Ch 2. Basic Structures: Sets, Functions

Ch 9. Relations

Relations

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 9. Relations

9.1 Relations and Their Properties

9.2 n-ary Relations and Their Applications

9.3 Representing Relations

9.4 Closures of Relations

9.5 Equivalence Relations

9.6 Partial Orderings

A *relation* R is a set of ordered pairs. That is,

$$R \subseteq \{(a,b) \mid a \in A \wedge b \in B\}$$

A is called the *domain* of R and B is called the *range or codomain* of R.

Definition. Let A1, A2, ..., An be sets. An *n-ary relation* on these sets is a subset of A1 × A2 × ··· × An. The sets A1, A2, . . . , An are called the *domains* of the relation, and n is called its *degree*.

- ☛ If we just say a relation, we mean a *binary relation*.

Example 1. Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where

A is the airline,
N is the flight number,
S is the starting point,
D is the destination, and
T is the departure time.

Its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then

$$(\text{Nadir}, 963, \text{Newark}, \text{Bangor}, 15{:}00) \in R.$$

The degree of this relation is 5.

[redacted]

A *n-ary relation* on A is a subset of A $\times$ A $\times$ .... $\times$ A, that is A^n.

Definition

A *relation on a set S* is a subset of $S \times S$. If R is a relation on S, we say that "a is related to b" if $(a, b) \in R$, which we sometimes write as $a\,R\,b$. If $(a, b) \notin R$, then a is not related to b; in symbols, $a \not{R}\, b$.

Examples [redacted]

- The symbols $=$, $<$, $>$, $\leq$, $\geq$ all define relations on $\mathbf{Z}$ (or on any set of numbers). For example, if $S = \{1, 2, 3\}$, then the relation on S defined by $<$ is the set $\{(1, 2), (1, 3), (2, 3)\}$.
- Let P be the set of all people, living or dead. For any $a, b \in P$, let $a\,R\,b$ if a and b are (or were) brothers. Then R is a relation on P, and the ordered pair $(\text{Cain}, \text{Abel}) \in R$.
- Let W be the set of all web pages. Then

$$L = \{(a, b) \in W \times W \mid a \text{ has a link to } b\}$$

is a relation on W. In other words, $a\,L\,b$ if page a links to page b.

All the relations in the previous slide except for Example 1 were *binary relations*, i.e. sets of pairs.

Example 2. Let R be the relation on **N** × **N** × **N** consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$.

Then

$(1,2,3) \in R$

but

$(2,4,3) \notin R$.

The degree of this relation is 3.

Its domains are all equal to the set of natural numbers.

A more complicated relation

Example: the equivalence modulo n relation

Let $a, b \in \mathbb{N}$ If, for some $n \in \mathbb{N}$ $n \mid (a - b)$, we say that "a is equivalent to b modulo n." The notation for this relation is

i.e. "n divides (a - b)"

$$a \equiv b \mod n.$$

(also written $a \equiv_n b$)

For example, $1, 4, 7, 10, 13, \ldots$ are all equivalent modulo 3. Notice that

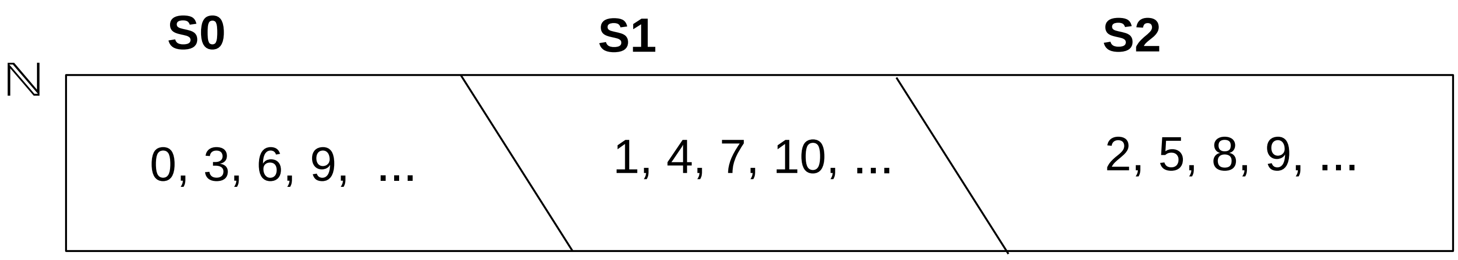
$$\begin{aligned} a \equiv b \mod n \quad &\Leftrightarrow \quad n \mid (a - b) && \text{- by definition of } \equiv \\ &\Leftrightarrow \quad a - b = kn \text{ for some } k \in \mathbb{N} && \text{- by definition of } \mid \\ &\Leftrightarrow \quad a = b + kn \text{ for some } k \in \mathbb{N} && \text{- by arithmetic} \end{aligned}$$

so adding any multiple of n to a number b gives a number that is equivalent to b modulo n.

0, 3, 6, 9, ... are all equivalent modulo 3
because if they are divided by 3 they all have the same remainder 0.
Call this set **S0**.

1, 4, 7, 10, ... are all equivalent modulo 3
because if they are divided by 3 they all have the same remainder 1.
Call this set **S1**.

2, 5, 8, 11, ... are all equivalent modulo 3
because if they are divided by 3 they all have the same remainder 2.
Call this set **S2**.



Quiz 11-1

Let a, b$\in\mathbb{Z}$ and n $\in\mathbb{N}$. Which of the following is NOT equivalent to the other formulas?

(a) $a \equiv_n b$

(b) $a \equiv b \bmod n$

(c) $n \mid (a - b)$

(d) $(n + b) \mid a$

(e) $\exists k\in\mathbb{N}\ (a - b = kn)$

(f) $\exists k\in\mathbb{N}\ (a = b + kn)$