Ch 6. Counting

Basic Counting Techniques

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

6.1 The Basics of Counting

6.2 The Pigeonhole Principle

6.3 Permutations and Combinations

6.4 Binomial Coefficients and Identities

How many different ways are there to do such and such things?

- How many ways are there to take CS courses when certain CS courses are prerequisites to other CS courses?
- How many different ways can this map be colored such that no neighboring countries are colored in the same color?
- How many ways can a team comprising of 6 developers and 4 testers be chosen from among 50 developers and 38 testers?

Basic Counting Techniques

- Addition
- Multiplication
- Mixing Addition and Multiplication

Addition

The Addition Principle

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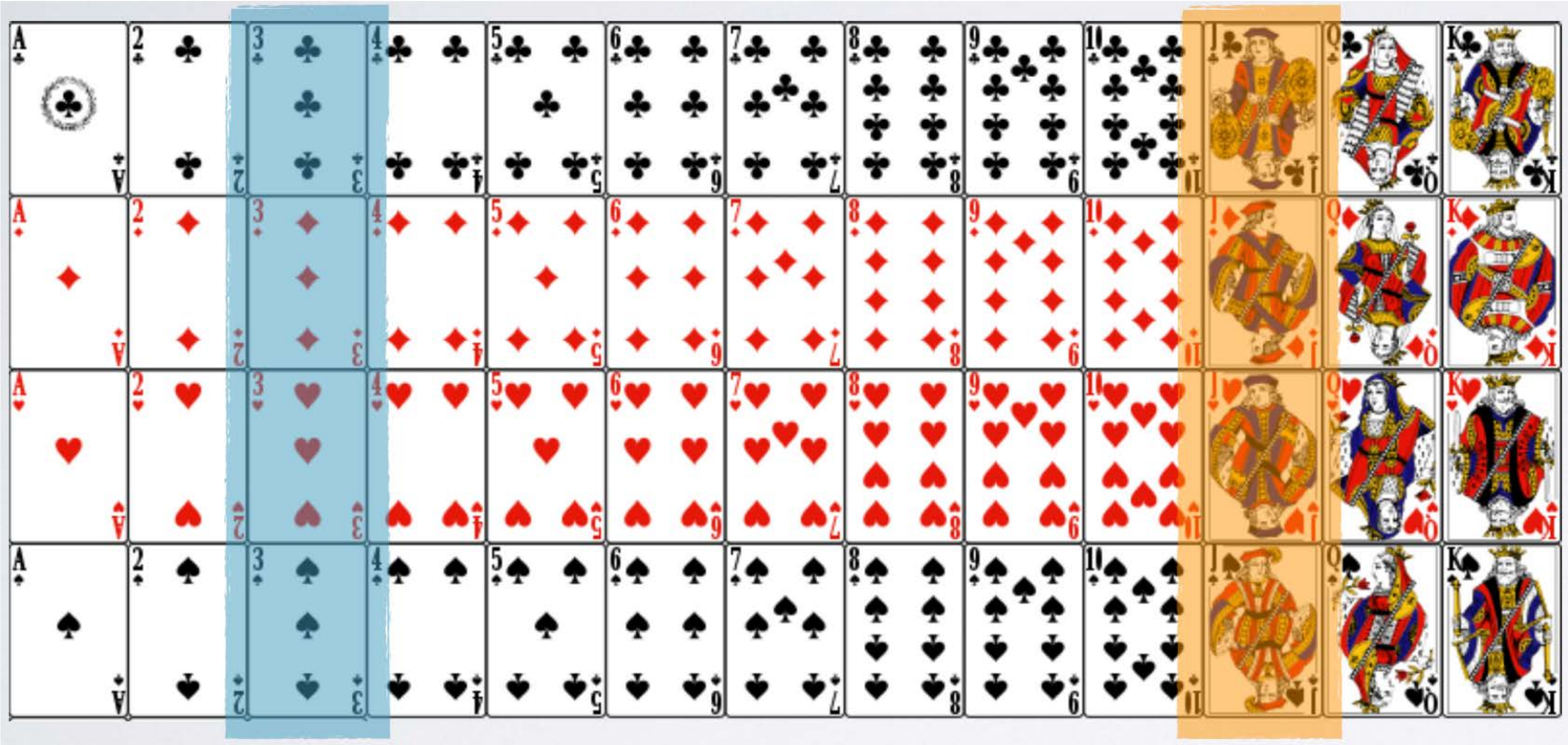
Solution

Since it is impossible to take both a car and a bicycle to work, these are disjoint sets. Thus Ray has $5 + 3 = 8$ *choices.*

How many different ways are there to draw a Jack or a Three from a well shuffled full deck of cards?

