CS 204 Discrete Mathematics

Homework 8

Sample Solutions

RECURSIVE DEFINITIONS

1. Give a recursive definition for the set X of all binary strings with an even number of 0's.

Solution)

The set X of all binary strings (strings with only 0's and 1's) having an even number of 0's is defined as follows.

B₁. λ is in X.

B₂. 1 is in X.

R₁. If x is in X, so is $0x0$.

R₂. If x and y are in X, so is xy.

(Answers may vary.)

2. The following recursive definition defines a set ℤ of ordered pairs.

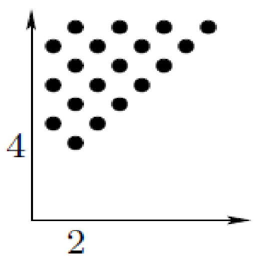
B. (2, 4) is in ℤ.

R1. If (x,y) is in ℤ with x < 10 and y < 10, then (x+1,y+1) is in ℤ.

R2. If (x,y) is in ℤ with x > 1 and y < 10, then (x-1, y+1) is in ℤ.

Plot these ordered pairs in the xy-plane.

Solution)



3. Give a recursive definition for the set X of even integers (including both positive and negative even integers).

Solution)

B. $0 \in E$.

R. If $n \in E$ so are $n+2$ and $n-2$.

4. Let S be a set of sets with the following recursive definition.

B. $\varnothing \in$S.

R. If X $\subseteq$ S, then X$\in$S.

(a) List three different elements of S.

(b) Explain why S has infinitely many elements.

Solution)

(a) $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}, \emptyset\}$

(b) There are infinitely many elements of the form $\{\{\cdots\{\emptyset\}\cdots\}\}$, for example.

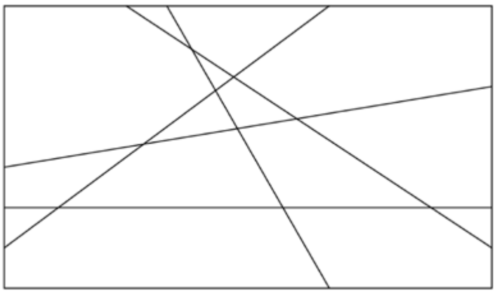
Structural Induction

1. A line map is defined as follows:

   B. A blank rectangle is a line map.

   R. A line map with a straight line drawn all the way across it is a new line map.

Here is an example of a line map.



(a) Prove by induction that a line map with n distinct lines has at least n+1 regions.
(b) Prove by induction that a line map with n distinct lines has at most 2^n regions.
(c) Part (a) gives a lower bound on the number of regions in a line map. For example, a line map with five lines must have at least six regions. Give an example of a line map that achieves this lower bound, that is, draw a line map with five lines and six regions.
(d) Part (b) says that a line map with three lines can have at most eight regions. Can you draw a line map with three lines that achieves this upper bound? Do so, or explain why you can't.

Solution)

(a) *Proof.* (Induction on the number of lines.) If a line map contains 0 lines, then it is just a blank rectangle containing 1 region, and $1 \geq 0 + 1$; the number of regions is at least the number of lines plus 1. Assume as inductive hypothesis that any line map with $k - 1$ lines has at least k regions, for some $k > 0$. Let M be a line map with k lines. Remove one line from M, call it l. By inductive hypothesis, the resulting map M' has k regions. Now put back l. Since l crosses the whole rectangle, it must pass through at least one region of M', dividing this region into two regions. Hence M as at least one more region than M', so M has at least $k + 1$ regions, as required. □

(b) *Proof.* (Induction on the number of lines.) If a line map contains 0 lines, then it is just a blank rectangle containing 1 region, and $1 \leq 2^0 = 1$. Assume as inductive hypothesis that any line map with $k - 1$ lines has at most 2^{k-1} regions, for some $k > 0$. Let M be a line map with k lines. Remove one line from M, call it l. By inductive hypothesis, the resulting map M' has at most 2^{k-1} regions. Now put back l. Every region of M' that l passes through gets divided into two regions. At most, l passes through all 2^{k-1} regions of M', so the number of regions in M is at most $2 \cdot 2^{k-1} = 2^k$ regions, as required. □

(c) Five lines and six regions:

(d) A line map M with two lines will form at most four regions. Suppose for contradiction that a third line l passes through all four regions. This third line must therefore cross over three borders between regions of M. But there are only two lines forming the regions of M, so l must intersect some line twice, a contradiction. Therefore a third line must pass through at most 3 regions of M, forming at most 7 regions total.

2. Define a Q-sequence recursively as follows.

Basis Case. $<x, 4-x>$ is a Q-sequence (of length 2) for any real number x.
Recursive Case. If $<x_1, x_2, ..., x_{j-1}, x_j>$ and $<y_1, y_2, ..., y_{k-1}, y_k>$ are Q-sequences, so is

$$< x_1 - 1, x_2, ..., x_{j-1}, x_j, y_1, y_2, ..., y_{k-1}, y_k - 3 >$$

(, of which the length is j+k).

Use structural induction to prove that the sum of the numbers in any Q-sequence is 4.

Solution)
Let S be a Q-sequence and P(S) be the statement that S satisfies the property "the sum of the numbers in S is 4".

