CS 204 Discrete Mathematics Fall 2020

Homework 7

October 31, 2020

MATHEMATICAL INDUCTION

1. (12 pts) Consider the following recursively defined function

$$f(m,n) = \begin{cases} n + 1 & \text{if } m = 0 \\ f(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ f(m-1, f(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

a) What is the value of f(3,4)?

b) Prove by mathematical induction that $f(3,n) = 2^{n+3} - 3$.

2. (10 pts) Consider the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \le 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2. \end{cases}$$

Prove that $H(2n) = H(2n - 1) = n$ for all $n \ge 1$.

3. (10 pts) Consider the following recurrence relation:

$$C(n) = \begin{cases} 0 & \text{if } n = 0 \\ n+3\cdot C(n-1) & \text{if } n > 0. \end{cases}$$

Prove by induction that $C(n) = \dfrac{3^{n+1} - 2n - 3}{4}$ for all $n \ge 0$.

4. (10 pts) Let

$$f(m, n) = \begin{cases} 5 & \text{if } m = n = 1 \\ f(m-1,n) + 2 & \text{if } n = 1 \text{ and } m > 1 \\ f(m, n-1) + 2 & \text{if } n > 1 \end{cases}$$

Prove by mathematical induction that

$$f(m,n) = 2(m+n) + 1 \text{ for all, } m, n \in \mathbb{N}^+.$$

(Hint: First, define $(x1,y1) <= (x2,y2)$ iff $x1 < x2$ or ($x1 = x2$ and $y1 < y2$). Then use $(m,n) = (1,1)$ as the basis case.)

5. (15 pts)

Find a recurrence relation and initial conditions for the number of ways to go up a flight of stairs if stairs can be climbed one, two, or three at a time.

6. (15 pts)

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis Step) $(0,0) \in S$

Recursive Step) If $(a,b) \in S$, then $(a,b+1) \in S$, $(a+1, b+1) \in S$, and $(a+2, b+1) \in S$.

(a) List the elements of S produced by the first four applications of the recursive definition.

(b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \leq 2b$ whenever $(a, b) \in S$.