Ways to draw a Jack: 4,   Ways to draw a Three: 4

4 + 4 = 8



Acknowledgement: Dr. Mine Çetinkaya-Rundel, Duke University

The Generalized Addition Principle
(The Inclusion-Exclusion Principle)

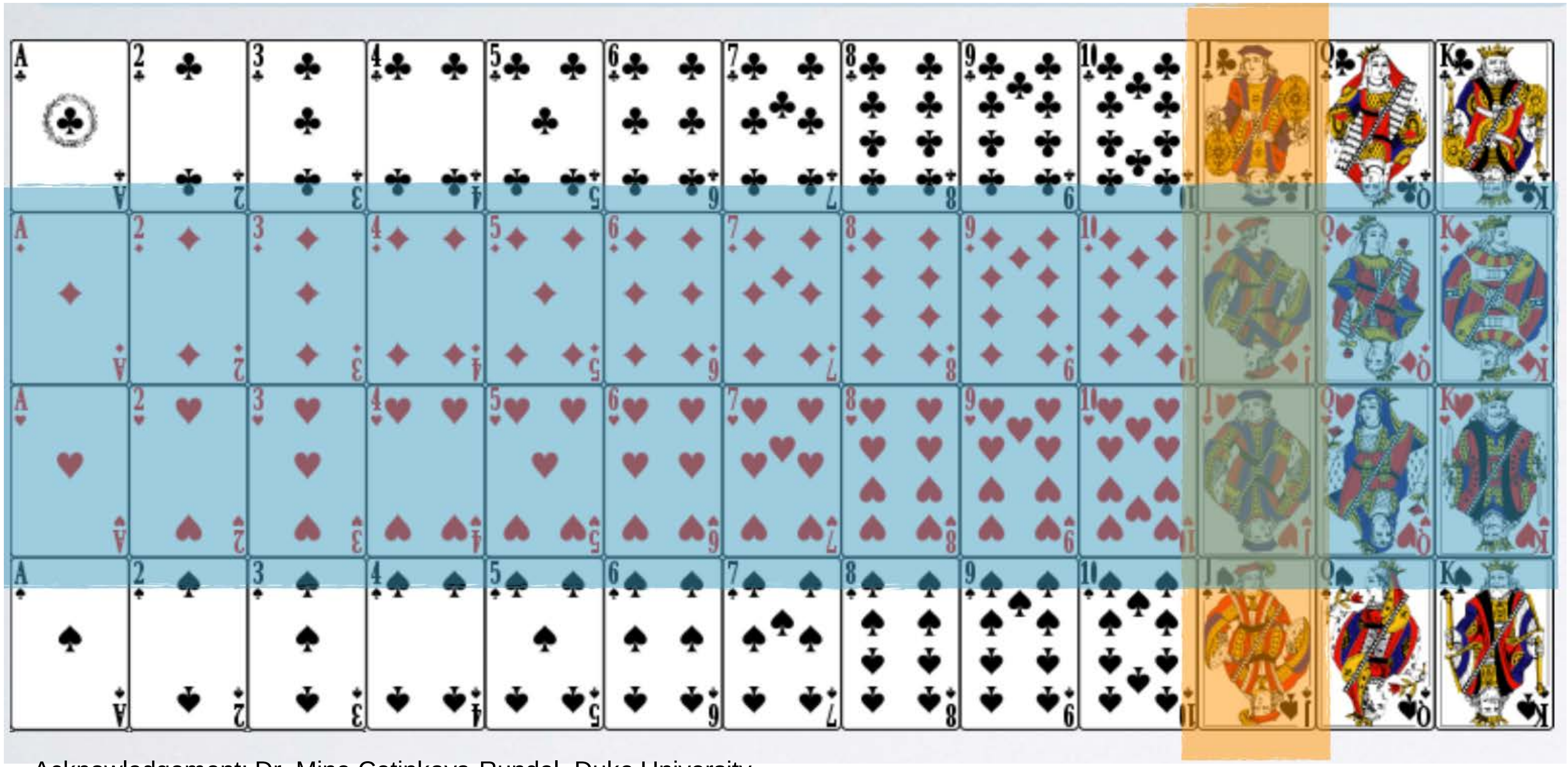
Suppose A and B are finite sets.

