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Homework 5 - Proof

1) a) $3\cdot 1+5\cdot 5=3+25=28=7\cdot 4$ where we found that $3\cdot 1+5\cdot 5=7\cdot k$ for some $k\in\mathbb{Z}$ $(k=4) \Rightarrow \boxed{1\lhd 5}$

$3\cdot 3+5\cdot 1=9+5=14=7\cdot 2 \Rightarrow 3\cdot 3+5\cdot 1=7\cdot k$ for some $k\in\mathbb{Z}$ $(k=2)$, then $\boxed{3\lhd 1}$

$3\cdot 0+5\cdot 7=35=7\cdot 5$, or $3\cdot 0+5\cdot 7=7\cdot k$ for some $k\in\mathbb{Z}$ $(k=5)$ hence, $\boxed{0\lhd 7}$

b) Take $a=1, b=5, c=3, d=1 \Rightarrow$ we proved in a) that $a\lhd b$ and $c\lhd d$ are true, then suppose $a\cdot c\lhd b\cdot d$ is satisfying $\Rightarrow 3\lhd 5$ - true, or according to the definition $3\cdot 3+5\cdot 5=9+25=34=7k$ for some $k\in\mathbb{Z}$, but it is surely wrong as $\frac{34}{7}\notin\mathbb{Z} \Rightarrow$ hence, $a\cdot c\lhd b\cdot d$ is not satisfying meaning that we found specific counterexample ■

2) a) According to "Definition", the term "scalene" has been given a property and we know that "Definition"s are implicitly regarded as if and only if, although they are written as if-then $\Rightarrow$ Therefore, we can interpret our "Definition" as - A triangle is scalene if and only if all of its sides have different lengths. Now, coming back to part a), we were given $\triangle ABC=$ scalene triangle. Then, from

our "Definition", we can conclude that all of the sides of $\triangle ABC$ have different lengths $\Rightarrow$ $\boxed{\text{Conclusion is valid}}$

Note: A definition is a statement that describes the meaning of a new term, which implies that the statement is always true. A theorem is only true when a certain set of axioms is true

b) In this case, situation is different. "Theorem" implies: If a triangle is a right triangle that is not isosceles, then that triangle is scalene.
Given $\triangle ABC$ - scalene triangle, it's not necessary for $\triangle ABC$ to be a right triangle that is not isosceles. It can be any triangle for which all of its sides are having different lengths. $\boxed{\text{Conclusion is not valid}}$ ■

3) a) $(\forall n)(\exists x)(\exists y)(\exists z)\, P(n,x,y,z)$ or $(\forall n)(\exists x)(\exists y)(\exists z)(x^n+y^n=z^n)$

b) Using universal and existential negation rules consecutively

$\neg\left((\forall n)(\exists x)(\exists y)(\exists z)\, P(n,x,y,z)\right) \Leftrightarrow (\exists n)\neg\left((\exists x)(\exists y)(\exists z)\, P(n,x,y,z)\right)$
$\Leftrightarrow (\exists n)(\forall x)\neg\left((\exists y)(\exists z)\, P(n,x,y,z)\right) \Leftrightarrow (\exists n)(\forall x)(\forall y)\neg\left((\exists z)\, P(n,x,y,z)\right)$
$\Leftrightarrow (\exists n)(\forall x)(\forall y)(\forall z)\left(\neg P(n,x,y,z)\right)$. Therefore, negation of predicate logic statement from (a) is $(\exists n)(\forall x)(\forall y)(\forall z)(\neg P(n,x,y,z))$ or $(\exists n)(\forall x)(\forall y)(\forall z)(x^n+y^n \neq z^n)$

c) Our key point for finding a counterexample to the given statement in part a) is to find a specific integer n such that, for any positive integers $x, y, z \Rightarrow x^n + y^n \neq z^n$

So, firstly we have to come up with specific n, and then prove that for any positive integers $x, y, z \Rightarrow x^n + y^n$ will be different from z^n. (here, n - positive integer)
Or, equivalently $\Rightarrow$ prove that there does not exist positive integers x, y, z such that $x^n + y^n$ will be equal to z^n
(finding counterexample to (a) means $(\exists n)(\forall x)(\forall y)(\forall z)(\neg P(x,y,z))$ is true) ■

4) a) Let x be a xamel. If x is a borfin, then x has been schlumpfed

b) Let x be a xamel. If x is not a borfin, then x has not been schlumpfed

c) Part (b) is logically equivalent to given theorem. The first sentence is remained as it's and when we give the contrapositive of the theorem, it becomes logically equivalent to it. Specifically, this is contraposition rule and it means $p \to q$ is logically equivalent to $\neg q \to \neg p$

