Ch 2. Basic Structures: Sets, Functions
Ch 9. Relations

Functions

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Acknowledgement
- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 2. Basic Structures: Sets, Functions

Functions

1. Definition
2. One-to-One and Onto Functions
3. Function Composition

1. Definition

Definition

A *function* from a set X to a set Y is a relation such that it assigns a single element of Y to every element of X. If f is such a function, we write

$$f : X \longrightarrow Y$$

and we denote the element of Y assigned to $x \in X$ by $f(x)$.

Examples

A simple function and its diagram

Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3, 4\}$. The formula $f(x) = x + 1$ defines a function $f : X \longrightarrow Y$. For this function, $f(1) = 2$, $f(2) = 3$ and $f(3) = 4$.

$$X \longrightarrow Y$$

Examples

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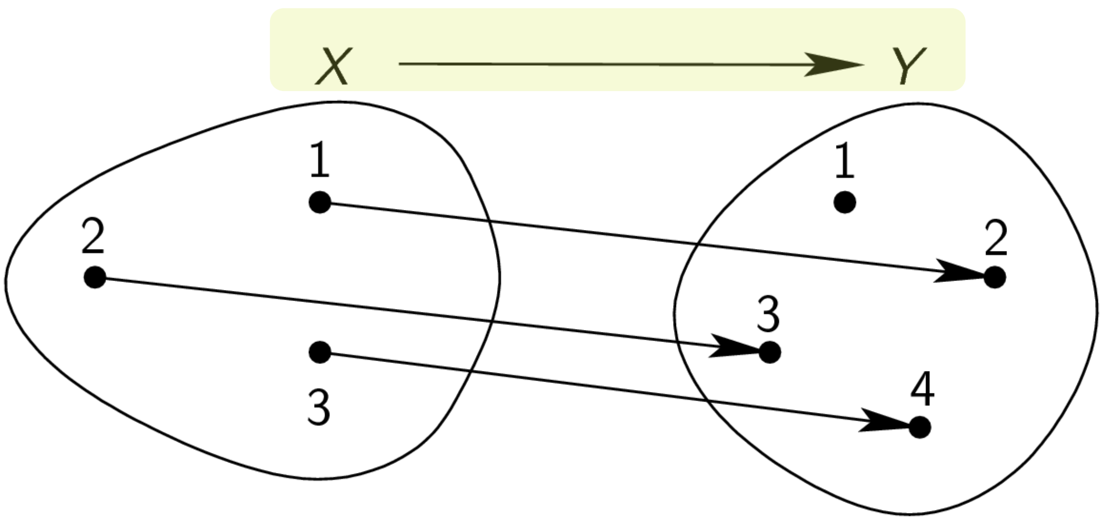
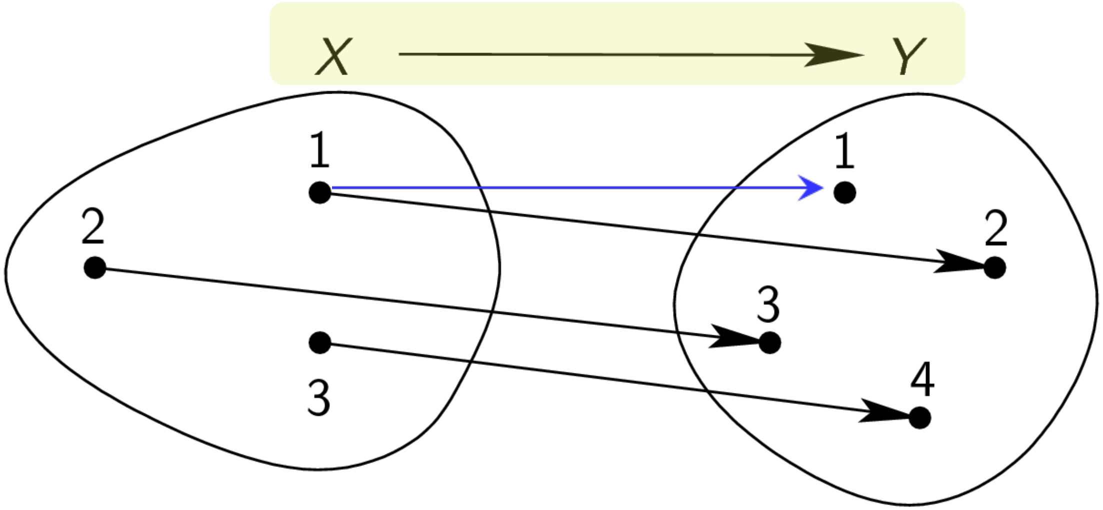


Examples

A relation that is not a function



Examples

- The formula $f(x) = x^2 - 3x + 2$ defines a function $f : \mathbf{R} \longrightarrow \mathbf{R}$.
- Let W be the set of all words in this book, and let L be the set of all letters in the alphabet. Define a function $f : W \longrightarrow L$ by setting $f(w)$ equal to the first letter in the word w.

