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Predicate Logic - Part II

1) a)

Statements	Reasons	
$\neg(\exists x)(R(x)\wedge B(x))$	given	1
$(\forall x)\neg(R(x)\wedge B(x))$	Existential negation, 1	2
$(\forall x)(\neg R(x)\vee\neg B(x))$	De Morgan's Law, 2	3
$(\forall x)(R(x)\rightarrow\neg B(x))$	implication, 3	4

Now, we'll prove the other direction

Statements	Reasons
1) $(\forall x)(R(x)\rightarrow\neg B(x))$	given
2) $(\forall x)(\neg R(x)\vee\neg B(x))$	implication, 1
3) $(\forall x)\neg(R(x)\wedge B(x))$	De Morgan's Law, 2
4) $\neg(\exists x)(R(x)\wedge B(x))$	Existential negation, 3

Hence, we proved that $\neg(\exists x)(R(x)\wedge B(x))\Leftrightarrow$
$\Leftrightarrow(\forall x)(R(x)\rightarrow\neg B(x))$ or $\boxed{S_1\Leftrightarrow S_2}$

b) There does not exist a right triangle that has an obtuse angle

c) A right triangle does not have an obtuse angle ■

2) a) Assume domain $\rightarrow$ integers, that is, given domain is $\mathbb{Z}$, or all integers. $P(x) =$ x is even

$Q(x) =$ x is odd

$(\exists x)(P(x) \wedge Q(x))$ means there exist an integer x such that x is even and x is odd. However, we know integer can never be both even and odd $\Rightarrow$ $(\exists x)(P(x) \wedge Q(x))$ is false. $(\exists x)(P(x))$ means there exist an integer x such that x is even, which is obviously true (take $x=2$) $\Rightarrow$ $(\exists x)(P(x))$ is true

$(\exists x)(Q(x))$ means there exists an integer which is odd; definitely true (take integer as equal to 1) $\Rightarrow$ $(\exists x)(Q(x))$ is true. Hence, $\boxed{(\exists x)P(x) \wedge (\exists x)Q(x) \Rightarrow \text{is always True}}$

whereas $\boxed{(\exists x)(P(x) \wedge Q(x)) \text{ is always False}}$. This concludes these statements are not logically equivalent

b) $(\exists x)(P(x) \vee Q(x))$ means there exist an integer x such that x is even or x is odd, which is definitely true as we can pick $x=2$ for example

Since $P(x) \vee Q(x)$ is disjunction $\Rightarrow$ $\boxed{(\exists x)(P(x) \vee Q(x)) \rightarrow \text{True}}$

$(\exists x)P(x) \rightarrow$ means there exist integer which is even; definitely true, as we can pick integer equal to 10

$\boxed{(\exists x)P(x) \rightarrow \text{True}}$ Similarly, $(\exists x)Q(x) \rightarrow$ means there exist x-integer for which x is odd $\Rightarrow$ pick $x=1$ and it becomes true $\boxed{(\exists x)Q(x) \rightarrow \text{True}}$ Therefore we've

$\boxed{(\exists x)P(x) \vee (\exists x)Q(x) \rightarrow \text{True}}$ because both of them are True

This concludes our example from part (a) satisfies the equivalence $(\exists x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$

3) a) Pick $x=0$ and $0^2>0$ is false $\Rightarrow$ meaning this inequality is not true for all real numbers x $\boxed{\text{False}}$

b) Choose $x=\sqrt{3}$, then $x^2-2=3-2=1$, meaning there exist x such that $x^2-2=1 \Rightarrow \boxed{\text{True}}$

c) If there exist $x \in R$ such that $x^2+2=1$, then $\underbrace{x^2+1=0}_{x^2=-1}$ for some real number x. However, we know there does not exist such solution in the domain of real numbers ($x=\pm i$ is solution, where $i=\sqrt{-1}$) It only exists in complex numbers $\Rightarrow \boxed{\text{False}}$

d) For any real number x, choose $y=4-x^2$. It's obvious that y will be real number and we've

$x^2+y=4$, meaning for all real numbers x, we can find real number y (particularly, $y=4-x^2$ which is a real number) such that $x^2+y=4 \Rightarrow \boxed{\text{True}}$

e) Assume that there exist a real number y such that, for all real numbers x, $x^2+y=4$. As it's true for all $x \in \mathbb{R}$, we pick $x=0 \Rightarrow y=4$. But if we choose $x=1 \Rightarrow y+1=4$ implies $y=3$, which is contra-diction as we had $y=4 \Rightarrow \boxed{X}$ Hence, there should not exist such real number y $\boxed{\text{False}}$ ■

