1 Let f and g be any two bounded functions on $[a, b]$.

2+4+2+2 points

(a) Let π be any partition of $[a, b]$. Prove that

$$L(f+g,\pi) \geq L(f,\pi) + L(g,\pi).$$

(b) Using the result of (a), prove that

$$L(f+g) \geq L(f) + L(g).$$

(c) Using the result of (b), prove that if $f, g \in R[a, b]$, then also $f+g \in R[a, b]$.

(c) Give an example that equality of (b) does not holds.

Solution. (a) Since

$$\inf_{x_{j-1}\leq x\leq x_j} \{f(x)+g(x)\} \geq \inf_{x_{j-1}\leq x\leq x_j} f(x) + \inf_{x_{j-1}\leq x\leq x_j} g(x),$$

We have

$$L(f+g,\pi) = \sum_{j=1}^{p} m_{f+g,j}\Delta x_j \geq \sum_{j=1}^{p} m_{f,j}\Delta x_j + \sum_{j=1}^{p} m_{g,j}\Delta x_j = L(f,\pi) + L(g,\pi).$$

(b) For any two partitions π_1 and π_2 on $[a, b]$, with **Theorem 6.2.1**, we can obtain

$$L(f+g,\pi_1\vee\pi_2) \geq L(f,\pi_1\vee\pi_2) + L(g,\pi_1\vee\pi_2) \geq L(f,\pi_1) + L(g,\pi_2).$$

Therefore,

$$L(f+g) = \sup_{\pi\in\Pi[a,b]} L(f+g,\pi) \geq L(f,\pi_1) + L(g,\pi_2).$$

By taking supremum among π_1 and π_2, finally we have

$$L(f+g) \geq L(f) + L(g),$$

as desired.

Remark. Since

$$\sup_{\pi\in\Pi[a,b]} (L(f,\pi) + L(g,\pi)) \leq \sup_{\pi\in\Pi[a,b]} L(f,\pi) + \sup_{\pi\in\Pi[a,b]} L(g,\pi),$$

we cannot obtain the result directly form (a).

(c) By using same method of (a) and (b), we can obtain

$$U(f) + U(g) \geq U(f+g).$$

Thus,

$$U(f) + U(g) \geq U(f+g) \geq L(f+g) \geq L(f) + L(g).$$

However, if $f, g \in R[a,b]$, then

$$U(f) = L(f), \text{ and } U(g) = L(g).$$

Therefore, $U(f+g) = L(f+g)$, which implies $f+g \in R[a,b]$.

(d) Let f and g be functions on $[a,b]$ such that

$$f = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases} \qquad \text{and} \qquad g = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then $L(f) = L(g) = 0$, but $L(f+g) = 1$.