Anar Rzayev   Discrete Mathematics   20190788

Homework 8- Recursive Definitions

1) Define a set X of all binary strings with an even number of zeros as follows:

B1. λ is in X

B2. 1 is in X

R1. If x is in X, so is $0x0$

R2. If x and y are in X, so is xy

2) $(2,4) \in f \Rightarrow (3,5) \in f$ and $(1,5) \in f$

$(3,5) \in f \Rightarrow (4,6) \in f$ and $(2,6) \in f$

$(1,5) \in f \Rightarrow (2,6) \in f$ ✓, so up to now, we found $(2,4), (1,5), (3,5), (2,6), (4,6) \in f$. Continuing on it

$(2,6) \in f \Rightarrow (3,7)$ and $(1,7)$ are in f. Similarly, $(4,6) \in f \Rightarrow (5,7)$ and $(3,7)$ are in f. Hence, we get $(2,4), (1,5), (3,5), (2,6), (4,6), (1,7), (3,7), (5,7) \in f$

$(1,7) \in f \Rightarrow (2,8) \in f$

$(3,7) \in f \Rightarrow (4,8)$ and $(2,8)$ are in f

$(5,7) \in f \Rightarrow (6,8)$ and $(4,8)$ are in f. From this, $(2,4), (1,5), (3,5), (2,6), (4,6), (1,7), (3,7), (5,7), (2,8), (4,8), (6,8) \in f$

$(2,8) \in f \Rightarrow (3,9)$ and $(1,9)$ are in f

$(4,8) \in f \Rightarrow (5,9)$ and $(3,9)$ are in f

$(6,8) \in f \Rightarrow (7,9)$ and $(5,9)$ are in f

Therefore, $\underline{(1,9), (3,9), (5,9), (7,9) \in f}$. Going on from this

$(1,9) \in f \Rightarrow (2,10)$ is in f

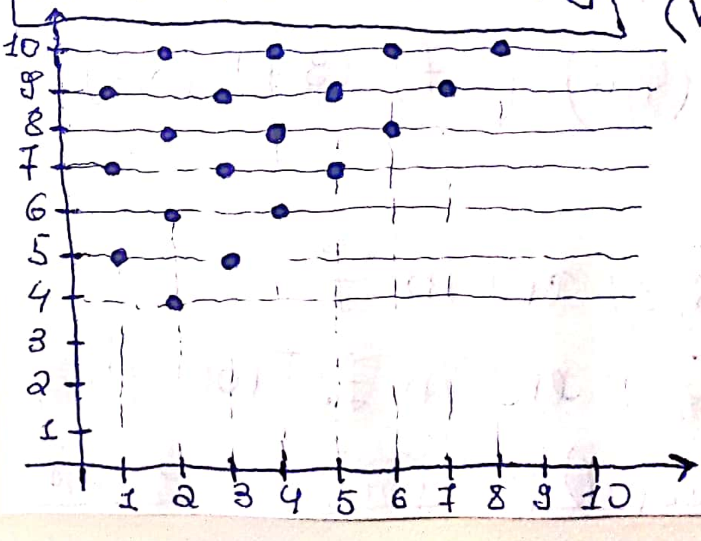
$(3,9) \in f \Rightarrow (2,10)$ and $(4,10)$ are in f

$(5,9) \in f \Rightarrow (4,10)$ and $(6,10)$ are in f

$(7,9) \in f \Rightarrow (6,10)$ and $(8,10)$ are in f. Thus, we find $\underline{(2,10), (4,10), (6,10), (8,10) \in f}$. Since we can't continue from this case, we conclude that <u>the set f is</u>

$$\boxed{f = \{(2,4), (1,5), (3,5), (2,6), (4,6), (1,7), (3,7), (5,7), (2,8), (4,8), (6,8), (1,9), (3,9), (5,9), (7,9), (2,10), (4,10), (6,10), (8,10)\}}$$

, all elements of f are presented (we used R_1 and R_2 in each case)



It's important to note that we were always using R_1 and R_2 for each element in f, whenever possible. Due to inequality constraint, this is the final outcome of f ■

3) Define a set X of even integers (Both positive and negative) as follows:

