Ch 2. Basic Structures: Sets, Functions

Ch 9. Relations

Order Relations

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 9. Relations

Order Relations

1. Partial Order
2. Total Order
3. Well Order
4. Lexicographical Order

Acknowledgement
[Stanat 77] Donald F. Stanat, David F. McAllister, *Discrete Mathematics in Computer Science*, Prentice Hall, 1977.

1. Partial Order

Definition

A relation R on a set S is a *partial ordering* if it satisfies all three of the following properties.

1. *Reflexivity.* For any $a \in S$, $a \, R \, a$.
2. *Transitivity.* For any $a, b, c \in S$, if $a \, R \, b$ and $b \, R \, c$, then $a \, R \, c$.
3. *Antisymmetry.* For any $a, b \in S$, if $a \, R \, b$ and $b \, R \, a$, then $a = b$. That is, if aRb is true, then bRa is never true unless a = b.

What is the difference from the definition of equivalence relation?

[redacted]

Definition

A relation R on a set S is a *partial ordering* if it satisfies all three of the following properties.

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3. *Antisymmetry.* For any $a, b \in S$, if $a \, R \, b$ and $b \, R \, a$, then $a = b$.

If a set X has a partial ordering $\preceq$ on it, we say that $(X, \preceq)$ is a partially ordered set, or *poset*, for short.

Examples of posets

- If S is a set of numbers, then $\leq$ defines a partial ordering on S.
- Let S be any set, and let $\mathcal{P}(S)$ be the power set of S. Then $\subseteq$ defines a partial ordering on $\mathcal{P}(S)$.
- The "divides" relation $|$ defines a partial ordering on $\mathbb{N}^+$

Example

$\mathbb{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

Then the partial order

$(\mathbb{P}(\{1,2,3\}, \subseteq)$

$= \{(\emptyset, \emptyset), (\emptyset,\{1\}), (\emptyset,\{2\}), (\emptyset,\{3\}), (\emptyset,\{1,2\}), (\emptyset,\{2,3\}), (\emptyset,\{1,3\}), (\emptyset,\{1,2,3\}),$

$(\{1\},\{1\}), (\{1\},\{1,2\}), (\{1\},\{1,3\}), (\{1\},\{1,2,3\}),$

$(\{2\},\{2\}), (\{2\},\{1,2\}), (\{2\},\{2,3\}), (\{2\},\{1,2,3\}),$

$(\{3\},\{3\}), (\{3\},\{2,3\}), (\{3\},\{1,3\}), (\{3\},\{1,2,3\}),$

$(\{1,2\},\{1,2\}), (\{1,2\},\{1,2,3\}),$

$(\{2,3\},\{2,3\}), (\{2,3\},\{1,2,3\}),$

$(\{1,3\},\{1,3\}), (\{1,3\},\{1,2,3\}),$

$(\{1,2,3\},\{1,2,3\})$

$\}$

How many elements are in $(\mathbb{P}(\{1,2,3\}, \subseteq)$? 27

Hasse diagrams

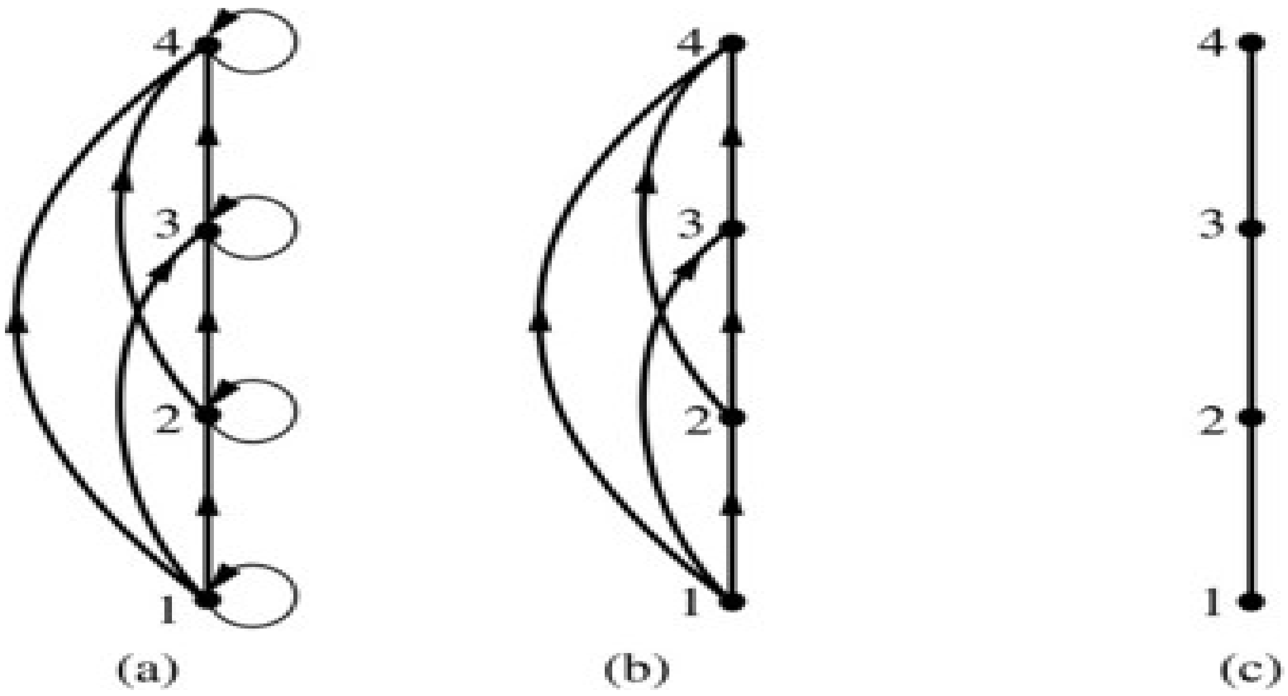
A *finite* partial order can be *represented* as a Hasse diagram.

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

Hasse diagram for $(X, \preceq)$ consists of a label, or *node*, for each element in the set, along with lines connecting related nodes. More specifically, if x, y are distinct elements of X with $x \preceq y$, and there are no elements z such that $x \prec z \prec y$, then there should be an upward sloping line from node x to node y in the Hasse diagram.

