Analysis I. Midterm Exam. 9:00-11:20am May 6, 2020

name: student id#: Score:

Guidelines for the exam:

(1) Answer the following 10 questions. 10 points each. All items have the same weight.
(2) Short answers are preferred.
(3) You are allowed to use lecture videos, books, and notes. Use any theorem in the book.
(4) Discussion is not allowed. If you get helped from a person, you will get F grade for the course.
(5) It is online exame. If there is typo in the exam, point it out and fix it by yourself.
(6) Exam ends at 11:20am. Scan your exam and upload (or email) it by 11:40am as you did in quizes.

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{c} = (c_1, \cdots, c_n)$, and $g_1(t) = f(t, c_2, \cdots, c_n)$. Prove the followings.
  (a) If f is continuous at $\mathbf{x} = \mathbf{c}$, $g_1(t)$ is continuous at $t = c_1$.
  (b) If $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$ for any sequence $\mathbf{x}_k$ that converges to $\mathbf{c}$, $f(\mathbf{x})$ is continuous at $\mathbf{c}$.

(2) Let $f : [a, b] \to [a, b]$ be continuous. Prove or disprove the followings.
  (a) There exists at least one $x \in [a, b]$ such that $f(x) = x$. (Fixed point theorem)
  (b) The image $f([a, b])$ is closed.
  (c) The inverse image $f^{-1}([c, d])$ is connected for any $[c, d] \subset [a, b]$.

(3) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove or disprove the followings.
  (a) For any $\epsilon > 0$, there exists a piecewise constant function $s : [a, b] \to \mathbb{R}$ such that $\|f - s\|_\infty < \infty$.
  (b) If f is uniformly continuous and strictly monotone, inverse function f^{-1} is also uniformly continuous.

(4) (a) Let $f_k : (a, b) \to \mathbb{R}$ be a Cauchy sequence in $C_\infty((a, b))$, i.e., in continuous and bounded function space. Show that f_k converges to a continuous function (i.e., prove the theorem).
  (b) Prove or disprove that the limit is uniformly continuous if f_k are all uniformly continuous.

(5) (a) Prove or disprove that a set $A = \{f \in C_\infty(\mathbb{R}) : \|f\|_\infty \le 1\}$ is closed.
  (b) Prove or disprove that the set A always have a cluster point (i.e., A is compact).

(6) The followings are all false. Find counterexamples.
  (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is closed, $f(A)$ is closed.
  (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is open, $f(A)$ is open.
  (c) If A_i are open, $\cap_{i=1}^\infty A_i$ is open or the empty set.
  (d) If $A_i \ne \emptyset$, are closed, and $A_1 \supset A_2 \supset \cdots$, then $\cap_{i=1}^\infty A_i \ne \emptyset$.

(7) Let $\liminf x_k = a$ and $\limsup x_k = b$. Show the followings directly from definitions.
  (a) Show that, if $a = b \in \mathbb{R}$, x_k is a Cauchy sequence.
  (b) Prove or disprove that, if $b < \infty$ and $\epsilon > 0$, there is $k_0 \in \mathbf{N}$ such that $x_k > b - \epsilon$ for all $k > k_0$.
  (c) Prove or disprove that $a \le b$.

(8) Let $C_1, C_2 \subset \mathbb{R}^n$ be two disjoint and closed sets.
  (a) Give the definition for the distance between the two sets.
  (b) Find an example that the distance of the two disjoint and closed sets is zero.
  (c) Show that, if C_1 is bounded, there exist two open sets U_1 and U_2 such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$, and $U_1 \cap U_2 = \emptyset$.

(9) (a) Prove or disprove that $f(x, y) = \frac{x^2 y}{x^2+y^2}$ is continuous at $(x, y) = (0, 0)$ if we set $f(0, 0) = 0$.
  (b) Show that $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is continuous, but not uniformly continuous.

(10) (a) Let $X := \{f : N \to \{0, 1\}\}$ be the collection of all functions defined on natural numbers which have values of 0 or 1. Prove or disprove that the set is uncountable.
  (b) Prove that a sequence x_k has a cluster point if $x_k \in (0, 1)$ for all k (Do not use Heine-Borel).