Basis Step) $P(<x, 4 - x>)$ is true because $x + (4 - x) = 4$.
Inductive Step)
For induction hypothesis, suppose that $S1 = <x_1, x_2, \dots , x_m>$ and $S2 = <y_1, y_2, \dots , y_n>$ are Q-sequences and P(S1) and P(S2) are true.
Consider the sequence $S3 = < x_1 - 1, x_2, \dots , x_m, y_1, y_2, \dots , y_n - 3>$.
By induction hypothesis, $x_1 + x_2 + \dots + x_m = 4$ and $y_1 + y_2 + \dots + y_n = 4$.
So $(x_1 + x_2 + \dots + x_m) + (y_1 + y_2 + \dots + y_n) = 8$ and thus
$(x_1 - 1) + x_2 + \dots + x_m + y_1 + y_2 + \dots + (y_n - 3) = 4$.

Since S3 in the Inductive Step is the Q-sequence obtained from S1 and S2 by the recursive definition of Q-sequence, by the structural induction, P(S) is true for any Q-sequence S.

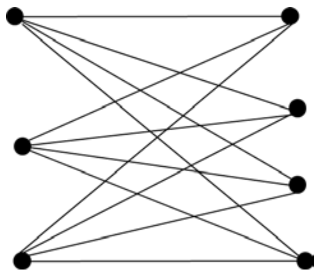
RECURSIVE ALGORITHMS & RECURRENCE RELATIONS

1. Write an iterative algorithm to compute F(n), the n-th Fibonnaci number.

Solution)

$i \leftarrow 2$
$L \leftarrow 1$
$F \leftarrow 1$
while $i < n$ do
  $F \leftarrow F + L$
  $L \leftarrow F - L$
  $i \leftarrow i + 1$

2. The complete bipartite graph $K_{m,n}$ is the simple undirected graph with m+n vertices split into two sets V1 and V2 (|V1| = m, |V2| = n) such that vertices x,y share an edge if and only if x ∈ V1 and y ∈ V2 . For example $K_{3,4}$ is the following graph.



(a) Find a recurrence relation for the number of edges in $K_{3,n}$.
(b) Find a recurrence relation for the number of edges in $K_{n,n}$.

Solution)

(a) Let $E(n)$ be the number of edges in $K_{3,n}$. Since each new vertex requires three new edges,

$$E(n) = \begin{cases} 3 & \text{if } n = 1 \\ 3 + E(n-1) & \text{if } n > 1 \end{cases}$$

(b) Let $F(n)$ be the number of edges in $K_{n,n}$.

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \\ F(n-1) + 2n - 1 & \text{if } n > 1 \end{cases}$$

Grading guideline)

The answer must include the case of n=1 (or n=0) and n > 1 (or n > 0). If students missed it, then they get a half point.

3. Consider the following recurrence relation:

$$G(n) = \begin{cases} 1 & \text{if } n = 0 \\ G(n-1) + 2n - 1 & \text{if } n > 0. \end{cases}$$

(a) Calculate G(0), G(1), G(2), G(3), G(4), and G(5).

(b) Guess at a closed-form solution for G(n) using sequence of differences.

(c) Prove that your guess is correct.

Solution)

(a) $1, 2, 5, 10, 17$

(b) First differences: $1, 3, 5, 7$. Second differences: $2, 2, 2$. So a quadratic formula is suggested. Guess: $f(n) = n^2 + 1$.

(c) *Proof.* (Induction on n.) Let $f(n) = n^2 + 1$.

Base Case: If $n = 0$, the recurrence relation says that $G(0) = 1$, and the formula says that $f(0) = 0^2 + 1 = 1$, so they match.

Inductive Hypothesis: Suppose as inductive hypothesis that

$$G(k-1) = (k-1)^2 + 1$$

for some $k > 0$.

Inductive Step: Using the recurrence relation,

$$\begin{aligned} G(k) &= G(k-1) + 2k - 1, \text{ by the second part of the recurrence relation} \\ &= (k-1)^2 + 1 + 2k - 1, \text{ by inductive hypothesis} \\ &= k^2 - 2k + 1 + 1 + 2k - 1 \\ &= k^2 + 1 \end{aligned}$$

so, by induction, $G(n) = f(n)$ for all $n \geq 0$. □

4. Consider the following recurrence relation:

$$L(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ L(n-1) + L(n-2) & \text{if } n > 2. \end{cases}$$

Let α and β be the constants that are used to compute the Fibonacci numbers as below:

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$

Prove that $L(n) = \alpha^n + \beta^n$ for all $n \in \mathbb{N}$. Use strong induction.

Solution)

Use the equalities $\alpha^2 = 1 + \alpha$ and $\beta^2 = 1 + \beta$.

Proof. First, note that $L(1) = 1 = \alpha + \beta$, and $\alpha^2 + \beta^2 = (\alpha + 1) + (\beta + 1) = \alpha + \beta + 2 = 3 = L(2)$. Suppose as inductive hypothesis that $L(i) = \alpha^i + \beta^i$ for all $i < k$, for some $k > 2$. Then $L(k) = L(k-1) + L(k-2) = \alpha^{k-1} + \beta^{k-1} + \alpha^{k-2} + \beta^{k-2} = \alpha^{k-2}(\alpha+1) + \beta^{k-2}(\beta+1) = \alpha^{k-2}(\alpha^2) + \beta^{k-2}(\beta^2) = \alpha^k + \beta^k$, as required. □