Note: Converse interchanges given 2 statements: $p \to q$ transforms into $q \to p$ (if q, then p). Contrapositive interchanges and negates $p \to q$ transforms into $\neg q \to \neg p$ ■

5) According to Axiom 3, there should be exactly four points. Then, considering Axiom 4, there does not exist 3 points which are on the same line. So, we can always have 3 distinct points which do not lie on the same line. Using Axiom 1, we know that for each pair of distinct points x and y, there is a unique line passing through both x and y. Then, we take arbitrary 3 points and we know they do not lie on the same line. Taking each pair from those 3 points, we know there'll exist unique line connecting those pairs (we'll have 3 pairs, and correspondingly, we'll get 3 lines). So, we concluded that it's <u>always possible</u> to have 3 distinct points, not lying on the same line, such that a line <u>passes</u> through each pair of points; in other words, $\boxed{\text{a triangle exists}}$

<u>Note:</u> "line" = line segment

"point" = end point line segment

A point "is on" a line when the point is one of the endpoints of the line ∎

6) By Definition 1.10 in David J. Hunter's book, An integer x divides an integer y if there is some integer k such that $y = kx \Rightarrow$ Considering a, b, c - are integers and using the fact $a|b$, (a divides b) it means there is some $k_1 \in \mathbb{Z}$ such that $b = k_1 \cdot a$ Similarly, knowing $b|c$ means b divides c and there's some $k_2 \in \mathbb{Z}$ for which $c = k_2 \cdot b$ Plugging b into new equation, $c = k_2 \cdot b = k_2 \cdot (k_1 a) =$ (associativity) $= (k_1 \cdot k_2) a \Rightarrow c = (k_1 \cdot k_2) a$ for some $k_1, k_2 \in \mathbb{Z}$. Since Axiom 1.1 in that book says: If a and b are integers, so are $a+b$ and $a \cdot b \Rightarrow$ then, $k_1 k_2$ - integer and using "divides" definition rule, we conclude $\boxed{a|c}$

As we see, the above proof is a direct proof ■

7) Let a, b - be rational numbers. From the definition of rational number, we find $a = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$ and $b = \frac{r}{s}$ for some $r, s \in \mathbb{Z}$ with $s \neq 0 \Rightarrow a + b =$ $= \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$. Using Axiom 1.1 in that book, p, s - integers $\Rightarrow ps \in \mathbb{Z}$ and q, r - integers $\Rightarrow qr \in \mathbb{Z}$. Similarly $ps + qr \in \mathbb{Z}$ and q, s - integers $\Rightarrow qs \in \mathbb{Z}$. If $qs = 0$, then either $q = 0$ or $s = 0$, but we knew q and s were different from zero. Therefore, $qs \neq 0$ should be satisfying. Combining $\Rightarrow$

$a+b=\dfrac{ps+qr}{qs}$ with $ps+qr, qs \in \mathbb{Z}$ and $qs \neq 0$

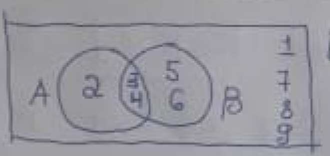
Using definition of "rational numbers", we conclude that

$\boxed{a+b \text{ is a rational number}}$ ■

Note: r-rational number if and only if there exist $a, b \in \mathbb{Z}$ such that $r=\frac{a}{b}$ and $b \neq 0$

HW 5- Sets

1)



a) $(A \cup B)' = \{1, 7, 8, 9\}$

b) $A \cap B = \{3, 4\}$

$A = \{2, 3, 4\}$

then according to Cartesian product rule for $A \cap B$ and A

$(A \cap B) \times A = \{(3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}$

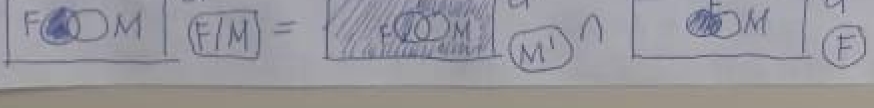
c) $B \setminus A = \{5, 6\}$, then according to power set rule for $B \setminus A$

$P(B \setminus A) = \{\varnothing, \{5\}, \{6\}, \{5,6\}\}$

2) a) Initially, let's translate the first sentence into language of set theory: freshmen who aren't math majors

F = set of all freshmen, M = set of all math majors

Ideally, this is equivalent to $F \setminus M$, but we can't use "difference" notation, so we transform it into $F \cap M'$



