Reinforcement Learning

Introduction to Artificial Intelligence with Mathematics
Lecture Notes

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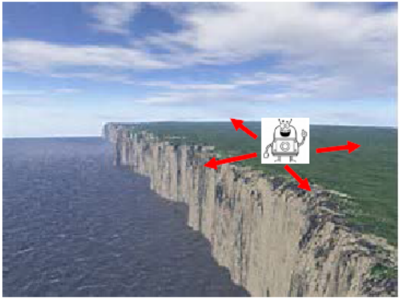
Introduction

Reinforcement Learning (RL) is concerned with how software agents ought to take actions in an environment so as to maximize some notion of cumulative reward.

(positive, negative)



Robot → agent

understanding environment

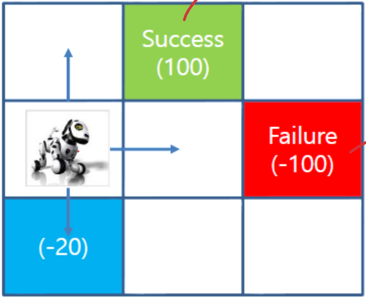
(some trials)

Figure: Agent and Environment

- Markov Decision Process

Example

(1,2) destination state



(2,3)

Figure: example

Robot moves to right ⇝ state (2,2)

- The environment or state $\{(1,1),(1,2),\cdots,(3,3)\}$ is (stochastically) changed according to the action that the agent takes
- The set of actions {up, down, left, right}
- The set of rewards $\{-100,-20,0,100\}$

- Agent - Environment Interaction

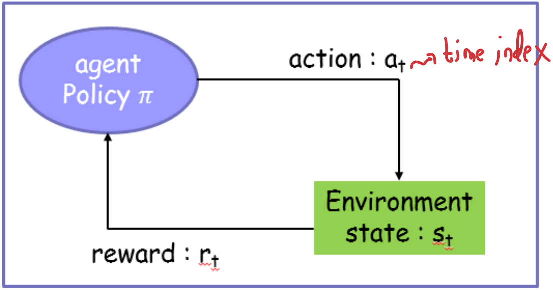


Figure: Markov Decision Process

- The agent tries to maximize the cumulative reward
- Reinforcement learning (RL) is learning from interaction
    - approach to sequential decision making

Overview of Markov Decision Process

- Markov Decision Process (MDP) consists of
  - a set of decision epochs or stages t time index of process
    - finite/infinite horizon, discrete/continuous
  - state space S
    - discrete: finite/countable, continuous
    - s_t: the state at stage t
  - action set A → all possible actions
  - (stationary) transition probability matrix $p(j|s, a)$ → of next stage j
    - $p : S \times A \rightarrow S$ → invariant
  - (stationary and bounded) reward function
    - $R : S \times A \rightarrow \mathbb{R}$
    - $r(s, a)$: the reward when the state is s and the action a is taken
  - policy π → set of actions
    - $\pi : S \rightarrow A$
    - $\pi(s)$: the action taken at state s
  - MDP is sometimes called a controlled Markov chain.

- at stage t
  - observe the state s_t
  - select an action based on the state s_t
  - obtain the resulting reward $r_t(s, a)$
- Influence diagram of MDP

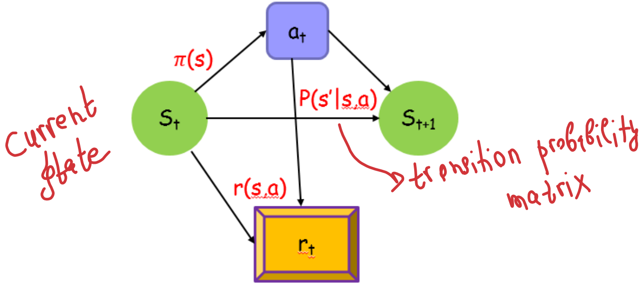


Figure: Influence diagram of MDP

Basic Assumptions invariants in time

1. stationary rewards $r(s, a)$ and transition probabilities $p(j|s, a)$
2. bounded rewards, i.e., $|r(s, a)| \leq M < \infty$ for all $a \in A$ and $s \in S$
3. discounting with λ $(0 \leq \lambda < 1)$ discounting vector
4. discrete state spaces S which is finite or countable

Remarks on policies

A decision rule d_t: $S \to A$ at stage t.