Then there are $|A|+|B|-|A\cap B|$ ways to choose an element from $A \cup B$.

How many different ways are there to draw a Jack or a red card from a well shuffled full deck of cards?

Ways to draw a Jack: 4, Ways to draw a red card: 26, Ways to draw a red Jack: 2

$4 + 26 - 2 = 28$



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Multiplication

The Multiplication Principle

Let A and B be finite sets. The number of elements (i.e., ordered pairs) in $A \times B$ is $|A| \cdot |B|$. So there are $|A| \cdot |B|$ ways to choose two items in sequence, with the first item coming from A and the second item from B.

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Example: Ray owns 5 bicycles and 3 cars. He plans to ride a bicycle to and from work, and then take one of his cars to go to a restaurant for dinner. How many different ways can he do this?

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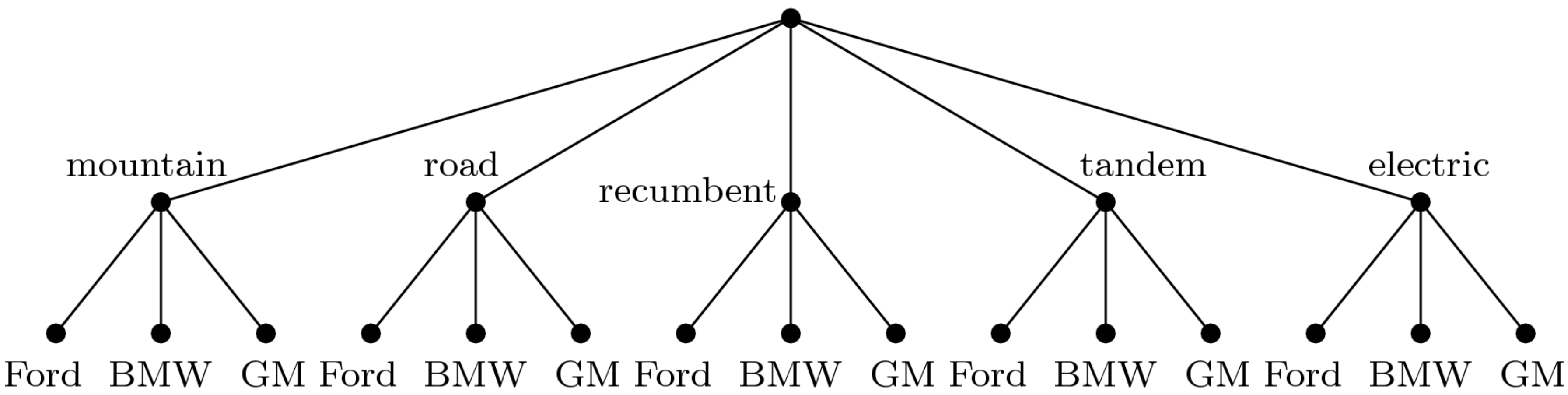
Solution

Ray is making two choices in sequence, so he is forming an ordered pair of the form (bicycle, car). Thus there are $5 \cdot 3 = 15$ *ways possible.*

Using decision trees to count



Note that in the Multiplication Principle choosing one from B is independent from choosing one from A.

More examples of multiplication

Example: How many strings of length 3 can be formed from a 26-symbol alphabet?

