Ch 1. The Foundations: Logic and Proofs

Logic and Mathematics

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Historically,

$$\text{Logic} \underset{\boxed{\text{We want to make logic mathematical.}}}{\overset{\text{is the basis of}}{\rightleftarrows}} \text{Mathematics}$$

Historically,

Logic $\xrightarrow{\text{is the basis of}}$ Mathematics

Logic $\longleftarrow$ Mathematics

We want to make logic mathematical.

$\Downarrow$

Formal System

Historically,

Logic $\xrightarrow{\text{is the basis of}}$ Mathematics

We want to make logic mathematical.

Gödel proved in 1929 Ph.D. Dissertation

Successful
<= Completeness
of the Formal System
for Predicate Logic

Formal System

Historically,

Logic — is the basis of → Mathematics

← We want to make logic mathematical.

Gödel proved in 1929 Ph.D. Dissertation

Successful

<= Completeness
of the Formal System
for Predicate Logic

Gödel proved in 1931

Unsuccessful

<= Incompleteness
of Formalizing
Arithmetic

Formal System

Incompleteness Theorem [Gödel 1931]

Any number-theoretic formal system (i.e. a consistent formal system to contain arithmetic) is incomplete (i.e. there is a true statement or a theorem of arithmetic that cannot be proved within the system).

Kurt Gödel

(1906 – 1978)

Historically,

is the basis of

Logic $\longrightarrow$ Mathematics

We want to make logic mathematical.

Formal System => Automated Computation

Enabled by Modern Computer

=> Linking Logic and Mathematics

- The Role of Definitions in Mathematics
- Other Types of Mathematical Statements
- Counterexamples
- Axiomatic Systems

Definitions in mathematics

Dictionary definitions are usually *descriptive*.

Example: "mortadella" - Any of several types of Italian sausage.

- Definitions in mathematics are *stipulative*.
  (= demand or require)

Example:

Definition

Two lines are *parallel* if they have no points in common.

Does this agree with your *concept image*?

- No. It specifies what exactly we "shall" mean when we use the word "parallel".

Using definitions: even and odd

Definition

An integer n is *even* if $n = 2k$ for some integer k.

Definition

An integer n is *odd* if $n = 2k + 1$ for some integer k.

To show that "17 is odd" we note that $17 = 2 \cdot 8 + 1$.

(Let k = 8. Then 17 = 2 • 8 + 1. Therefore 17 is odd.)

How can you show that 10 is even?

Definitions are reversible

A definition of the form

[Object] x is [defined term] if [defining property about x].

means

$$(\forall x)\ (\mathrm{D}(x) \leftrightarrow \mathrm{P}(x))$$

where

$$\begin{aligned} D(x) &= x \text{ is [defined term]} \\ P(x) &= \text{[defining property about } x\text{]} \end{aligned}$$

- D: definiendum

- P: definiens

Example

$$\forall n(Even(n) \leftrightarrow \exists k(n = 2 \times k))$$

Mathematical statements

- Definitions
- Axioms (a.k.a. postulates)
- Theorems
- Corollaries
- Lemmas
- Propositions : statements
- Claims
- Conjectures

Theorems

Theory: the set of all statements that can be proved from a set of axioms.

Axiomatic systems

- Axioms
- Undefined terms

An ***axiomatic system*** consists of axioms and undefined terms.

Axioms of the Formal Number Theory

(S1) $x = y \rightarrow (x = z \rightarrow y = z)$

(S2) $x = y \rightarrow x' = y'$

(S3) $0 \neq x'$

(S4) $x' = y' \rightarrow x = y$

(S5) $x + 0 = x$

(S6) $x + y' = (x + y)'$

(S7) $x \cdot 0 = 0$

(S8) $x \cdot (y') = (x \cdot y) + x$

(S9) For any well-formed formula $\mathcal{A}(x)$ of S,

$$\mathcal{A}(0) \Rightarrow (((\forall x)\, \mathcal{A}(x) \Rightarrow \mathcal{A}(x')) \Rightarrow (\forall x)\, \mathcal{A}(x))$$

Axioms of the Formal Set Theory

Extensionality Axiom If two sets have exactly the same members, then they are equal.

$$\forall A\ \forall B\ [\forall x\ (x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$$

Empty Set Axiom There is a set having no members.

