1 5 points

Define a subset of $\mathbb{R}^2$ as

$$S = \{(x, y) : y = \sin(1/x), x > 0\}$$

Is S a closed set on $\mathbb{R}^2$? Explain your answer.

Solution. S is not a closed set on $\mathbb{R}^2$. **(+1 points)** Consider a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of $\mathbb{R}^2$ such that $\mathbf{x}_n = (1/\pi n, 0)$. First, note that $\sin(\pi n) = 0$ for all n, so $\mathbf{x}_n \in S$ for all n. In addition, we can easily check that $\mathbf{x} = (0, 0)$ is a limit point of $\{\mathbf{x}_n\}_{n=1}^{\infty}$, but $\mathbf{x} \notin S$. **(+3 points)** Therefore, by **Theorem 2.2.4**, S is not a closed set on $\mathbb{R}^2$. **(+1 points)**

2 5 points

Let X and Y be two nonempty connected subsets of $\mathbb{R}^n$ such that $X \cap Y \neq \emptyset$. Prove that $Z = X \cup Y$ is also a connected subset of $\mathbb{R}^n$.

Solution. We will assume Z is disconnected and draw out a contradiction. Since Z is disconnected, there exist two nonempty, disjoint open sets U and V such that $Z \subseteq U \cup V$, and $Z \cap U \neq \emptyset$ and $Z \cap V \neq \emptyset$. **(+1 points)**

Since $Z \subseteq U \cup V$, $X \subseteq U \cup V$ and $Y \subseteq U \cup V$.If $X \cap U \neq \emptyset$ and $X \cap V \neq \emptyset$, then X is disconnected by the definition, thus $X \cap U = \emptyset$ or $X \cap V = \emptyset$. This implies $X \subseteq U$ or $X \subseteq V$. By using same argument, also $Y \subseteq U$ or $Y \subseteq V$. **(+2 points)**

However, since $X \cap Y \neq \emptyset$, X and Y are both contained in U or both contained in V. Thus, $Z = X \cup Y \subseteq U$ or $Z = X \cup Y \subseteq V$. This implies $Z \cap U = \emptyset$ or $Z \cap V = \emptyset$, which is a contradiction. **(+2 points)**