Ch 1. The Foundations: Logic and Proofs

Predicate Logic - 1

Sungwon Kang

Acknowledgement
- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 1. The Foundations: Logic and Proofs

1.1 Propositional Logic
1.2 Applications of Propositional Logic
1.3 Propositional Equivalences
1.4 Predicates and Quantifiers
1.5 Nested Quantifiers
1.6 Rules of Inference
1.7 Introduction to Proofs
1.8 Proof Methods and Strategy

With the propositional logic, we cannot make the following valid inference:

Socrates is a human.
All humans die.

Socrates dies.

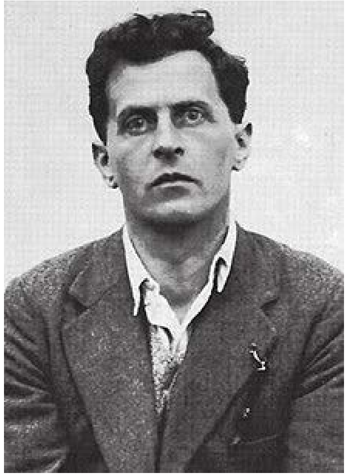
because it would translate to

$$\frac{\begin{array}{l} p \\ q \end{array}}{r}$$

“*The limits of my language* mean

the limits of my world.”

Tractatus Logico-Philosophicus, 1922



Ludwig Wittgenstein
(1889 – 1951)

Predicates

Definition

A *predicate* is a declarative sentence whose T/F value depends on one or more variables. In other words, a predicate is a declarative sentence with variables, and after those variables have been given specific values, the sentence becomes a statement.

Example:

A sentence has a subject and a predicate.

$$
\begin{aligned}
P(x) &= \text{“}x \text{ is even”} \\
Q(x, y) &= \text{“}x \text{ is heavier than } y\text{”}
\end{aligned}
$$

are predicates. The statement $P(8)$ is true, while the statement $Q(\text{feather}, \text{brick})$ is false.

Predicates

Definition

A *predicate* is a declarative sentence whose T/F value depends on one or more variables. In other words, a predicate is a declarative sentence with variables, and after those variables have been given specific values, the sentence becomes a statement.

Domain (Universe) of a variable: the set of values that a variable can take.

In the previous example,
for P(x), the domain of x $\rightarrow$ the set of integers;
for Q(x,y) the domain of x and y $\rightarrow$ the set of physical objects.

Quantifiers

A *quantifier* indicates whether some or all elements of the domain satisfy the predicate.

- Universal quantifier: $\forall$
- Existential quantifier: $\exists$

A sentence either asserts or denies existence of a subject or subjects that satisfies a given predicate or the opposite of the predicate.

Quantifiers

A *quantifier* indicates whether some or all elements of the domain satisfy the predicate.

- Universal quantifier: $\forall$
- Existential quantifier: $\exists$

The statement

$$(\forall x)P(x)$$

says that $P(x)$ is true *for all* x in the domain.

The statement

$$(\exists x)P(x)$$

says that *there exists* an element x of the domain such that $P(x)$ is true; in other words, $P(x)$ is true *for some* x in the domain.

Connection between Propositional Logic and Predicate Logic (1/3)

Suppose that S = {Larry, Joe, Moe} and we want to say that all elements in S are tall.

To assert the statement that

All the elements of a given set satisfy a property.

we can take either of the following two approaches:

1) We introduce a predicate Tall(__) for "__ is tall" and state:

$\text{Tall(Larry)} \wedge \text{Tall(Joe)} \wedge \text{Tall(Moe)}$

or, in short, $\forall x\ \text{Tall}(x)$

2) Alternatively, we can let p represent "Larry is tall", q "Joe is tall" and r "Moe is tall" and write:

$p \wedge q \wedge r$

So both $\forall x\ \text{Tall}(x)$ and $p \wedge q \wedge r$ express the statement "All elements in S are tall" under the interpretations above, respectively.

