CS 204 — Discrete Mathematics — Anar Rzayev

# Relations

1) a) $R_1 \cup R_2 = \{(1,2), (2,3), (3,4), (1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$

b) $R_1 \cap R_2 = \{(1,2), (2,3), (3,4)\}$

c) $R_1 - R_2 = \emptyset$

d) $R_2 - R_1 = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$ ■

2) a) $(x,y) \in R \iff x+y=0$ on the set of all real numbers

For every element $a$, $(a,a) \in R$ is not true. If $(a,a) \in R$, then $2a=0$, or $a=0$ where $a$ is any real number $\boxed{\times}$ $\boxed{\text{not reflexive}}$ For all $a, b$ - real numbers, $a+b=0$ means $(a,b) \in R$ and $b+a=0$ is true $\Rightarrow (b,a) \in R$ should also be true
$(a,b) \in R \iff a+b=0 \iff b+a=0 \iff (b,a) \in R$, so $\boxed{\text{symmetric}}$
For all real numbers $a, b$, if $(a,b) \in R$ and $(b,a) \in R$, then $a=-b$, where $b$ is any real number. Choosing $b \neq 0 \Rightarrow a=-b \neq b$, hence $\boxed{\text{not antisymmetric}}$ For all real $a, b, c$ if $(a,b) \in R$ and $(b,c) \in R$, then $a+b=0$ and $b+c=0 \Rightarrow b=-a=-c$, $\underline{a=c}$ and $\underline{a+c=2a}$. Choosing $a \neq 0$, we can find $\underline{b=-a,\ c=a}$ and $\underline{a+c=2a \neq 0}$ with $(a,b)$ and $(b,c) \in R$

Therefore, $(a,c) \in R$ is not true since $a+c \neq 0$ $\boxed{\text{not transitive}}$

b) $(x,y) \in R \Leftrightarrow x = \pm y$ For any real number $a$, $(a,a) \in R$ since $a = \pm a$ (specifically, $a=a$ ✓) $\boxed{\text{reflexive}}$ For any real numbers $a, b$

$(a,b) \in R \Leftrightarrow a = \pm b \Leftrightarrow b = \pm a \Leftrightarrow (b,a) \in R$, where $a = \pm b$ means $a=b$ or $a=-b \Rightarrow b=a$ or $b=-a$, meaning $b = \pm a$ $\boxed{\text{symmetric}}$ For such $a, b$; if $(a,b) \in R$ and $(b,a) \in R$, then $a = \pm b$ and $b = \pm a$; the conclusion is "$a=b$ will not be necessary"

$\underline{b=-a}$ with $\underline{a \neq 0} \Rightarrow a=-b$ and $b=-a$, but $b \neq a$ $\boxed{\text{not antisymmetric}}$

For all $a, b, c$ - real, $(a,b) \in R$ and $(b,c) \in R$ means $a = \pm b$ and $b = \pm c \Rightarrow \pm b$ is same as $\pm(\pm c)$ or just $\pm c \Rightarrow$ If $a=b$, $b=c \Rightarrow a=c$ / $b=-c \Rightarrow a=-c$. If $a=-b$, $b=c \Rightarrow a=-c$ / $b=-c \Rightarrow a=c$. Basically, in all cases, $\underline{a = \pm c}$ or $\underline{(a,c) \in R}$ $\boxed{\text{transitive}}$

c) $(x,y) \in R \Leftrightarrow x-y \in \mathbb{Q}$ For any real $a$, $a-a = 0 \in \mathbb{Q} \Rightarrow (a,a) \in R$, or $\boxed{\text{reflexive}}$ If $(a,b) \in R$, then we find

$(a,b) \in R \Leftrightarrow a-b \in \mathbb{Q} \Leftrightarrow -(a-b) \in \mathbb{Q} \Leftrightarrow b-a \in \mathbb{Q} \Leftrightarrow (b,a) \in R$ where $x \in \mathbb{Q}$ is equivalent to $-x \in \mathbb{Q}$ $\boxed{\text{symmetric}}$ If $(a,b) \in R$ then $a-b \in \mathbb{Q}$, and $(b,a) \in R \Rightarrow b-a \in \mathbb{Q}$, basically same as $a-b \in \mathbb{Q}$

But, $a=b$ is not necessary $\Rightarrow$ Take $a=1, b=0$ $\boxed{\text{not antisymmetric}}$

