CS 204 Discrete Mathematics Fall 2020

Homework 5

Due date October 10, 2020

PROOF

1. (10 pts) Consider the following definition of the "$\lhd$" symbol.

Definition. Let x and y be integers. Write $x \lhd y$ if $3x + 5y = 7k$ for some integer k.

(a) Show that $1 \lhd 5$, $3 \lhd 1$, and $0 \lhd 7$.

(b) Find a counterexample to the following statement:

If $a \lhd b$ and $c \lhd d$, then $a \cdot c \lhd b \cdot d$.

2. (10 pts) Let the following statements be given.

Definition. A triangle is *scalene* if all of its sides have different lengths.

Theorem. A triangle is scalene if it is a right triangle that is not isosceles.

Suppose ΔABC is a scalene triangle. Which of the following conclusions are valid? Why or why not?

(a) All of the sides of ΔABC have different lengths.

(b) ΔABC is a right triangle that is not isosceles.

3. (12 pts) Let P(n,x,y,z) be the predicate "$x^n + y^n = z^n$."

(a) Write the following statement in predicate logic, using positive integers as the domain.

For every positive integer n, there exist positive integers x, y, and z such that $x^n + y^n = z^n$.

(b) Formally negate your predicate logic statement from part (a). Simplify so that no quantifier lies within the scope of a negation.

(c) In order to produce a counterexample to the statement in part (a), what, specially, would you have to find among x, y, z and n?

4. (12 pts) Consider the following theorem.

Theorem. Let x be a wamel. If x has been schlumpfed, then x is a borfin.

Answer the following questions.

(a) Give the converse of this theorem.

(b) Give the contrapositive of this theorem.

(c) Which statement, (a) or (b), is logically equivalent to the Theorem?

5. (10 pts) In the four-point geometry discussed in class, do triangles exist? In other words, is it possible to have three distinct points, not on the same line, such that a line passes through each pair of points? Why or why not?

6. (10 pts) Give a direct proof.

Let a, b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

7. (10 pts) Prove that the rational numbers are closed under addition. That is, prove that, if a and b are rational numbers, then a + b is a rational number.

SETS

1. (9 pts) Let A = {2, 3, 4}, B = {3, 4, 5, 6}, and suppose the universal set if U = {1, 2, 3, 4, 5, 6, 7, 8, 9}. List all the elements in the following sets.

(a) $(A \cup B)'$

(b) $(A \cap B) \times A$

(c) $\mathcal{P}(B \setminus A)$

2. (10 pts) Let the following sets be given. The universal set for this problem is the set of all students at some university.

F = the set of all freshmen.

S= the set of all seniors.

M = the set of all math majors.

C = the set of all CS majors.

(a) Using only the symbols F, S, M, C, $| \quad |$, $\cap$, $\cup$, $'$, and $>$, translate the following statement into the language of set theory.

There are more freshmen who aren't math majors than
there are senior CS majors.

(b) Translate the following statement in set theory into everyday English.

$$(F \cap M) \subseteq C$$

3. (10 pts) Two sets are called *disjoint* if they have no elements in common, i.e., if their intersection is the empty set. Prove that finite sets A and B are disjoint if and only if $|A| + |B| = |A \cup B|$. Use the definition of $\varnothing$ and the inclusion-exclusion principle discussed in class in your proof.

4. (10 pts) Let X be a finite set with $|X| > 1$. What is the difference between $P1 = X \times X$ and $P2 = \{S \in \mathcal{P}(X) \mid |S| = 2\}$? Which set, P1 or P2, has more elements?

5. (12 pts)

Let W be the set of words in the English language and w1, w2 $\in$W.

(a) Define a relation R on W by

$$w1 \ R \ w2 \Leftrightarrow w1 \text{ and } w2 \text{ have a letter in common}$$

Which properties of equivalence relations does R satisfy? Explain.

(b) Define a relation S on W by

$$w1 \ S \ w2 \Leftrightarrow w1 \text{ has at least as many letters as } w2$$

Which properties of equivalence relations does S satisfy? Explain.

6. (12 pts)

Define a relation on $\mathbb{Z}$ by a R b if $a^2 = b^2$.

(a) Prove that R is an equivalence relation

(b) Describe the equivalence classes (with respect to the relation R above).