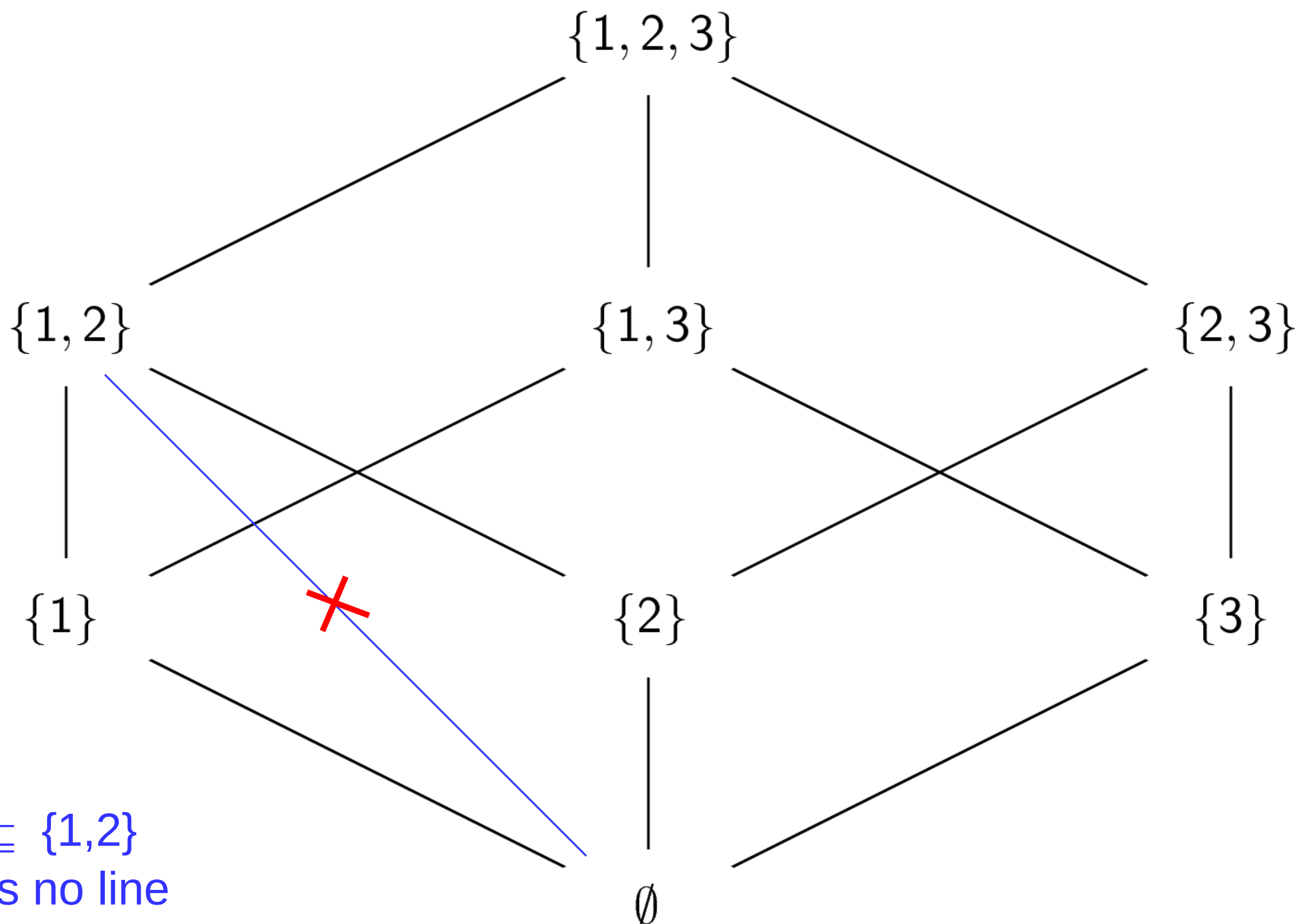
Hasse diagram is minimal.

Constructing a Hasse Diagram



A partial ordering is shown in (a) of the figure above.
The loops due to the reflexive property are deleted in (b).
The edges that must be present due to the transitive property are deleted in (c).
The Hasse diagram for the partial ordering (a), is depicted in (c).

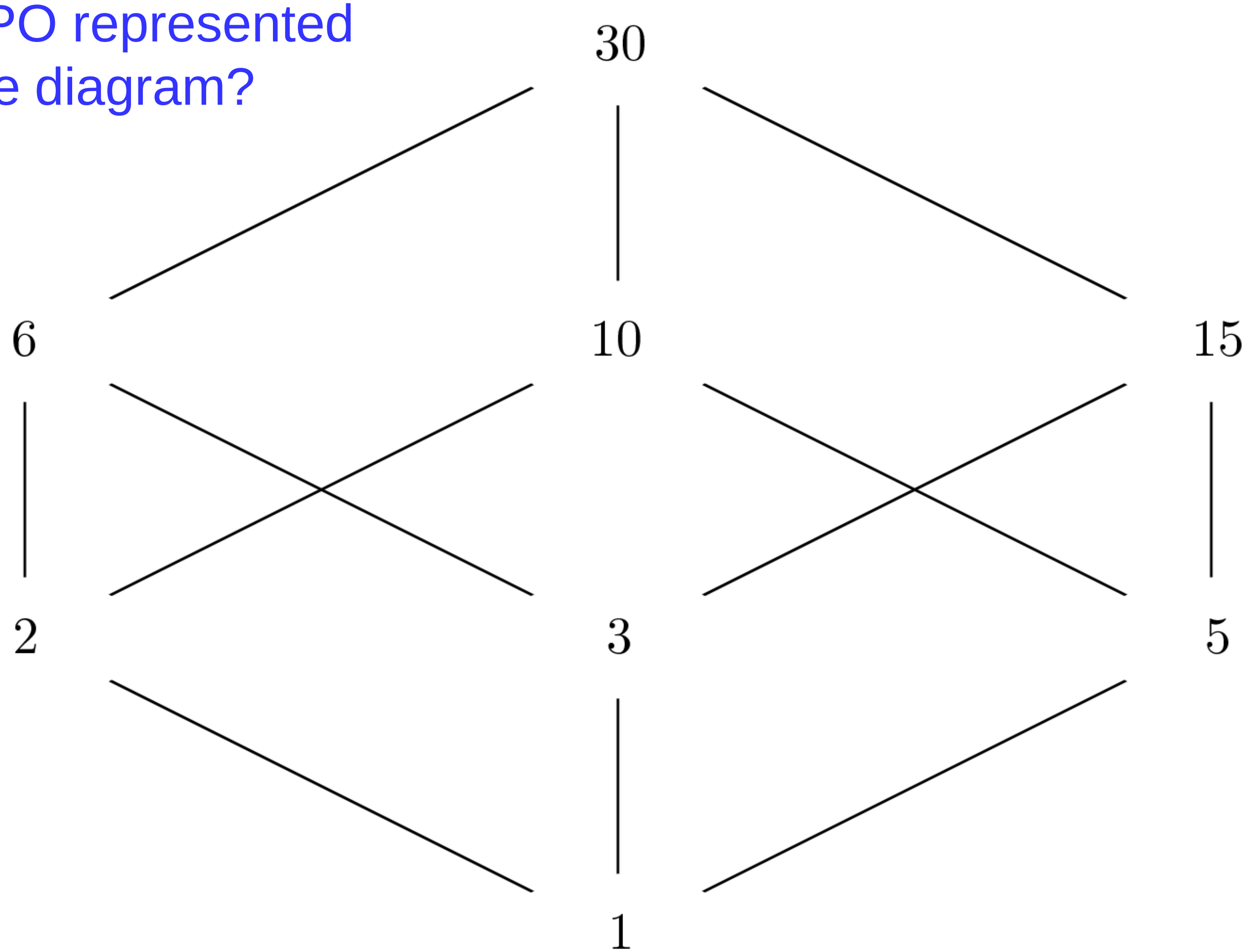
Hasse diagram for $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$



$\varnothing \subseteq \{1\} \subseteq \{1,2\}$
So there is no line
between $\varnothing$ and {1,2}.