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Solution

There are three choices to be made in sequence: the first letter, the second letter, and the third letter. We have 26 options for each choice. Therefore the total number of length 3 strings is $26 \cdot 26 \cdot 26 = 26^3 = 17,576$.

Example: How many different binary strings of length 24 are there?

consisting of 0's and 1's

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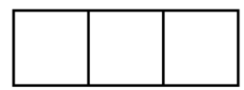
Solution

$$\underbrace{2 \cdot 2 \cdots 2}_{24 \text{ two's}} = 2^{24} = 16,777,216.$$

Decision trees with restrictions

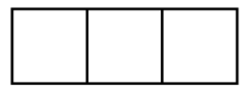
Example: How many designs of the form



are possible, if each square must be either red, green, or blue, and no two adjacent squares may be the same color?

Note that choosing the next one is dependent on which have been chosen in the previous choosing.

Example: How many designs of the form



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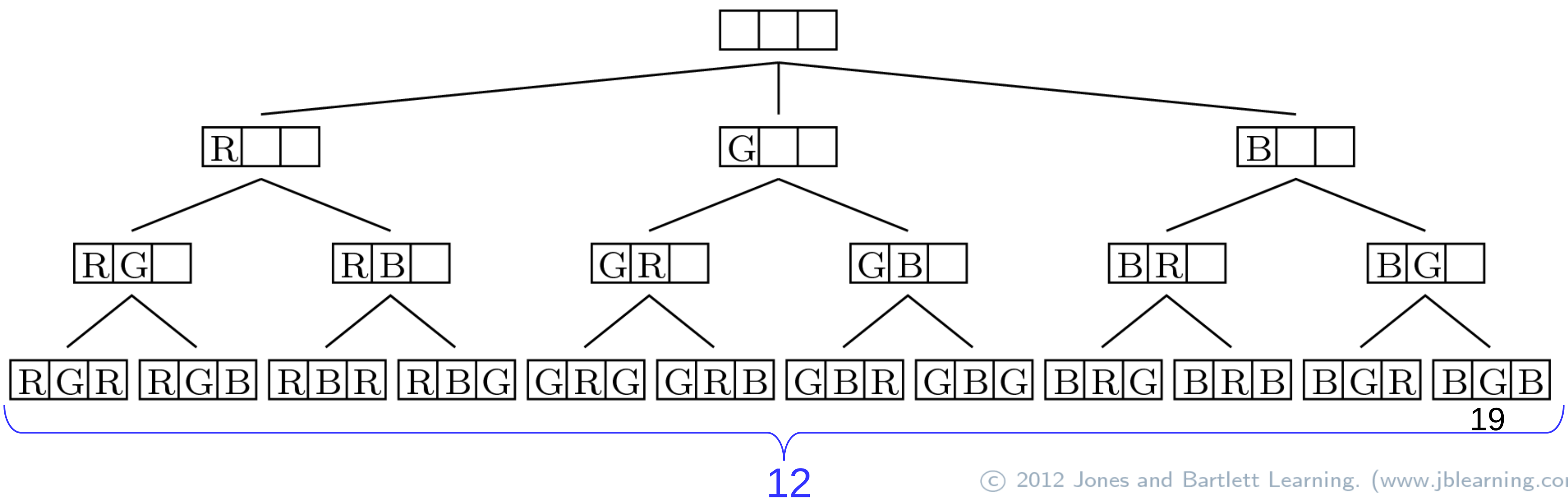
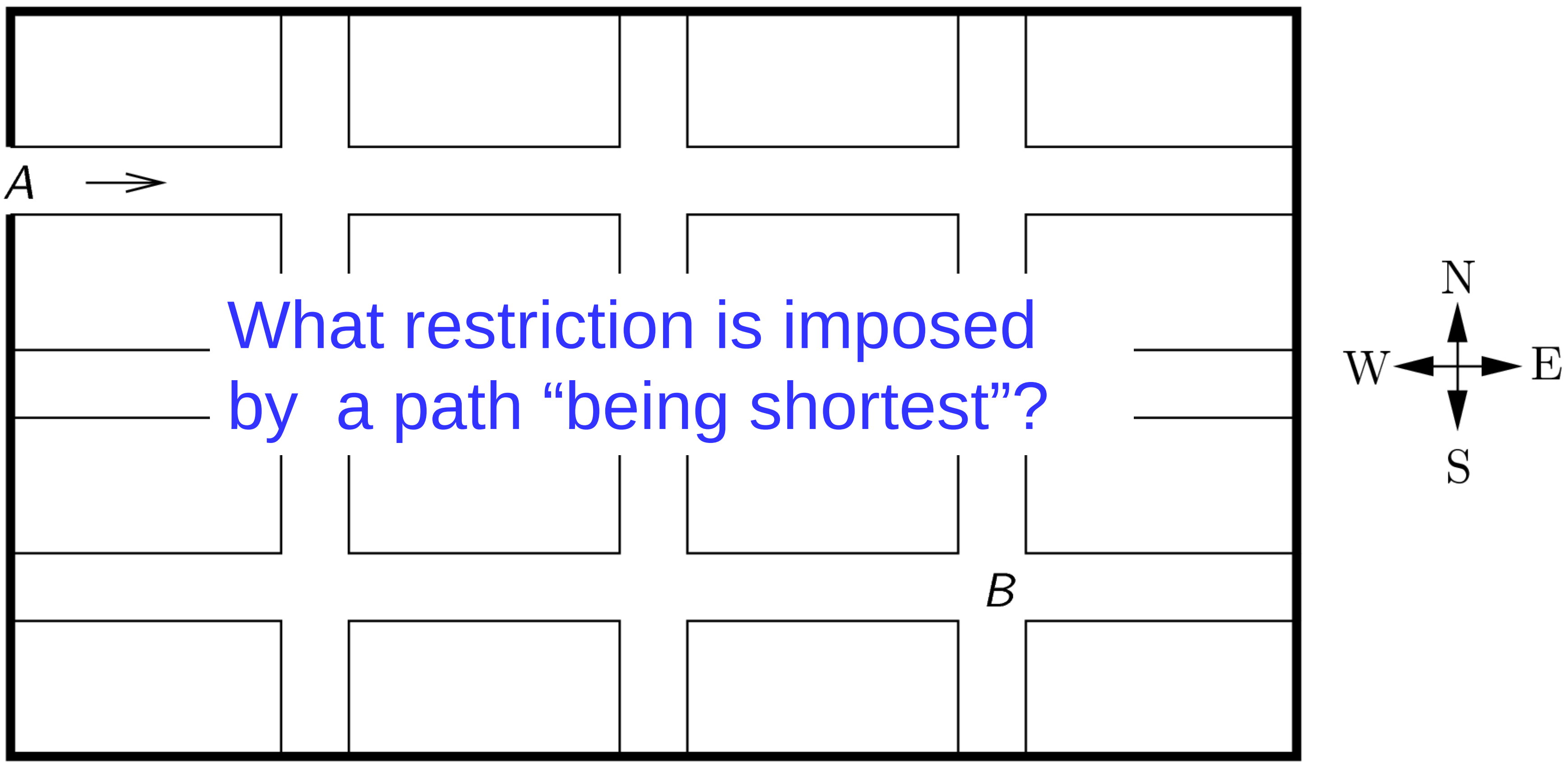