Connection between Propositional Logic and Predicate Logic (2/3)

Suppose that S = {Larry, Joe, Moe} and we want to say that some elements in S are tall.

Similarly for the statement that

There exists in a given set an element that satisfies a property.

we can take either of the following two approaches:

1) We introduce a predicate Tall(__) for "__ is tall" and state:

$\text{Tall(Larry)} \vee \text{Tall(Joe)} \vee \text{Tall(Moe)}$

or, in short, $\exists x\ \text{Tall}(x)$

2) Alternatively, we can let p represent "Larry is tall", q "Joe is tall" and r "Moe is tall" and write:

$p \vee q \vee r$

So both $\exists x\ \text{Tall}(x)$ and $p \vee q \vee r$ express the statement "Some elements in S are tall" under the interpretations above, respectively.

Connection between Propositional Logic and Predicate Logic (3/3)

- But for large sets enumeration becomes unwieldy.

- For infinite sets, it is impossible to express in propositional logic.

  $\forall$x Tall(x) should be expanded to

  Tall(p1) $\wedge$Tall(p2) $\wedge$ Tall(p2) $\wedge$ …

  $\exists$x P(x) should be expanded to

  Tall(p1) $\vee$ Tall(p2) $\vee$ Tall(p3) $\vee$ …

- However, the expanded expressions are not valid propositions.

Scoping

- A quantifier has its scope.

  In $\forall x$ *A(x)*, the scope of the quantifier "$\forall x$" is *A(x)*.

  In $\exists x$ *A(x)*, the scope of the quantifier "$\exists x$" is *A(x)*.

Example

$\forall x\ (P(x) \wedge Q(x) \rightarrow R(x))$

$\forall x\ (P(x) \wedge Q(x)) \rightarrow R(x)$ -- different

$(\forall x\ P(x)) \wedge (Q(x) \rightarrow R(x))$ -- different

Bound and Free Variables (1/2)

- An occurrence of a variable x in a formula A is said to be *bound* (or *as a bound variable*), if the occurrence is in a quantifier $\forall x$ or $\exists x$ or in the scope of a quantifier $\forall x$ or $\exists x$ (with the same x); otherwise, *free* (or *as a free variable*).
- A variable x which occurs as a free variable (briefly, occurs free) is A is called a *free variable of* A, and A is then said to *contain x as a free variable* (briefly, to *contain x free*); and likewise for bound variables.

Bound and Free Variables (2/2)

Example 1. $\forall x\ (\forall y\ (\exists z\ (\ P(x,y,z)\ )))$ -- variables x, y, z are bound

Example 2. $\forall x\ (\forall y\ (\ P(x,y,z)\ ))$ -- variable z is free

- A variable may be both free and bound in the same expression

  (but an occurrence of a variable **cannot** be both free and bound)

Example 3. $\forall x(x > y) \land \exists y(y > 0)$ -- y is both free and bound

Example 4. $\forall x\ (P(x) \land \forall x\ (Q(x) \rightarrow R(y)) \land Q(x))$

-- “y” is a free variable

-- scope hole in middle of expression

Renaming

- Bound variables may be changed (within their scope) provided that the new variable does not make *a free variable* bound.

Example 1. $\forall x(P(x) \rightarrow R(x))$

may be changed to $\forall y(P(y) \rightarrow R(y))$

Example 2. $\forall x\ (P(x) \wedge (\forall x\ (Q(x) \rightarrow R(y))) \wedge Q(x))$

may be changed to

$\forall z\ (P(z) \wedge (\forall x\ (Q(x) \rightarrow R(y))) \wedge Q(z))$

but NOT to

$\forall y\ P(y) \wedge (\forall x\ (Q(x) \rightarrow R(y))) \wedge Q(y))$

- Note the similarity to local variables in programs

Translating predicate logic expression

Example: Let the domain be the set of all cars.

$$
\begin{aligned}
P(x) &= \text{“}x \text{ gets good mileage.”} \\
Q(x) &= \text{“}x \text{ is large.”}
\end{aligned}
$$

$(\forall x)(Q(x) \to \neg P(x))$ translates as

?

