#### **CS204: Discrete Mathematics**

# Ch 1. The Joundations: Logic and Proofs Predicate Logic-5 Proof Examples

#### Sungwon Kang

#### Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides



Socrates is a human.

All humans dies.

Socrates dies.

**Example 2a**:  $\exists x \forall y \ P(x,y) \Rightarrow \forall y \exists x \ P(x,y)$ 

?

**Example 2b**:  $\forall y \exists x \ P(x,y) \Rightarrow \exists x \forall y \ P(x,y)$ 

Socrates is a human.

All humans dies.

Socrates dies.

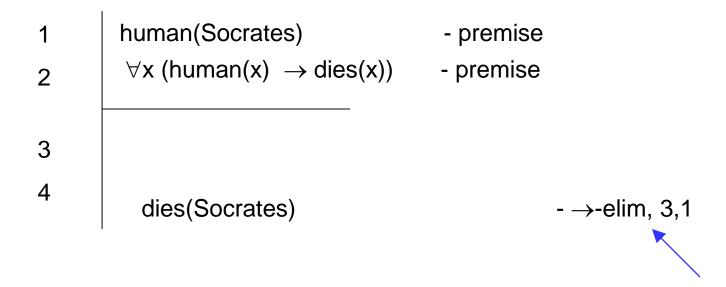
human(\_\_): "\_\_\_ is human" dies(\_\_): " \_\_\_ dies"

1 human(Socrates) - premise 2  $\forall x \text{ (human(x)} \rightarrow \text{dies(x))}$  - premise 3 dies(Socrates)

Socrates is a human.

All humans dies.

Socrates dies.



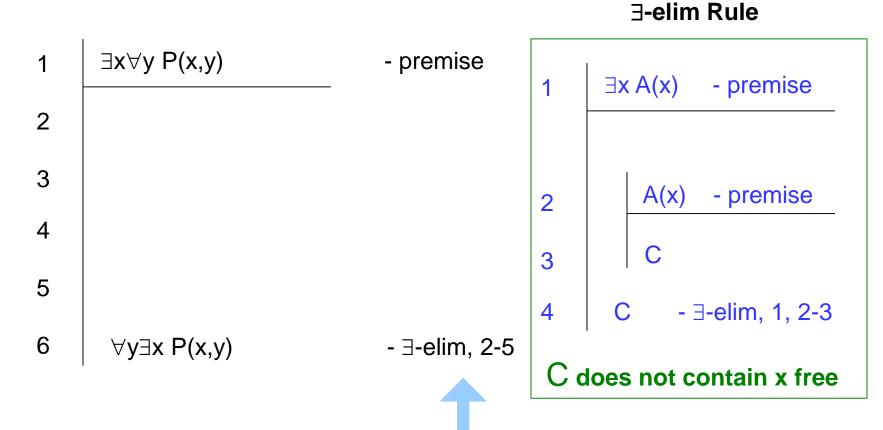
Socrates is a human.

All humans dies.

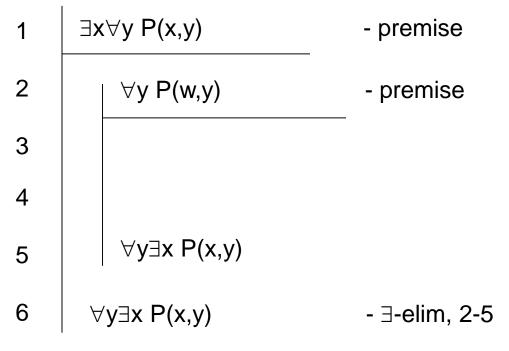
Socrates dies.

