Ch 10. Graphs

Graph Theory 1

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Graph Theory

1. Graphs: Formal Definitions
2. Relations and Graphs
3. Isomorphisms of Graphs
4. Degree of a Node
5. Paths and Circuits

1. Graphs: formal definitions

A graph is a pair consisting of a finite set of vertices and a finite set of edges connecting the vertices.

☛ A graph is finite !

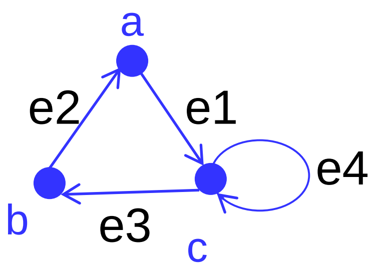
Directed Graph

Definition

A *directed graph* G is a finite set of vertices V_G and a finite set of edges E_G, along with a function $i : E_G \longrightarrow V_G \times V_G$. For any edge $e \in E_G$, if $i(e) = (a, b)$, we say that edge e *joins* vertex a to vertex b.

Directed Graph

G1:



G1 = < {a, b, c},
{e1, e2, e3, e4}
{ (e1, (a,c)),
(e2, (b,a)),
(e3, (c,b)),
(e4, (c,c))}
>

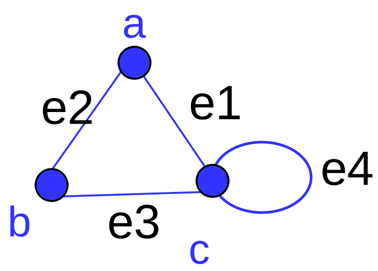
Undirected Graph

Definition

An *undirected graph* G is a finite set of vertices V_G and a finite set of edges E_G, along with a function $i : E_G \longrightarrow V_G \times V_G$. For any edge $e \in E_G$, if $i(e) =$ {a,b}, we say that vertices a and b are *joined* by edge e, or equivalently, *e joins a to b and e joins b to a*. (Here it is possible that $a = b$; if $i(e) =$ {a}, then e is a loop *joining* a to itself.)

Undirected Graph

G2:



G1 = < {a, b, c},
{e1, e2, e3, e4}
{ (e1, {a,c}),
(e2, {b,a}),
(e3, {c,b}),
(e4, {c})}
>

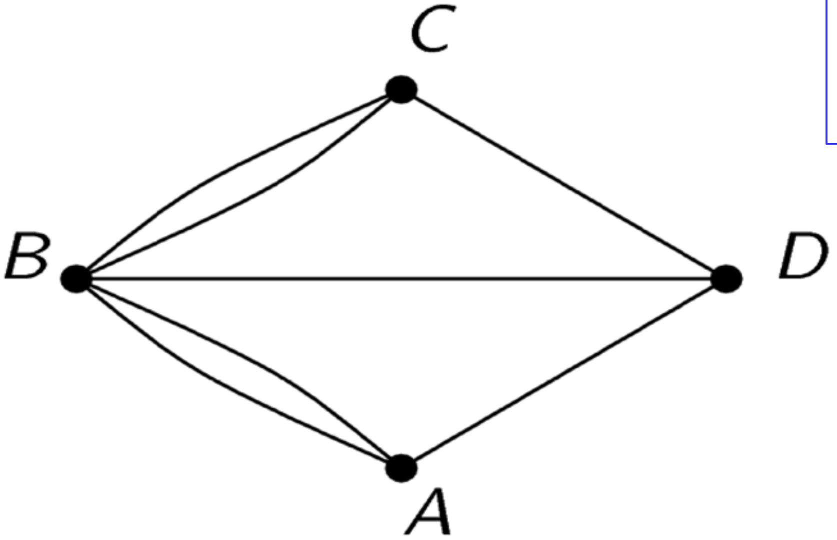
"{ }" instead of "( )".
Why?

Multigraph

A graph that has multiple edges between the same two nodes is called a *multigraph*.

Example The following is a graph model for the Könisberg Bridge problem.