A policy π specifies the decision rules to be used at all decision epochs.

$$\pi = (d_1, d_2, \cdots).$$

There are different types of decision rules.

- Markovian and randomized (MR) $d_t : S \to \mathcal{P}(A)$
- Markovian and deterministic (MD) $d_t : S \to A$

We call a policy *stationary* if it uses the same decision rule for all stages, i.e.,

$$\pi = (d, d, d, \cdots).$$

Basic Setting

Let

- X_t: the state at stage t
- Y_t: the action taken at stage t

For a Markovian and deterministic decision rule d_t at stage t

$$Y_t = d_t(X_t) \text{ for } d_t \in D^{MD}$$

For a Markovian and randomized decision rule d_t,

$$P\{Y_t = a\} = q_{d_t(X_t)}(a) \text{ for } d_t \in D^{MR}$$

where $q_{d_t(s)}(\cdot)$ is a probability distribution for the action taken at state s.

The values of a policy in infinite horizon models

- The expected total reward of a policy π

$$v^{\pi}(s) = \lim_{N \to \infty} E_s^{\pi} \left[\sum_{t=1}^{N} r^{\pi}(X_t, Y_t) \right], \; s_1 = s \in S$$

- The expected total discounted reward of a policy π

$$v_{\lambda}^{\pi}(s) = \lim_{N \to \infty} E_s^{\pi} \left[\sum_{t=1}^{N} \lambda^{t-1} r^{\pi}(X_t, Y_t) \right], \; s_1 = s \in S$$

- The average reward or gain of a policy π

$$v^{\pi}(s) = \lim_{N \to \infty} \frac{1}{N} E_s^{\pi} \left[\sum_{t=1}^{N} r^{\pi}(X_t, Y_t) \right], \; s_1 = s \in S$$

The objective of the expected total discounted reward

- The objective of the expected total discounted reward

Find an optimal policy π^* that maximizes the expected total discounted reward

$$v_\lambda^\pi(s) = E^\pi\left[\sum_{t=1}^{\infty} \lambda^{t-1} r(X_t, Y_t)\right], \; s_1 = s \in S$$

(called the value function of a policy π). Here, λ is a discounted factor in $[0, 1)$.

Remarks:

- If the rewards are stochastic, the expectation of each reward is considered.
- Why a discount factor?
  - future rewards are not worth as much as current reward
  - there is a possibility that the process is terminated with probability $1-\lambda$
  - mathematically tractable

Recall that

$$v_\lambda^\pi(s) = E^\pi \left[\sum_{t=1}^{\infty} \lambda^{t-1} r(X_t, Y_t) \right], \; s_1 = s \in S$$

Which is given by

$$v_\lambda^\pi(s) \quad = \quad \sum_{t=1}^{\infty} \sum_{j \in S} \sum_{a \in A} \lambda^{t-1} r(j, a) P^\pi \{X_t = j, Y_t = a | X_1 = s\},$$

Policy Evaluation

For $d \in D^{MD}$, define

$$r_d(s) = r(s, d(s)), p_d(j|s) = p(j|s, d(s)).$$

For $d \in D^{MR}$, define

$$r_d(s) = \sum_{a \in A} q_{d(s)}(a) r(s, a), p_d(j|s) = \sum_{a \in A} q_{d(s)}(a) p(j|s, a).$$

r_d and P_d are the corresponding vector and matrix to $r_d(s)$ and $p_d(j|s)$, respectively.

Let $\pi = (d_1, d_2, d_3, \cdots) \in \Pi^{MR}$, $P_\pi^0 = I$, and $P_\pi^{(t)}$ be a matrix whose (s, j)-component is given by

$$P_\pi^{(t)}(j|s) = P^\pi\{X_{t+1} = j | X_1 = s\}.$$

Then $P_\pi^{(t)} = P_{d_1} P_{d_2} \cdots P_{d_t}$.

