Group 1 HW 5 (8th April, 2021)

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$\boxed{2.73}$ a) Since $\mathbb{R}^n$ has Bolzano-Weierstrass property

$\therefore$ Any bounded infinite subset of $\mathbb{R}^n$ has a limit point

$\therefore$ T has a limit point $x_0 \in \mathbb{R}^n$

Claim $\text{dist}(x_0, S) = 0$

If $\text{dist}(x_0, S) > 0$, then $N'(x_0, \text{dist}(x_0, S)) \cap S = \emptyset$

$\therefore N'(x_0, \text{dist}(x_0, S)) \cap T = \emptyset$

$\therefore x_0$ is not a limit point of T

$\therefore \forall r > 0,\ N'(x_0, r) \cap S \neq \emptyset$

Case 1: $\forall r > 0,\ N(x_0, r) \cap S^c \neq \emptyset$ $\therefore x_0$ is boundary point of S

Since S is closed, $x_0 \in S$

Case 2: $\exists r > 0,\ N(x_0, r) \cap S^c \neq \emptyset$ $\therefore N(x_0, r) \subset S$ $\therefore x_0 \in S$

b.i) Let $\emptyset = S \subseteq \mathbb{R}^n$; S is unbounded

$\therefore \forall M > 0, \exists x \in S ;\ |x| > M$

Let x_1 be arbitory element of S

For $n \geq 2$, Let $x_n \in S$ be element such that $|x_n| > |x_{n-1}| + 1$

Let $T = \{x_n \mid n \in \mathbb{N}\}$ $\therefore$ T is infinite subset of S

Clearly, T has no limit point ($\because \forall x \in T,\ N'(x, \frac{1}{2}) \cap T = \emptyset$)

b.ii) Let $\emptyset = S \subseteq \mathbb{R}^n$; S is bounded and every infinite subset of S has a limit point

Let x_0 be limit point of S $\therefore \forall \varepsilon > 0 : S \cap N'(x_0, \varepsilon) \neq \emptyset$

Let x_1 be arbitory element of $S - \{x_0\}$

For $n \geq 2$, Let $x_n \in S$ be element such that $x_n \in S \cap N'(x_0, \frac{1}{2}\|x_{n-1} - x_0\|)$

$\therefore \|x_1 - x_0\| > 2\|x_2 - x_0\| > 4\|x_3 - x_0\| > \dots$

Let $T = \{x_n \mid n \in \mathbb{N}\}$ $\therefore$ T is infinite subset of S with x_0 as only limit point

$\boxed{3.12}$ a) $\lim_{x\to c} \frac{\sin(x-c)}{(x^2-c^2)} = \lim_{x\to c}\left(\frac{(x-c) - \frac{1}{3!}(x-c)^3 + \frac{1}{5!}(x-c)^5 - \dots}{(x^2-c^2)}\right)$ by taylor series

$= \lim_{x\to c}\left(\frac{1}{x+c}\left(1 - \frac{1}{3!}(x-c)^2 + \frac{1}{5!}(x-c)^4 - \dots\right)\right)$

$= \frac{1}{\lim_{x\to c}(x+c)}\left[1 - \frac{1}{3!}\lim_{x\to c}(x-c)^2 + \frac{1}{5!}\lim_{x\to c}(x-c)^4 - \dots\right]$ since all limit exists

$= \frac{1}{2c}\left[1 - 0 + 0 - 0 + \dots\right]$

$= \frac{1}{2c}$

b) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0}\left(\frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right)}{x^2}\right)$ by taylor series

$= \lim_{x\to 0}\left(\frac{1}{2} - \frac{1}{4!}x^2 + \frac{1}{6!}x^6 - \dots\right)$

$= \frac{1}{2} - \frac{1}{4}\lim_{x\to 0}x^2 + \frac{1}{6}\lim_{x\to 0}x^4 - \dots$ since all limit exists