Example: Let the domain be the set of all cars.

$$\begin{aligned} P(x) &= \text{“}x \text{ gets good mileage.”} \\ Q(x) &= \text{“}x \text{ is large.”} \end{aligned}$$

$(\forall x)(Q(x) \to \neg P(x))$ translates as

For all cars x, if x is large, then x does not get good mileage.

or

All large cars get bad mileage.

or

There aren't any large cars that get good mileage.

Translating natural language sentences

Example: Let the domain be the set of integers.

Let $P(x) =$ "x is even."

The sum of an even number with an odd number is odd.

translates as

?

[redacted]

Example: Let the domain be the set of all integers.

Let $P(x) =$ "x is even."

The sum of an even number with an odd number is odd.

translates as

$$(\forall x)(\forall y)[(P(x) \wedge \neg P(y)) \rightarrow (\neg P(x+y))]$$

Literally,

for all integers x and for all integers y, if x is even and y is not even, then $x + y$ is not even.

Example:

Translate the following natural language sentences into predicate logic formulas:

a $<$ b: a is less than b$\wedge$

(a) Some subset of the set of natural numbers does not have the *greatest* element. For example, { 6, 8, 10, 12, . . . } does not have the greatest element.

$\exists S(S \subseteq \mathbb{N} \wedge \neg\exists x \{ x \in S \wedge \forall y(y \in S \rightarrow y \leq x)\})$

(b) Every subset of the set of natural number has the *greatest lower bound*. For example, { 6, 8, 10, 12, . . . } has 6 as the greatest lower bound.

$\forall S(S \subseteq \mathbb{N} \rightarrow \exists x \{ x \in S \wedge$ x is a lower bound of S $\wedge$
x is the greatest lower bound of S$\}$)

=

$\forall S(S \subseteq \mathbb{N} \rightarrow \exists x \{ x \in S \wedge \forall y(y \in S \rightarrow x \leq y)$
$\wedge \; \forall z[(\forall y(y \in S \rightarrow z \leq y) \wedge z < x) \rightarrow z \notin S] \; \})$

Order matters

In general,

$$(\forall y)(\exists x)G(x,y)$$

is different from

$$(\exists x)(\forall y)G(x,y).$$

For example, $G(x,y) =$ "$x > y$."

Exercise 1:
Translate these two formulas into natural language sentences.

Exercise 2:
Which is true?
Which is false?

Two common constructions (= Idioms)

1. "All $\langle$blanks$\rangle$ are $\langle$something$\rangle$."

   Example Every student in this class has studied calculus.

2. "There is a $\langle$blank$\rangle$ that is $\langle$something$\rangle$."

   Example Some student in this class has visited Mexico.

Two common constructions (= Idioms)

1. "All ⟨blanks⟩ are ⟨something⟩." translates as

$$\textbf{(1)} \quad (\forall x)(P(x) \to Q(x)).$$

2. "There is a ⟨blank⟩ that is ⟨something⟩." translates as

$$\textbf{(2)} \quad (\exists x)(P(x) \wedge Q(x)).$$

Two common constructions (= Idioms)

Example 1. All baseball players are rich.

Interpretation
Domain of x: all people
P(x): x is a baseball player.
Q(x): x is rich.

Example 2. Some oysters taste funny.

Interpretation
Domain of x: shellfish
P(x): x is an oyster.
Q(x): x tastes funny.

Two common constructions (= Idioms)

Example 1. All baseball players are rich.

Interpretation
Domain of x: all people
P(x): x is a baseball player.
Q(x): x is rich.

translates to $(\forall x)(P(x) \rightarrow Q(x))$

Example 2. Some oysters taste funny.