Suppose for real $a, b, c \Rightarrow (a,b) \in R$ and $(b,c) \in R$. This means

$a-b\in\mathbb{Q}$ and $b-c\in\mathbb{Q}$. Since sum of two rational numbers is also a rational number (we proved this claim in previous HW) then, $(a-b)+(b-c)=\underline{(a-c)\in\mathbb{Q}}\Rightarrow\underline{(a,c)\in R}$, so $\boxed{\text{transitive}}$

d) $(x,y)\in R\Leftrightarrow x=2y$. For any real $a$, $(a,a)\in R$ means $a=2a$, which is not true for $a\neq 0$ (choose $a=1$) $\boxed{\text{not reflexive}}$

For real $a,b\Rightarrow$ If $(a,b)\in R$, then $a=2b$ and $b=\frac{1}{2}a\neq 2a$, if we choose $a\neq 0\Rightarrow b\neq 2a$ and $(b,a)\notin R$ (choose $a=2, b=1$) $(2,1)\in R$ but $(1,2)\notin R$ $\boxed{\text{not symmetric}}$. If $(a,b)\in R$ and $(b,a)\in R\Rightarrow a=2b$ and $b=2a$, then $a=2\cdot b=2\cdot 2a=4a$ or $\underline{a=0}\Rightarrow\underline{b=0}$. This implies $\underline{a=b=0}$ or $\boxed{\text{antisymmetric}}$ For real $a,b,c\Rightarrow$ assume $(a,b)\in R$ and $(b,c)\in R$. Then $a=2b$ and $b=2c\Rightarrow a=2\cdot b=2\cdot 2c=4c$, $\underline{a=4c}$ and $\underline{\text{choosing } c\neq 0}$ $\underline{a\neq 2c}$ or $\underline{(a,c)\notin R}$ (for $c\neq 0$, choose $b=2c$ and $a=4c$).

$\boxed{\text{not transitive}}$

e) $(x,y)\in R\Leftrightarrow xy\geq 0$. For any real $a$, $a^2\geq 0\Leftrightarrow a\cdot a\geq 0$ $\Leftrightarrow(a,a)\in R$, since $a^2\geq 0$ for each real $a\Rightarrow\underline{(a,a)\in R}$ $\boxed{\text{reflexive}}$

For real $a,b\Rightarrow(a,b)\in R\Leftrightarrow ab\geq 0\Leftrightarrow ba\geq 0\Leftrightarrow(b,a)\in R$, so if $(a,b)\in R$ is true, then $(b,a)\in R$ is also true $\boxed{\text{symmetric}}$

if $(a,b)\in R$ and $(b,a)\in R\Rightarrow$ just $ab\geq 0$ will hold, and it does not mean $a=b$ should be true (pick $a=1, b=2$) $\boxed{\text{not antisymmetric}}$

Suppose for real $a, b, c \Rightarrow (a,b) \in R$ and $(b,c) \in R$ are both true

Then $ab \geq 0$ and $bc \geq 0 \Rightarrow$ If we pick $\underline{a=-1, b=0, c=1}$, then $(-1,0) \in R$ since $-1 \cdot 0 = 0 \geq 0$ and $(0,1) \in R$ since $0 \cdot 1 = 0 \geq 0$

But $a \cdot c = -1 \cdot 1 = -1 < 0 \Rightarrow \underline{(a,c) = (-1,1) \notin R}$. This implies

$\boxed{\text{not transitive}}$ ■

3) a) $S \circ R$ is constructed using all ordered pairs in $R$ and ordered pairs in $S$, where the second element of the ordered pair in $R$ agrees with the first element of the ordered pair in $S$. For example, the ordered pair $(1,2)$ in $R$ and $(2,1)$ in $S$ produce the ordered pair $(1,1)$ in $S \circ R$

