1 (4 points) We say that F is Lipschitz continuous on $[a, b]$ when there exists some constant $C > 0$ such that all $x, y \in [a, b]$ satisfies

$$|F(x) - F(y)| \leq C|x - y|.$$

Let F be in $BV(a, b)$.

(a) Prove that if V_F is Lipschitz continuous in $[a, b]$, then F is also Lipschitz continuous on $[a, b]$.

(b) Find an example of F so that V_F is differentiable at some $c \in (a, b)$ but F is not differentiable at c.

Solution. (a) Suppose V_F is Lipschitz continuous on $[a, b]$ with the constant C. Let $x, y \in [a, b]$. Without loss of generality, we assume $x > y$. Then, choosing the trivial partition $\{y, x\}$,

$$|F(x) - F(y)| \leq V(F; y, x) = V_F(x) - V_F(y) \leq C|x - y|.$$ **(+2 points)**

(b) Let $F(x) = |x|$ be defined for $x \in [-1, 1]$. Then, $V_F(x) = x + 1$ is differentiable at $x = 0$ but F is not. **(+2 points)**

2 Set $I_n = \left[0, \frac{1}{n}\right]$. Define a function $f : [0,1] \to \mathbb{R}$ by

6 points

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad \text{where } f_n(x) = \begin{cases} \dfrac{x^2}{2^{n-1}} & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases}$$

(a) Find the values of $h(x) = f(x) - \lim\limits_{y \to x^+} f(y)$ for all $x \in [0,1)$.

(b) Prove or disprove that f is of bounded variation.
(Hint: Is there a function g such that g and $f+g$ are monotone increasing?)

Solution. Let $J_n = I_n \backslash I_{n+1} = \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Then, $f(0) = 0$, and

$$f(x) = \sum_{k=1}^{n} \frac{x^2}{2^{k-1}} = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} x^2 = (2 - 2^{1-n})x^2$$

whenever $x \in J_n$ for some $n \in \mathbb{N}$. **(+1 points)**

(a) f is continuous in each J_n. Also, $0 \le f(x) \le 2x^2$, so f is continuous at 0. Thus, we have

$$h(x) = 0 \quad \text{if } x \notin \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\}. \text{ **(+1 points)**}$$

In addition, for each $n \in \mathbb{N}$,

$$h\left(\frac{1}{n+1}\right) = f\left(\frac{1}{n+1}\right) - \lim_{x \in J_n,\, x \to \frac{1}{n+1}} f(x) = \frac{1}{(n+1)^2 \times 2^n}. \text{ **(+1 points)**}$$

(b) $f(x)$ is monotone increasing at $x = 0$ (because $f(x) \ge 0$) and in each J_n. More importantly, if we define

$$g(x) = \begin{cases} 0 & \text{when } x \in J_1, \\ -\displaystyle\sum_{k=2}^{n} h\left(\frac{1}{k}\right) & \text{when } x \in J_n \text{ for some } n \ge 2, \\ -100 & \text{if } x = 0, \end{cases}$$

then g is monotone increasing, and $f+g$ is monotone increasing at $x = \frac{1}{n+1}$ for each $n \in \mathbb{N}$. **(+2 points)** Therefore, g and $f+g$ are monotone increasing, so f is of bounded variation by Theorem 5.4.4. **(+1 points)**

- The function g can be chosen differently. For example, -100 in the solution can be replaced by any number less than or equal to $-\sum_{k=2}^{\infty} h(1/k)$. Or, you can just use the countably infinite sum $g(x) = \sum_{y<x} h(y)$.