Intuitively, we observe from previous three pictures that $F \setminus M = F \cap M'$, but let's prove it mathematically: Assume $x \in (F \setminus M)$, then from definition of it $\Rightarrow (x \in F) \wedge (x \notin M)$ is true. Since $x \notin M$ means $x \in M'$ from its definition, we get $(x \in F) \wedge (x \in M')$ or $x \in (F \cap M')$ is true. Therefore, $\boxed{x \in (F \setminus M) \Rightarrow x \in (F \cap M')}$ Now, let $x \in (F \cap M')$. From the intersection, $(x \in F) \wedge (x \in M')$ is true. Since $x \in M'$ means $x \notin M \Rightarrow (x \in F) \wedge (x \notin M)$ is true $\Rightarrow x \in (F \setminus M)$. Henceforth $\boxed{x \in (F \cap M') \Rightarrow x \in (F \setminus M)}$ All this say $F \setminus M = F \cap M'$ and $\boxed{\text{first sentence} \rightarrow F \cap M'}$ Second sentence is "senior CS majors" and translated to $\boxed{S \cap C}$ Now, using cardinality for comparisons of # of students, we conclude $\boxed{|F \cap M'| > |S \cap C|}$

b) $F \cap M$ means freshmen who are math majors. Then $(F \cap M) \subseteq C$ states "Freshmen who are math majors are subset of CS majors". Alternatively, we have $\Rightarrow$ All students who are freshmen and math majors are part of CS majors

Freshmen who are math majors are part of CS majors
Freshmen who are math majors are portions (members) of CS majors ■

3) Assume finite sets A and B are disjoint. From the definition of "disjoint", they have no common elements $|A \cap B| = |\emptyset| = 0$, since $A \cap B = \emptyset$ and cardinality of an empty set is zero $\Rightarrow$ Using the inclusion-exclusion principle $|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B|$ from $|A \cap B| = 0 \Rightarrow$ $|A \cup B| = |A| + |B|$. Hence, $\boxed{A, B\text{-disjoint} \Rightarrow |A \cup B| = |A| + |B|}$

Now, let $|A \cup B| = |A| + |B|$, then implementing inclusion-exclusion principle, $|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B|$ or $\boxed{|A \cap B| = 0}$ that means $A \cap B$ has cardinality zero and implies A, B-share no common elements $\Rightarrow A \cap B = \emptyset$

From the definition of "disjoint", we can imply that Two sets are called disjoint iff they've no elements in common $\Rightarrow$ A, B-disjoint $\Rightarrow$ $\boxed{|A \cup B| = |A| + |B| \Rightarrow A, B\text{-disjoint}}$

Combining last 2 results, it is obviously becomes true that finite sets A and B are disjoint iff $|A| + |B| = |A \cup B|$

■

4) X - finite set, $|X| > 1$ and according to the definition of Cartesian Product, $P_1 = X \times X$ is the set of all ordered pairs (a, b), where $a, b \in X$. So basically, P_1 consists of ordered pairs and if $|X| = n > 1$, then # of such pairs are equal to $n^2 \Rightarrow \boxed{|P_1| = n^2}$ because we have n possibilities for first X and similarly n possibilities for second X. The multiplication rule gave us the desired answer n^2. Specifically, $X = \{a_1, a_2, ..., a_n\}$ are elements of X, then

$$P_1 = X \times X = \left\{ \begin{matrix} (a_1, a_1), & (a_1, a_2), ..., & (a_1, a_n) \\ (a_2, a_1), & (a_2, a_2), ..., & (a_2, a_n) \\ & \vdots & \\ (a_n, a_1), & (a_n, a_2), ..., & (a_n, a_n) \end{matrix} \right\}$$

P_1 is just set of ordered pairs, using elements from set X

When it comes to $P_2 = \{ S \in P(X) \mid |S| = 2 \}$, it is just about taking subsets from X which have cardinality 2. $P(X)$ is a power set of X, and it is the set of all subsets of $X \Rightarrow$ Then, from these subsets, we choose those who have cardinality $= 2$. Specifically, $X = \{a_1, a_2, ..., a_n\}$

$$P_2 = \{ \{a_1, a_2\}, \{a_1, a_3\}, ..., \{a_1, a_n\}, \{a_2, a_3\}, ..., \{a_2, a_n\}, ..., \{a_{n-1}, a_n\} \}$$

P_2 is just set of some sets (from set X), which have cardinality $= 2$

$$|P_2| = \underbrace{(n-1) + (n-2) + ... + 1}_{\text{(basic combinatorial fact)}} = \frac{(n-1)n}{2} = \underbrace{\binom{n}{2}}_{\text{(n choose 2)}}, \quad \boxed{|P_2| = \frac{n(n-1)}{2}}$$