In this course, we consider only the graphs that are not multigraphs.



Relations and directed graphs

- Graph $\leftrightarrow$ Relation
  - That is, a graph is just a relation and a relation can be modelled as a graph.

Definition

Let R be a relation on a set X. The *directed graph* associated with (X, R) is the graph whose vertices correspond to the elements of X, with a directed edge from vertex x to vertex y whenever $x\,R\,y$.

[redacted]

For example, consider the "$|$" relation on the set $X = \{2, 3, 4, 6\}$.

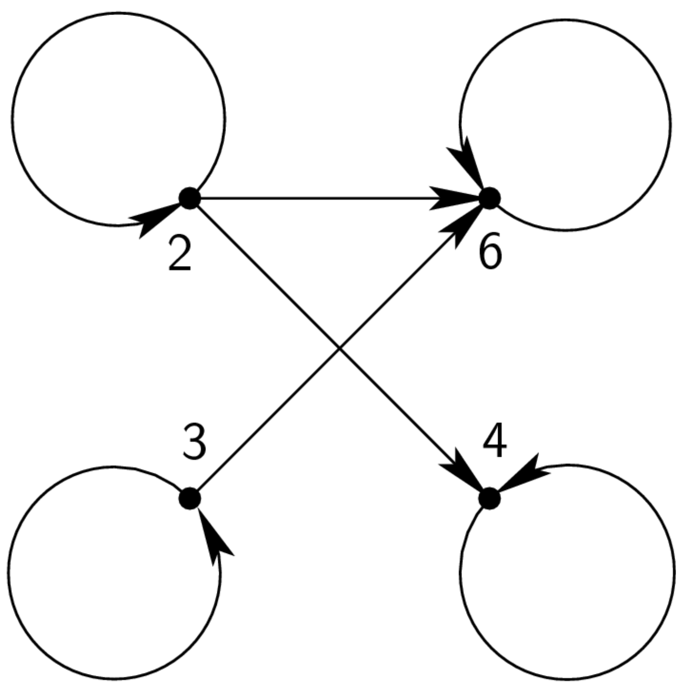
The “divides” relation

R = <X, | >
 = < {2,3,4,6},
 {(2,2),(3,3),(4,4),
 (6,6),(2,6),(2,4),(3,6)}
 >

[redacted]

For example, consider the "|" relation on the set $X = \{2, 3, 4, 6\}$.

R = <X, | >
= < {2,3,4,6},
{(2,2),(3,3),(4,4),
(6,6),(2,6),(2,4),(3,6)}
>



Relations and undirected graphs

Symmetric relations can be modelled with undirected graphs.

Definition

Let R be a relation on a set X, and suppose $x \, R \, y \rightarrow y \, R \, x$ for all $x, y \in X$. The *undirected graph* associated with (X, R) is the graph whose vertices correspond to the elements of X, with an (undirected) edge joining any two vertices x and y for which $x \, R \, y$.

Define a relation on the set $X = \{2, 3, 4, 6\}$

For example, R = < {2,3,4,6}, {(2,2),(3,3),(4,4),(6,6), (2,3),(3,2),(2,4),(4,2), (2,6),(6,2),(3,4),(4,3), (3,6),(6,3),(4,6),(6,4)} >