$= \frac{1}{2}$

c) $\lim_{x\to 1} \frac{[(1+x)^{1/2} - (1-x)^{1/2}]}{x} = \frac{[(1+\lim_{x\to 1}x)^{1/2} - (1-\lim_{x\to 1}x)^{1/2}]}{\lim_{x\to 1}x} = \frac{(1+1)^{1/2} - (1-1)^{1/2}}{1} = \sqrt{2}$

d) For $L \geq 0$, $\exists \varepsilon = 1$, $\forall \delta > 0$, $\exists x = -\delta/2$ $\therefore f(x) - L = -\frac{\delta}{2} - 1 - L < 1$

such that $|x-0| = \frac{\delta}{2} < \delta$ but $|f(x) - L| > \varepsilon = 1$

For $L < 0$, $\exists \varepsilon = 1$, $\forall \delta > 0$, $\exists x = \delta/2$ $\therefore f(x) - L = \frac{\delta}{2} + 1 - L > 1$

such that $|x-0| = \frac{\delta}{2} < \delta$ but $|f(x) - L| > \varepsilon = 1$

$\therefore \forall L \in \mathbb{R}, \exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R} : |x-c| < \delta$ and $|f(x) - L| \geq \varepsilon$

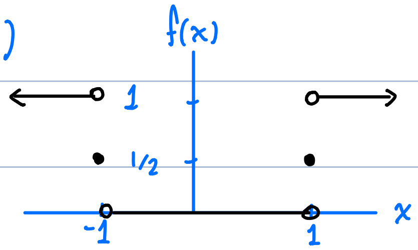
$\therefore \forall L \in \mathbb{R}$ $\lim_{x\to c} \neq L$

$\therefore$ limit doesn't exist

3.15 a) Let $c \in \mathbb{R}$ ∴ $\lim\limits_{x\to c} f_k(x) = \lim\limits_{x\to c} \frac{1}{1+x^{2k}} = \frac{1}{1+(\lim\limits_{x\to c} x)^{2k}} = \frac{1}{1+c^{2k}} = f(c)$

∴ $\forall c \in \mathbb{R}$, $f_k(x)$ is continuous ∴ $f_k(x)$ is continuous on $\mathbb{R}$

b)



at $c \in (\infty,-1)\cup(1,\infty)$, $\forall \varepsilon>0, \exists \delta = |c|-1, \forall x \in \mathbb{R}: |x-c|<|c|-1 \rightarrow |c|-|x| \leq |x-c| < |c|-1$

∴ $|x| > 2|c|-1 > 1$ ∴ $|f(x)-1| = 0 < \varepsilon$

∴ at $c \in (\infty,-1)\cup(1,\infty)$, $\lim\limits_{x\to c} f(x) = f(c) = 1$ ∴ continuous

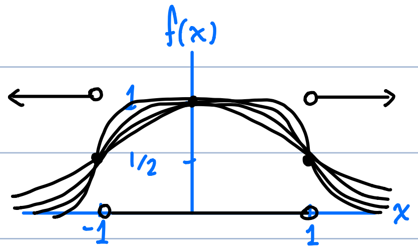
at $c \in (-1,1)$, $\forall \varepsilon>0, \exists \delta = 1-|c|, \forall x \in \mathbb{R}: |x-c|<1-|c| \rightarrow |x|-|c| \leq |x-c| < 1-|c|$

∴ $|x| < 1$ ∴ $|f(x)-0| = 0 < \varepsilon$

∴ at $c \in (-1,1)$, $\lim\limits_{x\to c} f(x) = f(c) = 0$ ∴ continuous

at $c = \pm 1$, $\lim\limits_{x\to c} f(x)$ doesn't exist ∴ discontinuous

c)



d) For $|x|<1$, $\forall \varepsilon>0, \exists k_0 = \frac{1}{2}\log_{|\frac{1}{x}|}\left(\max\left(\frac{1-\varepsilon}{\varepsilon}, 1\right)\right), \forall k>k_0: |y_k - 1| < \varepsilon$

∴ $\begin{cases} \text{Case } \varepsilon \geq 1,\ k_0 = \frac{1}{2}\log_{|\frac{1}{x}|} 1 = 0 \ \therefore\ |y_k - 1| = \left|\frac{1}{1+x^{2k}} - 1\right| < 1 < \varepsilon \\ \text{Case } \varepsilon < 1 \ \therefore\ \text{If } k>k_0 = \frac{1}{2}\log_{|\frac{1}{x}|}\left(\max\left(\frac{1-\varepsilon}{\varepsilon}, 1\right)\right) \geq \frac{1}{2}\log_{|\frac{1}{x}|}\left(\frac{1-\varepsilon}{\varepsilon}\right) \\ y_k = \frac{1}{1+x^{2k}} = \frac{1}{1+\frac{1}{|\frac{1}{x}|^{2k}}} > \frac{1}{1+\frac{1}{|1/x|^{\log_{|1/x|}(\frac{1-\varepsilon}{\varepsilon})}}} = \frac{1}{1+\frac{1}{\frac{1-\varepsilon}{\varepsilon}}} = 1-\varepsilon \\ \therefore\ 1-\varepsilon < y_k = \frac{1}{1+x^{2k}} < 1 < 1+\varepsilon \ \therefore\ |y_k - 1| < \varepsilon \end{cases}$