$S \circ R = \{(1,1), (1,1), (1,2), (2,1), (2,2), (2,2)\}$ or just

$\boxed{S \circ R = \{(1,1), (1,2), (2,1), (2,2)\}}$

b) Using the above method, we can recursively find $R^2, R^3, \dots$

$R = \{(1,1), (1,2), (1,3), (2,3), (2,4), (3,1), (3,4), (3,5), (4,2), (4,5),$
$(5,1), (5,2), (5,4)\}$

$R^2 = R \circ R = \{\cancel{(1,1)}, \cancel{(1,2)}, \cancel{(1,3)}, \cancel{(1,3)}, \cancel{(1,4)}, \cancel{(1,1)}, \cancel{(1,4)}, \cancel{(1,5)},$
$\cancel{(2,1)}, \cancel{(2,4)}, \cancel{(2,5)}, \cancel{(2,2)}, \cancel{(2,5)}, \cancel{(3,1)}, \cancel{(3,2)}, \cancel{(3,3)},$
$\cancel{(3,2)}, \cancel{(3,5)}, \cancel{(3,1)}, \cancel{(3,2)}, \cancel{(3,4)}, \cancel{(4,3)}, \cancel{(4,4)}, \cancel{(4,1)}, \cancel{(4,2)},$
$\cancel{(4,4)}, \cancel{(5,1)}, \cancel{(5,2)}, \cancel{(5,3)}, \cancel{(5,3)}, \cancel{(5,4)}, \cancel{(5,2)}, \cancel{(5,5)}\}$

So, $R^2=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(5,4),(5,5)\}$

Now, using the fact that $R^3=R^2\circ R$, we will find

$R^3=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,4),(4,5),(4,3),(5,1),(5,2),(5,3),(5,4),(5,5)\}$

Similarly, from the formula $R^4=R^3\circ R$ and $R^5=R^4\circ R$

$R^4=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ and

$R^5=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ Notice that $\boxed{R^3=R^4=R^5}$ and all $R^2, R^3, R^4$ and $R^5$ are presented in this page ■

4) a) From the definition of "Union of Relations", $R \cup S = \{(a,b) \mid aRb \text{ or } aSb\}$, provided that $a, b \in A$. Since $R$ and $S$ are reflexive relations on the set $A$, for any $a \in A \Rightarrow \underline{aRa \text{ is true}}$ and for any $b \in A$, $bRb$ is true. Similarly, $\underline{\forall a \in A \Rightarrow aSa \text{ is true}}$. Taking

$$a \in A \Rightarrow \left.\begin{array}{l} aRa - \text{true} \\ aSa - \text{true} \end{array}\right\} \begin{array}{l} aRa \text{ or } aSa - \text{true} \\ (\text{"true" or "true"} - \text{"true"}) \end{array}$$

This implies $\boxed{\forall a \in A \Rightarrow (a,a) \in R \cup S}$ So, $\boxed{R \cup S \rightarrow \text{reflexive}}$

b) In the similar way, $R \cap S = \{(a,b) \mid aRb \text{ and } aSb\}$, provided that $a, b \in A \Rightarrow$ Since $R, S$ - reflexive relations on set $A$,

$$\forall a \in A \Rightarrow \left.\begin{array}{l} aRa - \text{true} \\ aSa - \text{true} \end{array}\right\} \begin{array}{l} aRa \text{ and } aSa \longrightarrow \text{true} \\ (\text{"true" and "true"} \longrightarrow \text{"true"}) \end{array}$$

$\boxed{\forall a \in A \Rightarrow (a,a) \in R \cap S}$ or just $\boxed{R \cap S \rightarrow \text{reflexive}}$

c) Initially, we'll use a famous formula for XOR:

$$A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$$

| $A$ | $B$ | $\neg A$ | $\neg B$ | $A \wedge \neg B$ | $\neg A \wedge B$ | $(A \wedge \neg B) \vee (\neg A \wedge B)$ | $A \oplus B$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | F | F | F | F |
| T | F | F | T | T | F | T | T |
| F | T | T | F | F | T | T | T |
| F | F | T | T | F | F | F | F |

It's easily seen that the formula above is correct, and we proved it using truth table. Now, when it comes to relation it is crucial to point out necessary definitions for that:

<u>Symmetric difference:</u> $A \oplus B$, all elements in $A$ or in $B$, but not in both. Then, symmetric difference of two relations can be seen as $\to$ Let $\underline{a \in A}$, since $R$ and $S$ is reflexive, $\underline{(a,a) \in R}$ and $\underline{(a,a) \in S}$. From here, it's easily seen $(a,a)$ is in both $R$ and $S$, which concludes from the definition that $\underline{(a,a) \notin R \oplus S}$. Therefore, we found $\boxed{\forall a \in A \Rightarrow (a,a) \notin R \oplus S}$ or just $\boxed{R \oplus S \text{ is irreflexive}}$

d) By the definition, $R - S = \{(a,b) \mid aRb \text{ and not } aSb\}$

From the reflexivity of $R, S \Rightarrow aRa \to$ true

So, $\boxed{\begin{array}{l} \text{not } aSa \to \text{false} \\ \quad aRa \to \text{true} \end{array}}$ $\quad aSa \to$ true