Examples

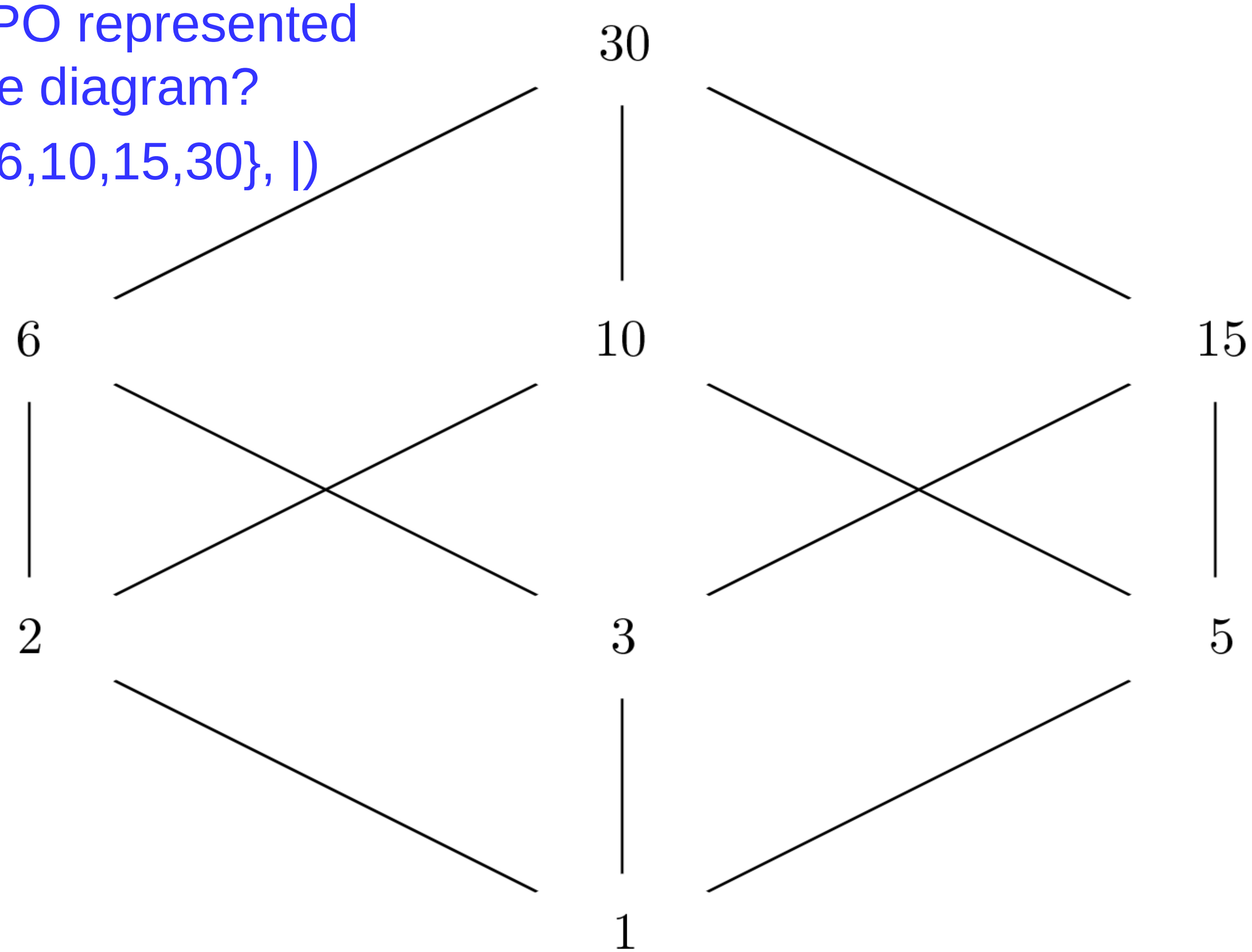
What is the PO represented by this Hasse diagram?



Examples

What is the PO represented by this Hasse diagram?

☛ ({1,2,3,5,6,10,15,30}, |)



Meet and join

Let $(X, \preceq)$ be a poset. For any elements $a, b \in X$, define the *meet* of a and b (denoted $a \wedge b$) to be the greatest lower bound of a and b, if such a lower bound exists. In other words, $a \wedge b$ is an element of X with the following properties.

1. $a \wedge b \preceq a$ and $a \wedge b \preceq b$.
2. If some $x \in X$ satisfies $x \preceq a$ and $x \preceq b$, then $x \preceq a \wedge b$.

It follows from property 2 that if the meet of two elements exists, then it is unique. Similarly, we can define the *join* of $a, b \in X$ to be the element $a \vee b \in X$ satisfying

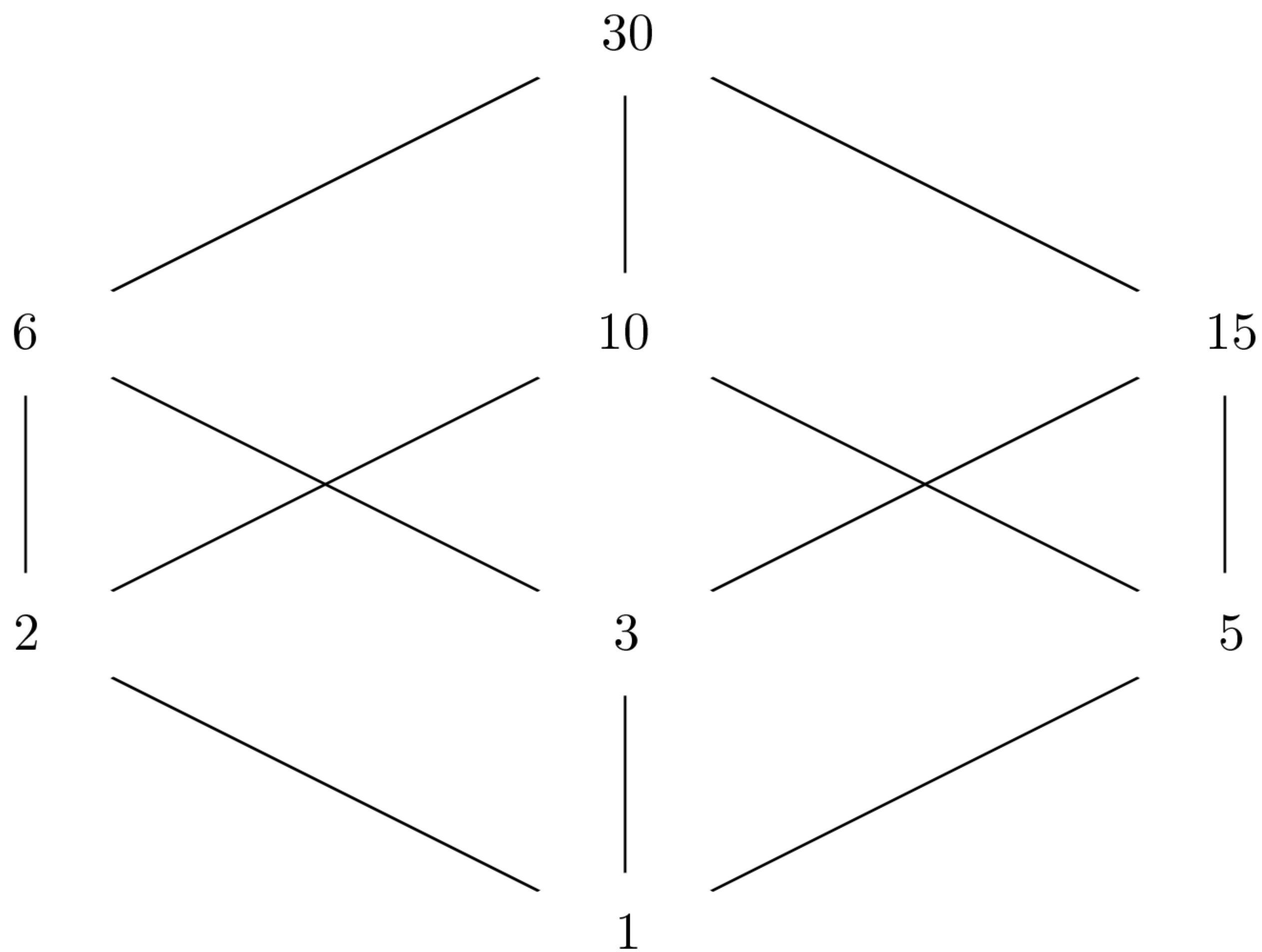
1. $a \preceq a \vee b$ and $b \preceq a \vee b$.
2. If some $x \in X$ satisfies $a \preceq x$ and $b \preceq x$, then $a \vee b \preceq x$.

if such an element exists.

(Join is the least upper bound of two elements.)

