Name: Emil Gasimov

ID: 20180849

PIN: 1234

Problem 1.

a) Let x_1, x_2 be any numbers in $[a, b]$ with $x_1 < x_2$.

Then

~~$v^+(f; a, x_1) = \sup\left\{\sum_{j=1}^{k}(\Delta f_j)^+ : \pi : \Pi[a, x_1]\right\}$~~

$$v^+(f; a, x_2) = \sup\left\{\sum_{j=1}^{k}(\Delta f_j)^+ : \pi \in \Pi[a, x_2]\right\} \geq$$

$$\geq \left[\sum_{j=1}^{k'}(\Delta f_j)^+ : \pi' \in \Pi[a, x_1]\right] + \max(0, f(x_2) - f(x_1))$$

for any ~~π~~ partition π' of $[a, x_1]$. Taking the supremum:

$$v^+(f; a, x_2) \geq v^+(f; a, x_1) + \max(0, f(x_2) - f(x_1)) \geq$$

$$\geq v^+(f; a, x_1) \Rightarrow v^+ \text{ is monotone increasing}$$

Similarly,

$$v^-(f; a, x_2) = \sup\left\{\sum_{j=1}^{k}(\Delta f_j)^- : \pi \in \Pi[a, x_2]\right\} \geq$$

$$\geq \left[\sum_{j=1}^{k'}(\Delta f_j)^- : \pi' \in \Pi[a, x_1]\right] + \max(0, f(x_1) - f(x_2))$$

for any $\pi' \in \Pi[a, x_1]$. Taking the supremum:

$$v^-(f; a, x_2) \geq v^-(f; a, x_1) + \max(0, f(x_1) - f(x_2)) \geq$$

$$\geq v^-(f; a, x_1). \Rightarrow v^- \text{ is monotone increasing.}$$

b) For any $\pi \in \Pi[a, x]$ (where $x \in (a, b)$ is any real number):

① $\displaystyle\sum_{j=1}^{n} |\Delta f_j| \geq \sum_{j=1}^{n} (\Delta f_j)^+$, since $|a| \geq \max(0, a)$

② $\displaystyle\sum_{j=1}^{n} |\Delta f_j| \geq \sum_{j=1}^{n} (\Delta f_j)^-$, since $|a| \geq \max(0, -a)$.

① implies $V(f; a, x) \geq V^+(f; a, x)$;

② implies $V(f; a, x) \geq V^-(f; a, x)$. (for any $x \in (a, b)$).

c) Suppose V_f is cont. at c. ~~Then~~ ~~So~~ Given $\varepsilon > 0$, there exists a $\delta > 0$ s.t ~~$[a, b] \cap$~~ if $x \in [a, b] \cap N(c, \delta)$, then $|V_f(x) - V_f(c)| < \varepsilon$. For any $x \in [a, b] \cap [c, c+\delta)$, one has

$$|V_f(x) - V_f(c)| < \varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon,$$

because ~~$f(x) - f(c)$~~ ~~$V_f(x$~~ $V_f(x) \geq V_f(c) + |f(c) - f(x)|$. This means that f is cont. from the right at c. Similarly, for any $x \in [a, b] \cap [c-\delta, c]$, one has

$$|V_f(x) - V_f(c)| < \varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon,$$

since $V_f(c) \geq V_f(x) + |f(x) - f(c)|$. So, f is cont. from the left at c. ~~Thus f is co~~ Thus $f^-(c) = f(c) = f^+(c) \Rightarrow$

f is cont. at c.

d) Clearly

$$\sum_{j=1}^{r} |\Delta f_j| = \sum_{j=1}^{r} (\Delta f_j)^{+} + \sum_{j=1}^{r} (\Delta f_j)^{-},$$

for any partition $\pi \in \Pi[a, b]$.

Let π_1 and π_2 be any two parts in $\Pi[a, b]$.

Then $\cancel{V(f, \pi_1 \vee \pi_2) \geq V}$

$$V(f, \pi_1 \vee \pi_2) = \sum_{\pi_1 \vee \pi_2} |\Delta f_j| = \sum_{\pi_1 \vee \pi_2} (\Delta f_j)^{+} +$$

$$+ \sum_{\pi_1 \vee \pi_2} (\Delta f_j)^{-} \geq \sum_{\pi_1} (\Delta f_j)^{+} + \sum_{\pi_2} (\Delta f_j)^{-} . \Longrightarrow$$

Problem 2.

b) f is cont. on $[a,b] \Rightarrow f$ is unif. cont. on $[a,b]$ (since $[a,b]$ is compact). Given $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|s-t| < \delta \Rightarrow$

$|f(s) - f(t)| < \varepsilon / |g(b) - g(a)|$

Let π_0 be any part. with gauge $< \delta$, and let π be any refinement of π_0. ~~For any~~ Suppose that $\pi = (a = x_0, x_1, \dots x_{p-1}, x_p = b)$. Then f is cont. on each $[x_{j-1}, x_j]$ (for $j = 1, 2, \dots, p$), so f ~~assumes its~~ assumes its max. and min. values on $[x_{j-1}, x_j] \Rightarrow$ there exist s_j and t_j s.t