For instance,

$$\begin{aligned}
(P_\pi^{(1)})_{sj} &= P^\pi\{X_2 = j | X_1 = s\} \\
&= \sum_{a \in A} P^\pi\{X_2 = j, d_1(s) = a | X_1 = s\} \\
&= \sum_{a \in A} P^\pi\{X_2 = j | d_1(s) = a, X_1 = s\} P^\pi\{d_1(s) = a | X_1 = s\} \\
&= \sum_{a \in A} q_{d_1(s)}(a) p(j|s, a) \\
&= (P_{d_1})_{sj} \text{ (the } (s, j)\text{-th element of } P_{d_1})
\end{aligned}$$

$$
\begin{aligned}
(P_\pi^{(2)})_{sj} &= P^\pi\{X_3 = j | X_1 = s\} \\
&= \sum_{a\in A} P^\pi\{X_3 = j, d_1(s) = a | X_1 = s\} \\
&= \sum_{a\in A} P^\pi\{X_3 = j | d_1(s) = a, X_1 = s\} P^\pi\{d_1(s) = a | X_1 = s\} \\
&= \sum_{a\in A} q_{d_1(s)}(a) \sum_{i\in S}\sum_{b\in A} P^\pi\{X_3 = j, X_2 = i, d_2(i) = b | d_1(s) = a, X_1 = s \\
&= \sum_{a\in A} q_{d_1(s)}(a) \sum_{i\in S}\sum_{b\in A} P^\pi\{X_3 = j | X_2 = i, d_2(i) = b, X_1 = s, d_1(s) = a \\
&\qquad\qquad\qquad\qquad P^\pi\{X_2 = i, d_2(i) = b | d_1(s) = a, X_1 = s\} \\
&= \sum_{a\in A} q_{d_1(s)}(a) \sum_{i\in S}\sum_{b\in A} P^\pi\{X_3 = j | X_2 = i, d_2(i) = b\} \\
&\quad P^\pi\{d_2(i) = b | X_2 = i, d_1(s) = a, X_1 = s\} P^\pi\{X_2 = i | d_1(s) = a, X_1 = \\
&= \sum_{i\in S}\sum_{a\in A} q_{d_1(s)}(a) p(i|s,a) \sum_{b\in A} q_{d_2(i)}(b) p(j|i,b) \\
&= \sum_{i\in S} (P_{d_1})_{si}(P_{d_2})_{ij} = (P_{d_1}P_{d_2})_{sj}
\end{aligned}
$$

Note that the value function v_λ^π of a policy π is given as follows:

$$\begin{aligned}
v_\lambda^\pi(s) &= \sum_{t=1}^{\infty}\sum_{j\in S}\sum_{a\in A}\lambda^{t-1}r(j,a)P^\pi\{X_t=j,Y_t=a|X_1=s\}\\
&= \sum_{t=1}^{\infty}\sum_{j\in S}\sum_{a\in A}\lambda^{t-1}r(j,a)P^\pi\{Y_t=a|X_t=j,X_1=s\}\\
&\qquad\qquad P^\pi\{X_t=j|X_1=s\}\\
&= \sum_{t=1}^{\infty}\sum_{j\in S}\sum_{a\in A}\lambda^{t-1}r(j,a)q_{d_t(j)}(a)P^\pi\{X_t=j|X_1=s\}\\
&= \sum_{t=1}^{\infty}\sum_{j\in S}\lambda^{t-1}\sum_{a\in A}r(j,a)q_{d_t(j)}(a)P^\pi\{X_t=j|X_1=s\}\\
&= \sum_{t=1}^{\infty}\sum_{j\in S}\lambda^{t-1}r_{d_t}(j)P^\pi\{X_t=j|X_1=s\}\\
&= \sum_{t=1}^{\infty}\lambda^{t-1}(P_\pi^{(t-1)}r_{d_t})_s.
\end{aligned}$$

It then follows that

$$
\begin{aligned}
v_\lambda^\pi &= \sum_{t=1}^{\infty} \lambda^{t-1} P_\pi^{(t-1)} r_{d_t} \\
&= r_{d_1} + \lambda P_{d_1} r_{d_2} + \lambda^2 P_{d_1} P_{d_2} r_{d_3} + \cdots \\
&= r_{d_1} + \lambda P_{d_1} (r_{d_2} + \lambda P_{d_2} r_{d_3} + \cdots) \\
&= r_{d_1} + \lambda P_{d_1} v_\lambda^{\pi'}
\end{aligned}
$$

where $\pi' = (d_2, d_3, \cdots)$.

