Ch 10. Graphs

Graph Theory 2

Sungwon Kang

Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Graph Theory

1. Graphs: Formal Definitions
2. Relations and Graphs
3. Isomorphisms of Graphs
4. Degree of a Node
5. Paths and Circuits
6. An Exercise on Graphs and Counting

3. Isomorphism of Graphs

Definition

Let G be a graph with vertex set V_G and edge set E_G, and let H be a graph with vertex set V_H and edge set E_H. Then G is *isomorphic* to H if there are one-to-one correspondences

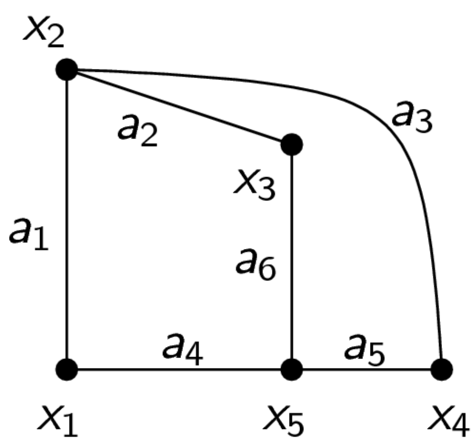
$$\underline{\alpha : V_G \longrightarrow V_H} \quad \text{and} \quad \underline{\beta : E_G \longrightarrow E_H}$$

such that, for any edge $e \in E_G$, e joins vertex v to vertex w if and only if $\beta(e)$ joins vertex $\alpha(v)$ to vertex $\alpha(w)$. In this case, we write $G \cong H$.

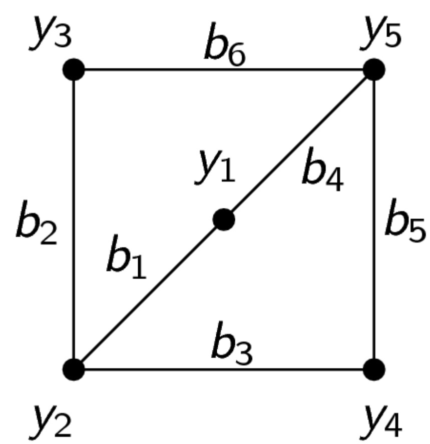
- Isomorphic graphs are the same graph with different naming of nodes and edges.

Check that these graphs are isomorphic

G1:



G2:



Check that these graphs are isomorphic

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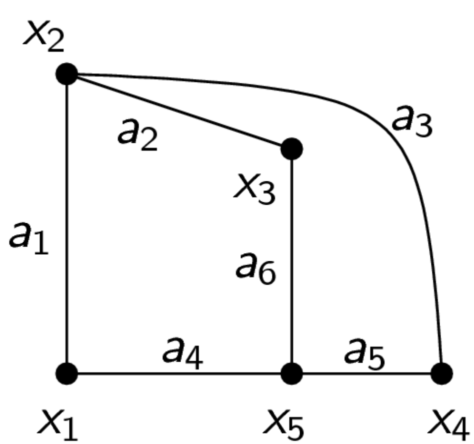
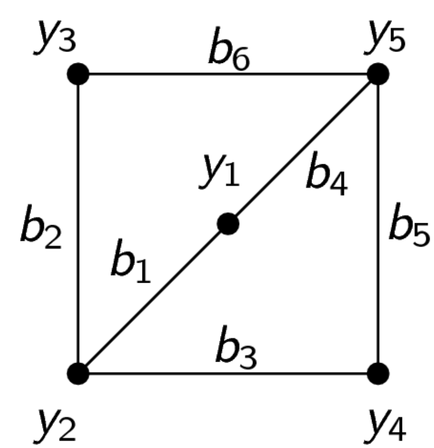
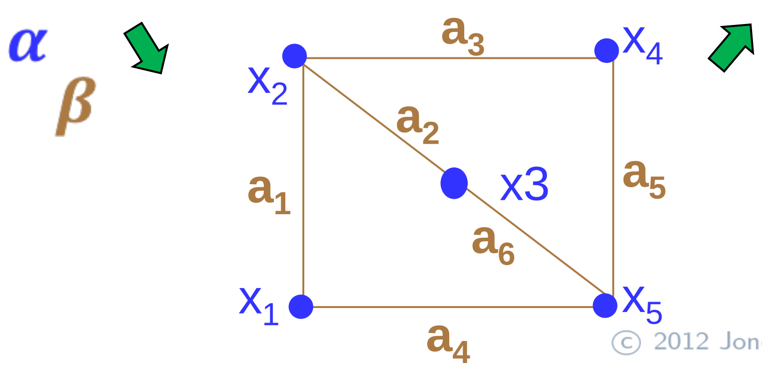


G2:





Intermediate Graph

Theorem

Let G and H be graphs without multiple edges, with vertex sets V_G and V_H, respectively. For vertices x and y, write $x\,R\,y$ if an edge joins x to y. If there is a one-to-one correspondence $f : V_G \longrightarrow V_H$ with the property that