Example: paths through a neighborhood

Example: How many different shortest paths are there from A to B?



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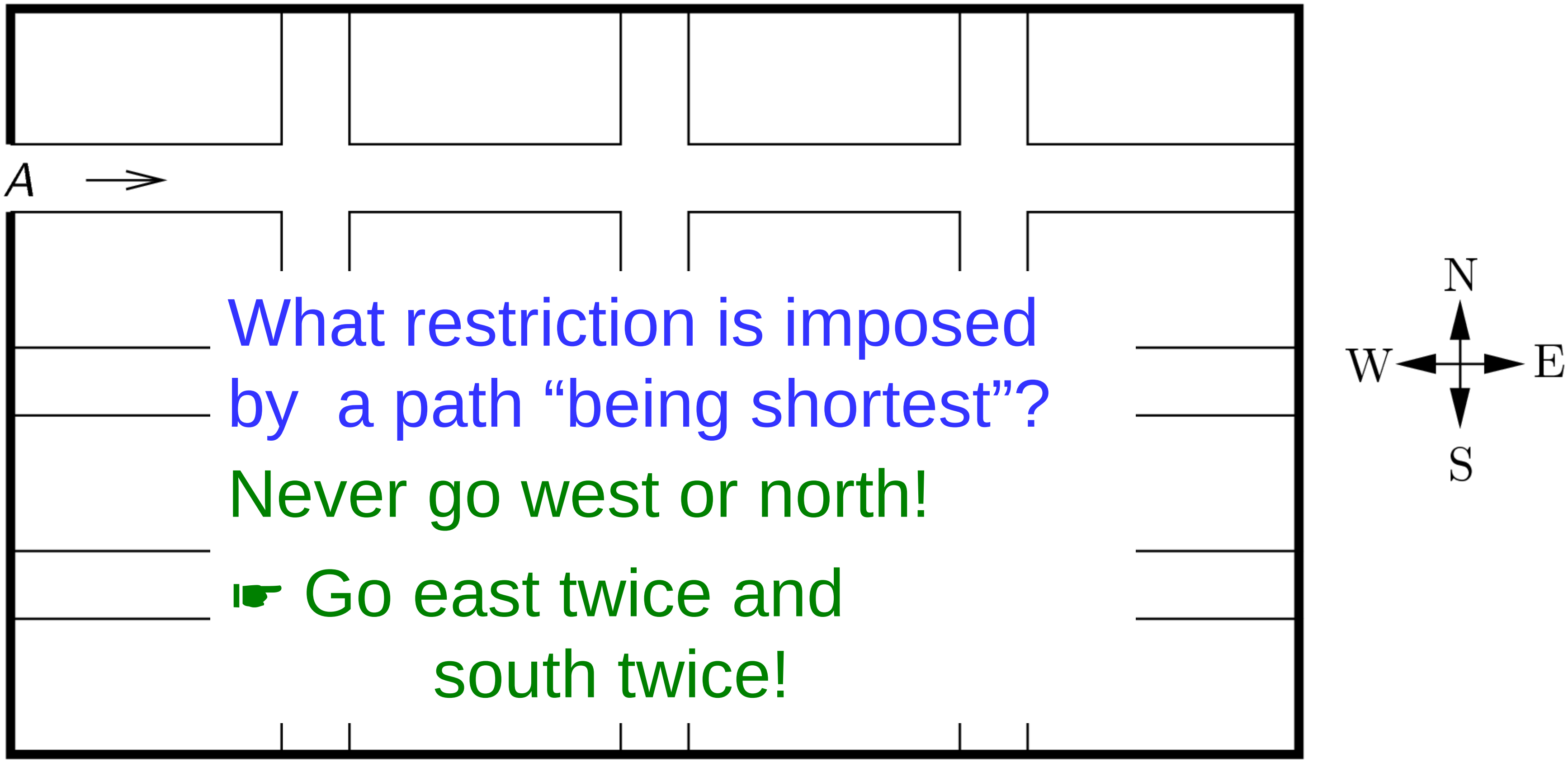
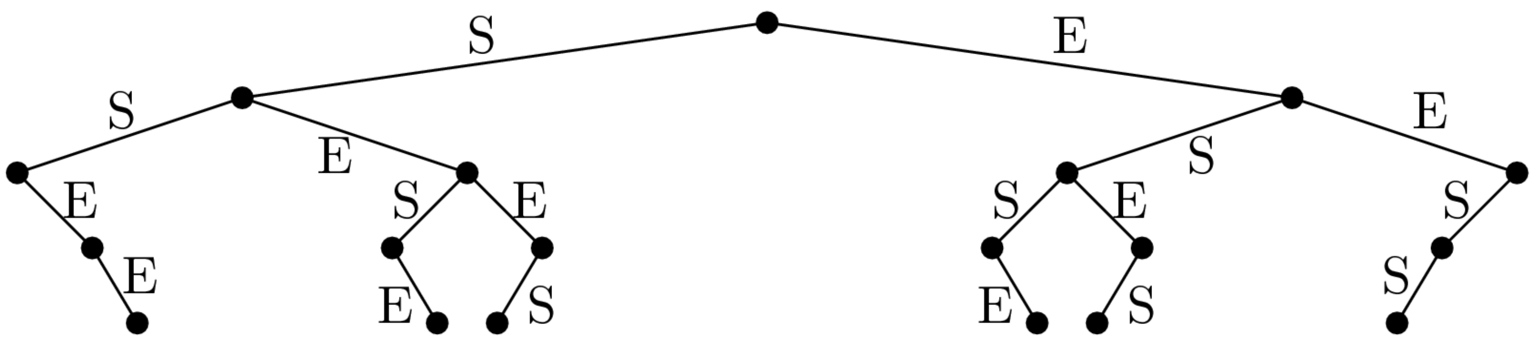