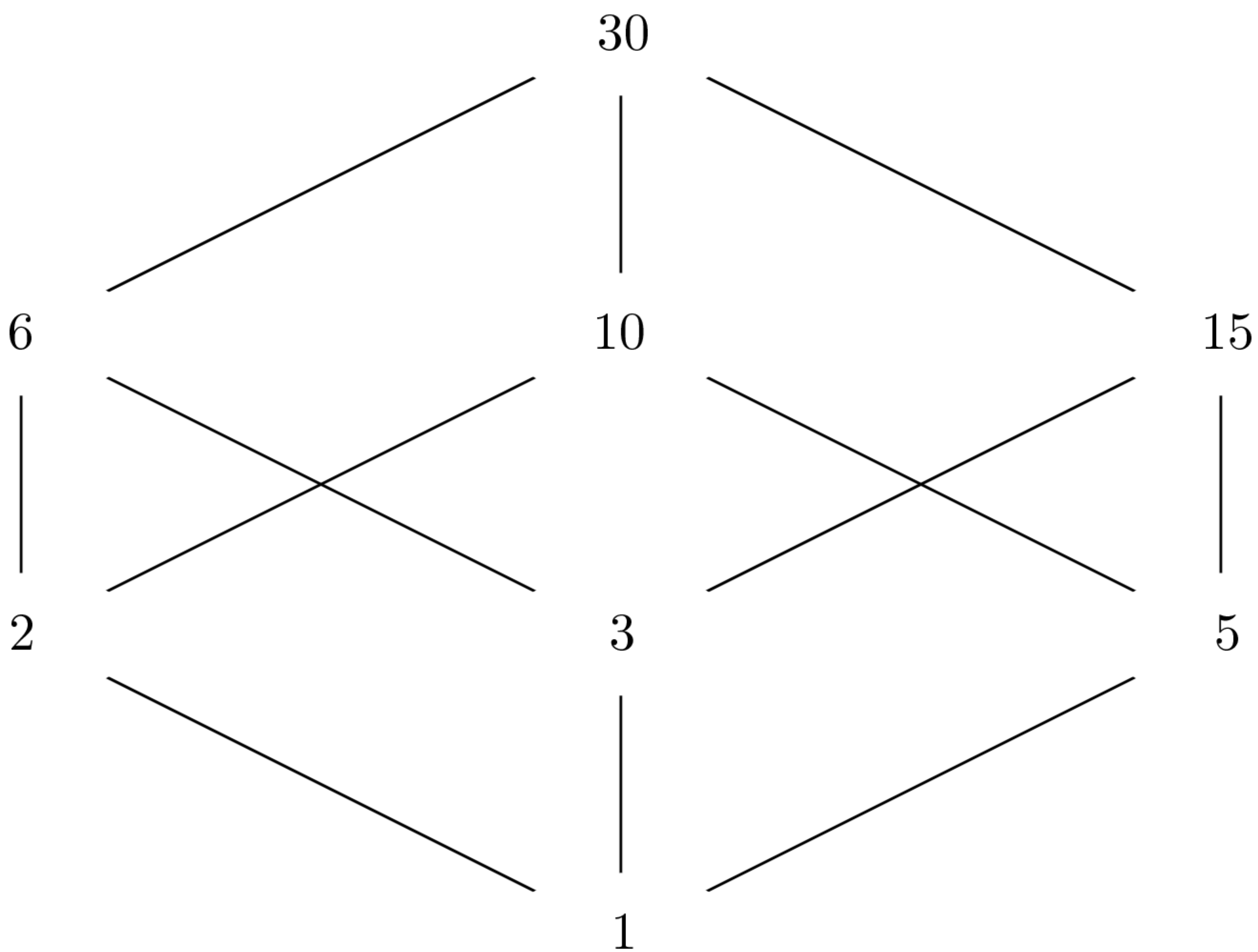
Examples

$3 \wedge 2 = ?$ $3 \vee 2 = ?$ $15 \vee 10 = ?$ $15 \wedge 2 = ?$



Examples

$3 \wedge 2 = 1$ $\quad$ $3 \vee 2 = 6$ $\quad$ $15 \vee 10 = 30$ $\quad$ $15 \wedge 2 = 1$



Fact. If the meet of two elements exists, then it is unique.

Why?

Let $c1 = a \wedge b$ and $c2 = a \wedge b$.

Then $c1 \leq c2$ and $c2 \leq c1$. But $\leq$ is antisymmetric.

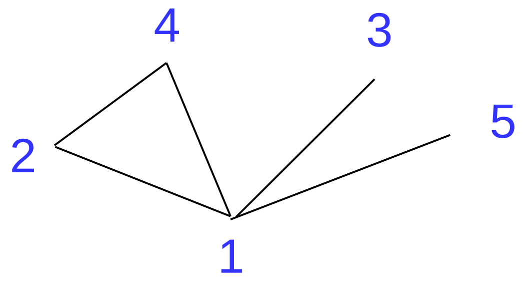
Fact. If the join of two elements exists, then it is unique.

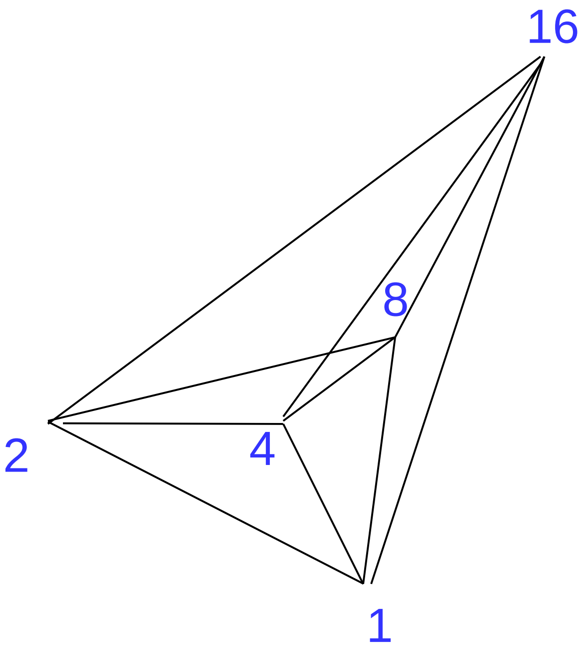
Lattice

Definition A poset in which every pair of elements has a meet and a join is called a *lattice*.

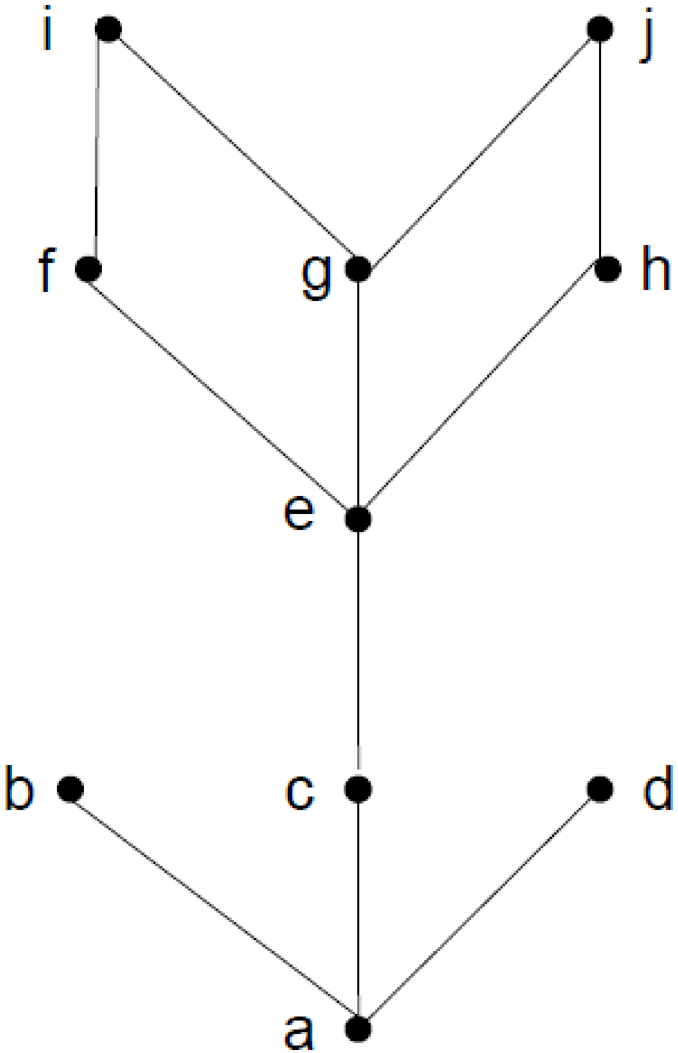
Exercise. Determine whether the posets ({1,2,3,4,5}, |) and ({1,2,4,8,16},|) are lattices.