Example f("element") = "e" = f("elf")

Examples

- The formula $f(x) = x^2 - 3x + 2$ defines a function $f : \mathbf{R} \longrightarrow \mathbf{R}$.
- Let W be the set of all words in this book, and let L be the set of all letters in the alphabet. Define a function $f : W \longrightarrow L$ by setting $f(w)$ equal to the first letter in the word w.
- Let F be the set of all non-empty finite sets of integers, so $F \subseteq \mathcal{P}(\mathbf{Z})$. Define a function

$$s : F \longrightarrow \mathbf{Z}$$

by setting $s(X)$ to be the sum of all the elements of X. For example, $s(\{1, 2, 3\}) = 6$.

Examples Let $x \in \mathbb{R}$.

$\lceil x \rceil$: the ceiling function returns the smallest integer $\geq x$

$\lfloor x \rfloor$: the floor function returns the largest integer $\leq x$

$\lceil 2.4 \rceil = 3$, $\lfloor 2.4 \rfloor = 2$

2. One-to-one and Onto functions

One-to-one functions

Definition

A function $f : X \longrightarrow Y$ is *injective* (or *one-to-one*) if, for all a and b in X, $f(a) = f(b)$ implies that $a = b$. In this case we say that f is a *one-to-one mapping* from X to Y.

I.e., if $a \neq b$, then $f(a) \neq f(b)$ or "Different elements map to different elements".

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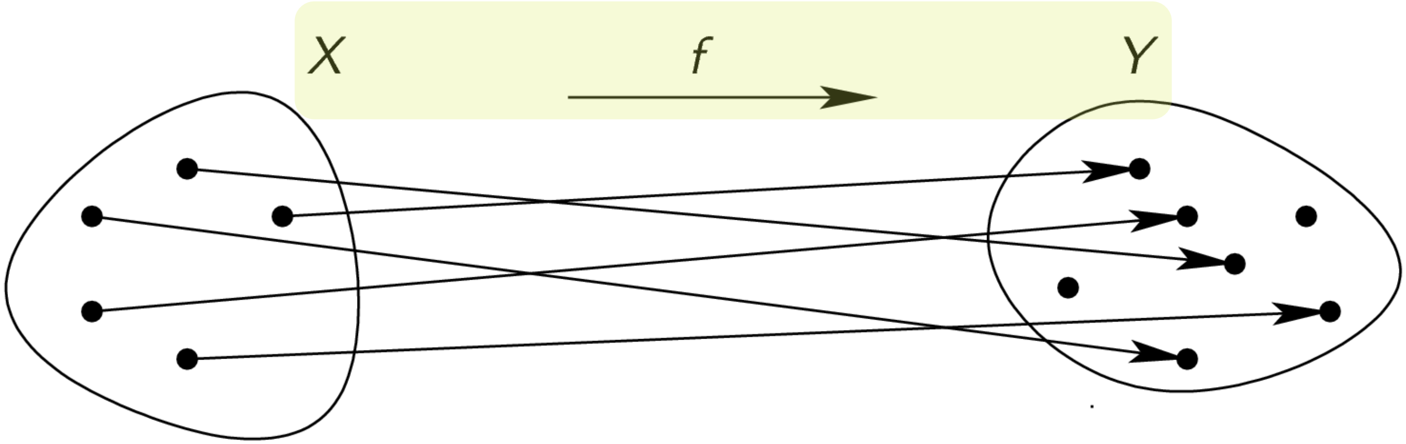

Proving one-to-one

Prove that the function $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ defined by $f(x) = 2x + 1$ is one-to-one.

Proof.

Direct proof?

or

Proof by proving contraposition?

Proving one-to-one

Prove that the function $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ defined by $f(x) = 2x + 1$ is one-to-one.

Proof.