For $|x|>1$, $\forall \varepsilon>0, \exists k_0 = \frac{1}{2}\log_{|x|}\left(\max\left(\frac{1-\varepsilon}{\varepsilon}, 1\right)\right), \forall k>k_0: |y_k| < \varepsilon$

∴ $\begin{cases} \text{Case } \varepsilon \geq 1,\ k_0 = \frac{1}{2}\log_{|x|} 1 = 0 \ \therefore\ |y_k| = \left|\frac{1}{1+x^{2k}}\right| < 1 < \varepsilon \\ \text{Case } \varepsilon < 1 \ \therefore\ \text{If } k>k_0 = \frac{1}{2}\log_{|x|}\left(\max\left(\frac{1-\varepsilon}{\varepsilon}, 1\right)\right) \geq \frac{1}{2}\log_{|x|}\left(\frac{1-\varepsilon}{\varepsilon}\right) \\ y_k = \frac{1}{1+x^{2k}} < \frac{1}{1+|x|^{\log_{|x|}(\frac{1-\varepsilon}{\varepsilon})}} = \frac{1}{1+\frac{1-\varepsilon}{\varepsilon}} = \varepsilon \\ \therefore\ -\varepsilon < 0 < y_k < \varepsilon \ \therefore\ |y_k| < \varepsilon \end{cases}$

For $|x|=1$, $y_k = \frac{1}{1+x^{2k}} = \frac{1}{1+1} = \frac{1}{2}$; $\forall k \in \mathbb{N}$

$\therefore \forall x \in \mathbb{R}$ $\{y_k\}$ converges

e) From d) $\lim_{k \to \infty} f_k(x) = \begin{cases} 1 & ; |x| < 1 \\ 1/2 & ; |x| = 1 \\ 0 & ; |x| > 1 \end{cases} \neq f_0(x)$

2.68 (e). Let C_1 and C_2 are compact subsets of $\mathbb{R}^n$. Let V be an open cover of $C = C_1 \cup C_2$. Now, since $C_1 \subset C$, then V is also an open cover of C_1. Similarly, V is an open cover of C_2. By compactness of C_1, there is a finite subcollection of an open cover V that covers C_1. Let $V_1, V_2, ---, V_k$ be this finite subcover of C_1. By the same argument, there is a finite subcollection of an open cover V that covers C_2. Let $(V_n, V_{n+1}, ---, V_t)$ be this finite subcover of C_2. Then, the finite subcollection $(V_1, V_2, --- V_k, V_n, V_{n+1}, --- V_t)$ covers $C_1 \cup C_2$. Thus, $C = C_1 \cup C_2$ is compact.

(f). Let C_1 and C_2 are compact subsets of $\mathbb{R}^n$. By theorem 2.5.5, both C_1 and C_2 are closed. By theorem 2.2.3, their intersection $C = C_1 \cap C_2$ is closed. Then, C is a closed subset of a compact set C_2. Let A be an open cover of C. Then, $V = (C_2 - C) \cup A$ will be an open cover of C_2. Since C_2 is compact, there is a finite subcollection of V that covers C_2. By eliminating $(C_2 - C)$ from this finite subcollection, we get a finite subcover of A. Thus, $C = C_1 \cap C_2$ is compact.

$\boxed{2.69}$ Let X be a compact set in $\mathbb{R}^n$ and let $S \subseteq X$ be closed.

Suppose $C = \bigcup_{\alpha \in A} C_\alpha$ is an open cover of S.

Since S is closed, $S^c = X - S$ is open and then $S^c \cup C$ is an open cover of X. Since X is compact, there exists a finite sub cover of the open cover $S^c \cup C$ that covers X. So let $S^c \cup C_1 \cup \cdots \cup C_k$ be this finite subcover. Then, this finite subcover also covers S and since $S \cap S^c = \phi$, then $C_1 \cup \cdots \cup C_k$ will still cover S after removing S^c. So there is a finite subcover that covers S and hence, it is compact.