What restriction is imposed by a path "being shortest"?

Never go west or north!

☛ Go east twice and south twice!

Solution: decision tree with restrictions



Mixing addition and multiplication

Example: How many (nonempty) strings of length at most 3 can be formed from a 26-symbol alphabet?

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Solution

There are 26 strings of length 1, 26^2 *of length 2, and* 26^3 *of length 3. Since these cases are mutually exclusive, the total number of strings is* $26 + 26^2 + 26^3 = 18,278$.

Prototypical example: License plate problems

Example: Illinois license plates used to consist of either three letters followed by three digits or two letters followed by four digits. How many such plates are possible?

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Solution

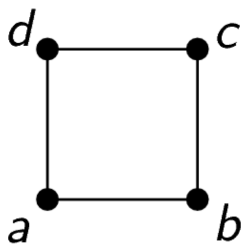
The two types of license plates can be thought of as two disjoint sets; the cases are mutually exclusive. The first case involves a choice of 3 letters (26^3) *followed by a choice of 3 digits* (10^3)*. For the second case, we first choose 2 letters* (26^2) *and then choose 4 digits* (10^4)*. Putting these together, we have a total of*

$$26^3 \cdot 10^3 + 26^2 \cdot 10^4 = 24,336,000$$

different possible license plates.

Example: graph coloring

Example: Using the four colors red, green, blue, and violet, how many different ways are there to color the vertices of the graph



so that no two adjacent vertices have the same color?

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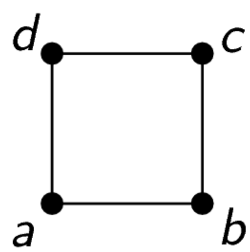

so that no two adjacent vertices have the same color?

a b d c

What is wrong with $4 \times 3 \times 3 \times 2$?

If $d = b$, then $c \neq 2$, but $c = 3$.

Solution: two disjoint cases

Case 1. Suppose b and d are different colors. Then, as above, we have four choices for a, three choices for b, and then two choices for d, since it must differ from both b and a. We are left with only two choices for c, for a total of $4 \cdot 3 \cdot 2 \cdot 2 = 48$ different colorings.

Case 2. Suppose b and d are colored the same color. Then we have 4 choices for a, and then 3 choices for the color that b and d share. There are then 3 choices for c, of a total of $4 \cdot 3 \cdot 3 = 36$ ways to color this case.

By the addition principle, the total number of colorings is $48 + 36 = 84$.

Are these two cases disjoint?

Quiz 17-1

How many different ways are there to draw a flush from a well shuffled full deck of cards? (A flush is a hand that contains 5 cards the same suit. Example, K♣ 10♣ 7♣ 6♣ 4♣.)

(a) 13 x 12 x 11 x 10 x 9 / (1 x 2 x 3 x 4 x 5) = 1,287

(b) 4 x 4 x 4 x 4 x 4 = 1,024

(c) 52 x 12 x 11 x 10 x 9 / (1 x 2 x 3 x 4 x 5) = 5,148

(d) 52 x 51 x 50 x 49 x 48 = 311,875,200

(e) (52 x 51 x 50 x 49 x 48) / (1 x 2 x 3 x 4 x 5) = 2,598,960

(f) None of the above. The correct answer is ( ).