Are these Hasse Diagrams?
Are these lattices?





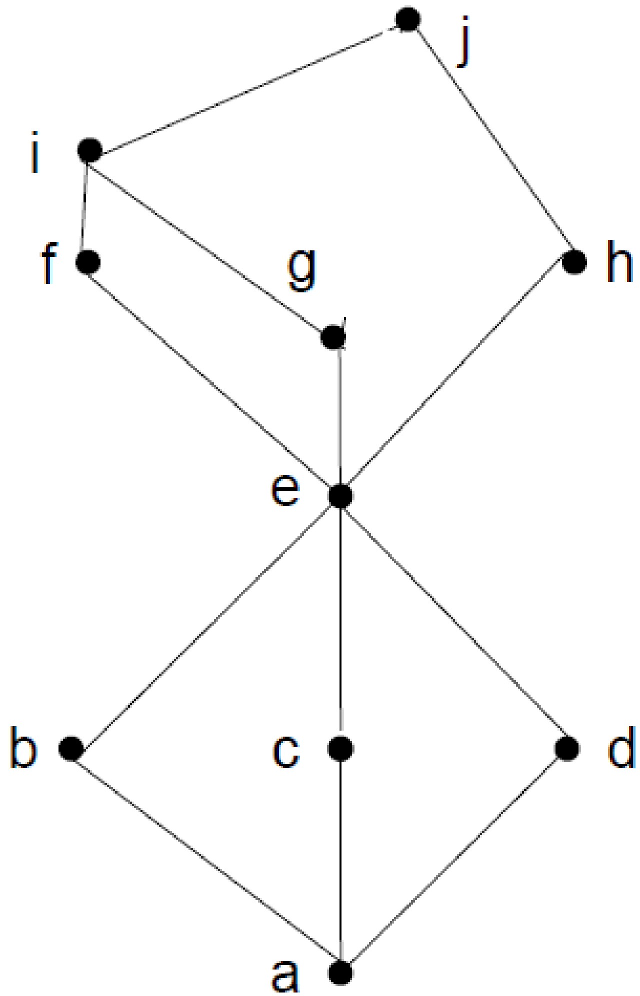
Is this a Hasse Diagram?



Is this a lattice?

No

Is this a Hass Diagram?



Is this a lattice?

In a lattice, *meet* and *join* satisfy the following properties:

For all a b, c, ∊ X.

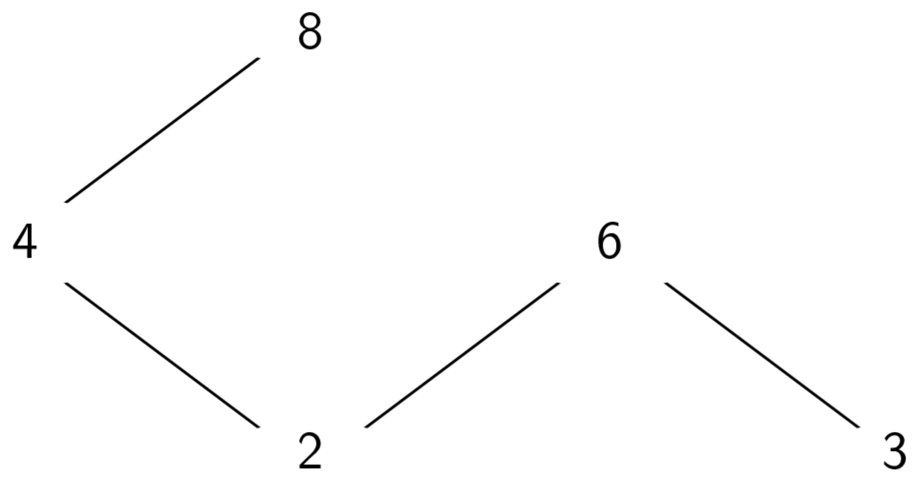
Commutativity: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$.

Absorption: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Hasse diagram for $(\{2,3,4,6,8\}, |)$

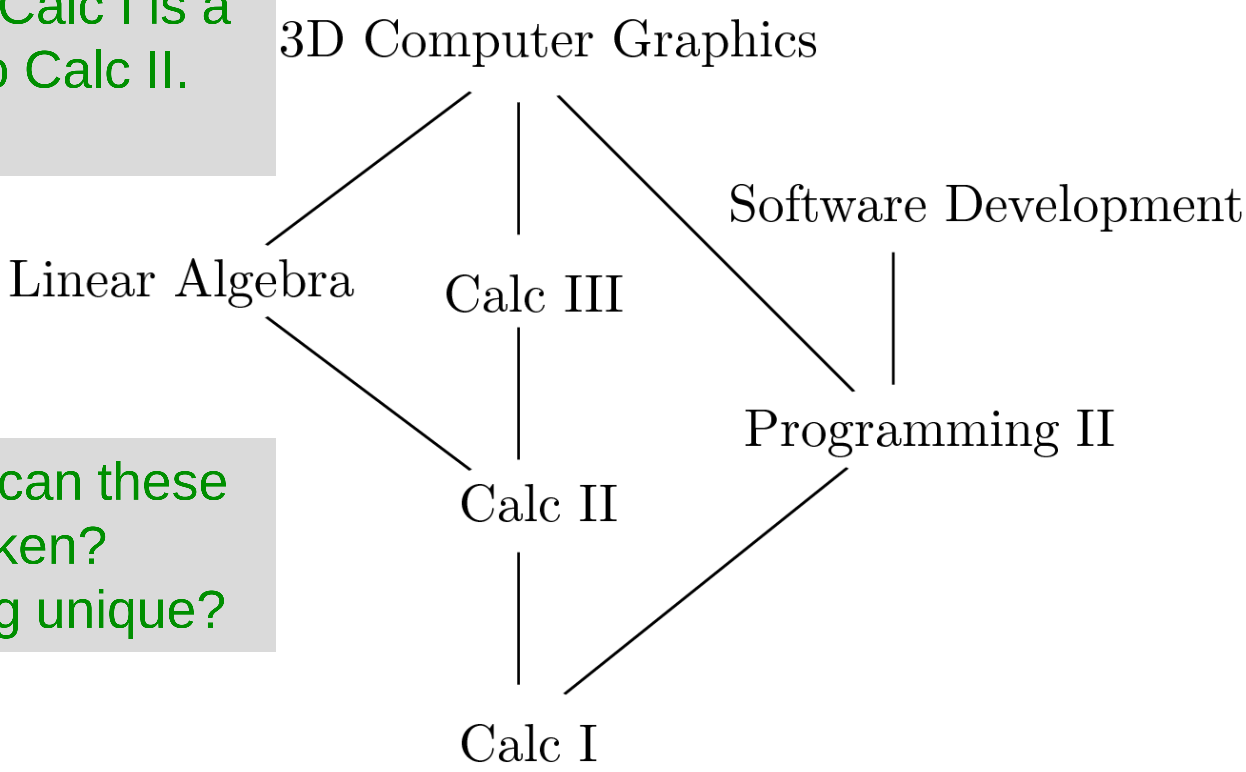
An element a is *minimal* if no other element is less than a.
An element a is *maximal* if no other element is greater than a.



- Minimal elements: **?**
- Maximal elements: **?**

Partial Order Example: dependencies among courses

For example, Calc I is a prerequisite to Calc II.