$\boxed{2.72}$ Let C_1 and C_2 be two disjoint compact subsets of $\mathbb{R}^n$.

Then, for every $\vec{x} \in C_1$, let $N(\vec{x}; \varepsilon_\alpha)$ be a neighborhood that does not intersect C_2.

similarly, for every $\vec{y} \in C_2$, let $N(\vec{y}; \varepsilon_\beta)$ be a neighborhood that doesn't intersect C_1.

Now, let $N_1 = \bigcup_{\alpha \in A} N(\vec{x}; \frac{\varepsilon_\alpha}{2})$ and

$$N_2 = \bigcup_{\beta \in B} N(\vec{y}; \frac{\varepsilon_\beta}{2})$$

Then, N_1 and N_2 are open sets such that $C_1 \subseteq N_1$ and $C_2 \subseteq N_2$.

To prove that $N_1 \cap N_2 = \phi$, let $\vec{v} \in N_1 \cap N_2$.
Then, $\vec{v} \in N(\vec{x}; \frac{\varepsilon_\alpha}{2})$ and $\vec{v} \in N(\vec{y}; \frac{\varepsilon_\beta}{2})$ for some $\alpha \in A$ and $\beta \in B$.

$$\Rightarrow \|\vec{x} - \vec{v}\| < \frac{\varepsilon_\alpha}{2} \quad \text{and} \quad \|\vec{y} - \vec{v}\| < \frac{\varepsilon_\beta}{2}.$$

However,

$$\|\vec{x} - \vec{y}\| = \|\vec{x} - \vec{v} + \vec{v} - \vec{y}\|$$
$$\leq \|\vec{x} - \vec{v}\| + \|\vec{v} - \vec{y}\| < \frac{\varepsilon_\alpha}{2} + \frac{\varepsilon_\beta}{2}.$$

If $\varepsilon_\alpha \leq \varepsilon_\beta$, then $\|\vec{x} - \vec{y}\| < \frac{\varepsilon_\beta}{2} + \frac{\varepsilon_\beta}{2} = \varepsilon_\beta$

and then, $\vec{x} \in N(\vec{y}; \varepsilon_\beta)$.

$\boxed{2.72}$ continued

similarly, if $\varepsilon_\beta \le \varepsilon_\alpha$, $\|\vec{x} - \vec{y}\| < \frac{\varepsilon_\alpha}{2} + \frac{\varepsilon_\alpha}{2} = \varepsilon_\alpha$.

Then, $\vec{y} \in N(\vec{x}; \varepsilon_\alpha)$.

But from our assumption, $N(\vec{x}; \varepsilon_\alpha)$ is a neighborhood of $\vec{x}$ that doesn't intersect C_2 and thus, it can't contain $\vec{y} \in C_2$. By the same argument $N(\vec{y}; \varepsilon_\beta)$ cannot contain $\vec{x} \in C_1$.

Thus, we have a contradiction. This proves that $\vec{v} \notin (N_1 \cap N_2)$. $\Rightarrow N_1 \cap N_2 = \phi$.

Now, for every $\vec{x} \in C_1$, let U_α and V_α be disjoint open sets such that $\vec{x} \in U_\alpha$ and $C_2 \subset V_\alpha$.

Then, the collection of open sets $\{U_\alpha\}$ covers C_1 and by compactness of C_1, there is a finite subcollection $U_{\alpha_1}, ---, U_{\alpha_k}$ that covers C_1.

on the other hand, let $V = V_{\alpha_1} \cap V_{\alpha_2} \cap --- \cap V_{\alpha_j}$.

Then, $U = U_{\alpha_1} \cup U_{\alpha_2} \cup --- \cup U_{\alpha_k}$ and V are disjoint. Therefore, U and V are disjoint open sets such that $C_1 \subseteq U$ and $C_2 \subseteq V$.