$f(s_j) = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$ and

$f(t_j) = \sup\{f(x) : x \in [x_{j-1}]\}$. ~~So Note that~~

Note that $|s_j - t_j| < |x_{j-1} - x_j| < \delta$. So, ~~$|f(s_j) - f(t_j)|$~~ ~~Then~~ $|f(s_j) - f(t_j)| < \dfrac{\varepsilon}{|g(b) - g(a)|}$

Then

$$\mathcal{U}(f, g, \pi) - \mathcal{L}(f, g, \pi) =$$

$$= \sum_{j=1}^{p} (f(t_j) - f(s_j)) \Delta g_j < \cancel{\sum_{j=1}^{p} \frac{\varepsilon \Delta g_j}{|g(b) - g(a)|}}$$

$$\le \cancel{\sum_{j=1}^{p} (f(t_j) - f(s_j)) |\Delta g} \sum_{j=1}^{p} \frac{\varepsilon \Delta g_j}{g(b) - g(a)} = \varepsilon,$$

under the assumption that g is monotone increasing.

So, if g is cont. and mon. incre. on $[a,b]$, then Riemann's cond. holds for f wrt $g \Rightarrow f \in RS[g, a, b]$.

However, if g is not mon. increasing, then we cannot guarantee that $f \in RS[g; a, b]$.

a) Since $g(x) = x$ is cont. and mono. incre. on $[a, b]$, f is in ~~RS[x; a, b~~ $RS[x; a, b]$ (accor. to part (b)) $\Rightarrow f \in R[a, b]$.

Problem 3.

a) $p_1 \to p_4$, $p_1 \to p_5$,

$p_2 \to p_1$, $p_2 \to p_4$, $p_2 \to p_5$,

$p_3 \to p_1$, $p_3 \to p_4$, $p_3 \to p_5$, $p_3 \to p_6$

$p_4 \to p_5$, $p_6 \to p_1$, $\cancel{p_6 \to p_5}$ $p_6 \to p_5$.

b) $p_1 \to p_2$, $p_1 \to p_3$, $p_1 \to p_6$

$p_2 \to p_3$, $p_2 \to p_6$,

$p_3 \to p_2$, $\cancel{p_3 \to}$

$p_4 \to p_1$, $p_4 \to p_2$, $p_4 \to p_3$, $p_4 \to p_6$

$p_5 \to p_1$, $p_5 \to p_2$, $p_5 \to p_3$, $p_5 \to p_4$, $p_5 \to p_6$,

$p_6 \to p_2$, $p_6 \to p_3$, $p_6 \to p_4$.

Problem 4.

~~Let x, y be any two real numbers. Then~~

Let $\varepsilon > 0$ and c be any real numbers. Then

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \Rightarrow$$

there exists a $\delta > 0$ such that

$$|x-c| < \delta \Rightarrow \left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \varepsilon.$$

So, $|x-c| < \delta$ implies

$$\left| \frac{f(x)-f(c)}{x-c} \right| < |f'(c)| + \varepsilon < 10 + \varepsilon.$$

Let $\delta' \le \min\left(\delta, \ \varepsilon/_{10+\varepsilon}\right)$. Then

$$|x-c| < \delta' \Rightarrow |x-c| < \delta \Rightarrow$$

$$\Rightarrow \left| \frac{f(x)-f(c)}{x-c} \right| < 10 + \varepsilon \Rightarrow$$

$$\Rightarrow |f(x)-f(c)| < (10+\varepsilon)|(x-c)| < (10+\varepsilon)\delta' \le$$

$$< (10+\varepsilon)\frac{\varepsilon}{10+\varepsilon} = \varepsilon.$$

~~This means that, by choosing $\delta' = \frac{\varepsilon}{10+\varepsilon}$~~

Problem 5.

~~There exists a $c \in [0,1]$ such that~~

$$\cancel{\int_a^b f^2(x)\,dx = f(c)\int}$$

$$\cancel{\int_0^1 f^2(x)\,dx = f(c) \times \int_0^1 f(x)\,dx.}$$

If $f, g \in R[a,b]$, then

$$\left[\int_a^b f(x)g(x)\,dx\right]^2 \leq \left[\int_a^b f^2(x)\,dx\right]\left[\int_a^b g^2(x)\,dx\right].$$

Let $m = \int_a^b f(x)g(x)\,dx$, $n = \int_a^b f^2(x)\,dx$, $k = \int_a^b g^2(x)\,dx$.

Let $a(t) = \int_a^b [tf(x) + g(x)]^2\,dx$, $\forall t \in R$. Then

$$0 \leq a(t) = \int^b t^2 n + 2tn + k \Rightarrow$$

$$t^2 n + 2tm + k \geq 0 \text{ for all } t \in R \Rightarrow$$

discri. $\Delta = b^2 - 4ac = 4m^2 - 4nk \leq 0 \Rightarrow$

$$\left[\int_a^b f(x)g(x)\,dx\right]^2 \leq \left[\int_a^b f^2(x)\,dx\right]\left[\int_a^b g^2(x)\,dx\right].$$

Then $f\ g(x) = 1$, $a = 0$, $b = 1$ ~~implies~~: :

$$\left[\int_0^1 f(x)\,dx\right]^2 \leq \left[\int_0^1 f^2(x)\,dx\right],$$

as desired. $\left(\text{because } \int_0^1 g^2(x)\,dx = \int_0^1 1\,dx = 1\right)$ $\square$.

