CS 204 Discrete Mathematics

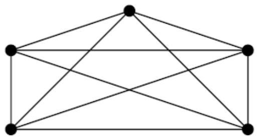
Homework 12

Sample Solutions

1. An undirected graph is called *complete* if every vertex shares an edge with every other vertex. Draw a complete graph on five vertices. How many edges does it have?

Solution)

Any such graph must have ten edges. For example,



2. Think of the Internet as one big graph, where each web page is a vertex and each link is an edge.

(a) Is this a directed graph? Why or why not?

(b) Is this graph connected? Why or why not?

(c) Is this graph complete? Why or why not?

(d) Is this graph simple? Why or why not? (A graph is called *simple* if it has no multiple edges or loops.)

(e) For a given web page p, what does the outdegree of p represent?

(f) For a given web page p, what does the indegree of p represent?

Solution)

(a) Yes. Links have a direction from a page to a page.

(b) No. There are pages with no links that are never linked to.

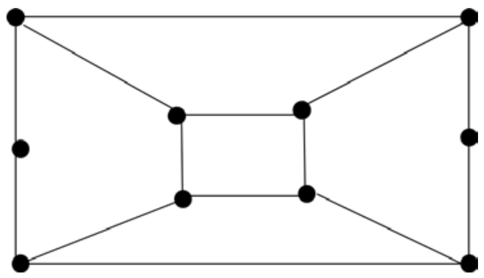
(c) No. There are lots of pages that don't link to each other.

(d) No. A page can link to itself.

(e) The outdegree is the number of links on page p.

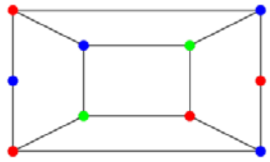
(f) The indegree is the number of pages that have links to p.

3. Color the vertices of the following graph so that no vertices of the same color share an edge. Use as few colors as possible. Explain why the graph cannot be colored using fewer colors. Be specific.



Solution)

The trapezoidal loops on the sides have five nodes, and since they must alternate colors, three colors are needed to complete these circuits. A possible three coloring is shown below.

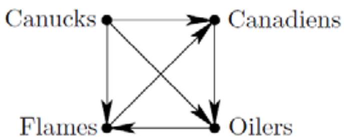


4. A round-robin tournament among four teams - Canadiens, Canucks, Flames, and Oilers - has the following results: Canucks defeat Canadiens; Canucks defeat Flames; Canucks defeat Oilers; Canadiens defeat Oilers; Flames defeat Canadiens; Oilers defeat Flames.

- Model these results with a directed graph, where ach vertex represents a team and each edge represents a game, pointing from the winner to the loser.
- Find a circuit in this graph.
- Explain why the existence of a circuit in such a graph makes it hard to rank the teams from best to worst.

Solution)

(a)

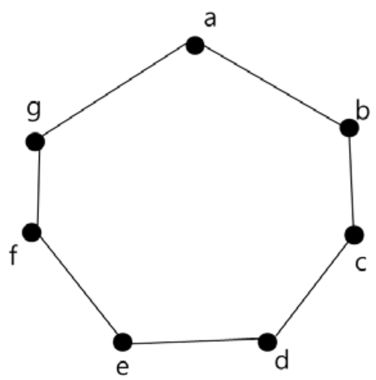


(b) Flames, Canadiens, Oilers, Flames.

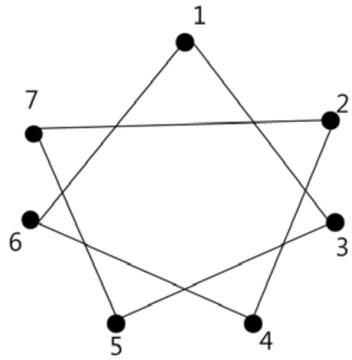
(c) The tournament does not define an ordering among the three teams in this circuit, because each team can claim to have beaten the team who beat the team they lost to.

5. Prove that the graph G1 and G2 are isomorphic.

G1:



G2:



Solution)

$\alpha(a) = 1$ $\beta(e1) = f1$ $\alpha(b) = 3$ $\beta(e2) = f2$ $\alpha(c) = 5$ $\beta(e3) = f3$ $\alpha(d) = 7$ $\beta(e4) = f4$ $\alpha(e) = 2$ $\beta(e5) = f5$ $\alpha(f) = 4$ $\beta(e6) = f6$ $\alpha(g) = 6$ $\beta(e7) = f7$

G1			α(n1)	α(n2)	Edge of G2 joining α(n1) and α(n2)	β(e)	Edge of G2 joining α(n1) and α(n2) = β(e)? (Answer with Yes/No)
n1	e	n2					
a	e1	b	1	3	f1	f1	Yes
b	e2	c	3	5	f2	f2	Yes
c	e3	d	5	7	f3	f3	Yes
d	e4	e	7	2	f4	f4	Yes
e	e5	f	2	4	f5	f5	Yes
f	e6	g	4	6	f6	f6	Yes
g	e7	a	6	1	f7	f7	Yes

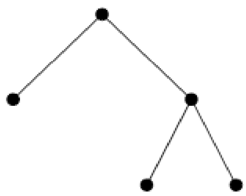
The two graphs G1 and G2 are isomorphic.