This concludes $aRa$ and not $aSa \to$ false

(true and false $\to$ false)

Hence, we proved that $\underline{\forall a \in A \Rightarrow aRa \text{ and not } aSa \to \text{false}}$ or $\underline{(a,a) \notin R - S.}$ $\boxed{\forall a \in A \Rightarrow (a,a) \notin R - S}$ This implies that

$\boxed{R - S \text{ is irreflexive}}$

e) Take $\underline{a \in A}$, and since $R, S$ are reflexive relations, $(a,\underline{a}) \in R$ and $(\underline{a},a) \in S$. From the definition of $S \circ R$, we can infer that $\underline{(a,a) \in S \circ R}$ because underlined numbers are same $\Rightarrow$ $\boxed{\forall a \in A,\ (a,a) \in S \circ R}$ $\boxed{S \circ R \text{ is reflexive}}$

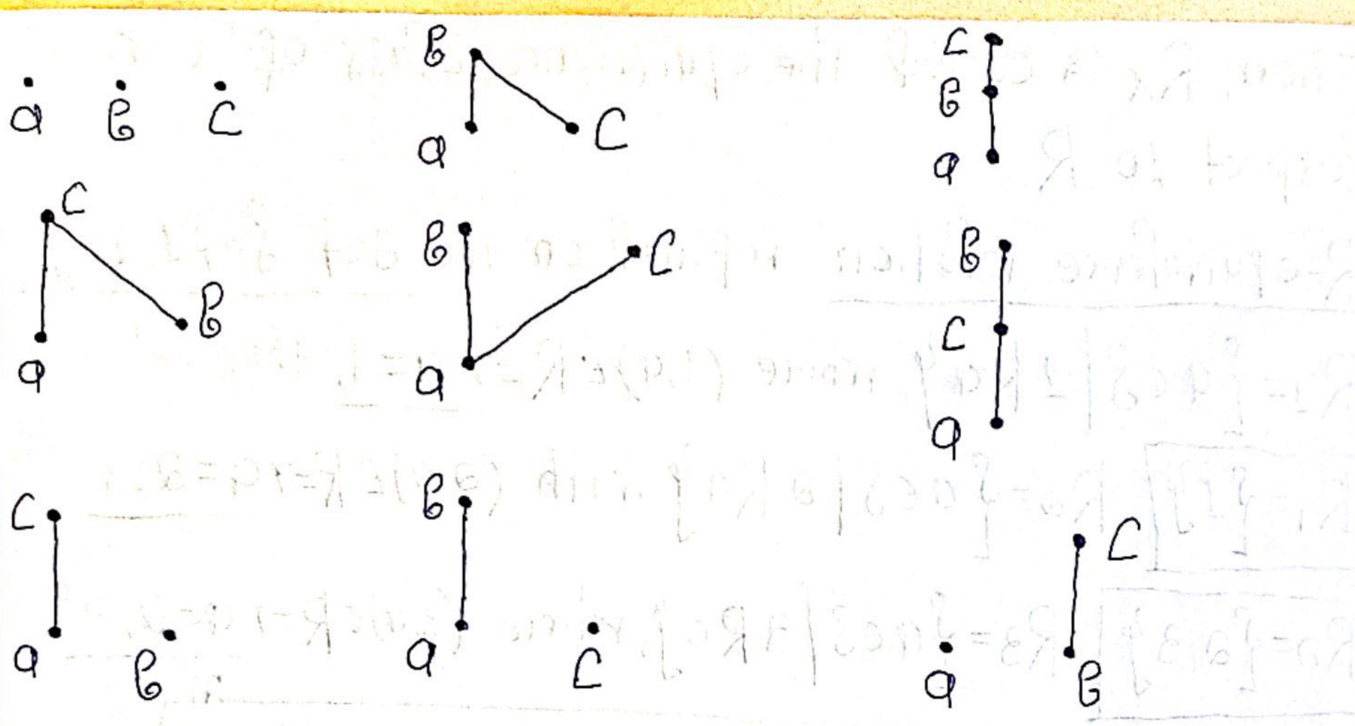
## Ordered Relations

1)