The value function for a stationary policy

When $\pi = d^\infty = (d, d, d, \cdots)$, i.e., π is stationary,

$$
v_\lambda^{d^\infty} = r_d + \lambda P_d v_\lambda^{d^\infty}.
$$

Bellman Equations (Optimality Equations)

Let V denote the set of bounded real valued functions on S and we use the supnorm on V. The corresponding matrix norm is

$$\|M\| = \sup_{s \in S} \sum_{j \in S} |M(j|s)|$$

Note that, for all $v \in V$ and $d \in D^{MR}$,

$$r_d + \lambda P_d v \in V.$$

For $v \in V$, define a linear transformation L_d by

$$L_d v = r_d + \lambda P_d v.$$

Then we know that $L_d : V \to V$ and $v_\lambda^{d^\infty}$ is a fixed point of L_d in V.

Theorem 1

For any stationary policy $\pi = d^\infty$ with $d \in D^{MR}$, $v_\lambda^{d^\infty}$ is the unique solution in V of

$$v = L_d v.$$

Moreover, $v_\lambda^{d^\infty}$ is written as

$$v_\lambda^{d^\infty} = (I - \lambda P_d)^{-1} r_d$$

where $(I - M)^{-1} = \sum_{n=0}^{\infty} M^n$.

Proof: We use the following corollary.

Corollary 2

Let Q be a bounded linear transformation on a Banach Space V, and suppose that the spectral radius satisfies $\rho(Q) < 1$. Then $(I - Q)^{-1}$ exists and satisfies

$$(I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n$$

Since $\|P_d\| = 1$ and $\lambda = \|\lambda P_d\| \geq \rho(\lambda P_d)$, $(I - \lambda P_d)^{-1}$ exists. From $v = L_d v = r_d + \lambda P_d v$, we obtain

$$v = (I - \lambda P_d)^{-1} r_d = \sum_{t=1}^{\infty} \lambda^{t-1} P_d^{t-1} r_d = v_\lambda^{d^\infty}.$$

We now consider the following system of equations, called the optimality equations or Bellman equations.

$$v(s) = \sup_{a \in A} \left\{ r(s,a) + \lambda \sum_{j \in S} p(j|s,a) v(j) \right\}$$

For $v \in V$, define an operator $\mathcal{L}$ on V by

$$\mathcal{L}v = \sup_{d \in D^{MR}} \{ r_d + \lambda P_d v \}.$$

When the supremum is attained for all $v \in V$, we define L by

$$Lv = \max_{d \in D^{MR}} \{ r_d + \lambda P_d v \}.$$

The Bellman equations are given by

$$\mathcal{L}v = v \text{ or } Lv = v.$$

Proposition 3

For all $v \in V$*,*

$$\sup_{d \in D^{MR}} \{r_d + \lambda P_d v\} = \sup_{d \in D^{MD}} \{r_d + \lambda P_d v\}$$

Proof: Since $D^{MD} \subset D^{MR}$, $\geq$ is trivial. To prove the $\leq$ part, for $v \in V$, $d \in D^{MR}$ and observe that, for all $s \in S$

$$\begin{aligned} &\sup_{a \in A} \left\{ r(s,a) + \lambda \sum_{j \in S} p(j|s,a) v(j) \right\} \\ &\geq \sum_{a \in A} q_{d(s)}(a) \left\{ r(s,a) + \lambda \sum_{j \in S} p(j|s,a) v(j) \right\}. \end{aligned}$$

Let

$$v_\lambda^* = \sup_{\pi \in \Pi^{MR}} v_\lambda^\pi.$$

Theorem 4

Suppose that there exists a $v \in V$ *for which*

- $v \geq \mathcal{L}v$, *then* $v \geq v_\lambda^*$.
- $v \leq \mathcal{L}v$, *then* $v \leq v_\lambda^*$.
- $v = \mathcal{L}v$, *then* $v = v_\lambda^*$.