3.12)

a) Replace x by $x+c$

$$\Rightarrow \lim_{x\to 0} \frac{\frac{\sin(x-c)}{x-c} \to 1}{x+2c \to 2c} = \frac{1}{2c}$$

if $c \neq 0 \rightarrow$

if $c=0$ $\quad \lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x \to 1}{2x \to 0}$

doesn't exist

b) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} = \frac{\sin x}{2x} = \frac{1}{2}$

c) $\lim_{x\to 1} \frac{\sqrt{1+x^2}-\sqrt{1-x}}{x} = \frac{\sqrt{2}-0}{1} = \sqrt{2}$

d) since $\lim_{x\to 0^-} (x+\operatorname{sgn}(x)) = -1$

and

$\lim_{x\to 0^+} (x+\operatorname{sgn}(x)) = +1$

limit at $x=0$ doesn't exist.

3.21) a) Let $\varepsilon > 0$ be given. $\delta = \frac{\varepsilon^{\frac{1}{k}}}{K}$

Then, if $\|x-c\| < \frac{\varepsilon^{\frac{1}{k}}}{K} \Rightarrow |f(x)-f(c)| \leq K \cdot \|x-c\|^k <$

$< \varepsilon \Rightarrow f$ is continuous at c.

b) $k=1 \Leftrightarrow |f(x)-f(c)| \leq K|x-c|$

$\Leftrightarrow$

$-K|x-c|+f(c) \leq f(x) \leq K|x-c|+f(c)$ ✓

c) $k=2, c=0$ $\quad f(x) = x^2 + \overset{=a}{f(0)}$

$\Rightarrow |f(x)-f(0)| = x^2 \leq 1 \cdot |x|^2$

$k=1/2, c=1$ $\quad f(x) = \sqrt{|x-1|} + \overset{=a}{f(1)}$

$\Rightarrow |f(x)-f(1)| = \sqrt{|x-1|} \leq 1 \cdot |x-1|^{\frac{1}{2}}$

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13) $f(x) = \begin{cases} 0, & \text{if } x\text{-irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest terms with } q > 0 \end{cases}$

Assume there exist a rational point $c = \frac{m}{n}$ in lowest terms with $n > 0$ (If $c < 0$, put minus sign on m) such that f is continuous at point c. Therefore, $\lim_{x \to c} f(x) = f(c) = f\left(\frac{m}{n}\right) = \frac{1}{n}$, meaning that

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\left|f(x) - \frac{1}{n}\right| < \varepsilon$ whenever $|x - c| < \delta$

Take any $0 < \varepsilon < \frac{1}{n}$, and for those chosen values ε, there would exist corresponding values δ. Since we know there exist an irrational number between any 2 real numbers, it implies $\exists x_0$-irrational for which $c - \delta = \frac{m}{n} - \delta < x_0 < c + \delta = \frac{m}{n} + \delta$ or $|x_0 - c| < \delta$ becomes true $\Rightarrow$ since x_0-irrational, $\left|f(x_0) - \frac{1}{n}\right| = \frac{1}{|n|} = \frac{1}{n} < \varepsilon$ which is $\boxed{X}$ Hence, $\boxed{f \text{ is discontinuous at every rational point}}$ ✓

Claim: f is periodic with period $1 \leadsto f(x+n)=f(x)$ for all $n\in\mathbb{Z}$ and $x\in\mathbb{R}$

Proof: $\forall x\in\mathbb{R}\setminus\mathbb{Q} \Rightarrow x+n\in\mathbb{R}\setminus\mathbb{Q}$, so $f(x+n)=f(x)=0$

$\forall x\in\mathbb{Q}$, $\exists p\in\mathbb{Z}$ and $\exists q\in\mathbb{N}$ s.t. $x=\frac{p}{q}$ with $(p,q)=1$ $\boxed{\checkmark}$

$x+n=\frac{p+qn}{q}$ where $(p+qn,q)=(q,p)=1 \Rightarrow f(x+n)=\frac{1}{q}=f(x)$

- Since f is periodic with period $=1$ and $0\in\mathbb{Q}$, it suffices to check all irrational points in $I=(0,1)$

Let $\varepsilon>0$, $i\in\mathbb{N}$, $x_0\in I\setminus\mathbb{Q} \Rightarrow$ From archimedes, $\exists r\in\mathbb{N}$, $\frac{1}{r}<\varepsilon$, and $\exists k_i\in\mathbb{N} \Rightarrow 0<\frac{k_i}{i}<x_0<\frac{k_i+1}{i}$ for $i=1,\dots,r$