| | Partial Order | Total Order | Well order |
|---|---|---|---|
| $\langle N, < \rangle$ | False | False | False |
| $\langle N, \leq \rangle$ | True | True | True |
| $\langle \mathbb{Z}, \leq \rangle$ | True | True | False |
| $\langle R, \leq \rangle$ | True | True | False |
| $\langle P(N), \subset \rangle$ | False | False | False |
| $\langle P(N), \subseteq \rangle$ | True | False | False |
| $\langle P(\{a\}), \subseteq \rangle$ | True | True | True |
| $\langle P(\emptyset), \subseteq \rangle$ | True | True | True |

2) A $\underline{\text{minimal element}}$ is an element $x \in X$ such that there is no $a \in X$ with $a \preceq X$ and $X \neq a$ $\boxed{A = \{a, b, c\}}$

The vertices of the Hasse diagram are all elements in the set $\{a, b, c\}$. $a$ is a minimal element in the Hasse diagram, when there is no edge touching the vertex that slopes upwards (that's, there is no higher vertex that's connected to the vertex). We then note that all possible cases are below



There are 10 such possible Hasse diagrams

(a-minimal ⇒ (b,a) and (c,a) do not exist. Antisymmetry → partial order can't have both (b,c) and (c,b), but it can have one of them.)

## Equivalence Relations

1) A relation R on a set S is an equivalence relation if it satisfies all three of the following properties:

→ Reflexivity
→ Symmetry
→ Transitivity

For $3,2,4 \in S = \{1,2,3,4\}$, it's given $3R2$ and $2R4$, meaning that $(3,2)$ and $(2,4) \in R$, then from transitivity (if we assume R is an equivalence relation), $3R4$ should be true ⇒ $(3,4) \in R$ should satisfy

But we can see from R that $(3,4) \notin R$. Hence, it's $\boxed{\times}$

Set R does not define an equivalence relation on $\{1,2,3,4\}$

2) Let R be an equivalence relation on a set S. For any element $x \in S$, define $R_x = \{a \in S \mid xRa\}$, set of all elements related to x

Then, $R_x$ is called the equivalence class of $X$ with respect to $R$.

R-equivalence relation defined on the set $S=\{1,2,3\}$

$R_1=\{a \in S \mid 1 R a\}$, where $(1,a) \in R \Rightarrow a=1$, then we've $\boxed{R_1=\{1\}}$ $R_2=\{a \in S \mid 2 R a\}$ with $(2,a) \in R \Rightarrow a=2,3$

$\boxed{R_2=\{2,3\}}$ $R_3=\{a \in S \mid 3 R a\}$, where $(3,a) \in R \Rightarrow a=2,3$

$\boxed{R_3=\{3,2\}}$ Hence, $\boxed{R_1=\{1\},\ R_2=\{2,3\},\ R_3=\{3,2\}}$ are desired equivalence classes ■

3) If $X=\{1,2,3\}$, then $P(X)=\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

2 sets are related to each other when they have the same # of elements. This then implies the fact that equivalence classes are the sets containing subsets in $P(X)$ with the same # of elements

$\{\emptyset\}$

$\{\{1\},\{2\},\{3\}\}$

$\{\{1,2\},\{2,3\},\{1,3\}\}$

$\{\{1,2,3\}\}$

$\Rightarrow \boxed{\{\emptyset\},\ \{\{1\},\{2\},\{3\}\},\ \{\{1,2\},\{2,3\},\{1,3\}\},\ \{\{1,2,3\}\}}$ ■

4) a) Since $a^2=a^2$ for any $a\in \mathbb{Z}$, we have $aRa$, and this implies $\boxed{R \text{ is reflexive}}$ Suppose $aRb$, then $a^2=b^2$ or
(is true)
$b^2=a^2 \Rightarrow$ from the statement, it says $bRa$ should be true. Basically, $aRb \Leftrightarrow a^2=b^2 \Leftrightarrow b^2=a^2 \Leftrightarrow bRa$, so $\boxed{R \text{ is symmetric}}$ (If $aRb$ is true, then $bRa$ is also true)

