1 Suppose f is continuous on $[0,1]$. Prove that

5 points

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

Solution. Put $x \leftarrow \pi - x$ in the left hand side, then

$$\int_0^\pi xf(\sin x)dx = \int_\pi^0 (-1)(\pi - x)f(\sin(\pi - x))dx = \int_0^\pi (\pi - x)f(\sin x)dx.$$

Thus,

$$\int_0^\pi xf(\sin x)dx - \int_0^\pi (\pi - x)f(\sin x)dx = 2\int_0^\pi xf(\sin x)dx - \pi\int_0^\pi f(\sin x)dx = 0,$$

which is equivalent to

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

2 Let $\{f_k\}$ be a sequence of continuously differentiable functions on $[a, b]$ such that

5 points

(i) $\lim_{k\to\infty} f_k = f_0$ pointwise on $[a, b]$, and

(ii) $\lim_{k\to\infty} f_k' = g$ uniformly on $[a, b]$.

Prove that f_0 is differentiable on $[a, b]$ and $f_0' = g$.

Solution. By condition (ii) and **Theorem 6.5.1**, g is integrable on $[a, b]$ and

$$\lim_{k\to\infty} \int_a^x f_k'(t)dt = \int_a^x g(t)dt.$$

By **Theorem 6.3.8**, it is equivalent to

$$\lim_{k\to\infty} (f_k(x) - f_k(a)) = \int_a^x g(t)dt.$$

Then by condition (i),

$$f_0(x) - f_0(a) = \int_a^x g(t)dt.$$

However, by condition (ii) and **Theorem 3.5.3**, g is continuous, and then by **Theorem 6.3.3 iii)**, $f_0(x)$ is differentiable and $f_0'(x) = g(x)$ for all $x \in [a, b]$.