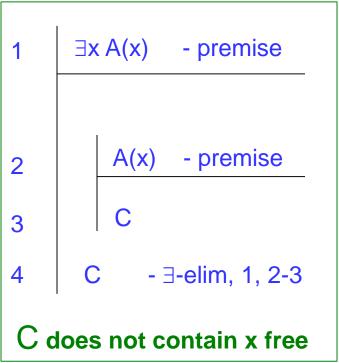
1 2	human(Socrates) $\forall x \text{ (human(x) } \rightarrow \text{dies(x))}$	- premi	
3	human(Socrates) → dies(Socrates)	ocrates)	- ∀-elim, 2 - →-elim, 3,1
-	alco(colates)		/ Cili 11, O, 1

```
1 ∃x∀y P(x,y) - premise
2 3 4 5 5 6 ∀y∃x P(x,y)
```

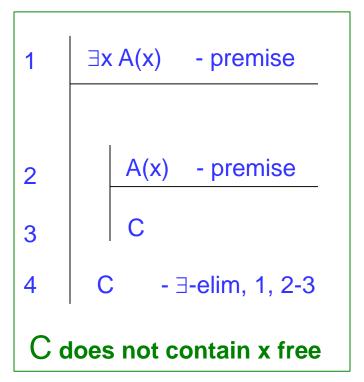


 $\forall y \exists x \ (x > y)$  does not contain x free





## 1 $\exists x \forall y \ P(x,y)$ - premise 2 $\forall y \ P(w,y)$ - premise 3 P(w,z) - $\forall$ -elim, 2 4 $\forall y \exists x \ P(x,y)$ 6 $\forall y \exists x \ P(x,y)$ - $\exists$ -elim, 2-5

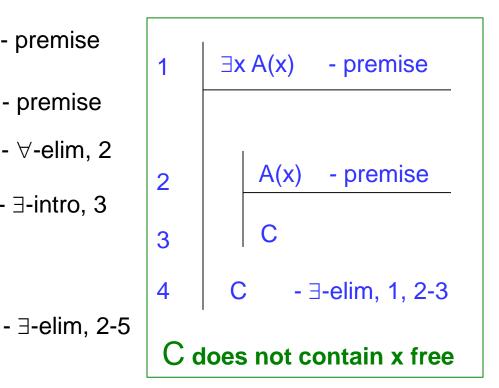


 $\forall y \exists x P(x,y)$ 

6

1 
$$\exists x \forall y \ P(x,y)$$
 - premise  
2  $\forall y \ P(w,y)$  - premise  
3  $P(w,z)$  -  $\forall$ -elim, 2  
4  $\exists x \ P(x,z)$  -  $\exists$ -intro, 3  
5  $\forall y \exists x \ P(x,y)$ 



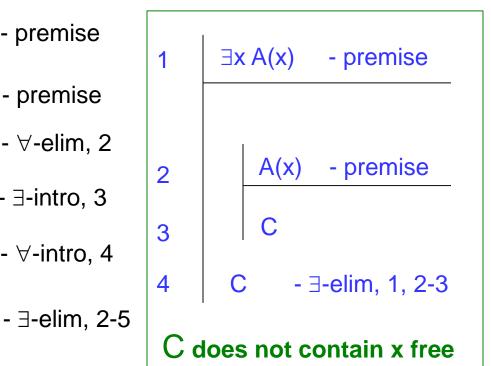


 $\forall y \exists x P(x,y)$ 

6

1 
$$\exists x \forall y \ P(x,y)$$
 - premise  
2  $\forall y \ P(w,y)$  - premise  
3  $P(w,z)$  -  $\forall$ -elim, 2  
4  $\exists x \ P(x,z)$  -  $\exists$ -intro, 3  
5  $\forall y \exists x \ P(x,y)$  -  $\forall$ -intro, 4





#### **Example 2a(A concrete example)**: $\exists x \forall y (x > y) \Rightarrow \forall y \exists x (x > y)$

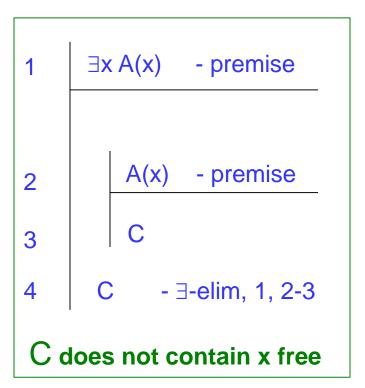
If w > any number, there is some number (i.e. w) > any number.