Let $a, b \in \mathbf{Z}$ and suppose $f(a) = f(b)$. Then

$$\begin{aligned} 2a + 1 &= 2b + 1 \\ 2a &= 2b \\ a &= b \end{aligned}$$

We have shown that $f(a) = f(b)$ implies that $a = b$, i.e., that f is one-to-one. $\square$

Onto functions

Definition

A function $f : X \longrightarrow Y$ is *surjective* (or *onto*) if, for all b $\in Y$, there exists an a $\in X$ such that $f(a) = b$. In this case we say that *f maps X onto Y*.

An *image* of a function f:X→Y is the set of all values in Y that f can take.

If f is onto, then the image of f is the same as Y.

Onto functions

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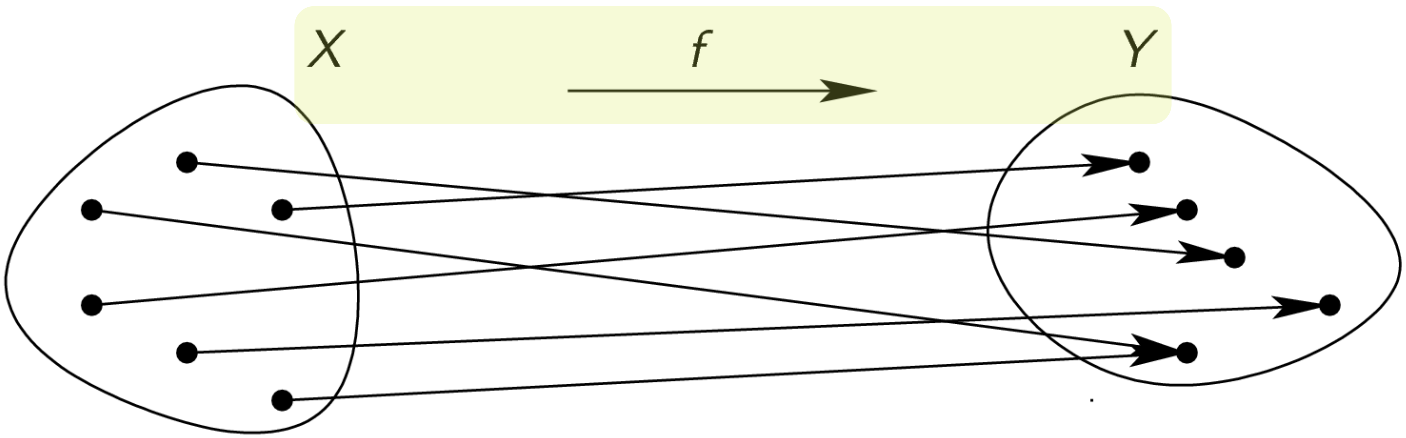


Proving onto

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $f : \mathbf{R} \longrightarrow \mathbf{Z}$ be defined by $f(x) = \lfloor x \rfloor$. Prove that f maps $\mathbf{R}$ onto $\mathbf{Z}$.

Proof.

Proving onto

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $f : \mathbf{R} \longrightarrow \mathbf{Z}$ be defined by $f(x) = \lfloor x \rfloor$. Prove that f maps $\mathbf{R}$ onto $\mathbf{Z}$.

Proof.

Let $n \in \mathbf{Z}$. Then, since $\mathbf{Z} \subseteq \mathbf{R}$, $n \in \mathbf{R}$ as well. But since n is an integer, $\lfloor n \rfloor = n$. Therefore $f(n) = n$. $\square$

This proof actually shows how to systematically find, for any $n \in \mathbb{Z}$, an element x in $\mathbb{R}$ that satisfies f(x) = n. **<=** A constructive proof

Disproving one-to-one and onto

Let $E = \{n \in \mathbf{Z} \mid n \text{ is even}\}$ and let $O = \{n \in \mathbf{Z} \mid n \text{ is odd}\}$. Define a function

$$f : E \times O \longrightarrow \mathbf{Z}$$

by $f(x, y) = x + y$. Is f one-to-one and/or onto? Prove or disprove.

Onto?

One-to-one?

Disproving one-to-one and onto

Solution

We first show that f is not onto. Suppose, to the contrary, that f is onto. Since $2 \in \mathbf{Z}$ is an element of the codomain, there is some ordered pair $(x, y) \in E \times O$ such that

$$f(x, y) = x + y = 2.$$

But since x is even and y is odd, $x + y$ is odd .

This contradicts that 2 is even.