In what order can these courses be taken?
Is this ordering unique?

That is, how can we extract total orders from a given partial order?

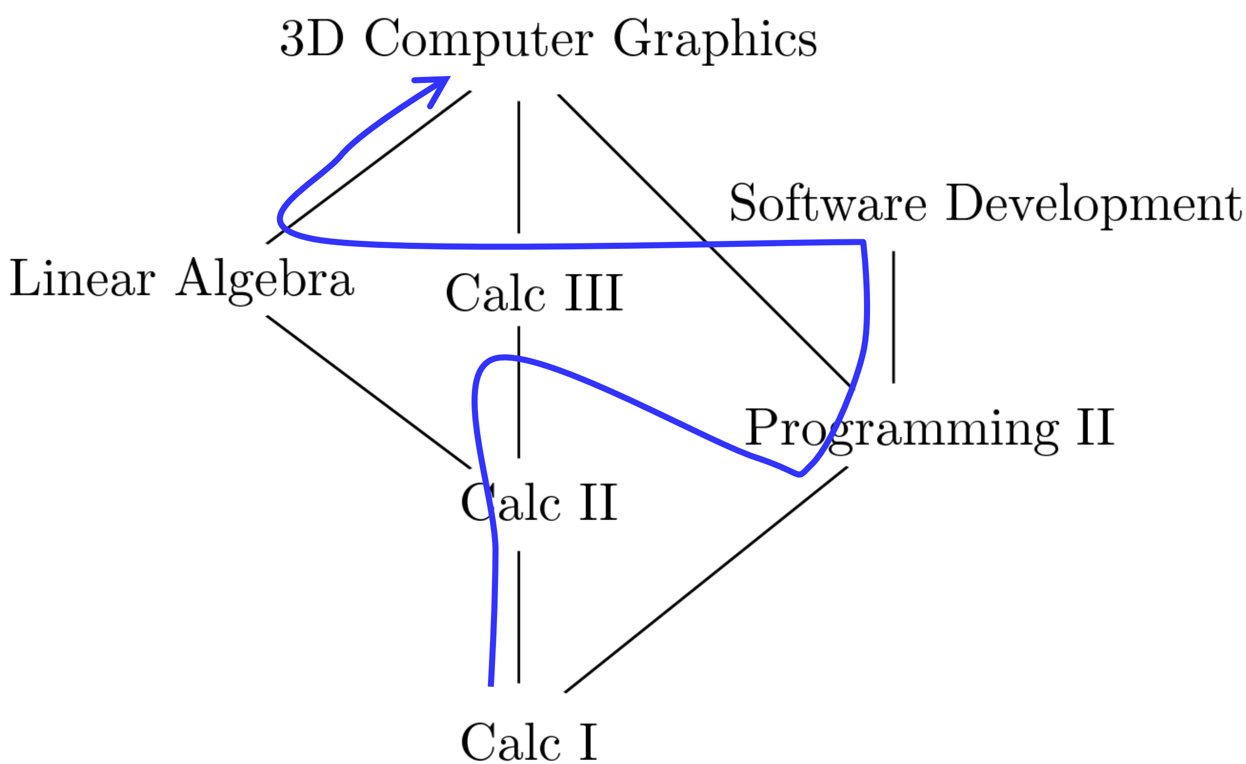
How to perform a topological sort

How can we extract total orders from a given partial order?

Repeat until you run out of elements:

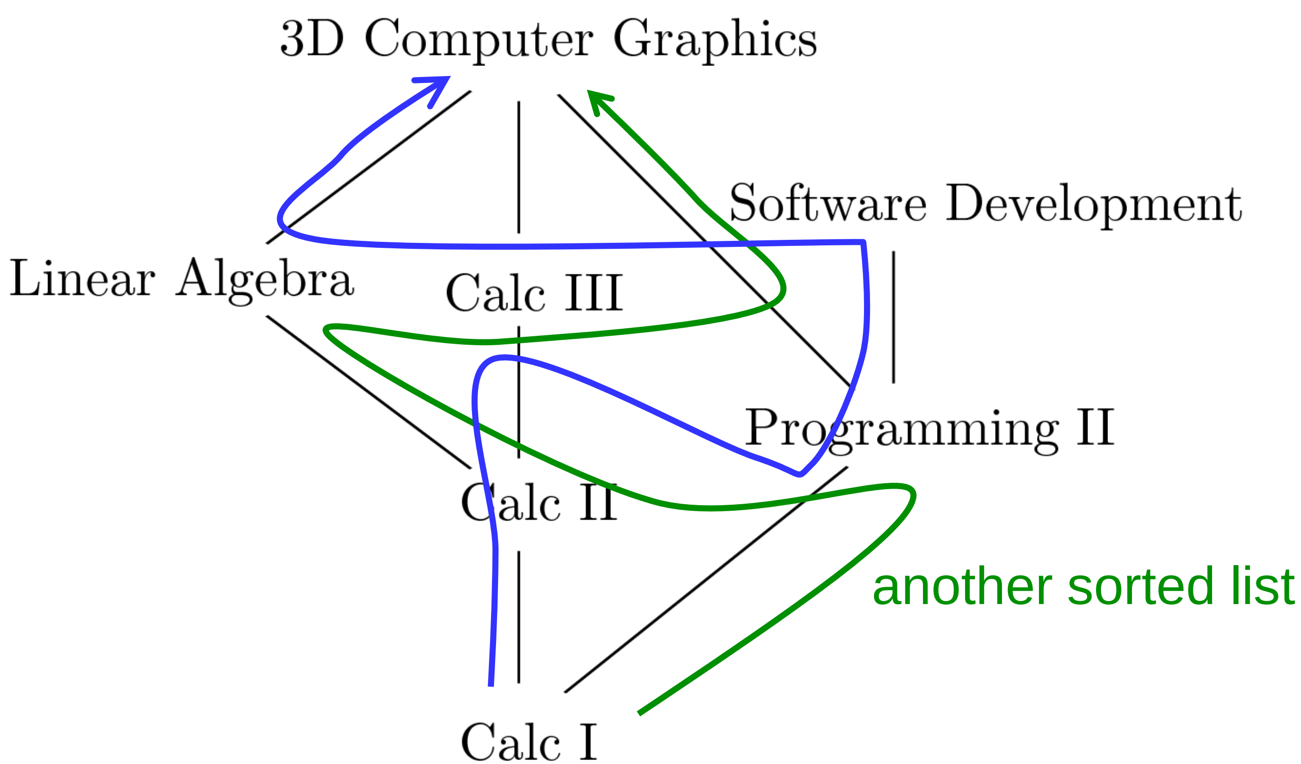
- Choose a minimal element in your Hasse diagram. (Note that you can always find a minimal element, but there might be more than one.) Add this element to the end of the list you are making.
- Remove the element from your Hasse diagram.

Partial Order Example: dependencies among classes



One possible topological sort: Calculus I, Calculus II, Calculus III, Programming II, Software Development, Linear Algebra, 3D Computer Graphics.

Partial Order Example: dependencies among classes



One possible topological sort: Calculus I, Calculus II, Calculus III, Programming II, Software Development, Linear Algebra, 3D Computer Graphics.