Therefore, for any number y, some number (for example, w) will be > y.

#### Recall that $\exists x \forall y \ P(x,y) \not\equiv \forall y \exists x \ P(x,y)$ .

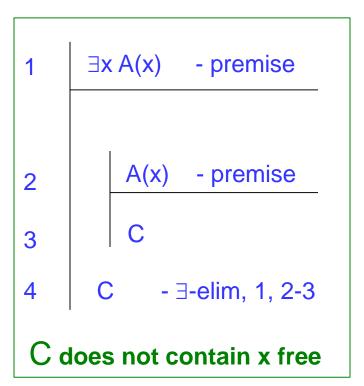
One failed approach Example 2b1:  $\forall y \exists x \ P(x,y) \stackrel{?}{\Rightarrow} \ \exists x \forall y \ P(x,y)$ 

1	∀y ∃x P(x,y)	- premise
2	∃x P(x,b)	- ∀-elim, 1
3		
4		
5	∀y P(x,y)	
6	∃x∀y P(x,y)	- ∃-intro, 5



## **Example 2b1**: $\forall y \exists x \ P(x,y) \stackrel{?}{\Rightarrow} \ \exists x \forall y \ P(x,y)$

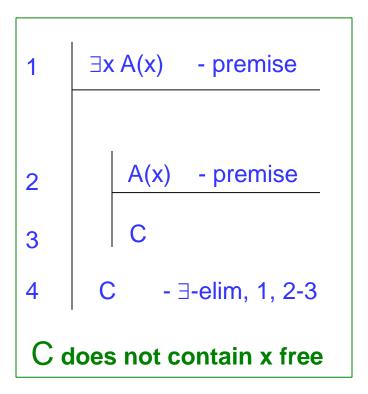
1 
$$\forall y \exists x P(x,y)$$
 - premise  
2  $\exists x P(x,b)$  -  $\forall$ -elim, 1  
3  $P(x,b)$  - premise  
4  $\forall y P(x,y)$  -  $\forall$ -intro, 3  
5  $\forall y P(x,y)$  -  $\exists$ -intro, 5



### **Example 2b1**: $\forall y \exists x \ P(x,y) \stackrel{?}{\Rightarrow} \ \exists x \forall y \ P(x,y)$

1 
$$\forall y \exists x P(x,y)$$
 - premise  
2  $\exists x P(x,b)$  -  $\forall$ -elim, 1  
3  $P(x,b)$  - premise  
4  $\forall y P(x,y)$  -  $\forall$ -intro, 3  
5  $\forall y P(x,y)$  -  $\exists$ -etim, 3-4  
6  $\exists x \forall y P(x,y)$  -  $\exists$ -intro, 5

#### ∃-elim Rule



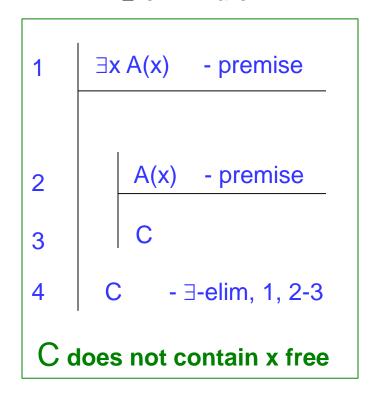
Line 5 cannot be justified by  $\exists$ -elim because  $\forall y \ P(x,y)$  contains x free. In line 4, x was only a constrained object of the domain.

In line 5, it would mean any object of the domain.