We next show that f is not one-to-one. Notice that

$$f(4, -3) = 1 = f(6, -5)$$

but $(4, -3) \neq (6, -5)$. This counterexample shows that f is not one-to-one.

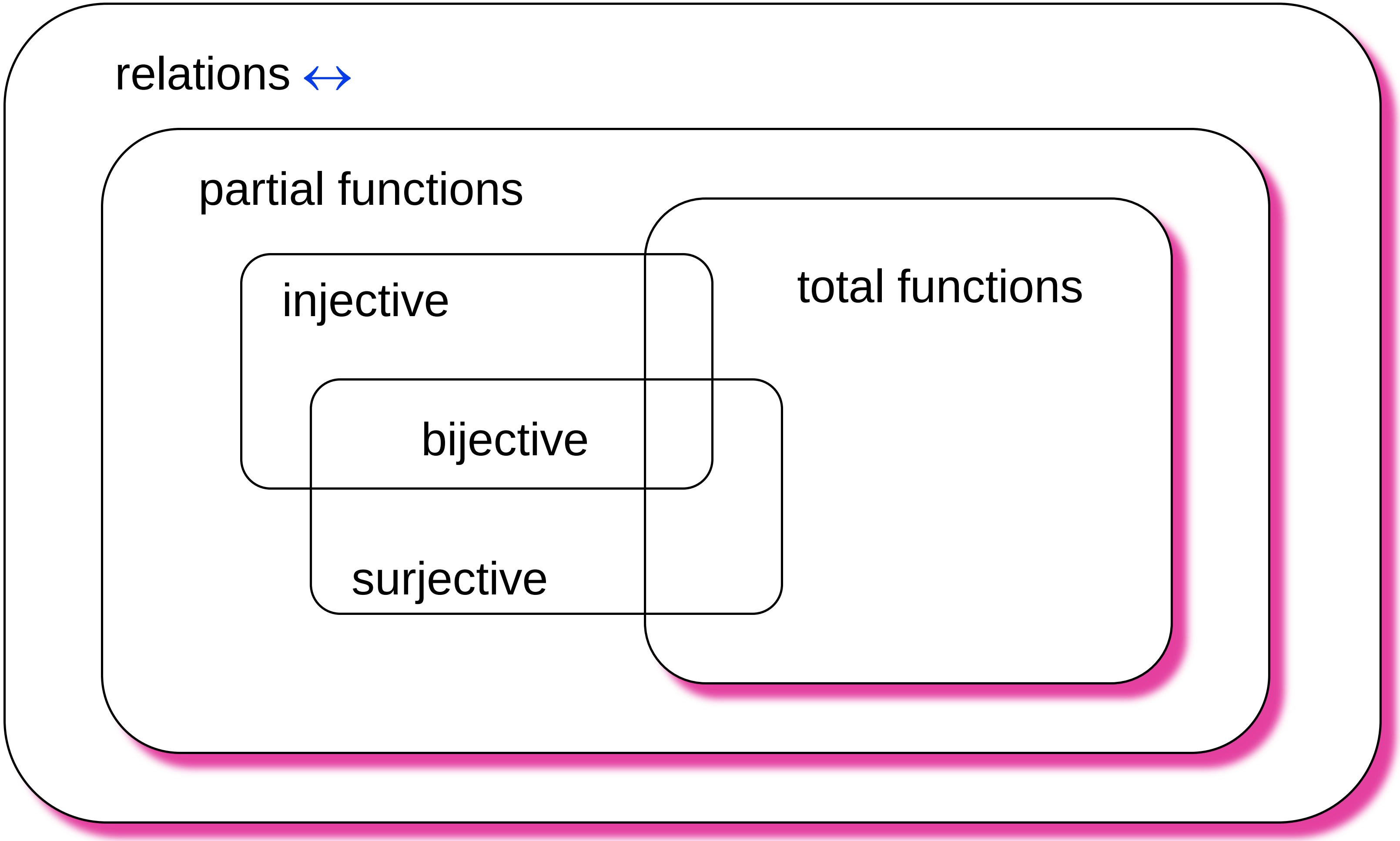
Special Cases (1/2)

Suppose f: A $\leftrightarrow$ B

1. f is a function defined for *all* values of A
   we say f is a "total" function, and write A $\rightarrow$ B
2. f is a function defined for some subset of A including $\varnothing$
   we say f is a "partial" function, and write A $\rightarrow\!\!\!\!\!\!+$ B
3. f is a function defined for a *finite set* of values of A
   we say f is a "finite" function, and write A $\rightarrow\!\!\!\!\!\!+\!\!\!\!+$ B
4. f is a function for which no element in ran(f) is associated with more than one element in dom(f)
   we say f is a "one-to-one" or "injective" function, and write
   A $\rightarrowtail$ B
5. f is a function whose range is B
   we say f is an "onto" or "surjective" function, and write
   A $\twoheadrightarrow$ B
6. f is both one-to-one and onto
   we say f is a "bijection", and write A $\rightarrowtail\!\!\!\!\!\twoheadrightarrow$B

Is bijection total?