Proof:

1. Choose $\pi = (d_1, d_2, \cdots) \in \Pi^{MR}$. Then,

$$v \geq \sup_{d \in D^{MR}} \{r_d + \lambda P_d v\} = \sup_{d \in D^{MD}} \{r_d + \lambda P_d v\},$$

so that

$$\begin{aligned} v &\geq r_{d_1} + \lambda P_{d_1} v \geq r_{d_1} + \lambda P_{d_1} r_{d_2} + \lambda^2 P_{d_1} P_{d_2} v \\ &\geq \cdots \\ &\geq r_{d_1} + \lambda P_{d_1} r_{d_2} + \cdots + \lambda^{n-1} P_{d_1} \cdots P_{d_{n-1}} r_{d_n} + \lambda^n P_\pi^{(n)} v. \end{aligned}$$

Hence, from $v_\lambda^\pi = r_{d_1} + \lambda P_{d_1} r_{d_2} + \lambda^2 P_{d_1} P_{d_2} r_{d_3} + \cdots$ we obtain

$$v - v_\lambda^\pi \geq \lambda^n P_\pi^{(n)} v - \sum_{k=n}^{\infty} \lambda^k P_\pi^{(k)} r_{d_{k+1}}.$$

Choose $\epsilon > 0$. Since $\|\lambda^n P_\pi^{(n)} v\| \leq \lambda^n \|v\|$ and $0 \leq \lambda < 1$, for sufficiently large n

$$-\frac{\epsilon}{2} e \leq \lambda^n P_\pi^{(n)} v \leq \frac{\epsilon}{2} e$$

where e is the column vector whose elements are all equal to 1. Moreover, for sufficiently large n

$$\sum_{k=n}^{\infty} \lambda^k P_\pi^{(k)} r_{d_{k+1}} \leq \sum_{k=n}^{\infty} \lambda^k M e = \frac{\lambda^n M e}{1-\lambda} < \frac{\epsilon}{2} e.$$

Therefore, for all $s \in S$

$$v(s) \geq v_\lambda^\pi(s) - \epsilon.$$

2. Since $v \leq \mathcal{L}v = \sup_{d \in D^{MD}} \{r_d + \lambda P_d v\}$, for each $s \in S$ there exists $a_s \in A$ such that

$$v(s) \leq r(s, a_s) + \lambda \sum_{j \in S} p(j|s, a_s) v(j) + \epsilon.$$

Choose $d(s) = a_s$. Then,

$$v \leq r_d + \lambda P_d v + \epsilon e, \text{ i.e.,}$$

$$(I - \lambda P_d) v \leq r_d + \epsilon e.$$

It then follows that

$$v \leq (I - \lambda P_d)^{-1} (r_d + \epsilon e) = v_\lambda^{d^\infty} + (1 - \lambda)^{-1} \epsilon e.$$

Banach Fixed Point Theorem

To go further, first observe that V is a Banach space.

We say that an operator $T : V \to V$ is a contraction mapping if there exists $\lambda (0 \leq \lambda < 1)$ such that $\|Tv - Tu\| \leq \lambda \|v - u\|$ for all $u, v \in V$.

Theorem 5 (Banach Fixed Point Theorem)

Suppose that V is a Banach space and $T : V \to V$ is a contraction mapping. Then

1. *there exists a unique v^* in V such that $Tv^* = v^*$; and*
2. *for arbitrary $v^{(0)}$ in V, the sequence $\{v^{(n)}\}$ defined by*

$$v^{(n+1)} = Tv^{(n)} = T^{n+1} v^{(0)},$$

converges to v^*.