Isomorphic posets

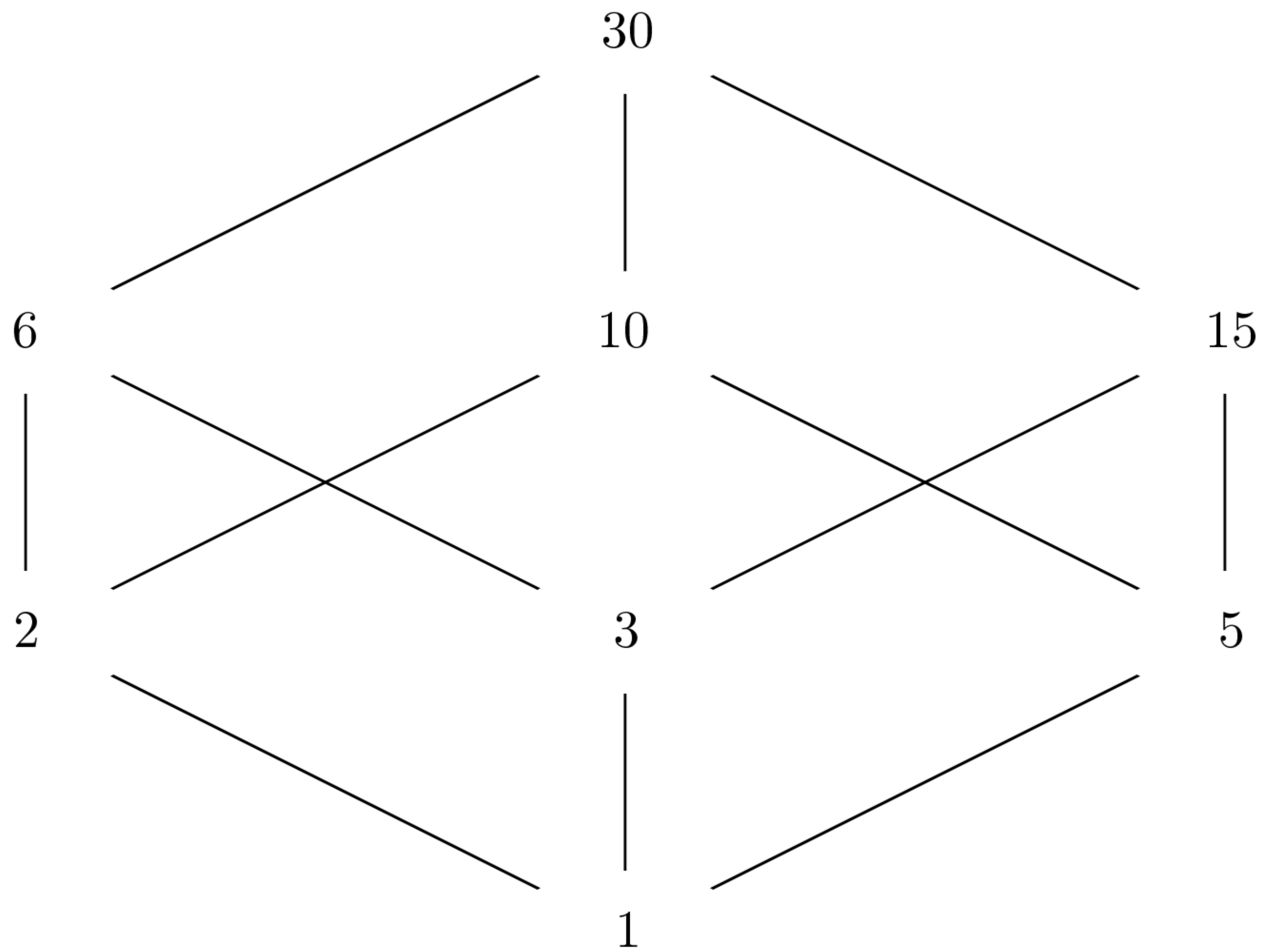
Definition

Let $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ be partially ordered sets. Then $(X_1, \preceq_1)$ is *isomorphic* to $(X_2, \preceq_2)$ if there is a one-to-one correspondence $f : X_1 \longrightarrow X_2$ such that

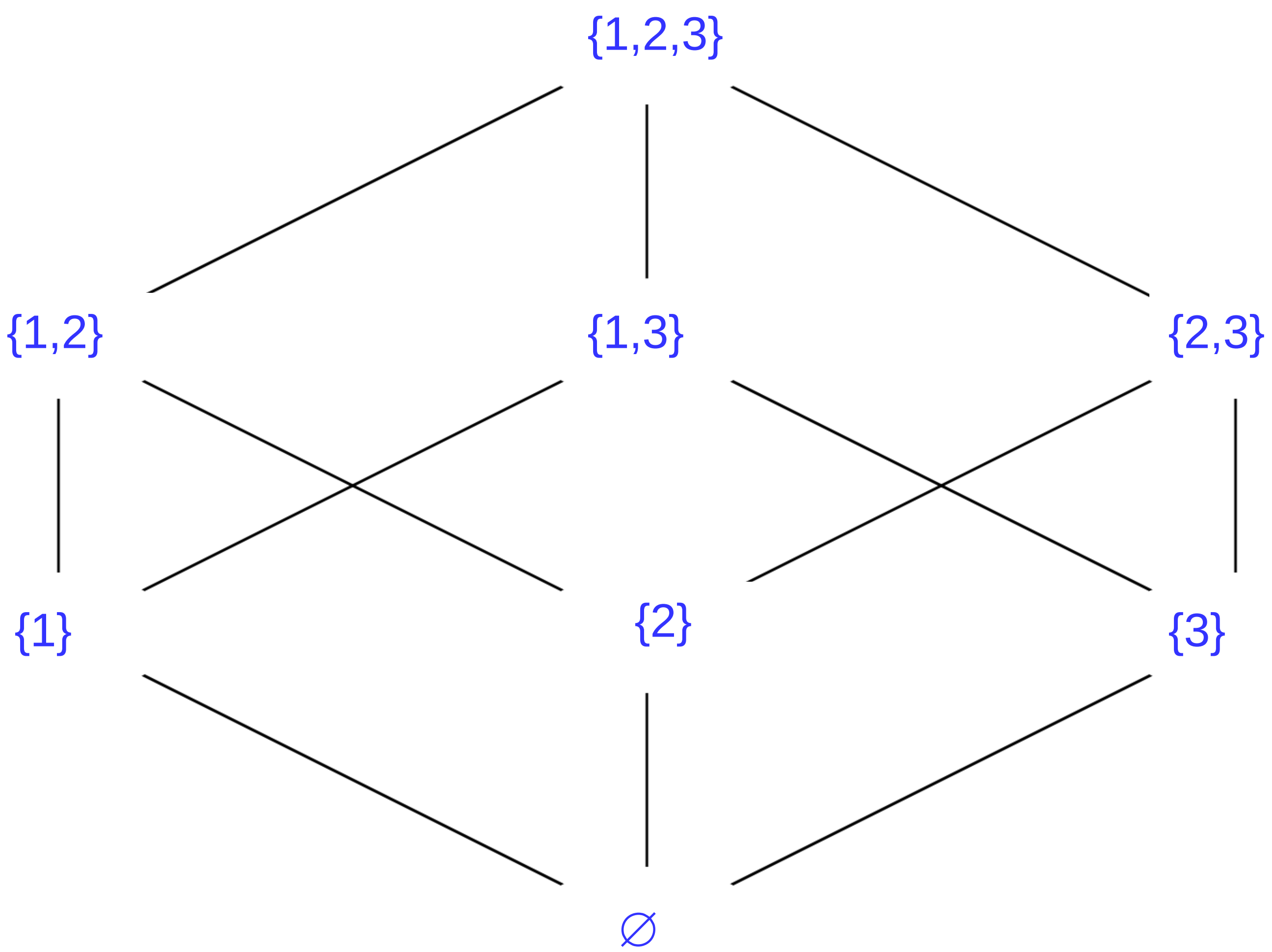
$$a \preceq_1 b \;\Leftrightarrow\; f(a) \preceq_2 f(b)$$

for all $a, b \in X_1$. In this case, we say that f is an *isomorphism* and we write $(X_1, \preceq_1) \cong (X_2, \preceq_2)$ to denote that these posets are isomorphic.