B. $0 \in X$

R1. If $x \in X$, so is $x+2$

R2. If $x \in X$, so is $x-2$

4) a) From the base case, $\emptyset \in S$. Since we know $\emptyset$ is an element of S, we get $\{\emptyset\} \subseteq S$ and thus, according to recursive case, $\{\emptyset\} \in S$. As $\emptyset$ and $\{\emptyset\}$ are elements of S, $\{\emptyset, \{\emptyset\}\} \subseteq S$, and accordingly due to recursive case, $\{\emptyset, \{\emptyset\}\} \in S$. Therefore, the three different elements of S are $\boxed{\{\emptyset, \{\emptyset\}\}, \{\emptyset\}, \emptyset}$

b) Assume to the sake of contradiction, that S has finitely many elements $x_1, x_2, \dots, x_n$. This means S has n number of elements. Note that $\{x_1\}$, $\{x_2\}, \dots, \{x_n\}$, $\{x_1, x_2, \dots, x_n\}$ are different $(n+1)$ subsets of S. These $(n+1)$ subsets then need to be elements in S, according to recursive case. Thus, it means S should contain at least $(n+1)$ eleme- nts

But, we had that S contains exactly n elements, contradicting our previous finding. This then implies that our assumption was wrong, meaning that $\boxed{S \text{ has infinitely many elements}}$ ■

Recursive Algorithms & Recurrence Relations

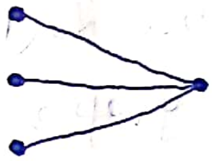
1) procedure iterative fibonacci(n: non-negative integer)

```
if n=0 then return 0
else
    x := 0
    y := 1
    for i := 1 to n-1
        z := x+y
        x := y
        y := z
    return y
```

{output is the n^{th} Fibonacci number} ■

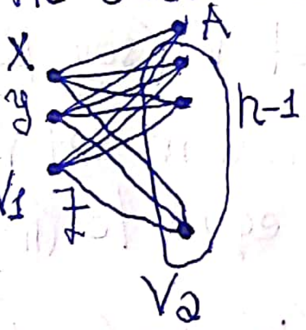
This procedure initializes x as $F(0)=0$ and y as $F(1)=1$. When the loop is traversed, the sum of x and y is assigned to the auxiliary variable z. Then x is assigned the value of y and y is assigned the value of the auxiliary variable z.

Therefore, after going through the loop the first time, it follows that x equals $F(1)$ and y equals $F(0)+F(1)=F(2)$. Furthermore, after going through the loop $(n-1)$ times, x equals $F(n-1)$ and y equals $F(n)$. Only $(n-1)$ additions have been used to find $F(n)$ with this iterative approach when $n \geq 1$ ∎

2) a) 

Assume the number of edges in the graph is denoted by $|K_{m,n}|$

Since we see from the above picture, $\boxed{|K_{3,1}|=3}$ is the base case for a recurrence relation



Assume we have the graph $K_{3,n-1}$ with $|V_1|=3$ and $|V_2|=n-1$. Similarly, number of edges will be equal to $|K_{3,n-1}|$

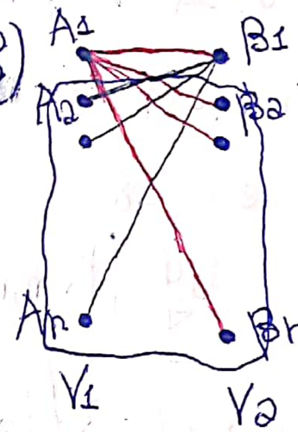
Now, we have graph $K_{3,n}$ with $|V_1|=3$, $|V_2|=n$, and # of edges $=|K_{3,n}|$. This implies there's an additi-onal vertex in V_2, and hence, there'll be 3 more edges connecting that vertex in V_2 with all 3 vertices in $V_1 \Rightarrow |K_{3,n}| = |K_{3,n-1}|+3$ for $\underline{n>1}$

$$\boxed{|K_{3,n}| = \begin{cases} 3, & \text{if } n=1 \\ |K_{3,n-1}|+3, & \text{if } n>1 \end{cases}}$$

(vertex A will have an edge with X, Y, Z, all vertices in V_1)

More formally, if $C(n)$ denotes the number of edges in the graph $K_{3,n}$, then we found that

$$\boxed{C(n)=\begin{cases}3, & \text{if } n=1\\ C(n-1)+3, & \text{if } n>1\end{cases}} \quad ⊛$$

b) 

Let $D(n)$ denote the number of edges in the graph $K_{n,n}$, where $|V_1|=|V_2|=n$ in the given picture. Since $D(n-1)$ is the # of edges on the graph $K_{n-1,n-1}$ we get that for the graph $K_{n-1,n}$ $(n-1)$ more edges should be added to $|K_{n-1,n-1}|$, because new vertex appeared on the set V_2 (say B_1) and it should be connected to all vertices in set V_1 (say $A_2, \dots, A_n$) where $|V_1|$ was initially equal to $(n-1)$.