(thus we don't require the same real number for each real number x to make the equation become true)

4) a) $(\forall x)(\neg P(x) \to (\exists y)(P(y) \wedge Q(y,x)))$

b)

Statements	Reasons	
$\neg(\forall x)(\neg P(x) \to (\exists y)(P(y) \wedge Q(y,x)))$	given	1
$(\exists x)\neg(\neg P(x) \to (\exists y)(P(y) \wedge Q(y,x)))$	Universal negation, 1	2
$(\exists x)\neg(\neg\neg P(x) \vee (\exists y)(P(y) \wedge Q(y,x)))$	Implication, 2	3
$(\exists x)\neg(P(x) \vee (\exists y)(P(y) \wedge Q(y,x)))$	Double negation, 3	4
$(\exists x)(\neg P(x) \wedge \neg(\exists y)(P(y) \wedge Q(y,x)))$	De Morgan's Law, 4	5
$(\exists x)(\neg P(x) \wedge (\forall y)\neg(P(y) \wedge Q(y,x)))$	Existential negation, 5	6
$(\exists x)(\neg P(x) \wedge (\forall y)(\neg P(y) \vee \neg Q(y,x)))$	De Morgan's Law, 6	7
$(\exists x)(\neg P(x) \wedge (\forall y)(P(y) \to \neg Q(y,x)))$	Implication, 7	8

Hence, we found that $\to$ $(\exists x)(\neg P(x) \wedge (\forall y)(P(y) \to \neg Q(y,x)))$

c) There is a non-prime x such that no prime y divides it ■ (There is a non-prime x such that for all primes y, y does not divide x)

5) We proceed the proof by contradiction. Let w, m, n be distinct lines, and suppose $w \parallel m$ and $m \parallel n \Rightarrow$ Suppose, to the contrary, that w is not parallel to n. By the definition of "parallel", this means there exist a point X on both w and n. Since $w \parallel m$, X should not be on m (otherwise, X is on m and X is on w would mean w and m are not parallel) According to Axiom 2, it says there exist a unique line such that X is on the line and that line is parallel to m. However, that contradicts the fact w and n are both distinct lines which contain point X and both are parallel to $m \Rightarrow \boxed{X}$

Hence, our assumption was wrong $\Rightarrow \boxed{w \parallel n}$ ■

6) a) $(\forall x)(\exists y)\, P(y, x)$

b) Using universal & existential negation rules,

$\neg(\forall x)(\exists y)\, P(y, x) \Leftrightarrow (\exists x)\neg(\exists y)\, P(y, x) \Leftrightarrow (\exists x)(\forall y)\neg P(y, x)$

Hence, we found negation of expression from (a) $\Rightarrow \boxed{(\exists x)(\forall y)\neg P(y, x)}$

c) Some number is greater than or equal to all numbers

There is a number that all numbers are less than or equal to that number ■

f) $\leftrightarrow$-intro

$$\frac{P \to Q, \; Q \to P}{P \leftrightarrow Q}$$

• The rule is that wherever instances of "$P \to Q$" and "$Q \to P$" appear on lines of a proof, "$P \leftrightarrow Q$" can validly be placed on a subsequent line

• The "biconditional introduction" rule may be written in sequent notation: $(P \to Q), (Q \to P) \vdash (P \leftrightarrow Q)$ where $\vdash$ is a metalogical symbol meaning that $P \leftrightarrow Q$ is a syntactic consequence when $P \to Q$ and $Q \to P$ are both in a proof

$\leftrightarrow$-elim

$$\frac{P \leftrightarrow Q}{P \to Q}$$

$\leftrightarrow$-elim

$$\frac{P \leftrightarrow Q}{Q \to P}$$

• The rule is that wherever an instance of "$P \leftrightarrow Q$" appears on a line of a proof, either "$P \to Q$" or "$Q \to P$" can be replaced on a subsequent line

• The "biconditional elimination" rule may be written in sequent notation: $(P \leftrightarrow Q) \vdash (P \to Q)$ and $(P \leftrightarrow Q) \vdash (Q \to P)$ where $\vdash$ is a metalogical symbol meaning that $P \to Q$, in the 1st case, and $Q \to P$ in the other are syntactic consequences of $P \leftrightarrow Q$ in some logical system