Finally, suppose $aRb$ and $bRc \Rightarrow a^2=b^2$ and $b^2=c^2$ are true, then $a^2=c^2$ is true. Equivalently, this means $aRc$
Hence, $aRc$ ✓ and $\boxed{R \text{ is transitive}}$ As all three conditions are satisfied, R is an equivalence relation

b) We proved R is an equivalence relation on $\mathbb{Z}$, and now we define equivalence classes wrt relation R:
$\forall x\in\mathbb{Z},\ R_x=\{a\in\mathbb{Z} \mid xRa\} \Rightarrow xRa \Leftrightarrow a^2=x^2 \Leftrightarrow a=\pm x$
So, $R_x=\{-x,x\}$ for $x\neq 0$ and $R_0=\{0\}$. Concluding that

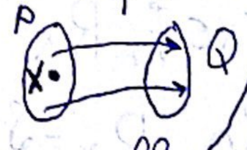
$$\boxed{\begin{gathered}\forall n\in\mathbb{Z} \Rightarrow R_n=\{-n,n\} \text{ for } n\neq 0\\ R_0=\{0\}\end{gathered}}$$

will be defined as equivalence classes of an integer for the relation R ■

## Functions

2) From the definition of total function, if $f: A \Leftrightarrow B$, we say f is a total function whenever f is a function defined for all values of A, and we write $A \rightarrow B$
P-set of people, Q-set of occupations and $f: P\rightarrow Q$ defined

by setting $f(p)$ equal to p's occupation. This means for each $x$ in $P$, $f(x)$ produces some value in $Q$, or in terms of people and occupations $\Rightarrow$ Each person should possess an occupation. If we assume that unemployment is not included as "occupation", meaning set of occupations include real job, then all people in $P$ should be employed, or have a job, for $f$ to be a total function. We do not include "unemployment" status in $Q$, since this does not provide correct view of occupations. Instead, if some person did not have a job, then there would not be a mapping from that person in $P$ to the set $Q$ of occupations (i.e. that point would be left on its own, without any correspondence to $Q$. (without any connections, or links to $Q$) (as point $X$ shows)




However, if we want $f$ to be a total function, all points in set $P$ should be mapped to some other point in $Q$. And considering set $Q$ has real-jobs (for example, doctor, ... businessman, ...) then we conclude $\boxed{\text{all people in } P \text{ are employed}}$ or it means $\boxed{\text{all people in } P \text{ should have an occupation}}$ since $f(p) =$ p's occupation and each $p \in P$ should mapped to some point in $Q$ ■ (Remember that as we work on functions nobody can have two occupations)

3) a) The function $f$ is one-to-one $\Leftrightarrow f(a)=f(b)$ implies that $a=b$ for all $a$ and $b$ in the domain

Let's take the sets $A=\{1,2\}$ and $B=\{2\}$

Note that largest element of $A$ is 2, and largest element of $B$ is 2. Then, this implies that the image of $A$ and $B$ are same $\rightarrow m(A)=m(B)$ $(2 \in S)$.

Knowing that $A \neq B$, and both $A, B$ - elements of $P(S)^*$ we conclude that $\boxed{m \text{ is not one-to-one}}$ ☒

b) The function $f$ is onto $\Leftrightarrow$ for $\forall b \in B$, there exist an element $a \in A$ such that $f(a)=b$, with $f: A \rightarrow B$

Let $x \in S$ be an arbitrary element of $S$. Note that $x$ is some integer from 0 to 5 $\Rightarrow$ It's easy to observe that $\{x\}$ is a non-empty subset of $S$, meaning that $\boxed{\{x\} \in P(S)^*}$ is true because of definition of $P(S)^*$

Furthermore, the largest element in $\{x\}$ is $x$; therefore $\boxed{m(\{x\})=x}$ In conclusion, we can say that there exist some set in $P(S)^*$ such that its image is $x$ for each $x \in S \Rightarrow \boxed{m \text{ is onto}}$ ☒ ■ (Remember that $m: P(S)^* \rightarrow S$)

4) a) The function f is <u>one-to-one</u> $\Leftrightarrow$ $f(a)=f(b)$ implies that $a=b$ for all $a,b$ in the domain

Let $\underline{\overset{(a,b\in\mathbb{Z})}{f(a)=f(b)}}$ $\Rightarrow (2a+3, a-4)=(2b+3, b-4)$ and since two ordered pairs are equal when the corresponding elements are equal $\Rightarrow$ $\left.\begin{matrix} 2a+3=2b+3 \\ a-4=b-4 \end{matrix}\right\}$ this implies $\underline{a=b}$. Henceforth, $\boxed{f(a)=f(b) \text{ concludes } a=b \text{ for all } a,b\in\mathbb{Z}}$