Special Cases (2/2)



One-to-one correspondence

Definition A total function f: $X \to Y$ that is surjective and injective is called a *one-to-one correspondence* (≠ one-to-one).

Example 1.
Consider the set of all natural numbers $\mathbb{N}$ and the set of all even numbers E. Is there a one-to-one correspondence between $\mathbb{N}$ and E?

Example 2.
Is there a one-to-one correspondence between $\mathbb{N}$ and the set of all rational numbers $\mathbb{Q}$?

Example 3.
Is there a one-to-one correspondence between $\mathbb{N}$ and the set of all real numbers $\mathbb{R}$?

Let $f\colon X \to Y$ be a one-to-one correspondence. The *inverse function* of f, denoted f^{-1}, is the function that assigns to an element $b \in Y$ the unique element $a \in X$ such that $f(a) = b$. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Example Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

Is f invertible?

Yes.

What is its inverse?

$f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

3. Function Composition

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then $g \circ f$ is a function from X to Z defined by $(g \circ f)(x) = g(f(x))$.

Example of function composition

Function composition

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then $g \circ f$ is a function from X to Z defined by $(g \circ f)(x) = g(f(x))$.

Example of function composition

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $f(x) = \lfloor x \rfloor$, and let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $g(x) = 3x$. Then for x = 2.4

$$(g \circ f)(2.4) = g(f(2.4)) = g(2) = 6$$

Function composition

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then $g \circ f$ is a function from X to Z defined by $(g \circ f)(x) = g(f(x))$.

Example of function composition

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $f(x) = \lfloor x \rfloor$, and let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $g(x) = 3x$. Then for x = 2.4

$$(g \circ f)(2.4) \quad = \quad g(f(2.4)) = g(2) = 6$$

and

$$(f \circ g)(2.4) \quad = \quad f(g(2.4)) = f(7.2) = 7$$

Function restriction

If $f : X \longrightarrow Y$ is some function, and $H \subseteq X$, then the restriction of f to H is the function

$$f|_H : H \longrightarrow Y$$

defined by taking $f|_H(x) = f(x)$.

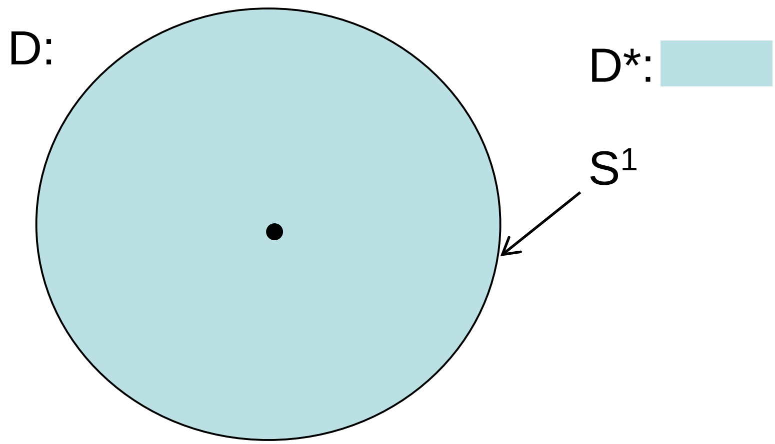
Example

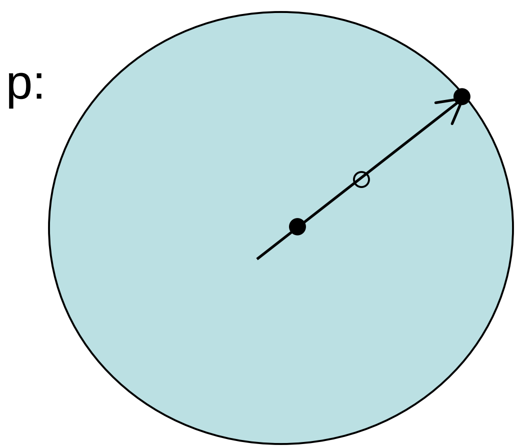
f (x,y) = x / y is a partial function with the type $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$f|_{\mathbb{R} \times (\mathbb{R} - \{0\})} = x / y$ is a total function.