Proof: For any $m \geq 1$

$$\begin{aligned}
\|v^{(n+m)} - v^{(n)}\| &\leq \sum_{k=0}^{m-1} \|v^{(n+k+1)} - v^{(n+k)}\| \\
&= \sum_{k=0}^{m-1} \|T^{n+k} v^{(1)} - T^{n+k} v^{(0)}\| \\
&\leq \sum_{k=0}^{m-1} \lambda^{n+k} \|v^{(1)} - v^{(0)}\| \\
&= \frac{\lambda^n (1 - \lambda^m)}{1 - \lambda} \|v^{(1)} - v^{(0)}\|.
\end{aligned}$$

For sufficiently large n, the RHS can be made arbitrarily small, which means that $\{v^{(n)}\}$ is Cauchy. Since V is complete, the sequence $\{v^{(n)}\}$ has a limit $v^* \in V$.

By the properties of norms and contraction mapping,

$$\begin{aligned} 0 &\leq \|Tv^* - v^*\| \leq \|Tv^* - v^{(n)}\| + \|v^{(n)} - v^*\| \\ &\leq \|Tv^* - Tv^{(n-1)}\| + \|v^{(n)} - v^*\| \leq \lambda\|v^* - v^{(n-1)}\| + \|v^{(n)} - v^*\|. \end{aligned}$$

By letting n go to ∞, the RHS goes to 0, which implies that v^* is a fixed point of T.

Now, let v^* and u^* be two fixed points of T. Then, we have

$$\|v^* - u^*\| = \|Tv^* - Tu^*\| \leq \lambda\|v^* - u^*\|$$

and it implies that $v^* = u^*$.

The existence of an optimal policy

W now show the existence and uniqueness of a solution v^*_λ of the Bellman equations. To this end, we first show that the operator

$$\mathcal{L}(v) := \sup_{a \in A} \left\{ r(\cdot) + \lambda \sum p(\cdot|\cdot) v(\cdot) \right\}$$

is a contraction mapping, i.e.,

$$\|\mathcal{L}(v) - \mathcal{L}(u)\| \leq \lambda \|v - u\|.$$

Proof of Contraction:
Consider u and v. Fix $s \in S$ and assume that $\mathcal{L}(v)(s) \geq \mathcal{L}(u)(s)$.
There exists $a_s \in A$ such that

$$r(s, a_s) + \lambda \sum_{j \in S} p(j|s, a_s)v(j) + \epsilon > \sup_{a \in A} \left\{ r(s, a) + \lambda \sum_{j \in S} p(j|s, a)v(j) \right\}$$

Then,

$$\begin{aligned} 0 &\leq \mathcal{L}(v)(s) - \mathcal{L}(u)(s) \\ &\leq \lambda \sum_{j \in S} p(j|s, a_s)v(j) + \epsilon - \lambda \sum_{j \in S} p(j|s, a_s)u(j) \\ &\leq \lambda \sum_{j \in S} p(j|s, a_s)\|v - u\| + \epsilon \\ &\leq \lambda\|v - u\| + \epsilon. \end{aligned}$$

Repeating the argument for $\mathcal{L}(v)(s) \leq \mathcal{L}(u)(s)$ shows that

$$\|\mathcal{L}(v) - \mathcal{L}(u)\| \leq \lambda\|v - u\|.$$

Since $\mathcal{L}$ is a contraction mapping on V, by applying Banach Fixed Point Theorem there exists the unique solution $v^{(f)}$ such that $\mathcal{L}(v^{(f)}) = v^{(f)}$. Moreover, by Theorem 4 we have

$$v^{(f)} = v_\lambda^*.$$

We also have the following theorem.

Theorem 6

A policy $\pi^* \in \Pi^{MR}$ is optimal iff $v_\lambda^{\pi^*}$ is a solution of the optimal equations.