Minimal distance of x_0 to its i^{th} lower and upper bounds equals $\Rightarrow d_i=\min\left\{\left|x_0-\frac{k_i}{i}\right|,\left|x_0-\frac{k_i+1}{i}\right|\right\}$

Assume δ to be the minimum of all finite d_i, $\delta=\min\limits_{1\le i\le r}\{d_i\}$ so that $\left|x_0-\frac{k_i}{i}\right|\geq\delta$ and $\left|x_0-\frac{k_i+1}{i}\right|\geq\delta$

for all $i=1,2,\dots,r \Rightarrow$ All these rational numbers $\frac{k_i}{i}$, $\frac{k_i+1}{i}$ are outside the δ-neighbd of x_0. If we let $x\in\mathbb{Q}\cap(x_0-\delta,x_0+\delta)$ with the unique representation $x=\frac{p}{q} \Rightarrow$ Then, necessarily $\boxed{q>r}$

and thus, $f(x)=\frac{1}{q}<\frac{1}{r}<\varepsilon$

Similarly, $\forall x$-irrational $\in I$, $f(x)=0=f(x_0)$, and thus if $\varepsilon>0$, then any choice of sufficiently small $\delta>0$ gives $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|=|f(x)|<\varepsilon$

$\boxed{f\text{-continuous on } \mathbb{R}\setminus\mathbb{Q}}$

20) f-real-valued function defined on all of $\mathbb{R}$

$f(x+y)=f(x)f(y)$ for $\forall x,y\in\mathbb{R}$

a) Plugging $x=y=\frac{z}{2}$ where z is an arbitrary real number $\Rightarrow f(z)=f^2\left(\frac{z}{2}\right)\geq 0$ or just $\boxed{f(z)\geq 0,\ \forall z\in\mathbb{R}}$

b) Suppose there exist $x_0\in\mathbb{R}$ such that $f(x_0)=0$

plugging $x=z-x_0$, $y=x_0$ where z-arbitrary real number

$f(z)=f(z-x_0)f(x_0)=0 \Rightarrow \boxed{f(z)=0,\ \forall z\in\mathbb{R}}$

c) Assume there exist $y_0\in\mathbb{R}$ such that $f(y_0)\neq 0$

plugging $x=y_0$, $y=0 \Rightarrow f(y_0)=f(y_0)f(0)$ or just having $f(y_0)(f(0)-1)=0$. Since $f(y_0)\neq 0 \Rightarrow \boxed{f(0)=1}$

d) f-continuous at $x=0 \Rightarrow \lim_{x\to 0} f(x)=f(0)$

If there would exist $a\in\mathbb{R}$ such that $f(a)=0$, then

From part b), we could imply $f(z)=0$ for any $z \in \mathbb{R}$ and this concludes the desired property for part d)

Now, assume there does not exist $a \in \mathbb{R}$ s.t. $f(a)=0$ $\exists b \in \mathbb{R} \Rightarrow f(b) \neq 0$. Hence, from previous $\Rightarrow$ $\boxed{f(0)=1}$ (part c)

$\lim_{x \to 0} f(x)=1 \iff \forall \varepsilon>0, \exists \delta>0, |f(x)-1|<\varepsilon$ whenever $|x|<\delta$

Fix any $c \in \mathbb{R}$, and choosing any $\varepsilon>0$, we know that $\dfrac{\varepsilon}{|f(c)|}>0$ with $f(c) \neq 0$ (otherwise, f would be identically zero on $\mathbb{R}$)

$\forall \varepsilon>0, \exists \delta_1>0 \Rightarrow \left|f(y)-1\right|<\dfrac{\varepsilon}{|f(c)|}$ whenever $|y|<\delta_1$

If we substitute those values y with $y=x-c$, where $x=y+c$ is a real number $\Rightarrow |f(x-c)-1|<\dfrac{\varepsilon}{|f(c)|}$ whenever $|x-c|<\delta_1$ for real number x. Setting $\begin{matrix} x \to x-c \\ y \to c \end{matrix}$ in the original equation, $\underline{f(x)=f(x-c)\,f(c)}$ or just

