Ch 5. Induction and Recursion

Mathematical Induction

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 5. Induction and Recursion

5.1 Mathematical Induction

5.2 Strong Induction and Well-Ordering

5.3 Recursive Definitions and Structural Induction

5.4 Recursive Algorithms

5.5 Program Correctness

Proof by Induction

1. The Principle of Mathematical Induction
2. Strong Induction

1. The First Principle of Mathematical Induction

To prove the statement

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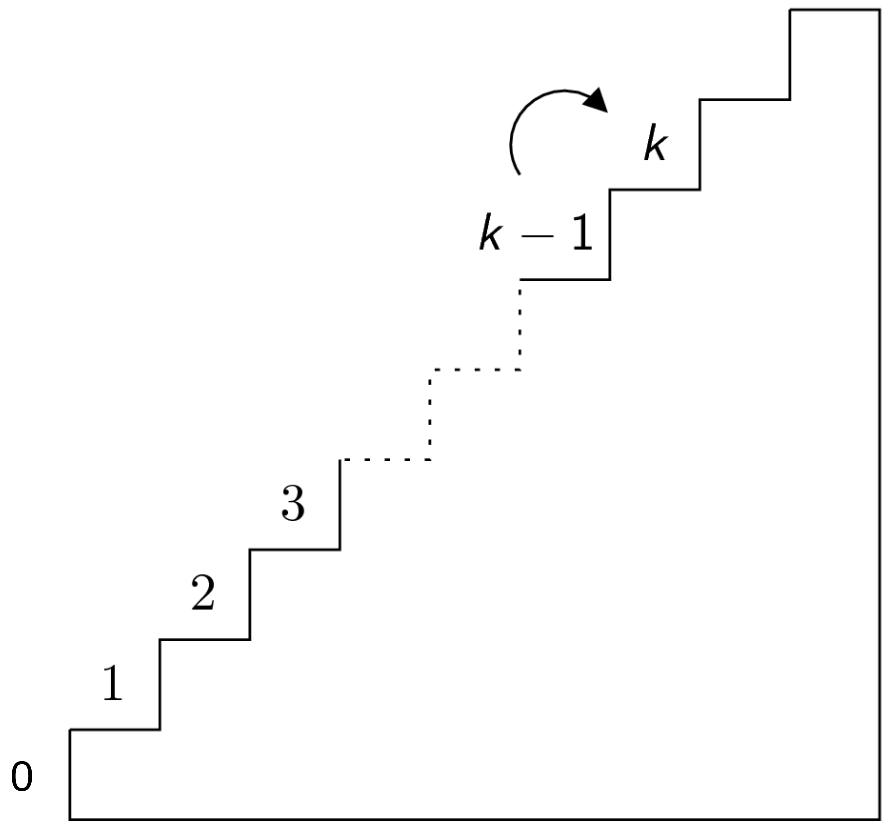
it suffices to prove

(Basis Step)	Statement(0) and
(Induction Step)	$\underbrace{\text{Statement}(k)}_{\text{Induction Hypothesis}} \Rightarrow$ Statement $(k+1)$ for $k \in \mathbb{N}$

Analogy: climbing a staircase



Examples of statements that are proved by induction

- The sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.
- A binary tree of height n has less than 2^{n+1} nodes.
- A convex n-gon has $\dfrac{n(n-3)}{2}$ diagonals.

What do these examples have in common?

☛ "For every natural number n ... "

Example: Adding the first n natural numbers

Theorem

For any $n \geq 1$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Apply the Principle of Mathematical Induction:

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To prove Statement(n): $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

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Apply the Principle of Mathematical Induction:

To prove Statement(n): $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Basis Case: Statement(1): $1 = \frac{1(1+1)}{2}$

Example: Adding the first n natural numbers

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For any $n \geq 1$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Apply the Principle of Mathematical Induction:

To prove
Statement(n): $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

Basis Case:
Statement(1): $1 = \dfrac{1(1+1)}{2}$

Inductive Case:
Statement($k-1$): $1 + 2 + 3 + \cdots + (k-1) = \dfrac{(k-1)(k-1+1)}{2}$

$\Downarrow$

Statement(k): $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$

Proof:

Base Case: If $n = 1$, then the sum of the first n natural numbers is 1, and $n(n+1)/2 = 1 \cdot 2/2 = 1$, so Statement(1) is true.

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for some $k > 1$.

Part of inductive step

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Inductive Step: Adding k to both sides of this equation gives

$$\begin{aligned} 1 + 2 + \cdots + (k-1) + k &= \frac{(k-1)(k-1+1)}{2} + k \\ &= \frac{(k-1)(k) + 2k}{2} \\ &= \frac{k(k+1)}{2} \end{aligned}$$

as required. $\square$

2. The Second Principle of Mathematical Induction

To prove the statement

"Statement(n), for every n $\in \mathbb{N}$ "

it suffices to prove

(Basis Step)	Statement(0) and
(Induction Step)	Statement(0) $\wedge$ Statement (1) $\wedge$ . . . $\wedge$ Statement (k) $\Rightarrow$ Statement (k+1), for k $\in \mathbb{N}$

Induction Hypothesis

This principle describes *strong induction*.

Theorem

Every integer $n \geq 2$ is either prime or the product of primes.

Definition

An integer p > 1 is called *prime* if and only if the only positive factors of p are 1 and p.

The first principle of mathematical induction does not work easily for this problem!

Theorem

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Proof.

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Part of inductive step

Inductive Step: If k is prime, we are done. If k is not prime, then $k = pq$ for some $p \geq 2$ and $q \geq 2$. And since $k = pq$, p and q are both less than k. By inductive hypothesis, p and q are both either prime or products of primes, so $k = pq$ is the product of primes. $\square$

Quiz 14-1

Which of the following is a proof technique that is different from the others?

(a) The second principle of mathematical induction

(b) The course-of-values induction

(c) The strong induction

(d) The structural induction