$$\exists B\ \forall x\ \ x \notin B.$$

Pairing Axiom For any sets u and v, there is a set having as members just u and v.

$$\forall u\ \forall v\ \exists B\ \forall x\ [x \in B \Leftrightarrow x = u \vee x = v]$$

Union Axiom For any sets a and b, there is a set whoe members are those sets belonging either to a or to b (or both)

$$\forall a\ \forall b\ \exists B\ \forall x\ [x \in B \Leftrightarrow x \in a \vee x \in b]$$

Power Set Axiom For any set a , there is a set whose members are exactly the subsets of a

$$\forall a\ \exists B\ \forall x\ [x \in B \Leftrightarrow x \subseteq a]$$

Subset Axioms For each formula P(x) not containing B,

$$\forall t_1 \ldots \forall t_k\ \forall c\ \exists B\ \forall x\ [x \in B \Leftrightarrow x \in c\ \&\ P(x)\ ]$$

The Axioms of Euclidean Plane Geometry

1. A straight line may be drawn between any two points.
2. Any terminated straight line may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.
5. For any given point not on a given line, there is exactly one line through the point that does not meet the given line.

Axiomatic system for four-point geometry

Examples.

Undefined terms: point, line, is on

Axioms:

1. For every pair of distinct points x and y, there is a unique line l such that x is on l and y is on l.
2. Given a line l and a point x that is not on l, there is a unique line m such that x is on m and no point on l is also on m.
3. There are exactly four points.
4. It is impossible for three points to be on the same line.

Simple four-point theorem and proof

Examples.

Theorem

In the axiomatic system for four-point geometry, there are at least two distinct lines.

Proof.

Simple four-point theorem and proof

Examples.