Problem 7.

$\mathcal{L}(f) \le \mathcal{U}(f)$ and $\mathcal{L}(g) \le \mathcal{U}(g)$ for any two bounded func.s f, g, so the middle ineq. holds trivially. ~~Let π_1 and π_2 be any part.s in $[a, b]$. So~~ ~~Lo~~

$\cancel{\mathcal{L}(f+g, \pi_1 \vee \pi_2)}$

~~Let π_1 and π_2 be any two parts. in $\Pi[a,b]$.~~

~~Then~~

$\cancel{\mathcal{L}(f) + \mathcal{L}(g) \geq \mathcal{L}}$

Let $\pi \in \Pi[a, b]$ be any part - $\pi = (x_0 = a, x_1, \ldots, x_{p-1}, x_p = b)$. Then, if $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$,

$m'_j = \inf\{g(x) : x \in [x_{j-1}, x_j]\}$, one has

$m''_j = \inf\{(f+g)(x) : x \in [x_{j-1}, x_j]\} \geq m_j + m'_j$.

So,

$\mathcal{L}(f+g) = \cancel{\sup\{(f+g)(x) : x} \sup\{\mathcal{L}(f+g, \pi)\} \le$

$\le \cancel{\sup\{\mathcal{L}(f, g)} \sup\{\mathcal{L}(f, \pi) : \pi \in \Pi[a, b]\} +$

$+ \sup\{\mathcal{L}(g, \pi) : \pi \in \Pi[a, b]\} = \mathcal{L}(f) + \mathcal{L}(g)$.

Similarly, if $\mathcal{M}_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$,

$\mathcal{M}'_j = \sup\{g(x) : x \in [x_{j-1}, x_j]\}$, and

$\mathcal{M}''_j = \sup\{(f+g)(x) : x \in [x_{j-1}, x_j]\}$, then

$\mathcal{M}_j + \mathcal{M}'_j \geq \mathcal{M}''_j \Rightarrow$

$\mathcal{U}(f) + \mathcal{U}(g) = \inf$

<u>Problem 10.</u> ~~Let $\pi = ($~~

Let $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$ and $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$. Then

$$\mathcal{L}(f, g, \pi) = \sum_{j=1}^{n} m_j (\Delta g_j)$$

and

$$\mathcal{U}(f, g, \pi) = \sum_{j=1}^{n} M_j (\Delta g_j).$$

Suppose that x' is inserted into $[x_{k-1}, x_k]$. Let $m' = \inf\{f(x) : x \in [x_{k-1}, x']\}$ and $m'' = \inf\{f(x) : [x', x_k]\}$.

So, $m_k \leq \min(m', m'')$. Note that

$$\mathcal{L}(f, g, \pi') = \sum_{j=1}^{k-1} + m'(g(x') - g(x_{k-1})) + m''(g(x_k) - g(x')) + \sum_{j=k+1}^{n} \geq \sum_{j=1}^{k-1} + m_k(g(x_k) - g(x_{k-1})) + \sum_{j=k+1}^{n} =$$

$$\geq \sum_{j=1}^{k-1} m_j \Delta g_j + m_k \Delta g_k + \sum_{j=k+1}^{n} m_j \Delta g_j =$$

$= \mathcal{L}(f, g, \pi)$.

Similarly, let $\mathcal{M}' = \sup\{f(x) : x \in [x_{k-1}, x']\}$, $\mathcal{M}'' = \sup\{f(x) : x \in [x', x_k]\}$. Then

$m_k \geq \max(\mathcal{M}', \mathcal{M}'') \Rightarrow$

$$\mathcal{U}(f, g, \pi') = \sum_{j=1}^{k-1} \mathcal{M}_j \Delta g_j + \mathcal{M}'(g(x') - g(x_{k-1})) +$$

$$+ \mathcal{M}''(g(x_k) - g(x')) + \sum_{j=k+1}^{n} \mathcal{M}_j \Delta g_j \leq$$

$$\leq \sum_{j=1}^{k-1} \mathcal{M}_j \Delta g_j + \mathcal{M}_k \Delta g_k + \sum_{j=k+1}^{n} \mathcal{M}_j \Delta g_j =$$

$= \mathcal{U}(f, g, \pi)$.

Problem 4.

b) Let $\varepsilon > 0$ be any real number. Suppose that $f(x)$ is unif. cont. on $\mathbb{R}$. Then there exists a $\delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

~~Then~~ Let x be any real number for which

$$e^x > \frac{\varepsilon}{e^{\delta/2} - 1} \Rightarrow e^{x+\delta/2} - e^x > \varepsilon.$$

Then $|e^{x+\delta/2} - e^x| > \varepsilon$, but $|x + \delta/2 - x| < \delta$, contra. So, $f(x)$ is not unif. cont. on $\mathbb{R}$.