Interpretation
Domain of x: shellfish
P(x): x is an oyster.
Q(x): x tastes funny.

translates to $(\exists x)(P(x) \wedge Q(x))$

Negation rules for Quantifiers

Equivalence	Name
$\neg[(\forall x)P(x)] \Leftrightarrow (\exists x)(\neg P(x))$	universal negation
$\neg[(\exists x)P(x)] \Leftrightarrow (\forall x)(\neg P(x))$	existential negation

Proofs of the Negation Rules for Quantifiers

Example: Domain = all car, revisited.

$$P(x) = \text{“}x \text{ gets good mileage.”}$$
$$Q(x) = \text{“}x \text{ is large.”}$$

Negate the statement “All large cars get bad mileage.”

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
n. $(\exists x)(P(x) \wedge Q(x))$	commutativity

Proofs of the Negation Rules for Quantifiers

Example: Domain = all car, revisited.

$$
\begin{aligned}
P(x) &= \text{"x gets good mileage."} \\
Q(x) &= \text{"x is large."}
\end{aligned}
$$

Negate the statement "All large cars get bad mileage."

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \to \neg P(x))$	universal negation , 1
n. $(\exists x)(P(x) \wedge Q(x))$	

Proofs of the Negation Rules for Quantifiers

Example: Domain = all car, revisited.

$$\begin{aligned} P(x) &= \text{"}x \text{ gets good mileage."} \\ Q(x) &= \text{"}x \text{ is large."} \end{aligned}$$

Negate the statement "All large cars get bad mileage."

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \to \neg P(x))$	universal negation, 1
3. $(\exists x)\neg(\neg Q(x) \vee \neg P(x))$	implication, 2
n. $(\exists x)(P(x) \wedge Q(x))$	

Proofs of the Negation Rules for Quantifiers

Example: Domain = all car, revisited.

$$\begin{aligned} P(x) &= \text{“}x \text{ gets good mileage.”} \\ Q(x) &= \text{“}x \text{ is large.”} \end{aligned}$$

Negate the statement "All large cars get bad mileage."

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \to \neg P(x))$	universal negation , 1
3. $(\exists x)\neg(\neg Q(x) \vee \neg P(x))$	implication , 2
4. $(\exists x)(\neg(\neg Q(x)) \wedge \neg(\neg P(x)))$	De Morgan's law, 3
n. $(\exists x)(P(x) \wedge Q(x))$	

Proofs of the Negation Rules for Quantifiers

Example: Domain = all car, revisited.

$$P(x) = \text{“}x \text{ gets good mileage.”}$$
$$Q(x) = \text{“}x \text{ is large.”}$$

Negate the statement “All large cars get bad mileage.”

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \rightarrow \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \rightarrow \neg P(x))$	universal negation , 1
3. $(\exists x)\neg(\neg Q(x) \vee \neg P(x))$	implication , 2
4. $(\exists x)(\neg(\neg Q(x)) \wedge \neg(\neg P(x)))$	De Morgan's law, 3
5. $(\exists x)(Q(x) \wedge P(x))$	double negation twice, 4
n. $(\exists x)(P(x) \wedge Q(x))$	

Proofs of the Negation Rules for Quantifiers

Example: Domain = all car, revisited.

$$\begin{aligned} P(x) &= \text{"}x \text{ gets good mileage."} \\ Q(x) &= \text{"}x \text{ is large."} \end{aligned}$$

Negate the statement "All large cars get bad mileage."

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \to \neg P(x))$	universal negation, 1
3. $(\exists x)\neg(\neg Q(x) \vee \neg P(x))$	implication, 2
4. $(\exists x)(\neg(\neg Q(x)) \wedge \neg(\neg P(x)))$	De Morgan's law, 3
5. $(\exists x)(Q(x) \wedge P(x))$	double negation twice, 4
6. $(\exists x)(P(x) \wedge Q(x))$	commutativity, 5

Quiz 05-1

Suppose that the domain of the variables is the set of natural numbers and G(x,y) represents “x > y”.

(1) Translate the following formula to a natural language sentence.

$(\forall x)(\exists y)\ G(x,y)$

(2) State whether the statement in (1) is true or false.

(3) Translate the following formula to a natural language sentence.

$(\exists y)\ (\forall x)\ G(x,y)$

(4) State whether the statement in (3) is true or false.