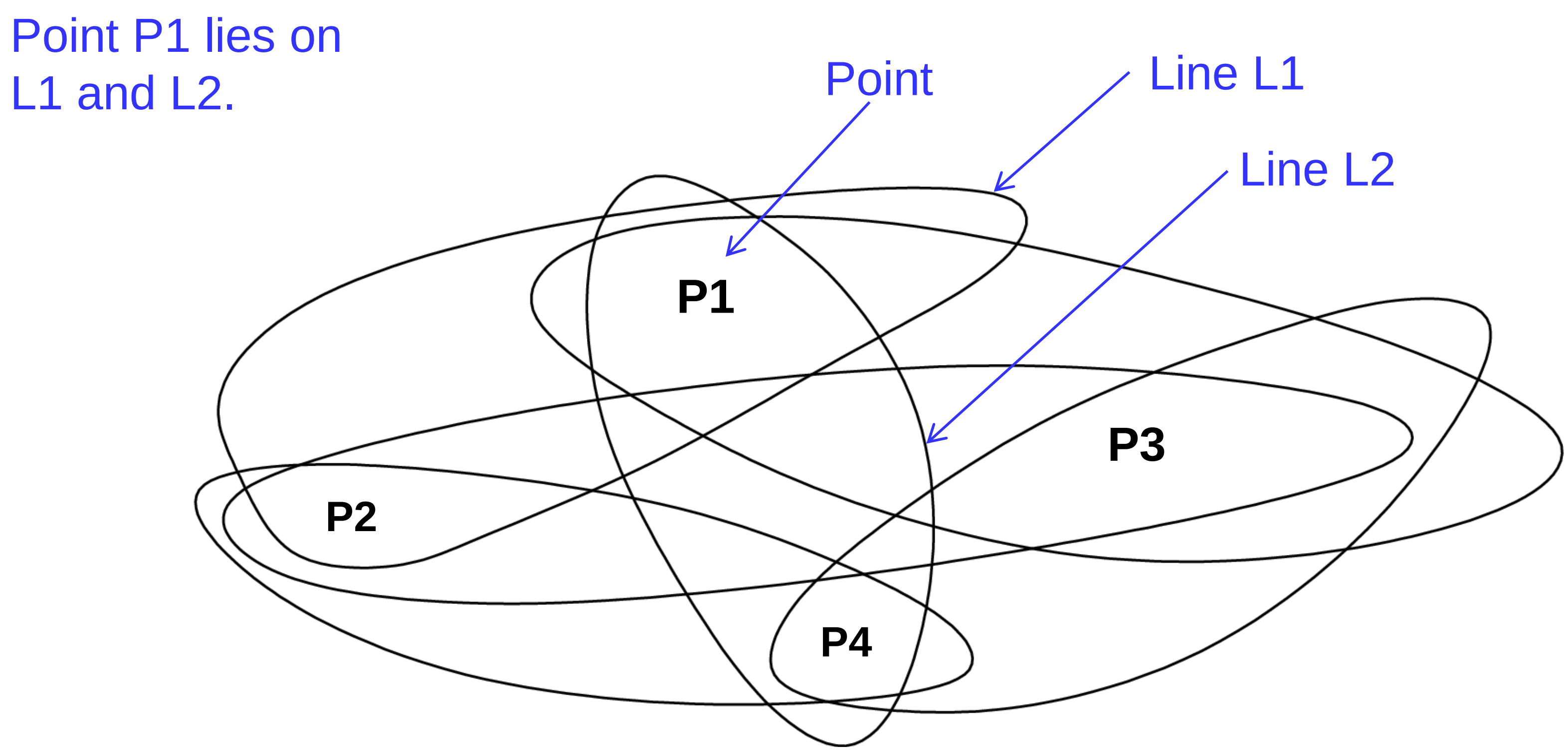
Theorem

In the axiomatic system for four-point geometry, there are at least two distinct lines.

Proof.

By Axiom 3, there are distinct points x, y, and z. By Axiom 1, there is a line l_1 through x and y, and a line l_2 through y and z. By Axiom 4, x, y, and z are not on the same line, so l_1 and l_2 must be distinct lines. $\square$

Model for four-point geometry



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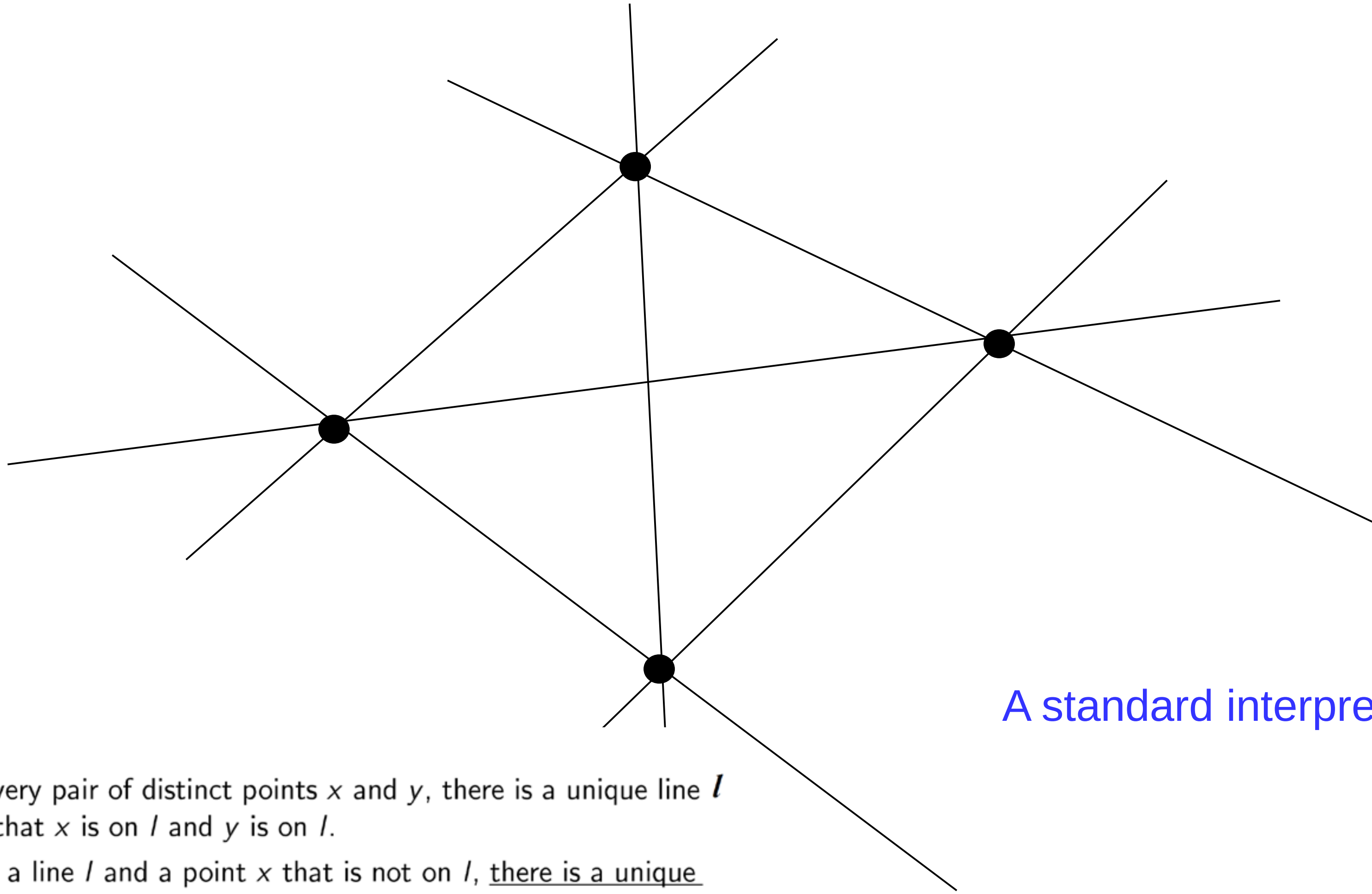

