CS 204 Discrete Mathematics

Homework 6

Sample Solutions

RELATIONS

1. Let R1 = {(1, 2), (2, 3), (3, 4)} and R2 = {(1, 1), (1, 2),(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)} be relations from {1, 2, 3} to {1, 2, 3, 4}. Find

a) R1 ∪ R2

b) R1 ∩ R2

c) R1 − R2

d) R2 − R1

Solution)

These are merely routine exercises in set theory. Note that $R_1 \subseteq R_2$.

a) $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\} = R_2$ **b)** $\{(1,2),(2,3),(3,4)\} = R_1$

c) $\emptyset$ **d)** $\{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

2. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where (x, y) ∈ R if and only if

a) x + y = 0

b) x = ±y

c) x − y is a rational number.

d) x = 2y

e) xy ≥ 0

Solution)

a) Since $1+1 \neq 0$, this relation is not reflexive. Since $x+y = y+x$, it follows that $x+y=0$ if and only if $y+x=0$, so the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1,-1) \in R$ and $(-1,1) \in R$, but $(1,1) \notin R$.

b) Since $x = \pm x$ (choosing the plus sign), the relation is reflexive. Since $x = \pm y$ if and only if $y = \pm x$, the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is transitive, essentially because the product of 1's and -1's is ± 1.

c) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x-y$ is rational, then so is $-(x-y) = y-x$. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x,y) \in R$ and $(y,z) \in R$, then $x-y$ and $y-z$ are rational numbers. Therefore their sum $x-z$ is rational, and that means that $(x,z) \in R$.

d) Since $1 \neq 2 \cdot 1$, this relation is not reflexive. It is not symmetric, since $(2,1) \in R$, but $(1,2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$. Thus the only time that (x,y) and (y,x) are both is R is when $x = y$ (and both are 0). This relation is clearly not transitive, since $(4,2) \in R$ and $(2,1) \in R$, but $(4,1) \notin R$.

3.

(a) Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$. Now let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)}, and let S be the relation {(2, 1), (3, 1), (3, 2), (4, 2)}. Find $S \circ R$.

(b) Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, . . . , are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$. Now let R be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), and (5, 4). Find R^2, R^3, R^4 and R^5.

Solution)

(a)

Since $(1,2) \in R$ and $(2,1) \in S$, we have $(1,1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1,1),(1,2),(2,1),(2,2)\}$.

(b)

We just apply the definition each time. We find that R^2 contains all the pairs in $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$ except $(2,3)$ and $(4,5)$; and R^3, R^4, and R^5 contain all the pairs.

4. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements. (Note that $R1 \oplus R2$ consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it.)

a) $R \cup S$ is reflexive.

b) $R \cap S$ is reflexive.

c) $R \oplus S$ is irreflexive.

d) $R - S$ is irreflexive.

e) $S \circ R$ is reflexive.

Solution)

a) Since R contains all the pairs (x,x), so does $R \cup S$. Therefore $R \cup S$ is reflexive.

b) Since R and S each contain all the pairs (x,x), so does $R \cap S$. Therefore $R \cap S$ is reflexive.

c) Since R and S each contain all the pairs (x,x), we know that $R \oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

d) Since R and S each contain all the pairs (x,x), we know that $R - S$ contains none of these pairs. Therefore $R - S$ is irreflexive.

e) Since R and S each contain all the pairs (x,x), so does $S \circ R$. Therefore $S \circ R$ is reflexive.

ORDERED RELATIONS

1. Fill in the following table describing the characteristics of the given ordered sets. Answer with T for True or F for False. (Note that "$\subset$" indicates proper containment, that is, for two sets A and B, $A \subset B$ if and only if $A \subseteq B \wedge A \neq B$.)

	Partial Order	Total Order	Well Order
$< N , < >$			
$< N , \leq >$			
$< Z , \leq >$			
$< R , \leq >$			
$< P(N) , \subset >$			
$< P(N) , \subseteq >$			
$< P(\{a\}), \subseteq >$			
$< P(\varnothing) , \subseteq >$			

Solution)

	Partial Order	Total Order	Well Order
$< N , < >$	F	F	F
$< N , \leq >$	T	T	T
$< Z , \leq >$	T	T	F
$< R , \leq >$	T	T	F
$< P(N) , \subset >$	F	F	F
$< P(N) , \subseteq >$	T	F	F
$< P(\{a\}), \subseteq >$	T	T	T
$< P(\varnothing) , \subseteq >$	T	T	T

EQUIVALENCE RELATIONS

1. Give a specific reason why the following set R does not define an equivalence relation on the set {1, 2, 3, 4}.

$$R = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (2,4), (4,2)\}$$

Solution) R contains 3R2 = (3,2) and 2R4 = (2,4) but does not contain (3,4). Therefore, the relation is not transitive and thus, not an equivalent relation.

2. The following set R defines an equivalence relation on the set {1, 2, 3}, where aRb means that (a,b) $\in$R.

$$R = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$$

What are the equivalence classes?

Solution)

$\{1\}, \{2, 3\}$

3. Let X be a finite set. For subsets, A, B $\in \mathcal{P}(X)$, let A *R* B if |A| = |B|. This is an equivalence relation on $\mathcal{P}(X)$. If X = {1, 2, 3}, list the equivalence classes.

Solution)

There are four equivalence classes:

$$\{\emptyset\}$$
$$\{\{1\}, \{2\}, \{3\}\}$$
$$\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$$
$$\{\{1, 2, 3\}\}$$

FUNCTIONS

1. Let A = {0, 1, 2} Find all total functions f: $A \to A$ for which $f^2(x) = f(x)$.
   (Note: $f^2 = f \circ f$)

(a) How many such functions are there?

(b) List all such functions.

Solution)

(a) There are 10 such functions.

(b) They consist of the following disjoint classes:

   (i) The constant functions f(x) = 0, f(x) = 1 and f(x) = 2:
      {(0,0), (1,0), (2,0)},
      {(0,1), (1,1), (2,1)},
      {(0,2), (1,2), (2,2)},

   (ii) The identify function f(x) = x.
      {(0,0), (1,1), (2,2)},

   (iii) The function which map two elements to themselves and the remaining elements to one of those two (For example, f(0) = 0, f(1) = 1 and f(2) = 1.):
      {(0,0), (1,1), (2,0)},
      {(0,0), (1,1), (2,1)},
      {(0,0), (1,0), (2,2)},
      {(0,0), (1,2), (2,2)},
      {(0,1), (1,1), (2,2)},
      {(0,2), (1,1), (2,2)},

2. Let P be a set of people, and let Q be a set of occupations. Define a function f: $P \to Q$ by setting f(p) equal to p's occupation. What must be true about the people in P for f to be a total function?

Solution)

Everybody must have some occupation, and nobody can have two occupations.

3. Let S = {0, 1, 2, 3, 4, 5}, and let $\mathcal{P}$(S)* be the set of all nonempty subsets of S. Define a function *m*: $\mathcal{P}$(S)* $\to$S by

$$m(\mathrm{H}) = \text{the largest element in H}$$

for any nonempty subset H $\subseteq$ S.

(a) Is *m* one-to-one? Why or why not?

(b) Does *m* map $\mathcal{P}$(S)* onto S? Why or why not?

Solution)

(a) No. For example, $m(\{0,3\}) = 3 = m(\{2,3\})$.

(b) Yes. $m(\{x\}) = x$ for all $x \in S$.