Existence of Optimal Policies

Given $v \in V$, call a decision rule $d_v \in D^{MD}$, *v-improving* if

$$d_v \in \operatorname{argmax}_{d \in D^{MD}} \{r_d + \lambda P_d v\}.$$

Equivalently,

$$r_{d_v} + \lambda P_{d_v} v = \max_{d \in D^{MD}} \{r_d + \lambda P_d v\}.$$

A decision rule $d^* \in D^{MD}$ is *conserving* (equivalently, v_λ^*-improving) if

$$L_{d^*} v_\lambda^* = L v_\lambda^* = v_\lambda^*.$$

Equivalently,

$$d^* \in \operatorname{argmax}_{d \in D^{MD}} \{r_d + \lambda P_d v_\lambda^*\}.$$

Suppose that d^* exists and is conserving. Then, the deterministic stationary policy $\pi = (d^*)^\infty = (d^*, d^*, \cdots)$ is the optimal policy.

Theorem 7

For all $\epsilon > 0$, there exists an ϵ-optimal deterministic stationary policy.

proof: For $\epsilon > 0$, choose $d_\epsilon \in D^{MD}$ such that

$$\begin{aligned} r_{d_\epsilon} + \lambda P_{d_\epsilon} v_\lambda^* &\geq \sup_{d \in D^{MD}} \{r_d + \lambda P_d v_\lambda^*\} - (1-\lambda)\epsilon e \\ &= v_\lambda^* - (1-\lambda)\epsilon e. \end{aligned}$$

So we have

$$r_{d_\epsilon} \geq (I - \lambda P_{d_\epsilon}) v_\lambda^* - (1-\lambda)\epsilon e.$$

Using $v_\lambda^{(d_\epsilon)^\infty} = (I - \lambda P_{d_\epsilon})^{-1} r_{d_\epsilon}$, we have

$$v_\lambda^{(d_\epsilon)^\infty} \geq v_\lambda^* - \epsilon e.$$

Hence, $(d_\epsilon)^\infty$ is ϵ-optimal.

Finding an optimal policy

There are three popular ways of finding an optimal policy in MDP.

- Value Iteration
- Policy Iteration
- Linear Programming

Their variants are

- Splitting Method
- Modified Policy Iteration
- etc.

Value Iteration

1. Start with an initial value function $v_0, \epsilon > 0$, and set $n = 0$.
2. For each $s \in S$, compute $v^{(n+1)}$ by

$$v^{(n+1)}(s) = \max_{a \in A} \left\{ r(s,a) + \lambda \sum_{j \in S} p(j|s,a) v^{(n)}(j) \right\}$$

3. If $\|v^{(n+1)} - v^{(n)}\| < \epsilon \frac{1-\lambda}{2\lambda}$, go to step 4. Otherwise, increment n by 1 and return to step 2.
4. For each $s \in S$, choose

$$d_\epsilon(s) = \text{argmax}_{a \in A} \left\{ r(s,a) + \lambda \sum_{j \in S} p(j|s,a) v^{(n+1)}(j) \right\}$$

and stop.

Convergence in Value Iteration

The value iteration algorithm finds a stationary policy that is ϵ-optimal within a finite number of iteration, i.e., $\|v_\lambda^{(d_\epsilon)^\infty} - v_\lambda^*\| < \epsilon$. Why?

First, observe that

$$\|v_\lambda^{(d_\epsilon)^\infty} - v_\lambda^*\| \leq \|v_\lambda^{(d_\epsilon)^\infty} - v^{(n+1)}\| + \|v^{(n+1)} - v_\lambda^*\|$$

Since $v_\lambda^{(d_\epsilon)^\infty}$ is a fixed point of L_{d_ϵ} and $Lv^{(n+1)} = L_{d_\epsilon} v^{(n+1)}$,

$$\begin{aligned}
\|v_\lambda^{(d_\epsilon)^\infty} - v^{(n+1)}\| &= \|L_{d_\epsilon} v_\lambda^{(d_\epsilon)^\infty} - v^{(n+1)}\| \\
&\leq \|L_{d_\epsilon} v_\lambda^{(d_\epsilon)^\infty} - Lv^{(n+1)}\| + \|Lv^{(n+1)} - v^{(n+1)}\| \\
&\leq \|L_{d_\epsilon} v_\lambda^{(d_\epsilon)^\infty} - L_{d_\epsilon} v^{(n+1)}\| + \|Lv^{(n+1)} - Lv^{(n)}\| \\
&\leq \lambda \|v_\lambda^{(d_\epsilon)^\infty} - v^{(n+1)}\| + \lambda \|v^{(n+1)} - v^{(n)}\|.
\end{aligned}$$

