CS 204 Discrete Mathematics

Homework

Sample Solutions

Basic Counting Techniques

1. Let A and B be finite sets, |A| = m and |B| = n. How many binary relations are there from A to B?

Solution)

A binary relation from A to B is a subset of A × B. There are $2^{|A \times B|} = 2^{|A| \times |B|} = 2^{mn}$ such subsets.

2. How many ways are there to rearrange the letters in INANENESS?

Solution)

9! / 3!2!2! = 15120

3. How many distinct ways are there to encode the decimal digits 0-9 as binary sequences of length 4? Consider only codes which represent different digits by different sequences.

Solution)

There are 16 binary sequences of length 4. Representation for the sequence of digits 0, 1, 2, ... , 9 can be chose in P(16,10) ways.

4. A string in {0,1}* has even parity if the symbol 1 occurs in the word an even number of times; otherwise, it has odd parity.

   (a) How many words of length n have even parity?
   (b) How many words of length n have odd parity?

Solution)

(a) $(1/2).2^n = 2^{n-1}$. This can be proved by induction.

(a) 2^{n-1}.

Selections and Arrangements

1. If you flip a coin 5 times, how many different ways can you get exactly 1 head? 2 heads? Find a formula for the number of ways of obtaining r heads with n flips of a coin.

Solution)
Exactly 1 head: C(5,1) = 5.
Exactly 2 heads: C(5,2) = 10.
Exactly r heads in n flips: C(n,r).

2. Prove the following equality for $n \geq 0$.

$$C(n+1, r) = C(n, r-1) + C(n, r)$$

Solution)
C(n, r-1) + C(n, r) = n!/(r-1)!(n-r+1)! + n!/r!(n-r)! = n!r/r!(n-r+1)! + n!(n-r+1)/r!(n-r+1)!
= n!(r+n-r+1)/r!(n-r+1)! = n!(n+1)/r!(n-r+1)! = (n+1)!/r!(n-r+1)!
= C(n+1,r) by the definition of combinations (for $n \geq 0$)

3. Possible grades for a class are A, B, C, D, and F. (No +/-'s.)
   (a) How many ways are there to assign grades to a class of seven students?
   (b) How many ways are there to assign grades to a class of seven students if nobody receives an F and exactly one person receives an A?

Solution)
(a) 5^7 = 78, 125
(b) $7 \cdot 3^6$ = 5, 103
Guideline) there is no partial point.

4. Show that if n is a positive integer, then
   $C(2n,2) = 2 \times C(n,2) + n^2$
   a) using a combinatorial argument. (A combinatorial argument, or combinatorial proof, is an argument that involves counting.)
   b) by algebraic manipulation.

Solution)

a) To choose 2 people from a set of n men and n women, we can either choose 2 men ($\binom{n}{2}$ ways to do so) or 2 women ($\binom{n}{2}$ ways to do so) or one of each sex ($n \cdot n$ ways to do so). Therefore the right-hand side counts the number of ways to do this (by the sum rule). The left-hand side counts the same thing, since we are simply choosing 2 people from $2n$ people.

b) $2\binom{n}{2} + n^2 = n(n-1) + n^2 = 2n^2 - n = n(2n-1) = 2n(2n-1)/2 = \binom{2n}{2}$

Guideline) In (a), if they do not use combinatorial argument, then they lose 7 pts.
In (b), if they do not prove by algebraic manipulation, then they lose 7 pts.

Counting & Counting with Functions

1. A small college offers 250 different classes. No two classes can meet at the same time in the same room, of course. There are twelve different time slots at which classes can occur. What is the minimum number of classrooms needed to accommodate all the classes?

Solution) The number of classes is |X| = 250, and the number of time slots is |C| = 12.
By the generalized pigeonhole principle, the number of required classrooms is

$\lceil 250/12 \rceil = 21.$

3. (15 pts)

(a) How many total functions are there from a set with three elements to a set with four elements?

(b) How many are one-to-one?

(c) How many are onto?

Solution)

(a) There are $4^3 = 64$ functions from a set with three elements to a set with four elements.

(b) There are $4 \cdot 3 \cdot 2 = 24$ one-to-one functions from a set with three elements to a set with four elements.

(c) There are no onto functions from a set with three elements to a set with four elements.

4. How many solutions are there to the equation x1 + x2 + x3 = 17, where x1, x2, and x3 are nonnegative integers with

a) x1 > 1, x2 > 3, and x3 > 3?

b) x1 < 6 and x3 > 5?

c) x1 < 4, x2 < 3, and x3 > 5

Solution)

a) 45

b) 57

c) 12

5. How many subsets of a set with ten elements

a) have fewer than five elements?

b) have more than seven elements?

c) have an odd number of elements?

Solution)

a) C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4)=386

b) C(10,8)+C(10,9)+C(10,10)=56

c) C(10,1)+ C(10,3)+ C(10,5)+ C(10,7)+ C(10,9)=512