Final Exam (MAS473)

Introduction to Artificial Intelligence with Mathematics

December 16, 2021: 9:00 a.m. $\sim$ 11:30 a.m.

1. (30 pts) Let $\mathbf{x}_i, 1 \le i \le n$, be an N dimensional input (column) vectors and $\mathbf{y}_i = \mathbf{W}\mathbf{x}_i$ where $\mathbf{W}$ is an $M \times N$ matrix for $M \le N$. To reconstruct $\mathbf{x}_i$ from $\mathbf{y}_i$, we consider the following problem:

$$\mathbf{x}_{i,E} = \mathbf{U}\mathbf{y}_i$$

where $\mathbf{U}$ is an $N \times M$ matrix.

(a) (10 pts) For two column vectors $\mathbf{a}$ and $\mathbf{b}$ and a matrix $\mathbf{B}$, derive the following.

$$\frac{\partial}{\partial \mathbf{B}}\mathbf{a}^\top \mathbf{B}\mathbf{b} := \left(\frac{\partial}{\partial B_{ij}}\mathbf{a}^\top \mathbf{B}\mathbf{b}\right), \quad \frac{\partial}{\partial \mathbf{B}}\mathbf{a}^\top \mathbf{B}^\top \mathbf{B}\mathbf{a} := \left(\frac{\partial}{\partial B_{ij}}\mathbf{a}^\top \mathbf{B}^\top \mathbf{B}\mathbf{a}\right)$$

where B_{ij} denotes the (i,j)-th element of $\mathbf{B}$.

(b) (10 pts) We assume that matrix $\mathbf{W}$ is fixed in (b). In this case, find the matrix $\mathbf{U}$ that minimizes $\mathcal{L} = \frac{1}{n}\sum_{i=1}^n ||\mathbf{x}_i - \mathbf{x}_{i,E}||^2$ by using the results in (a) where $||\cdot||$ denotes the Euclidean distance.

(c) (10 pts) Next, to find the optimal matrix $\mathbf{W}$ that minimizes $\mathcal{L}$ we assume that all rows of $\mathbf{W}$ are the transposed (orthonormal) eigenvectors of the covariance matrix $\mathbf{S} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^\top$, i.e.,

$$\mathbf{S}\mathbf{u}_j = \lambda_j \mathbf{u}_j, 1 \le j \le M, \quad \mathbf{u}_j^\top \mathbf{u}_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases}, 1 \le j, k \le M, \quad \mathbf{W} = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_M^\top \end{bmatrix}.$$

Using the result in (b), show that the eigenvectors in $\mathbf{W}$ should be those corresponding to the M largest eigenvalues of $\mathbf{S}$.

2. (30 pts) Consider a two-state Markov Decision Process. We assume stationary rewards and stationary transition probabilities. Let $\{s_1, s_2\}$ be the state space.

- In state s_1, we can choose one of two actions $a_{1,1}$ and $a_{1,2}$. When action $a_{1,1}$ is selected, we get a reward of 5 and the next state is s_1 with probability $1/2$ and s_2 with probability $1/2$. When action $a_{1,2}$ is selected, we get a reward of 10 and the next state is s_2 with probability 1.
- In state s_2, we have only one action a_2. When action a_2 is selected, we get a reward of -1 and the next state is s_2.

(a) (10 pts) Consider a decision rule d with $d(s_1) = a_{1,1}$ and $d(s_2) = a_2$ and a stationary policy $\pi_1 = d^\infty$. Compute the value function $v(s_1)$ and $v(s_2)$ for the expected total discounted reward of policy π_1 where λ is a discount factor.

(b) (10 pts) Let $\lambda = 0.9$. Find the decision rule d^* that is conserving and its corresponding value function $v_{0.9}^*(s_1)$ and $v_{0.9}^*(s_2)$.

(c) (10 pts) We want to characterize the optimal policy as a function of λ ($0 < \lambda < 1$). To this end, consider two decision rules d and δ where $\delta(s_1) = a_{1,2}$ and $\delta(s_2) = a_2$, and two stationary policies $\pi_1 = d^\infty$ and $\pi_2 = \delta^\infty$. Discuss which policy is optimal depending on the value of λ.

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