A standard interpretation

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Badda-Bing axiomatic system

Undefined terms: <u>badda, bing, hit</u>

Axioms:

1. Every badda hits exactly four bings.
2. Every bing is hit by exactly two baddas.
3. If x and y are distinct baddas, each hitting bing q, then there are no other bings hit by both x and y.
4. There is at least one bing.

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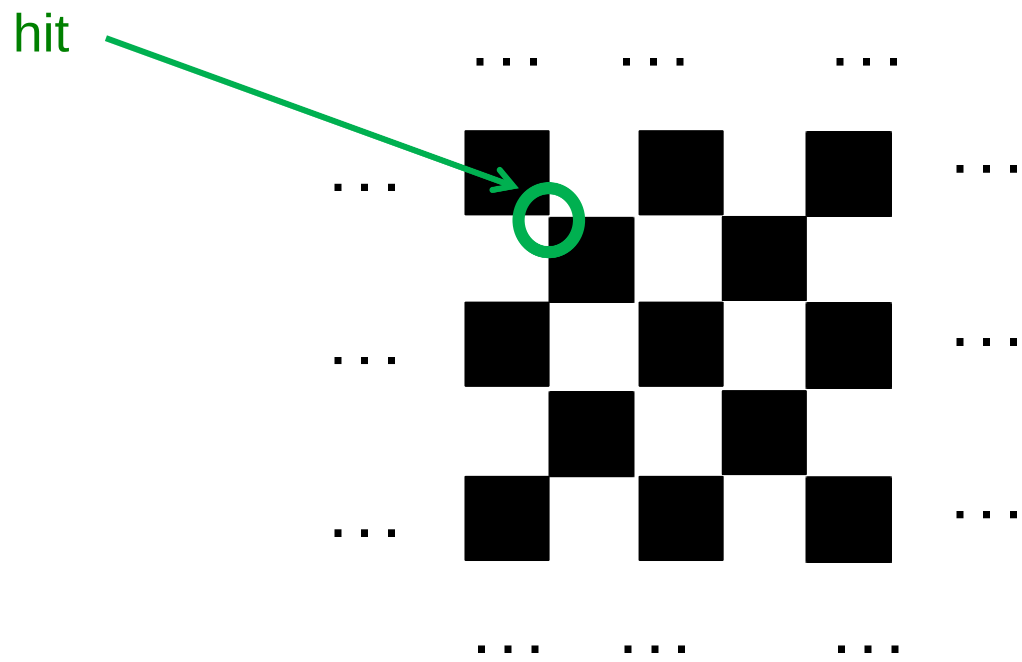
One possible interpretation:

badda: square
bing: corner of a square
hit: a square hits corner if the corner belongs to the square