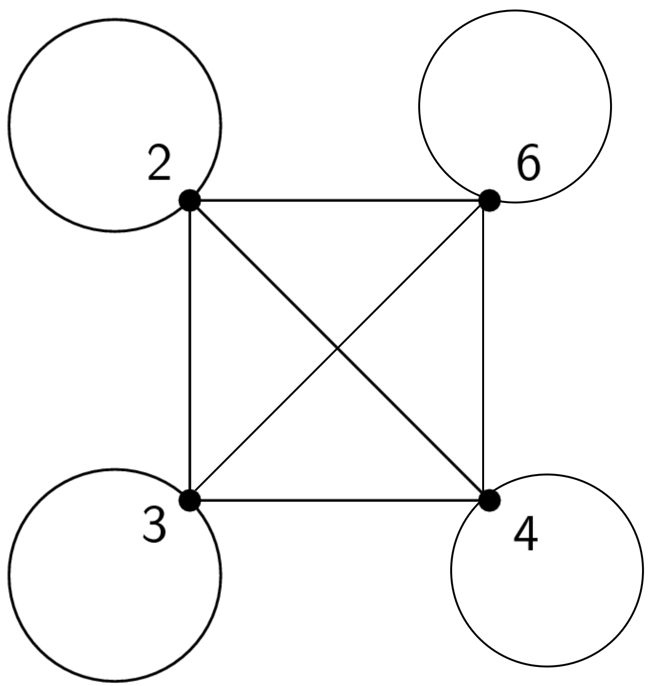
Is this a symmetric relation?

[redacted]

Define a relation on the set $X = \{2, 3, 4, 6\}$

For example, R = < {2,3,4,6}, {(2,2),(3,3),(4,4),(6,6), (2,3),(3,2),(2,4),(4,2), (2,6),(6,2),(3,4),(4,3), (3,6),(6,3),(4,6),(6,4)} >

G:



Exercise

Let $X = \{2, 3, 4, 5, 6, 7, 8\}$, and say that two elements $a, b \in X$ are related if $a \mid b$ and $a \neq b$. How can we represent this relation with a graph?

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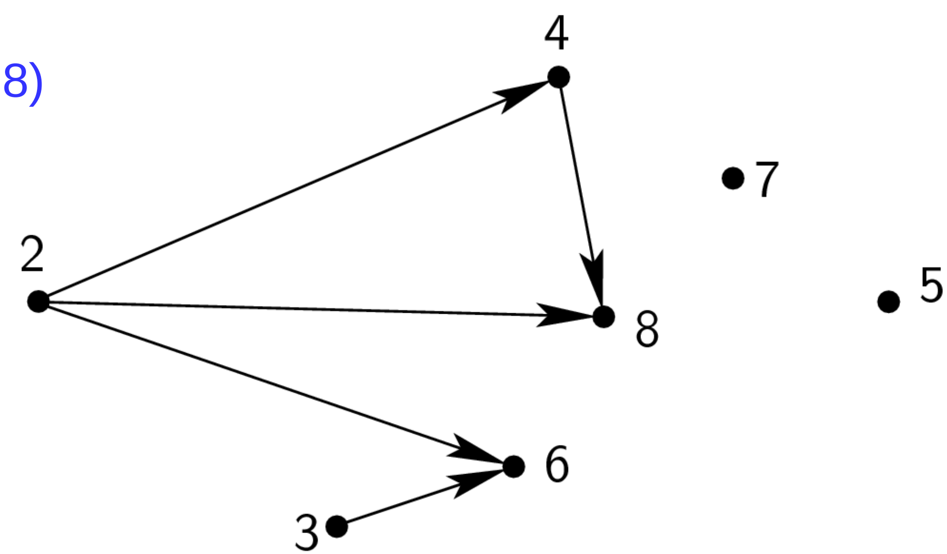
R = < {2,3,4,5,6,7,8},
{(2,4),(2,6), (2,8)
(3,6),
(4,8)}
>

[redacted]

Let $X = \{2,3,4,5,6,7,8\}$, and say that two elements $a, b \in X$ are related if $a \mid b$ and $a \neq b$. We can represent this relation with a directed graph: the elements of X are the vertices, and there is a directed edge from distinct vertices a and b whenever $a \mid b$.

R = < {2,3,4,5,6,7,8},
{(2,4),(2,6), (2,8)
(3,6),
(4,8)}
>



- Asymmetric relations and symmetric relations are represented with directed graphs and undirected graphs, respectively.

- Extracting relations from graphs can be done similarly.

Quiz 23-2

[1] What is the maximum number of edges that an undirected graph with four vertices can have?

[2] What is the maximum number of edges that a directed graph with four vertices can have?