So we have, if $\|v^{(n+1)} - v^{(n)}\| < \epsilon \frac{1-\lambda}{2\lambda}$

$$\|v_\lambda^{(d_\epsilon)^\infty} - v^{(n+1)}\| \leq \frac{\lambda}{1-\lambda} \|v^{(n+1)} - v^{(n)}\| < \frac{\epsilon}{2}.$$

Similarly,

$$\begin{aligned}\|v^{(n+1)} - v_\lambda^*\| &\leq \|v^{(n+1)} - Lv^{(n+1)}\| + \|Lv^{(n+1)} - Lv_\lambda^*\| \\ &\leq \lambda\|v^{(n+1)} - v^{(n)}\| + \lambda\|v^{(n+1)} - v_\lambda^*\|\end{aligned}$$

So,

$$\|v^{(n+1)} - v_\lambda^*\| \leq \frac{\lambda}{1-\lambda}\|v^{(n+1)} - v^{(n)}\| < \frac{\epsilon}{2}.$$

Therefore,

$$\|v_\lambda^{(d_\epsilon)^\infty} - v_\lambda^*\| \leq \|v_\lambda^{(d_\epsilon)^\infty} - v^{(n+1)}\| + \|v^{(n+1)} - v_\lambda^*\| < \epsilon.$$

Policy Iteration

1. Set $n = 0$ and select an arbitrary policy $d_0 \in D^{MD}$.
2. (policy evaluation)
   - Obtain $v^{(n)}$ by solving
   $$(I - \lambda P_{d_n})v^{(n)} = r_{d_n}.$$
3. (policy improvement)
   - Choose d_{n+1} to satisfy
   $$d_{n+1} \in \mathrm{argmax}_{d \in D^{MD}} \left\{ r_d + \lambda P_d v^{(n)} \right\}$$
   setting $d_{n+1} = d_n$ if possible.
4. If $d_{n+1} = d_n$, stop and set $d^* = d_n$. Otherwise, increase n by 1 and return to step 2.

Convergence in Policy Iteration

- For two successive value functions $v^{(n)}$ and $v^{(n+1)}$, we have

$$v^{(n+1)} \geq v^{(n)}$$

which can be shown as follows.

$$r_{d_{n+1}} + \lambda P_{d_{n+1}} v^{(n)} \geq r_{d_n} + \lambda P_{d_n} v^{(n)} = v^{(n)},$$

i.e.,

$$r_{d_{n+1}} \geq (I - \lambda P_{d_{n+1}}) v^{(n)}.$$

Hence

$$v^{(n+1)} = (I - \lambda P_{d_{n+1}})^{-1} r_{d_{n+1}} \geq v^{(n)}.$$

- The algorithm is terminated in a finite time when there are finite policies and states.
- The convergence is also proved for more general state and action spaces under some suitable assumption.

More on RL: Full Observability vs. Partial Observability

- Full Observability
  - An agent directly observes the environment state.
  - A Markov Decision Process is used in this case.
- Partial Observability
  - An agent indirectly observe the environment state.
    For instance, a poker playing agent only observes public cards.
  - In this case, the agent state is not identical to the environment state.
  - A Partially Observable Markov Decision Process is used in this case.
  - The agent constructs its own state representation.

Model based RL vs. Model free RL

- Model based RL
  - The environment is modeled by a mathematical model such as MDP.
  - Algorithms which use a model are called model-based methods.
  - Learing from experience: $P(j|s,a) \propto \#(s, a \to j)$
- Model free RL
  - A ground-truth model of the environment is usually not available to the agent.
  - It uses the experience to directly learn a value function (e.g., Q-learning).
  - The model-free methods are more popular and have been more extensively developed and tested than the model-based methods.

References

- M.L. Puterman, Markov Decision Processes Discrete Stochastic Dynamic Programming, Willy, 2005.
- R.S. Sutton and A.G. Barto, Reinforcement Learing, The MIT Press, 1998.