So, $|K_{n-1,n}|=|K_{n-1,n-1}|+(n-1)=D(n-1)+(n-1)$

$|K_{n-1,n}|=D(n-1)+(n-1)$. Now, for the graph $K_{n,n}$ $|V_1|=|V_2|=n$, and there's an additional vertex in set V_1 (say A_1). It should be connected with all vertices in set V_2 (say $B_1, B_2, \dots, B_n$) where $|V_2|=n$. Hence, $|K_{n,n}|=D(n)=|K_{n-1,n}|+n$. Combining the results,

$D(n)=|K_{n-1,n}|+n=D(n-1)+(n-1)+n=D(n-1)+2n-1$

So, for $n>1 \Rightarrow D(n)=D(n-1)+2n-1$ holds. It is crucial to point out that $n>1$ condition was very important in constructing this formula, especially for the existence of $D(n-1)$ and $K_{n-1,n-1}$. For the base case, 

$|K_{1,1}|=D(1)=1$. In conclusion,

$$\boxed{D(n)=\begin{cases}1, & \text{if } n=1\\ D(n-1)+2n-1, & \text{if } n>1\end{cases}}$$

■ Since $|V_1|=|V_2|=1$ for the case $n=1$, we got $|K_{1,1}|=1$, as shown above

3) a) from the definition of recurrence relation,

$\boxed{G(0)=1}$ $G(1)=G(0)+2-1=G(0)+1=1+1=2 \Rightarrow$ $\boxed{G(1)=2}$ $G(2)=G(1)+4-1=2+3=5 \Rightarrow$ $\boxed{G(2)=5}$

$G(3)=G(2)+6-1=5+5=10$, $\boxed{G(3)=10}$ for $n=4 \Rightarrow$

$G(4)=G(3)+8-1=10+7=17$, $\boxed{G(4)=17}$ Plugging $n=5$

$G(5)=G(4)+10-1=17+9=26$, $\boxed{G(5)=26}$ Henceforth plugging $n=0,1,2,3,4,5$ to given recurrence relation and using previous values for $G(n)$, we obtained that

$\boxed{G(0)=1,\ G(1)=2,\ G(2)=5,\ G(3)=10,\ G(4)=17, \text{ and } G(5)=26}$ These are the values we found by using the recurrence relation

b) $G(0)=1,\ G(1)=2,\ G(2)=5,\ G(3)=10,\ G(4)=17,\ G(5)=26$

$1 \quad 3 \quad 5 \quad 7 \quad 9$

$2 \quad 2 \quad 2 \quad 2$

From the above diagram, we first determined the diffe- -rence of any pair of consecutive terms, obtaining the values $1, 3, 5, 7, 9 \Rightarrow$ Since the differences were not constant, we again evaluated the second sequence of differences, finding $2, 2, 2, 2$

As the second differences are constant, this suggests that sequence has 2^{nd}- degree polynomial as closed- -form

$\boxed{G(n) = An^2 + Bn + C}$ $G(0) = C = 1$, $G(1) = A + B + C = A + B + 1 = 2 \Rightarrow A + B = 1$.

$G(2) = 4A + 2B + C = 4A + 2B + 1 = 5$, $4A + 2B = 4 \Rightarrow 2A + B = 2$

we find $A = 1$ and $B = 0 \Rightarrow$ $G(n) = n^2 + 1$ # Checking this for each values, $G(0)=1$, $G(1)=2$, $G(2)=5$, $G(3)=10$, $G(4)=17$, $G(5)=26$, as seen from this, all're correct

Hence, closed-form solution is $G(n) = n^2 + 1$.

c) To prove "$G(n) = n^2 + 1$ for all $n \geq 0$", we'll use the mathematical induction on n, with base case $n = 0$

Scanned with CamScanner

Let $P(n)$ be the statement "$G(n)=n^2+1$"

Basis step: $n=0 \Rightarrow G(0)=1$, which is true from the definition, thus $P(0)$ is true