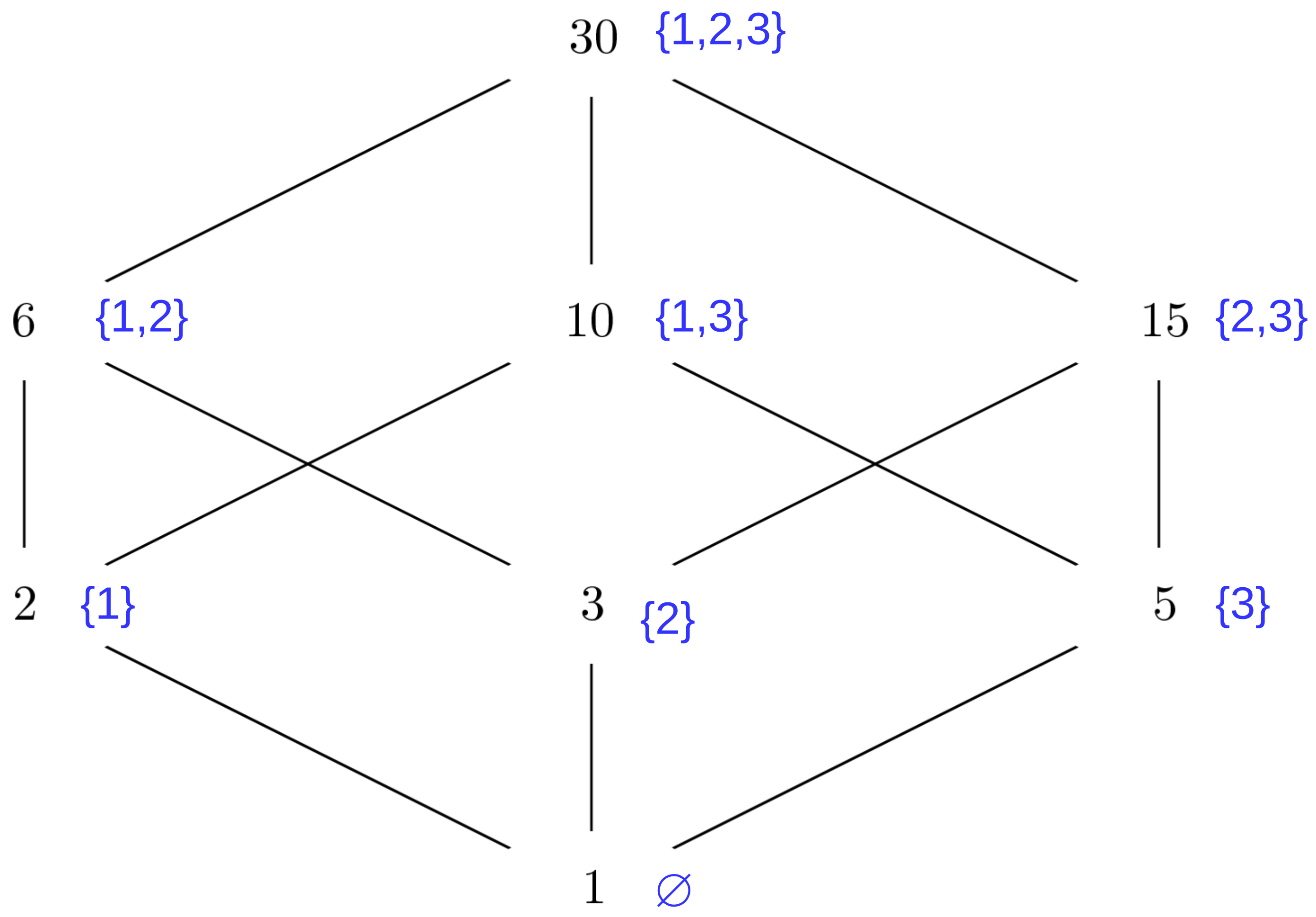
Example: $(\{1, 2, 3, 5, 6, 10, 15, 30\}, |)$ [redacted]



Example: [redacted] $(\mathcal{P}(\{1,2,3\}), \subseteq)$



Example: $(\{1,2,3,5,6,10,15,30\}, |) \cong (\mathcal{P}(\{1,2,3\}), \subseteq)$



2. Total Order

Definition Two elements of a poset are *comparable* if either a ≤ b or b ≤ a.

Definition Two elements of a poset are *incomparable* if neither a ≤ b nor b ≤ a.

Definition If a poset has no incomparable elements, it is a *total ordering*.

The following is an alternative definition of total ordering

Definition A partial order on a set A is a *linear* (or *simple*, or *total*) *order* if any a, b ∈ A are comparable.

If ≤ is a linear order on A, then < A, ≤ > is a *linearly ordered set*, or *chain*.

Example 1. <{1, 2, 3}, {<1,1>, <2,2>, <3,3>, <1,2> <2,3>, <1,3>}> is a total order.

Example 2. ($\mathbb{R}$, ≤) is a total order.

3. Well Order

Definition A binary relation R on A is a *well order* if R is a linear order and every nonempty subset of A has a least element.

The ordered pair (A, R) is called *a well ordered set*, and R is a *well ordering* of A.

Theorem 1. $(\mathbb{N}, \leq)$ is well ordered.

Examples

(a) Every finite linearly ordered set is well ordered.

(b) $(\mathbb{Z}, \leq)$ is not a well ordered set.

4. Lexicographical Order

Definition Let Σ be a finite alphabet with an associated alphabetic (linear) order. If $x,y \in \Sigma^*$, then $x \leq_L y$ in the *lexicographic ordering* of Σ^*if

(i) x is a prefix of y, or

(ii) x = zu and y = zv, where $z \in \Sigma^*$ in the longest prefix common to x and y, and the first symbol of u precedes the first symbol of v in the alphabetic order.

- The lexicographic ordering of is the usual “alphabetic” ordering used in dictionaries.

- The lexicographic order of Σ^* is a linear order but it is not a well order unless Σ consists of a single symbol.

  Why?

Let $\Sigma = \{a, b\}$. Then the lexicographic ordering is:

λ

a, aa, aaa, aaaa, ...

aaab, ...

ab, aab, aaba, ...

aabb, ...

aba, abaa, ...

abab, ...

abb, abba, ...

abbb, ...

b, ba, baa, baaa, ...

baab, ...

bb, bab, baba, ...

babb, ...

bba, bbaa, ...

bbab, ...

bbb, bbba, ...

bbbb, ...

Let Σ = {a, b}. Then the lexicographic ordering is:

λ

a, aa, aaa, aaaa, aaaaa, aaaaaa, aaaaaaa, ...

aaaaaab, ...

aaaaab, aaaaaba, ...

aaaab, aaaaba, ...

aaab, aaaba, ...

aaabb, ...

ab, aab, aaba, ...

aabb, ...

aba, abaa, ...

abab, ...

abb, abba, ...

abbb, ...

Does the following have the least element?

{b, ab, aab, aaab, aaaab, aaaaab, aaaaaab, ...}

b, ba, baa, baaa, ...

baab, ...

bb, bab, baba, ...

. . .

Examples

Let $\Sigma = \{a, b\}$, and let a precede b in the alphabetic order.

Then if x is any string in Σ^*, the immediate successor of x is xa.

The immediate predecessor of xa is x,

but there is no immediate predecessor f xb.

Moreover, the set {b, ab, aab, aaab, ...} has no least element,

since ach string $a^m b$ precedes any string $a^n b$ if $m > n$.

It follows that the lexicographic order is not a well order.

Quiz 12-1

Which of the following is NOT a total order ?

(a) $(\mathbb{N}, \leq)$

(b) $(\mathbb{Z}, \leq)$

(c) $(\mathbb{R}, \leq)$

(d) $(\mathbb{P}(\mathbb{N}), \subseteq)$

(e) $(\Sigma^*, \leq_L)$ where Σ is an alphabet and

$\leq_L$ is the lexicographic ordering on Σ^*