$|(f(x-c)-1)f(c)|=|f(x-c)f(c)-f(c)|=|f(x)-f(c)|$
$<\varepsilon$ whenever $|x-c|<\delta_1$. Hence, $\underline{\forall \varepsilon>0, \exists \delta_\varepsilon>0,}$ $\underline{\text{s.t. } |f(x)-f(c)|<\varepsilon \text{ whenever } |x-c|<\delta_\varepsilon}$, concluding that $\boxed{\lim_{x \to c} f(x)=f(c) \text{ for } \forall c \in \mathbb{R}}$ ✓ $\boxed{f \text{ is continuous on all of } \mathbb{R}}$ ✓

e) f-continuous and nonzero $\Rightarrow f(x)=a^x$, with $a=f(1)$

Let $g(x)=\log f(x)$ where $f(x)$-nonzero. Indeed, we proved that if $f(a)=0 \Rightarrow f$ is identically zero. Thus, $f(x)>0$ and $\boxed{g(x)\text{-continuous}}$ since $f(x)$-continuous ($g(x)$ is defined funct)

$f(x+y)=e^{g(x+y)}=f(x)f(y)=e^{g(x)}e^{g(y)} \Rightarrow \boxed{g(x+y)=g(x)+g(y)}$

$x=y=0 \Rightarrow g(0)=2g(0)$, $\boxed{g(0)=0}$ Let $q>0$ be a rational number

By repeated application of Cauchy's equation to $g(x+x+\dots+x)=g(\alpha x)$, we get $g(\alpha x)=\alpha g(x)$, $\alpha\in N$, $x\in Q$ (*)

$x\to\frac{x}{\alpha}$ and multiplying by $\frac{\beta}{\alpha} \Rightarrow \frac{\beta}{\alpha}g(x)=\beta g\left(\frac{x}{\alpha}\right)$ (**)

with $\beta\in N$. Since $g\left(\frac{\beta}{\alpha}x\right)=\frac{\beta}{\alpha}g(x)=\beta g\left(\frac{x}{\alpha}\right)$

Application of (*) to the LHS of (**) then gives

$g\left(\frac{\beta}{\alpha}x\right)=\frac{\beta}{\alpha}g(x)$ or just $g(qx)=qg(x)$, $q>0$, $q,x\in Q$

Hence, $g(q)=q\,g(1)=cq$ where $q\in Q^+$

$y=-x \Rightarrow f(-x)=-f(x)$ with $g(q)=-q\,g(1)$ for $q\in Q^-$

In conclusion, we get $\boxed{g(x)=g(1)x \text{ for any } x\in Q \text{ where } x\in Q,}$

Now, we show any solution must have the property that its graph $y=g(x)$ is dense in $\mathbb{R}^2$. Meaning that any disk in the plane (small) contains a point from the graph. From this, it's easy to prove the various conditions such as continuity

Suppose $\not\exists$ Pog $\Rightarrow g(q)=q, \forall q\in\mathbb{Q}$ and $g(\alpha)\neq\alpha$ for some $\alpha\in\mathbb{R}$

Let $g(\alpha)=\alpha+\delta$, with $\delta\neq 0$

We now show how to find a point in an arbitrary circle (center $=(x,y)$), radius $=r$ where $x,y,r\in\mathbb{Q}, r>0, x\neq y$

Put $\beta=\dfrac{y-x}{\delta}$ and choose rational $b\neq 0$ close to β with

$|\beta-b|<\dfrac{r}{2|\delta|}\Rightarrow$ Choose a rational a close to α with

$|\alpha-a|<\dfrac{r}{2|b|}\Rightarrow$ Now, put $\boxed{\begin{matrix}X=x+b(\alpha-a)\\ Y=g(X)\end{matrix}}$ Then, using the functional equation,

$Y=g(x+b(\alpha-a))=x+bg(\alpha)-bg(a)=y-\delta\beta+$
$+bg(\alpha)-bg(a)=y-\delta\beta+b(\alpha+\delta)-ba=y+b(\alpha-a)$
$+-\delta(\beta-b)=\underline{y+b(\alpha-a)-\delta(\beta-b)}$. Because of our choices, (above), the point (X,Y) is inside the circle ✓. Hence, $\boxed{g\text{-continuous}\Rightarrow g(x)=cx, \text{ where } c=f(1)}$

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Since $f(x)=e^{g(x)} \Rightarrow f(x)=e^{cx}=(e^c)^x$ where

$x=1 \Rightarrow e^c = f(1)$ or just $\boxed{f(x)=(f(1))^x}$ $\boxed{*}$ $\frac{V}{+}$