Trees

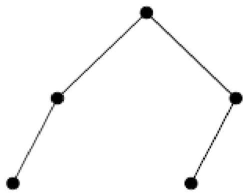
1. We gave a recursive definition of a binary tree in class. Suppose we modify this definition by deleting part B1, so that an empty tree is not a binary tree. Let's call a tree satisfying this revised definition a *T binary tree*.
(a) Give an example of a T binary tree with five nodes.
(b) Give an example of a binary tree with five nodes that is not a T binary tree.

Solution)

(a)



(b)



2. Consider the following list of numbers.

123, 684, 121, 511, 602, 50, 43

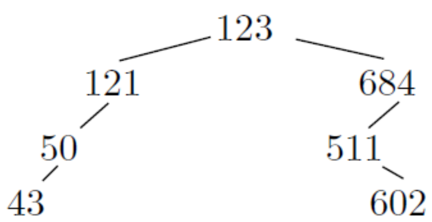
(a) Place the numbers, in the order given, into a binary search tree.

(b) The height of a binary search tree is the maximum number of edges you have to go through to reach the bottom of the tree, starting at the root. What is the height of the tree in part (a)?

(c) Reorder the numbers so that when they are put into a binary search tree, the height of the resulting tree is less than the height of the tree in part (a). Give both your new list and the search tree it produces.

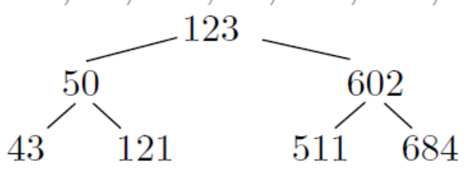
Solution)

(a)



(b) 3

(c) 123, 50, 602, 43, 121, 511, 684



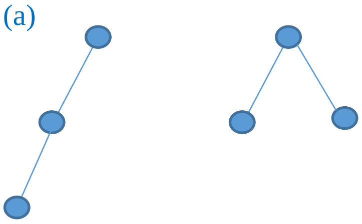
3.

(a) Draw all non-isomorphic rooted trees (i.e. trees with their unique roots) having three vertices.

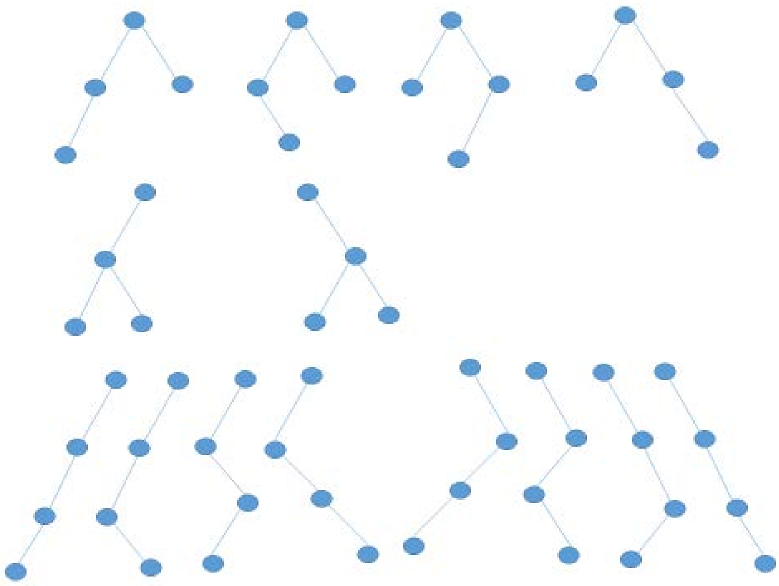
(b) Draw all non-isomorphic binary trees having four vertices.

Solution)

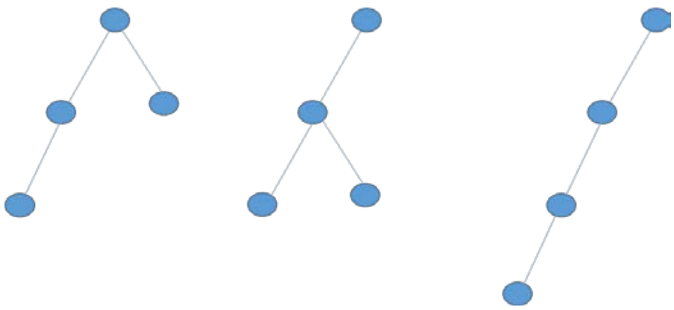
(a)



(b)



Note) The following would be the answer to the question "Draw all non-isomorphic trees that have four vertices and each node has at most two children."



4. Recall the definition of *full binary tree* discussed in class. Prove that if T is a full binary tree with i internal vertices (i.e. non-terminal nodes), then T has i+1 terminal vertices (i.e. leaf nodes) and 2i+1 total vertices.

Solution)

Because it is a full binary tree, each vertex has either two children or zero children. Therefore, every internal vertex has exactly 2 children. If there are I internal nodes, their total children are 2i. In the set of the internal nodes, there are i-1 nodes, except the root, being the children node at the same time.

Therefore, the number of the external nodes are 2i – (i-1) = i+1

The total vertices are (i+1) + i = 2i + 1

5.

Put the following words

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Cheddar  Swiss  Brie  Panela  Stilton  Mozzarella  Gouda
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into a binary search tree with the smallest height possible.

Solution)