Inductive step: Let $P(k-1)$ be true for some $k>0 \Rightarrow$

$G(k-1)=(k-1)^2+1=k^2-2k+1+1=k^2-2k+2 \Rightarrow G(k-1)=k^2-2k+2$

We have to prove that $P(k)$ is true. From the definition of recurrence relation, $k>0 \Rightarrow G(k)=G(k-1)+2k-1$, and using inductive hypothesis, $G(k-1)+2k-1=k^2-2k+2+2k-1=k^2+1 \Rightarrow G(k)=k^2+1$ is true, or $P(k)$ is true

Hence, by the principle of Mathematical Induction, $P(n)$ is true for all $n \geq 0 \Rightarrow \boxed{G(n)=n^2+1, \forall n \geq 0}$ ■

4) Take the equation $x^2-x-1=0$, where solutions are found as $x_1=\frac{1+\sqrt{(-1)^2-4(-1)}}{2}$ and $x_2=\frac{1-\sqrt{(-1)^2-4(-1)}}{2}$

$x_1=\frac{1+\sqrt{1+4}}{2}=\frac{1+\sqrt{5}}{2}=\alpha$ and $x_2=\frac{1-\sqrt{1+4}}{2}=\frac{1-\sqrt{5}}{2}=\beta$

This implies $\boxed{\alpha, \beta \text{ are roots of the equation } x^2=x+1}$ or just $\boxed{\alpha^2=\alpha+1,\ \beta^2=\beta+1}$ Then, we proceed by strong induction on n:

Base Case: If $n=1$ or $n=2$, the recurrence relation says that $L(1)=1$ and $L(2)=3$. Using the values $\alpha, \beta \Rightarrow$

Scanned with CamScanner

$\alpha+\beta=\frac{1+\sqrt{5}}{2}+\frac{1-\sqrt{5}}{2}=\frac{2}{2}=1 \Rightarrow \alpha+\beta=1$, or $L(1)=1=\alpha+\beta$

we find $\boxed{L(1)=\alpha^1+\beta^1}$ and $\alpha^2+\beta^2=(\alpha+1)+(\beta+1)$, from

previous page $\Rightarrow$ since we also found $\alpha+\beta=1$, we can

get $\alpha^2+\beta^2=\alpha+\beta+2=1+2=3$, $\boxed{\alpha^2+\beta^2=3}$ $L(2)=3 \Rightarrow$

$\boxed{L(2)=\alpha^2+\beta^2}$ so; the closed-form solution $L(n)=\alpha^n+\beta^n$ is correct for $n=1$ and $n=2$ ✓

Inductive Step: Using the recurrence relation,

$L(k)=L(k-1)+L(k-2)=(\alpha^{k-1}+\beta^{k-1})+(\alpha^{k-2}+\beta^{k-2})$,

By inductive hypothesis where it's as follows:

Inductive Hypothesis: Let $k>2$, suppose as inductive hypothesis that $L(i)=\alpha^i+\beta^i$ for all i such that $1\leq i<k$

Inductive Step: $k>2 \Rightarrow k\geq 3$ or $1\leq k-2<k-1<k$, then using recurrence relation and inductive hypothesis,

$L(k)=L(k-1)+L(k-2)=\alpha^{k-1}+\beta^{k-1}+\alpha^{k-2}+\beta^{k-2}$, since $k>2$

As $\alpha^2=\alpha+1$, $\beta^2=\beta+1 \Rightarrow \alpha^{k-1}+\alpha^{k-2}=\alpha^{k-2}(\alpha+1)=\alpha^{k-2}\cdot\alpha^2=\alpha^k$

and $\beta^{k-1}+\beta^{k-2}=\beta^{k-2}(\beta+1)=\beta^{k-2}\cdot\beta^2=\beta^k$, $L(k)=(\alpha^{k-1}+\alpha^{k-2})+(\beta^{k-1}+\beta^{k-2})=\alpha^k+\beta^k \Rightarrow L(k)=\alpha^k+\beta^k$ becomes true ✓

Hence, applying strong induction on n, we found $\boxed{L(n)=\alpha^n+\beta^n \text{ for } \forall n\in\mathbb{N}}$

5) We proceed by strong induction on n

Base case: If $n=1$ or $n=2$, the recurrence relation says that $a_1=10$ and $a_2=5 \Rightarrow$ since $5|10$, $5|5$, we get $\boxed{5|a_1}$ and $\boxed{5|a_2}$ Thus, $\boxed{5|a_n \text{ is true for } n=1 \text{ and } n=2}$ V