Another failed approach Example 2b2:  $\forall y \exists x \ P(x,y) \Rightarrow \exists x \forall y \ P(x,y)$ 

1	∀y ∃x P(x,y)	- premise
2	∃x P(x,b)	- ∀-elim, 1
3	P(x <sub>b</sub> ,b)	- premise
4	∀y P(x <sub>b</sub> ,y)	- ∀intro, 3
5	∃x∀y P(x,y)	- ∃-intro, 4
6	∃x∀y P(x,y)	- ∃-elim, 2, 3-5

#### ∃-elim Rule



b in line 2 is arbitrary b.

But x in the premise of line 3 may depend on b.

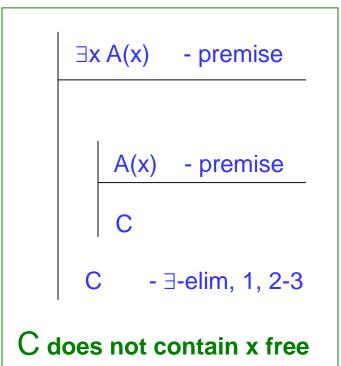
So it is written  $x_b$  in line 3 to syntactically indicate its dependency on b.

Although b in line 3 is arbitrary, we cannot infer line 4 because y in line 4 would be any y that has no dependency on b.

It is too strong an assertion that is not supported by line 3.

## Example 2b2(A concrete example): $\forall y \exists x (x > y)$ ? $\Rightarrow \exists x \forall y (x > y)$ $\exists$ -elim Rule

1 
$$\forall y \exists x (x > y)$$
 - premise  
2  $\exists x (x > b)$  -  $\forall$ -elim, 1  
3  $x > b$  - premise  
4  $\forall y (x > y)$  -  $\forall$ -intro, 3  
5  $\exists x \forall y (x > y)$  -  $\exists$ -intro, 4  
6  $\exists x \forall y (x > y)$  -  $\exists$ -elim, 2, 3-5



Let's consider what went wrong in line 4 with a concrete example where P(x,y) represents x > y.  $\exists x \forall y \ (x > y)$  does not contain x free. So it is okay to apply  $\exists$ -elim at line 6.

But line 3 means, "Let x be > b." Then x > any number (line 4) or existence of such x (line 5) can NOT be asserted as in the previous slide.

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# Soundness and Completeness of the Natural Deduction System

Natural Deduction is a "complete" predicate logic system in the sense that any valid inference can be proved using its 12 inference rules.

#### Soundness (= Consistency) Theorem

If there is a proof of a conclusion  $\varphi$  from a set of premises  $\Sigma$  using Natural Deduction (i.e.  $\Sigma \mid -\varphi$ ), then  $\varphi$  is true whenever  $\Sigma$  is true (i.e.  $\Sigma \mid =\varphi$ ).

**Proof Idea**) Each inference rule is valid. (In the case of propositional logic inference rules, we can check their validity with truth tables.) Then a sequence of applications of valid inference rules results in a conclusion that is true whenever the given assumptions are true.

#### **Completeness Theorem [Gödel 1930]**

If  $\varphi$  is true whenever a set of premises  $\Sigma$  is true (i.e.  $\Sigma \models \varphi$ ), then there is a proof of a conclusion  $\varphi$  from  $\Sigma$  using Natural Deduction (i.e.  $\Sigma \models \varphi$ ).



#### **Quiz 07-2**

Answer with "Yes" or "No".

[1] Does the set of inference rules of Gentzen's Natural Deduction have redundancy in the sense that without some rule of the system it can still be complete?

[2] Is "
$$\forall x \forall y P(x,y) \Rightarrow \forall y \forall x P(x,y)$$
" valid?

[3] Is "
$$\exists x \exists y \ P(x,y) \Rightarrow \exists y \exists x \ P(x,y)$$
" valid?

[4] Is "
$$\exists x \forall y \ P(x,y) \Rightarrow \forall y \exists x \ P(x,y)$$
" valid?

[5] Is "
$$\forall y \exists x \ P(x,y) \Rightarrow \exists x \forall y \ P(x,y)$$
" valid?