8) The given diagram contains 4 baddas, which are thus the 4 inner points that are touched by 4 line segments each

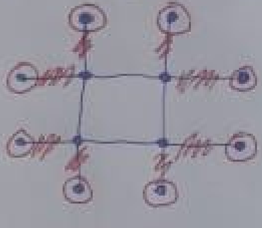
We see that each badda is hit by exactly 4 bings, because each of 4 inner points are touched by 4 line segments each $\Rightarrow$ $\boxed{\text{Axiom 1 is satisfied}}$

Now, we observe each bing is not hit by exactly 2 baddas, because there are 8 line segments that touch only 1 bing $\Rightarrow$ $\boxed{\text{Axiom 2 is not satisfied}}$

We see that 2 bings hit the same two baddas, as there's no more than 1 line segment between each pair of points $\Rightarrow$ $\boxed{\text{Axiom 3 is satisfied}}$

Finally, the diagram contains at least 1 bing, since diagram has 4 bings $\Rightarrow$ $\boxed{\text{Axiom 4 is satisfied}}$

Because axiom 2 is not satisfied $\Rightarrow$ $\boxed{\text{this's not a model for the system}}$



For Axiom 2,
these 8 points provide evidence that not every bing is hit by exactly 2 baddas
Individually, these 8 points prove that those bings are hit (individually) by exactly one baddas

For Axiom 3, it's just saying if two baddas intersect at bing ℓ, then there's no other bing hit by these two baddas. It is essentially saying no other intersection, which's obvious because if 2 lines intersect, they have only 1 common point and it infers that there is no other ling hit by x and y. If 2 lines have a common point, it should not have other common point, implying for distinct laddas x and y, each hitting ling $\ell \Rightarrow$ no other lings are hit by both x and y

g) a) $(\exists y)(H(y) \wedge IsTailOf(\forall t, y))$

b) Horses are animals: $(\forall y)(H(y) \rightarrow A(y))$

Horses' tails are tails of animals:

$(\forall x)(\forall y)((H(y) \wedge IsTailOf(x,y)) \rightarrow (A(y) \wedge IsTailOf(x,y)))$

c) Inference in (b) is valid, meaning that argument is $\boxed{\text{valid}}$

$$\frac{(\forall y)(H(y) \rightarrow A(y))}{(\forall x)(\forall y)((H(y) \wedge IsTailOf(x,y)) \rightarrow (A(y) \wedge IsTailOf(x,y)))}$$

d) We can formalize our predicate logic inference using natural deduction as follows:

$$\frac{\forall y(H(y)\to A(y))}{H(y)\to A(y)}\ \forall\text{-elim} \qquad \frac{[H(y)\land IsTaiPOf(x,y)]^{*}}{H(y)}\ \land\text{-elim} \qquad \frac{[H(y)\land IsTaiPOf(x,y)]^{*}}{IsTaiPOf(x,y)}\ \land\text{-elim}$$

$$\frac{H(y)\to A(y) \qquad H(y)}{A(y)}\ \to\text{-elim}$$

$$\land\text{-intro}\ \frac{A(y) \qquad IsTaiPOf(x,y)}{A(y)\land IsTaiPOf(x,y)}$$

$$\to^{*}\text{-intro}\ \frac{A(y)\land IsTaiPOf(x,y)}{(H(y)\land IsTaiPOf(x,y))\longrightarrow(A(y)\land IsTaiPOf(x,y))}$$

$$\forall\text{-intro}\ \frac{(H(y)\land IsTaiPOf(x,y))\longrightarrow(A(y)\land IsTaiPOf(x,y))}{\forall y((H(y)\land IsTaiPOf(x,y))\to(A(y)\land IsTaiPOf(x,y)))}$$

$$\forall\text{-intro}\ \frac{\forall y((H(y)\land IsTaiPOf(x,y))\to(A(y)\land IsTaiPOf(x,y)))}{(\forall x)(\forall y)((H(y)\land IsTaiPOf(x,y))\to(A(y)\land IsTaiPOf(x,y)))}$$

<u>Note:</u> $[A]^{*}$ stands for an assumption A discharged by the rule $*$

$[A]$ means that the hypotheses A is discharged

9) a) Note: If "x is a tail of a horse" means that x is a tail of every horse, then our predicate logic expression would be $(\forall y)(H(y) \to IsTailOf(x,y))$ However, we accepted that x is a tail of some horse, because of "a" article. Therefore, we implemented existential qualifier in first place