Inductive Hypothesis: Let $k>2$, suppose as inductive hypothesis that $5|a_i$ for all i such that $1\le i<k$ V

Inductive Step: Using the recurrence relation, since $k>2$ $\Rightarrow k\geq 3$ and $a_k=2a_{k-1}+3a_{k-2}$. As $5|a_{k-1}$ and $5|a_{k-2}$ because of $1\le k-2<k-1<k$ $(3\le k)$ and inductive hypothesis, we know that $\boxed{5|a_{k-2},\ 5|a_{k-1}}$ Obviously $5|a_{k-2}+a_{k-2}+a_{k-2}=3a_{k-2}$ since $a_{k-2}=5a$ for some $a\in\mathbb{Z}$, $3a_{k-2}=5\cdot(3a)$ or $\boxed{5|3a_{k-2}}$ ⊠ Similarly, $a_{k-1}=5b$ for some $b\in\mathbb{Z}$

$2a_{k-1}=5\cdot(2b)$ or $\boxed{5|2a_{k-1}}$ ⊠ with $5|a_{k-1}+a_{k-1}=2a_{k-1}$

As we had $a_k=2a_{k-1}+3a_{k-2}$ where $5|2a_{k-1}$ and $5|3a_{k-2}\Rightarrow 5|2a_{k-1}+3a_{k-2}=a_k$, finding $\boxed{5|a_k}$ ⊠ V

Hence, from the principle of strong induction on n, we obtained $\boxed{5|a_n \text{ for all } n\in\mathbb{N}}$ ⊠ ■

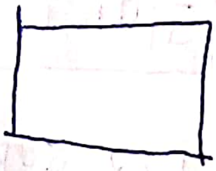
Note: If $5|x$ and $5|y$, then $5|x+y$ (Let $x=5a'$, $y=5b'$, $x+y=5(a'+b')$, or $5|x+y$)

Structural Induction

Proof by Induction

1) a) Let $P(n)$ be the statement "A line map with n distinct lines has at least $(n+1)$ regions"

<u>Basis step:</u> $n=0 \Rightarrow$



A line map with 0 distinct lines is a blank rectangle, so it has exactly 1 region where $1 \geq n+1 = 0+1 = 1$. Hence, $P(0)$ is true ✓

<u>Inductive step:</u> Assume for some $k > 0$, $P(k-1)$ is true

A line map with $(k-1)$ distinct lines has at least $(k-1)+1 = k$ regions ✓

We need to prove that $P(k)$ is true. Let's consider a line map with k distinct lines. When ignoring one of the lines a, then we know that there are at least k regions in the line map. When the line a is added to the line map, with $(k-1)$ distinct lines, then this line will divide at least one of the regions in two (since the line is distinct from all other lines) and thus at least 1 additional region is added, which results in at least $k+1$ regions in the line map.

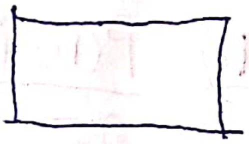
Thus, $P(k)$ is true. ✓

Scanned with CamScanner

Hence, in conclusion, from the principle of mathematical induction on n, we found $P(n)$ is true for all $n \in \mathbb{N}_0$

or, meaning $\boxed{\text{a line map with } n \text{ distinct lines has at least } (n+1) \text{ regions}}$

B) Proof By Induction: Let $P(n)$ be the statement "A line map with n distinct lines has at most 2^n regions"

Basis step: $n=0 \Rightarrow$  ; A line map with 0 distinct lines is a blank rectangle, and thus, it has exactly 1 region, where 1 is at most $2^n = 2^0 = 1 \Rightarrow P(0)$ is true ✓

Inductive step: Let $P(k-1)$ $(k \geq 1)$ be true $\Rightarrow$ A line map with $(k-1)$ distinct lines has at most 2^{k-1} regions. We need to prove that $P(k)$ is true. Let's consider a line map with k distinct lines. When ignoring one of the lines a, then we know that there are at most 2^{k-1} regions in the line map. When the line a is added to the line map with $(k-1)$ distinct lines, then this line will pass through at most k regions (one region between each pair of consecutive distinct lines ($k-2$ regions) and the 2 regions on either sides of all lines)

This line ϕ then divides at most k regions in two (as the line is distinct from all other lines) and thus, at most k additional regions are added, which results in at most $2^{k-1}+k$ regions in the line map. However, $2^{k-1}+k \leq 2^k$ and thus the line map has at most 2^k regions. Hence, P(k) is true ✓