Function restriction: example

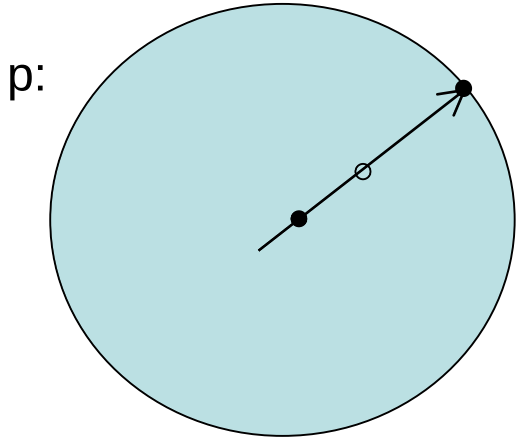
Let $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ and let $D^* = D \setminus \{(0, 0)\}$.
Let $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ Define a function $p : D^* \longrightarrow S^1$ by projecting straight out along a radius until you reach the boundary of the disk.

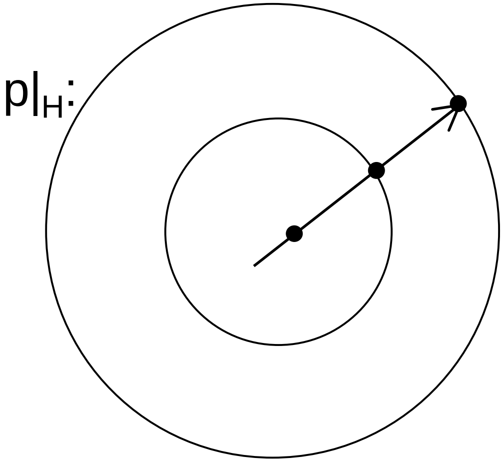




What is the formula for p?

Function restriction: example



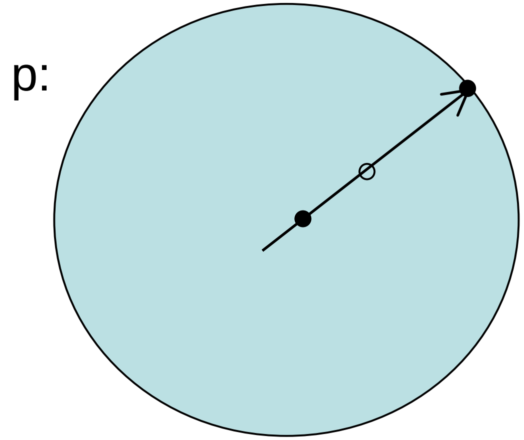


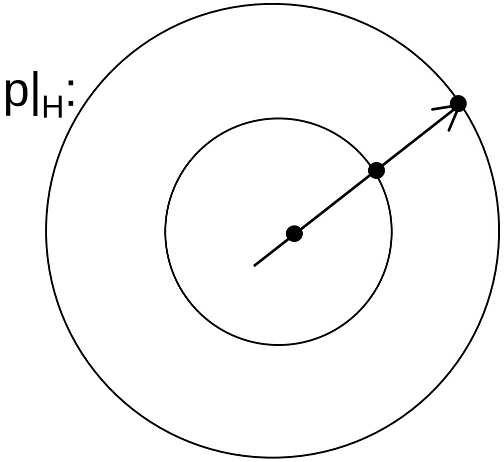
Getting a formula for p might be a little messy, but things can be cleaner if we consider a restriction. Let H be the circle of radius $\frac{1}{2}$:

$$H = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = \frac{1}{4}\}$$

Then $p|_H(x, y) =$?

Function restriction: example





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$$H = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = \frac{1}{4}\}$$

Then $p|_H(x, y) = (2x, 2y)$. (Why?)

Quiz 13-1

Which of the following is/are NOT true ?

(a) Any function is a partial function.
(b) A total function is a partial function.
(c) Some partial functions are total functions.
(d) A one-to-one function is a one-to-one correspondence.
(e) A one-to-one and onto function is a one-to-one correspondence.
(f) A one-to-one correspondence must be a total function.