Comparing P_1 and P_2, we observe that P_1 is just a set of ordered pairs from given set X (constructed from X) whereas P_2 is just a set of subsets of X, which have # of elements $= 2$. From one side, we have ordered pairs from the other side, we have sets. That's one of the key distinctions we should consider. Then, $|P_1| = n^2$ and $|P_2| = \frac{n(n-1)}{2}$ (# of elements in P_1 & P_2) is also main distinction for comparing P_1 & P_2

Since $n \geq 1$, $\frac{n^2-n}{2} < \frac{n^2}{2} < n^2 \Rightarrow \boxed{|P_2| < |P_1|}$ Therefore,

$\boxed{P_1 \text{ has more elements than } P_2}$ ■

5) a) For any $x \in W$ (set of words in the English Language) x and x have a letter in common, since they both are identical words and obviously share common letters. Then, xRx whenever x is from W, and therefore $\boxed{R \text{ is reflexive}}$ Next, suppose that xRy, so that x and y have a letter in common. Since we are comparing same words x & y, it's implying that y and x have a letter in common (nothing changed, same comparison's going). Therefore, yRx and it means $\boxed{R \text{ is symmetric}}$ Now, assume xRy and $yRz \Rightarrow$ that means

x and y have a common letter; y and z have a letter in common. However, we can't deduce x and z should have a common letter, since common letter for x & y can be different with common letter for y & z. As given domain is W, the set of words in the English language, we choose $\boxed{x = \text{pin},\ y = \text{people},\ z = \text{obey}}$ x and y have a common letter, which is "p"; y and z have a common letter → "o" (or "e") but x and z do not have a letter in common. Therefore, providing such a counterexample ⇒ $\boxed{R \text{ is not transitive}}$

Hence, $\boxed{R \text{ is not an equivalence relation}}$ considering that R is reflexive, symmetric, but not transitive

b) For any $x \in W$, x has at least as many letters as x since we are comparing # of letters of same words and it's obviously true ⇒ xSx whenever x is from W, and $\boxed{S \text{ is reflexive}}$ Next, suppose that xSy, so that # of letters in $x \geq$ # of letters in y ⇒ but it does not mean ySx, or # of letters in $y \geq$ # of letters in x

Take $\boxed{x = \text{apple},\ y = \text{pin}}$ xSy since # of letters in $x = 5 \geq$ # of letters in $y = 3$ with $x, y \in W$. However, we see # of letters in $y = 3 \geq$ # of letters in $x = 5$ is wrong meaning that ySx is wrong in this case. Providing such counterexample, we found $\boxed{S \text{ is not symmetric}}$ Finally, assume xSy and $ySz \Rightarrow$ that means

\# of letters in $x \geq^{\circledast}$ \# of letters in y; and

\# of letters in $y \geq^{\circledast}$ \# of letters in $z \Rightarrow$ using inequalities

\# of letters in $x \geq$ \# of letters in $y \geq$ \# of letters in z

from above $\circledast$ formulas $\Rightarrow$ then, $\boxed{\# \text{ of letters in } x \geq \geq \# \text{ of letters in } z}$

This means from the definition of relation S on W

$\boxed{xSz}$ So, we found that if xSy and $ySz \Rightarrow xSz$ implying

$\boxed{S \text{ is transitive}}$ So, $\boxed{S \text{ is not an equivalence relation}}$

Because S is reflexive, transitive, but not symmetric ■

6) a) Since $a^2 = a^2$ for any $a \in \mathbb{Z}$, we have aRa and this implies $\boxed{R \text{ is reflexive}}$ Suppose aRb, so that $a^2 = b^2 \Rightarrow$ then, $b^2 = a^2$ and from the definition, that says bRa. Therefore, $\boxed{R \text{ is symmetric}}$ Finally, suppose that aRb and $bRc \Rightarrow a^2 = b^2$ and $b^2 = c^2$, then $a^2 = c^2$ is true similarly, from the definition of relation R on $\mathbb{Z}$, aRc. Hence, $\boxed{R \text{ is transitive}}$ Because R is reflexive, symmetric, transitive, it is an equivalence relation

8) As we proved R is an equivalence relation on $\mathbb{Z}$, we define equivalence classes wrt the relation R as:

$\forall x \in \mathbb{Z}$, define $R_x = \{a \in \mathbb{Z} \mid xRa\}$

Since $xRa \Leftrightarrow x^2 = a^2$, this means $xRa \Leftrightarrow a = x$ or $a = -x$ (unless $x=0$). Plugging this result, $\boxed{R_x = \{-x, x\} \text{ if } x \neq 0 \ \forall x \in \mathbb{Z}}$ and $\boxed{R_0 = \{0\}}$ will be defined as equivalence classes of an integer for the relation R ■