Therefore, in conclusion, from the principle of mathematical induction, (on n), P(n) is true for all $n \geq 0$

Hence, $\boxed{\text{a line map with } n \text{ distinct lines has at most } 2^n \text{ regions}}$ ⊠

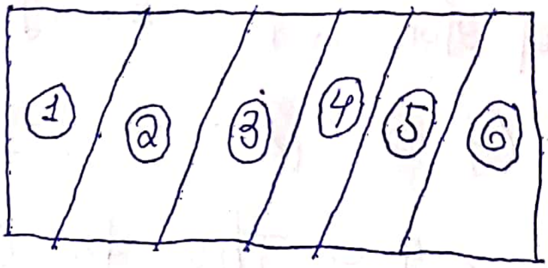
Note: $2^{k-1}+k \leq 2^k$ is true for all $k \geq 1$

$k=1 \Rightarrow 2^0+1=2 \leq 2^1$ ✓ Assume $2^{k-1}+k \leq 2^k$ is true for some $k=t \geq 1$, then $\cancel{2^{t-1}}+t \leq 2^t = \cancel{2^{t-1}}+2^{t-1}$, or $t \leq 2^{t-1}$. For $k=t+1$, we find that $2 \cdot 2^{t-1} \geq 2 \cdot t$ from inductive hypothesis, or just $2^t \geq 2t \geq t$ from $t \geq 1 \Rightarrow 2^t \geq t$, or $2^t+2^t=2^{t+1} \geq 2^t+t \Rightarrow 2^t+t \leq 2^{t+1}$

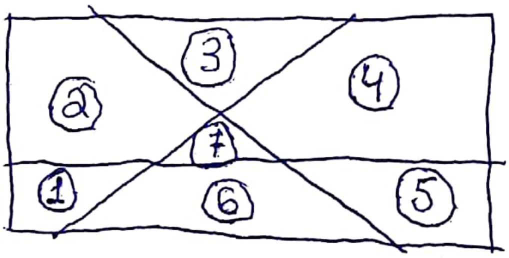
$2^t \geq 2t \geq t+1$ since $t \geq 1$, and $2^t+2^t=2^{t+1} \geq 2^t+(t+1)$

Hence, $2^{t+1-1}+(t+1)=2^t+(t+1) \leq 2^{t+1}$ becomes true, $k=t+1$ becomes true, and induction on k proves this property

c) We will obtain such a line map when the 5 lines are (approximately) parallel or when none of the lines intersect



d) It is not possible to draw such a map, because we obtain the most regions when all three lines intersect each other without falling together and we note that. We then get 7 regions instead of 8 regions (remember that $2^{3-1}+3=2^2+3=7<8=2^3$ inequality holds from part b), but with strict symbol)



It's impossible to obtain 8 regions, maximum is 7 as outlined in the proof for b) ■
(and also, $2^{3-1}+3<2^3$ is true)

2) Proof by Structural Induction: Base Case B:

Let x be a real number $\Rightarrow \langle x, 4-x\rangle$ is then a Q-sequence by the base case B, whose sum is $x+(4-x)=4$. Thus, all Q-sequences generated by the base case have a sum of 4 ✓

Recursive Case R: Let $\langle x_1, x_2, \dots, x_j\rangle$ and $\langle y_1, y_2, \dots, y_k\rangle$ be Q-sequences whose sum is 4 each $\Rightarrow$

Scanned with CamScanner

$x_1 + x_2 + \dots + x_j = 4$, $y_1 + y_2 + \dots + y_k = 4$

The recursive step then states that $\langle x_1 - 1, x_2, \dots, x_j, y_1, y_2, \dots, y_k - 3 \rangle$ is a Q-sequence, where sum of elements is

$(x_1 - 1) + x_2 + \dots + x_j + y_1 + y_2 + \dots + (y_k - 3) = (x_1 + \dots + x_j) - 1 + (y_1 + \dots + y_k) - 3 = 4 - 1 + 4 - 3 = 8 - 4 = 4$. Thus, the Q-sequence $\langle x_1 - 1, \dots, x_j, y_1, y_2, \dots, y_k - 3 \rangle$ then also has a sum of 4, which implies that all Q-sequences introduced by the recursive step also have a sum of 4

Henceforth, by the <u>principle of structural induction</u>,

$\boxed{\text{all Q-sequences have a sum of } 4}$ ■