CS 204 Discrete Mathematics

Homework 7

1. (12 pts) Consider the following recursively defined function

$$f(m,n) = \begin{cases} n + 1 & \text{if } m = 0 \\ f(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ f(m-1, f(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

(a) What is the value of f(3,4)?

(b) Prove by mathematical induction that $f(3,n) = 2^{n+3} - 3$.

Solution)

(a)

Values of *A*(*m*, *n*)

m \ *n*	**0**	**1**	**2**	**3**	**4**
0	1	2	3	4	5
1	2	3	4	5	6
2	3	5	7	9	11
3	5	13	29	61	125

(b)

Base case: If n = 0, then f(3, 0) = 5

Inductive hypothesis: Suppose as inductive hypothesis that f(3, k) = 2^(k+3) – 3 for some k > 0

Inductive step: f(3, k+1) = f(2, f(3, k)) = 2f(3, k) + 3 (as calculated in (a))

= 2^(k+4) – 6 +3 = 2^(k+4) – 3, as required.

2. (10 pts) Consider the following recurrence relation:

H(n)	= 0	if n ≤ 0
	= 1	if n = 1 or n = 2
	= H(n-1) + H(n-2) - H(n-3)	if n > 2.

Prove that H(2n) = H(2n - 1) = n for all n ≥ 1.

Solution)

Proof. (Induction on n.) By definition, $H(2\cdot 1) = 1 = H(2\cdot 1-1)$ and $H(2\cdot 0) = 0 = H(2\cdot 0-1)$. Note that $H(3) = H(2) + H(1) - H(0) = 1 + 1 - 0 = 2$ and $H(4) = H(3) + H(2) - H(1) = 2 + 1 - 1 = 2$, so $H(2 \cdot 2) = 2 = H(2 \cdot 2 - 1)$. Suppose as inductive hypothesis that $H(2k) = H(2k - 1) = k$ for all k such that $1 \leq k < n$, for some $n > 2$. Then

$$\begin{aligned}
H(2n) &= H(2n-1) + H(2n-2) - H(2n-3), \text{ by definition} \\
&= H(2n-1) + H(2(n-1)) - H(2(n-1)-1) \\
&= H(2n-1) + (n-1) - (n-1), \text{ by inductive hypothesis} \\
&= H(2n-1)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
H(2n-1) &= H(2n-2) + H(2n-3) - H(2n-4), \text{ by definition} \\
&= H(2(n-1)) + H(2(n-1)-1) - H(2(n-2)) \\
&= (n-1) + (n-1) - (n-2), \text{ by inductive hypothesis} \\
&= n
\end{aligned}$$

so $H(2n) = H(2n-1) = n$, as required. □

3. (10 pts) Consider the following recurrence relation:

$$C(n) = \begin{cases} 0 & \text{if } n = 0 \\ n+3\cdot C(n-1) & \text{if } n > 0. \end{cases}$$

Prove by induction that $C(n) = \dfrac{3^{n+1} - 2n - 3}{4}$ for all $n \geq 0$.

Solution)

Proof. (Induction on n.) Let $f(n) = (3^{n+1} - 2n - 3)/4$.

Base Case: If $n = 0$, the recurrence relation says that $C(0) = 0$, and the formula says that $f(0) = (3^1 - 2 \cdot 0 - 3)/4 = 0$, so they match.

Inductive Hypothesis: Suppose as inductive hypothesis that

$$C(k-1) = (3^k - 2(k-1) - 3)/4$$

for some $k > 0$.

Inductive Step: Using the recurrence relation,

$$\begin{aligned} C(k) &= k + 3 \cdot C(k-1), \text{ by the second part of the recurrence relation} \\ &= k + 3\left(\frac{3^k - 2(k-1) - 3}{4}\right), \text{ by inductive hypothesis} \\ &= \frac{4k}{4} + \frac{3^{k+1} - 6k + 6 - 9}{4} \\ &= \frac{3^{k+1} - 2k - 3}{4} \end{aligned}$$

so, by induction, $C(n) = f(n)$ for all $n \geq 0$. $\square$

4. (10 pts) Let

$$f(m, n) = \begin{cases} 5 & \text{if } m = n = 1 \\ f(m-1,n) + 2 & \text{if } n = 1 \text{ and } m > 1 \\ f(m, n-1) + 2 & \text{if } n > 1 \end{cases}$$

Prove by mathematical induction that

$$f(m,n) = 2(m+n) + 1 \text{ for all, } m, n \in \mathbb{N}^{+}.$$

(Hint: First, define (x1,y1) <= (x2,y2) iff x1 < x2 or (x1 = x2 and y1 < y2). Then use (m,n) = (1,1) as the basis case.)

Solution)

Double induction:

1. Prove f(1,1) is true
2. Prove f(m, 1) => f(m+1, 1)
3. Prove f(m, n) => f(m, n+1) for all natural numbers m