Ch 1. The Foundations: Logic and Proofs

Propositional Logic-4

Proof

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- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

What is the problem with truth table analysis?

- ☛ Table size grows rapidly.

What is the problem with truth table analysis?

- ☛ Table size grows rapidly.
  - ☛ Need propositional calculus

    (= language + inference rules).

      => Allows us to perform logical reasoning at a higher level.

Propositional Logic

Inference rules –> (Formal) Derivation rules

- Equivalence rules
- Other inference rules

Equivalence rules

"≡" and "⇔" reads "is equivalent to".
They are not propositional connectives.
They are meta symbols.

- A Special Kind of Inference Rule
  - ☛ Two statements **always** have the same truth value.

- With equivalence (written $A \equiv B$ or $A \Leftrightarrow B$), we can do the following:

(1) Given A, deduce B

(2) Given B, deduce A

(3) (Substitution) Given a statement containing statement A, deduce the same statement but with statement A replaced by statement B.

Equivalence rules (1/2)

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \rightarrow q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	distributivity of $\wedge$ over $\vee$ distributivity of $\vee$ over $\wedge$

Note that "$\Leftrightarrow$" is not part of a statement.

Equivalence rules (2/2)

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \rightarrow q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	distributivity of $\wedge$ over $\vee$ distributivity of $\vee$ over $\wedge$

Note that "$\Leftrightarrow$" is not part of a statement.

Additional Propositional Equivalence Rules

In the following table, we use abbreviations:

T: compound proposition that is a tautology

F: compound proposition that is a contradiction

Equivalence	Name
$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$	Identity laws
$p \vee T \Leftrightarrow T$ $p \wedge F \Leftrightarrow F$	Domination laws
$p \vee \neg p \Leftrightarrow T$ $p \wedge \neg p \Leftrightarrow F$	Negation laws

Equivalence rules

Example (Substitution)

If *Micah is not sick and Micah is not tired*, then Micah can play.

If *it is not the case that Micah is sick or tired*, then Micah can play.

What equivalence has been used here?

Inference rules

"$\Rightarrow$" reads "logically implies".
It is not a propositional connective.
It is a meta symbol like "$\equiv$".

With inference rules written in the form $A \Rightarrow B$, we can do the following:

- Given A, deduce B.

Inference rules

Inference	Name
$\left.\begin{array}{r} p \\ q \end{array}\right\} \Rightarrow p \wedge q$	conjunction
$\left.\begin{array}{r} p \\ p \rightarrow q \end{array}\right\} \Rightarrow q$	*modus ponens*
$\left.\begin{array}{r} \neg q \\ p \rightarrow q \end{array}\right\} \Rightarrow \neg p$	*modus tollens*
$p \wedge q \Rightarrow p$	simplification
$p \Rightarrow p \vee q$	addition

Note that "$\}$" and "$\Rightarrow$" are not part of a statement.

Inference rules

Example Our professor does not own a spaceship.
If our professor is from Mars, then our professor owns a spaceship.

Our professor is not from Mars.

Which inference rule has been used?

To prove the validity of the inference rules themselves:

Example

Prove:

$$\left.\begin{array}{r} p \\ p \rightarrow q \end{array}\right\} \Rightarrow q$$

Proof:

What if you are asked prove the validity of the inference rules themselves?

Example

Prove:

$$\left.\begin{array}{r} p \\ p \rightarrow q \end{array}\right\} \Rightarrow q$$

Proof: Approach using truth table.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Proof sequence

To show the validity of the following inference:

Prove:

premises

$$\left.\begin{array}{r} p \\ p \rightarrow q \\ q \rightarrow r \end{array}\right\} \Rightarrow r$$

conclusion

Proof: Approach using inference rules

Proof sequence

To show the validity of the following inference:

Prove:

$$\left.\begin{array}{r} p \\ p \to q \\ q \to r \end{array}\right\} \Rightarrow r$$

premises

conclusion

Proof:

Statements	Reasons
1. p	given
2. $p \to q$	given
3. $q \to r$	given
4.	