A model for badda-bing system



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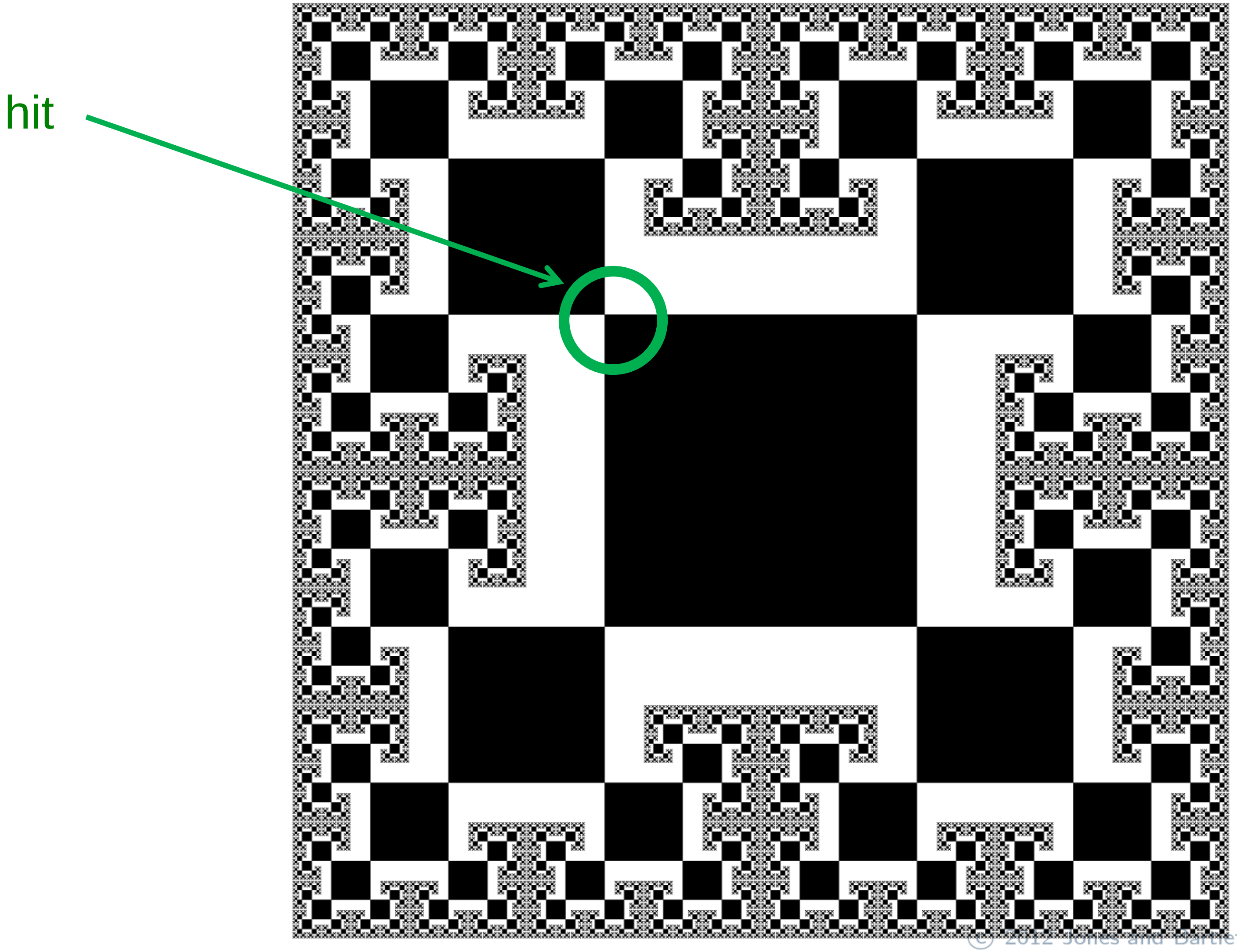
Model for badda-bing system using fractals



Axiomatic system

- All its theorems can be generated from the axioms with logical or mathematical inference rules.

- Can we capture all mathematical theorems in a finite set of axioms? (☛ Hilbert's Program in early 1920s.)
  - No !
  - This idea, although failed for the whole mathematics, provides an ideal way of organizing and developing computer programs

Quiz 08-1

Which of the following is FALSE ?

[1] A formal system can be inconsistent (or unsound).

[2] An inconsistent system has no model.

[3] A consistent system has one or more models.

[4] A consistent and complete system has a model.

[5] A consistent and incomplete system has no model.