$\Rightarrow$ $\boxed{f \text{ is one-to-one}}$ ☒

b) The function f is <u>onto</u> $\Leftrightarrow$ $\forall b\in B$, there exist an element $a\in A$ such that $f(a)=b$, where $f: A\to B$

<u>Let's consider $(5,5)\in\mathbb{Z}\times\mathbb{Z}$</u>. If $(5,5)$ is the image of some element in $\mathbb{Z}$ $\Rightarrow (2x+3, x-4)=(5,5)$ and two ordered pairs are equal when corresponding terms are equal $\Rightarrow$

$\left.\begin{matrix} 2x+3=5 \\ x-4=5 \end{matrix}\right\}$ $\left.\begin{matrix} x=1 \\ x=9 \end{matrix}\right\}$ $\boxed{\times}$ As seen, it requires to both have $x=1$ and $x=9$, which is wrong $\boxed{\times}$

This concludes that <u>(5,5) is not the image of some element in $\mathbb{Z}$</u> (remember that $(5,5)\in\mathbb{Z}\times\mathbb{Z}$), and therefore

$\boxed{f \text{ is not onto}}$ ☒ ■

1) a) Suppose $f(x)=y$, then $f^2(x)=f(f(x))=f(y)=f(x)=y$ $\Rightarrow f(y)=y$. In other words, $f$ must map any element in the image of $f$ to itself

We now have 3 cases: the image of $f$ can contain 1 element, 2 elements, or be the whole of $A$

- There are 3 functions where the image of $f$ contains 1 element. Each such $f$ certainly maps the single element in its image to itself
- There are 18 functions where the image of $f$ contains two elements. But in only 6 of these is each element in the image of $f$ mapped to itself
- There are 6 functions where the image of $f$ is the whole of $A$. But there's only one such function that maps each element of $A$ to itself

So, we find $3+6+1=10 \Rightarrow$ $\boxed{\text{There are 10 such functions}}$

b) $f(0)=0, f(1)=0, f(2)=0$ 1

$f(0)=1, f(1)=1, f(2)=1$ 2

$f(0)=2, f(1)=2, f(2)=2$ 3

$f(0)=0, f(1)=0, f(2)=2$ 4

$f(0)=0, f(1)=1, f(2)=0$ 5

$f(0)=0, f(1)=1, f(2)=1$ 6

$f(0)=0, f(1)=2, f(2)=2$ 7

$f(0)=1, f(1)=1, f(2)=2$ 8

$f(0)=2, f(1)=1, f(2)=2$ 9

10 $f(0)=0, f(1)=1, f(2)=2$

We listed all 10 possible cases

(1) $f(0)=0,\ f(1)=0,\ f(2)=0 \Rightarrow f^2(0)=f(0)=0,\ f^2(1)=f(0)=$
$=f(1)=0,\ f^2(2)=f(0)=f(2)=0$ ✓

(2) $f^2(0)=f(1)=1=f(0),\ f^2(1)=f(1),\ f^2(2)=f(1)=1=f(2)$ ✓

(3) $f^2(0)=f(2)=2=f(0),\ f^2(1)=f(2)=2=f(1),\ f^2(2)=f(2)$ ✓

(4) $f(0)=0,\ f(1)=0,\ f(2)=2 \Rightarrow f^2(0)=f(0),\ f^2(1)=f(0)=0=$
$=f(1),\ f^2(2)=f(2)$ ✓

(5) $f^2(0)=f(0),\ f^2(1)=f(1),\ f^2(2)=f(0)=0=f(2)$ ✓

(6) $f^2(0)=f(0),\ f^2(1)=f(1),\ f^2(2)=f(1)=1=f(2)$ ✓

(7) $f^2(0)=f(0),\ f^2(1)=f(2)=2=f(1),\ f^2(2)=f(2)$ ✓

(8) $f^2(0)=f(1)=1=f(0),\ f^2(1)=f(1),\ f^2(2)=f(2)$ ✓

(9) $f^2(0)=f(2)=2=f(0),\ f^2(1)=f(1),\ f^2(2)=f(2)$ ✓

(10) $f^2(0)=f(0),\ f^2(1)=f(1),\ f^2(2)=f(2)$ ✓