Proof sequence

To show the validity of the following inference:

Prove:

$$\left.\begin{array}{r} p \\ p \to q \\ q \to r \end{array}\right\} \Rightarrow r.$$

Proof:

Statements	Reasons
1. p	given
2. $p \to q$	given
3. $q \to r$	given
4. q	*modus ponens*, 1,2

Proof sequence

To show the validity of the following inference:

Prove:

$$\left.\begin{array}{r} p \\ p \to q \\ q \to r \end{array}\right\} \Rightarrow r.$$

Proof:

Statements	Reasons
1. p	given
2. $p \to q$	given
3. $q \to r$	given
4. q	*modus ponens*, 1,2
5. r	*modus ponens*, 4,3

Proof sequence: a sequence of statements and reasons to justify inferences.

Proof sequence: Strategy

Prove:

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Proof:

Proof sequence: Strategy

Prove:

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Proof:

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given

How to prove: Start with the given, see what you can deduce, end with what you are trying to prove.

Proof sequence: Strategy

Prove:

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Proof:

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given
n. q	

goal to prove

How to prove: Start with the given, see what you can deduce, end with what you are trying to prove.

Proof sequence: Strategy

Hint: Use the equivalence
$p \to q \equiv \neg p \vee q$

Prove:

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Proof:

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given
n-1. $\neg p \to q$	
n. q	*modus ponens*, **2, n-1**

How to prove: Start with the given, see what you can deduce, end with what you are trying to prove.

Proof sequence: Strategy

Prove:

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Proof:

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given
n-2. $\neg(\neg p) \vee q$	
n-1. $\neg p \rightarrow q$	implication, n-2
n. q	*modus ponens*, **2, n-1**

Proof sequence: Strategy

Prove:

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

Proof:

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given
3. $\neg(\neg p) \vee q$	double negation, 1
4. $\neg p \to q$	implication, 3
5. q	*modus ponens*, 4, 2

Forward-backward technique

Prove:

$$p \rightarrow q \quad \Rightarrow \quad \neg q \rightarrow \neg p$$

Proof:

Forward-backward technique

Prove:

$$p \to q \;\Rightarrow\; \neg q \to \neg p$$

Proof:

Statements	Reasons
1. $p \to q$	given
n. $\neg q \to \neg p$	

Forward-backward technique

Prove:

$$p \to q \;\Rightarrow\; \neg q \to \neg p$$

Hint: Use the implication equivalence

$p \to q \equiv \neg p \vee q$

Proof:

Statements	Reasons
1. $p \to q$	given
n. $\neg q \to \neg p$	

Forward-backward technique

Prove:

$$p \to q \;\Rightarrow\; \neg q \to \neg p$$

Hint: Use the implication equivalence

$p \to q \equiv \neg p \vee q$

Proof:

Statements	Reasons
1. $p \to q$	given
2. $\neg p \vee q$	implication equivalence, 1
n. $\neg q \to \neg p$	

Forward-backward technique

Prove:

$$p \to q \quad \Rightarrow \quad \neg q \to \neg p$$

Proof:

Statements	Reasons
1. $p \to q$	given
2. $\neg p \vee q$	implication equivalence, 1 $\downarrow$
n-1. $\neg(\neg q) \vee \neg p$	
n. $\neg q \to \neg p$	implication equivalence, n-1 $\uparrow$

Forward-backward technique

Prove:

$$p \to q \quad \Rightarrow \quad \neg q \to \neg p$$

Proof:

Statements	Reasons
1. $p \to q$	given
2. $\neg p \vee q$	implication equivalence, 1
n-2. $q \vee \neg p$	commutativity , 2
n-1. $\neg(\neg q) \vee \neg p$	double negation , n-2
n. $\neg q \to \neg p$	implication equivalence, n-1

Forward-backward technique

Prove:

$$p \to q \quad \Rightarrow \quad \neg q \to \neg p$$

Proof:

Statements	Reasons
1. $p \to q$	given
2. $\neg p \vee q$	implication
3. $q \vee \neg p$	commutativity
4. $\neg(\neg q) \vee \neg p$	double negation
5. $\neg q \to \neg p$	implication

Quiz 04-1

[1] What are the two components of propositional calculus?

[2] Which of the following is NOT true about the equivalence rules of the form “A $\equiv$ B” ?

(a) We can deduce A from B.

(b) We can substitute B for A.

(c) We can substitute A for B.

(d) “$\equiv$” is a symbol of propositional logic language .