CS 204 Discrete Mathematics

Homework 11

Sample Solutions

Integer random variables, Bernoulli trial; Expected value, Linearity of Expectation, Variance

1. A natual number n is chosen at random from the set {1, 2, 3, …, 99, 100}. Let D be the number of digits that n has. (So, for example, if n = 100 then D = 3 and if n = 98 then D = 2.) What is the expected value of D?

Solution)

$$1 \cdot \frac{9}{100} + 2 \cdot \frac{90}{100} + 3 \cdot \frac{1}{100} = 1.92$$

2. Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

Solution)

Letting X equal the number of heads ("successes") that appear, then X is a binomial random variable with parameters (n = 4, p = 1/2). Hence,

$P(X = 2) = 4C2 \times (1/2)^2 \times (1/2)^3$

$= 3/8$

Thus, only 32% of those persons whose test results are positive actually have the disease.

3. The final exam of a discrete mathematics course of a university consists of 50 true/false questions, each worth two points, and 25 multiple choice questions, each worth four points. (So the maximum score of the final exam is 200.) The probability that a student answers a true/false question correctly is 0.9 and the probability that a student answers a multiple-choice question correctly is 0.8. What is the expected score on the final?

Solution)

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4. Let X be the number appearing on the first die when two dice are rolled and let Y be the sum of the numbers appearing on the two dice. Show that $E(X) \times E(Y) \neq E(XY)$.

Solution)

X: The number on the first die

Y: The sum of numbers appearing on the two dice

$E(X) = (1+2+3+4+5+6)/6 = 7/2$

$E(Y) = \{(1+1)+(1+2)+...+(6+6)\}/(6x6) = /36$

$= \{(1+2+3+4+5+6) \text{ x}12\}/36$

$= 7$

$E(XY) = \{1 \text{ x } (2 + 3 + 4 + 5 + 6 + 7) + 2 \text{ x } (3+4+5+6+7+8) + 3 \text{ x } (4+5+6+7+8+9) +$

$4 \text{ x } (5+6+7+8+9+10) +5 \text{ x } (6+7+8+9+10+11) + 6 \text{ x } (7+8+9+10+11+12)\}/36$

$= \{27 + 66 + 117 + 180 + 255 + 342\}/36$

$= 987/36$

Now

$(7/2) \times 7 \neq 987/36 \approx 27.4$

Other Probability Problems

1. It is known that all items produced by a certain machine will be defective with probability 0.1, independently of each other. What is the probability that in a sample of three items, at most one will be defective?

Solution)

If X is the number of defective items in the sample, then X is a binomial random variable with parameters (3, 0.1). Hence, the desired probability is given by

$P[X=0] + P[X = 1] = C(3.0) \times 0.1^0 \times 0.9^3 + C(3,1) \times 0.1^1 \times 0.9^2 = 0.972.$

2. Consider two bags. The first contains two white and seven black balls, and the second contains five white and six black balls. We flip a fair coin and then draw a ball from the first bag or the second bag depending upon whether the outcome was heads or tails. What is the conditional probability that the outcome to the toss was heads given that a white ball was selected?

Solution)

Let W be the event that a white ball is drawn, and let H be the event that the coin comes up heads. The desired probability P(H|W) may be calculated as follows:

$$
\begin{aligned}
P(H|W) &= P(HW) / P(W) \\
&= P(W|H)P(H) / P(W) \\
&= P(W|H)P(H) / \{P(W|H)P(H) + P(W|\sim H)P(\sim H) \\
&= (2/9 + 1/2)/ (2/9 \times 1/2 + 5/11 \times 1/2) \\
&= 22/67
\end{aligned}
$$