CS 204 Discrete Mathematics Fall 2020

Homework 8

November 7, 2020

RECURSIVE DEFINITIONS

1. (10 pts) Give a recursive definition for the set X of all binary strings with an even number of 0's.

2. (10 pts) The following recursive definition defines a set $\mathbb{Z}$ of ordered pairs.

B. (2, 4) is in $\mathbb{Z}$.

R1. If (x,y) is in $\mathbb{Z}$ with $x < 10$ and $y < 10$, then (x+1,y+1) is in $\mathbb{Z}$.

R2. If (x,y) is in $\mathbb{Z}$ with $x > 1$ and $y < 10$, then (x-1, y+1) is in $\mathbb{Z}$.

Plot these ordered pairs in the xy-plane.

3. (8 pts) Give a recursive definition for the set X of even integers (including both positive and negative even integers).

4. (10 pts) Let S be a set of sets with the following recursive definition.

B. $\varnothing \in S$.

R. If $X \subseteq S$, then $X \in S$.

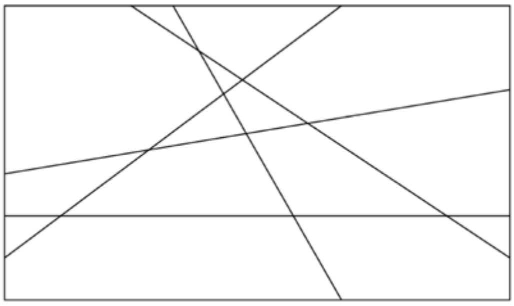
(a) List three different elements of S.

(b) Explain why S has infinitely many elements.

STRUCTURAL INDUCTION

1. (20 pts) A line map is defined as follows:
   B. A blank rectangle is a line map.
   R. A line map with a straight line drawn all the way across it is a new line map.

Here is an example of a line map.



(a) Prove by induction that a line map with n distinct lines has at least n+1 regions.
(b) Prove by induction that a line map with n distinct lines has at most 2^n regions.
(c) Part (a) gives a lower bound on the number of regions in a line map. For example, a line map with five lines must have at least six regions. Give an example of a line map that achieves this lower bound, that is, draw a line map with five lines and six regions.
(d) Part (b) says that a line map with three lines can have at most eight regions. Can you draw a line map with three lines that achieves this upper bound? Do so, or explain why you can't.

2. (15 pts) Define a Q-sequence recursively as follows.

Basis Case. <x, 4–x> is a Q-sequence (of length 2) for any real number x.
Recursive Case. If $<x_1, x_2, ..., x_{j-1},x_j>$ and $<y_1, y_2, ..., y_{k-1}, y_k>$ are Q-sequences, so is

$$< x_1 - 1, x_2, ..., x_{j-1},x_j, y_1, y_2, ..., y_{k-1}, y_k - 3 >$$

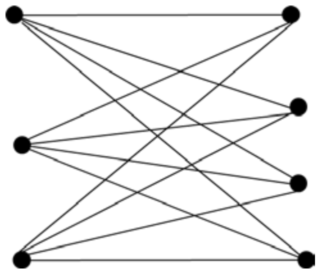
(, of which the length is j+k).

Use structural induction to prove that the sum of the numbers in any Q-sequence is 4.

RECURSIVE ALGORITHMS & RECURRENCE RELATIONS

1. (10 pts) Write an iterative algorithm to compute F(n), the n-th Fibonnaci number.

2. (10 pts) The complete bipartite graph $K_{m,n}$ is the simple undirected graph with m+n vertices split into two sets V1 and V2 (|V1| = m, |V2| = n) such that vertices x,y share an edge if and only if $x \in V1$ and $y \in V2$. For example $K_{3,4}$ is the following graph.



   (a) Find a recurrence relation for the number of edges in $K_{3,n}$.
   (b) Find a recurrence relation for the number of edges in $K_{n,n}$.

3. (15 pts) Consider the following recurrence relation:

$$G(n) = \begin{cases} 1 & \text{if } n = 0 \\ G(n-1) + 2n - 1 & \text{if } n > 0. \end{cases}$$

   (a) Calculate G(0), G(1), G(2), G(3), G(4), and G(5).
   (b) Guess at a closed-form solution for G(n) using sequence of differences.
   (c) Prove that your guess is correct.

4. (10 pts) Consider the following recurrence relation:

$$L(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ L(n-1) + L(n-2) & \text{if } n > 2. \end{cases}$$

Let α and β be the constants that are used to compute the Fibonacci numbers as below:

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

Prove that $L(n) = \alpha^n + \beta^n$ for all $n \in \mathbb{N}$. Use strong induction.

5. (12 pts)

Suppose that $a_1 = 10$, $a_2 = 5$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Prove that 5 divides a_n whenever n is a positive integer.