F) $\boxed{X}$

$\Rightarrow F'$ is not Riemann integrable ✓ $\boxed{X}$ +