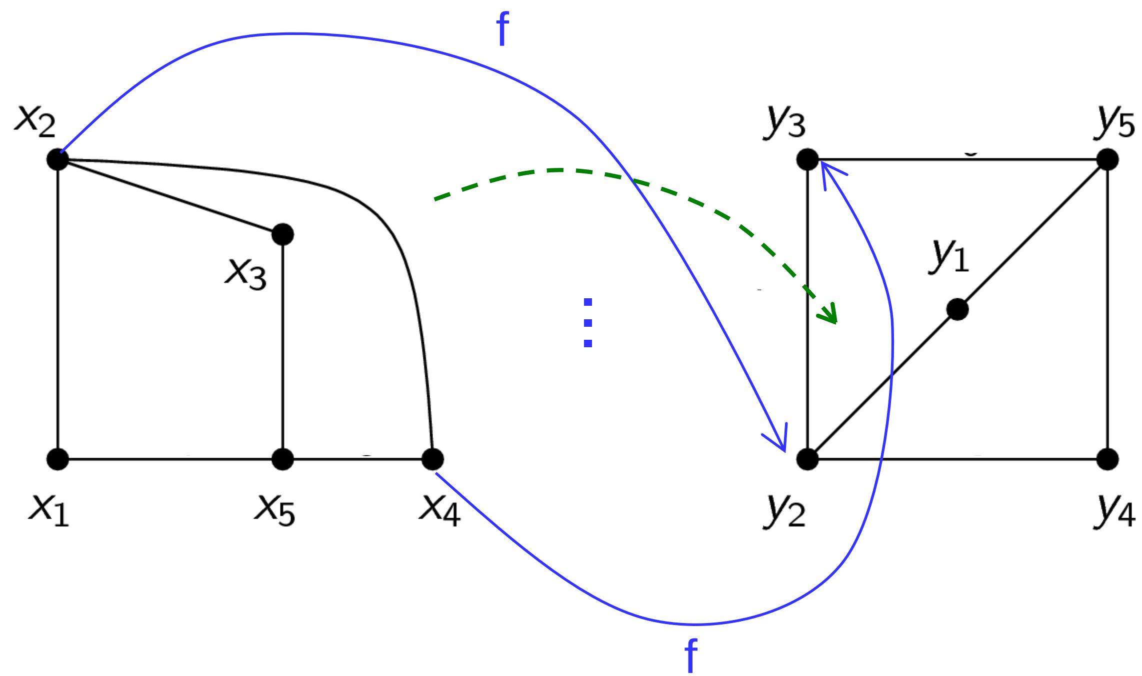
$$x\,R\,y \;\Leftrightarrow\; f(x)\,R\,f(y)$$

for all $x, y \in V_G$, then $G \cong H$.

☛ f exposes the same structure of two graphs.

[redacted]

Check that these graphs are isomorphic

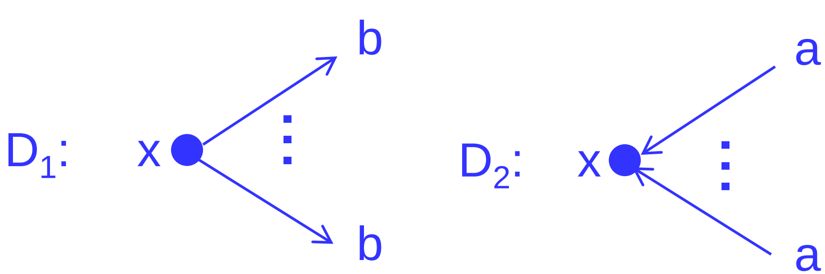


4. Degree of a Node

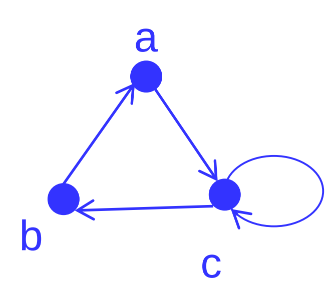
Degree - Directed Graph

Definition

Let G be a directed graph, and let $x \in V_G$ be a vertex. Let D_1 be the set of all edges $e \in E_G$ such that $\underline{i(e) = (x, b)}$ for some b, and let D_2 be the set of all edges $e \in E_G$ such that $\underline{i(e) = (a, x)}$ for some a. Then the *degree* of x is $|D_1| + |D_2|$.



Directed Graph



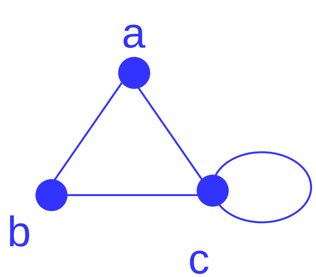
deg(a) = 2
deg(b) = 2
deg(c) = 4

Degree - Undirected Graph

Source: [Rosen 03]

Definition

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v). 1



deg(a) = 2
deg(b) = 2
deg(c) = 4

Euler's theorem on degrees

Theorem

Let G be an undirected graph. The sum of the degrees of the vertices of G equals twice the number of edges in G.

Proof.

Since each edge joins two vertices (or possibly a single vertex to itself), each edge contributes 2 to the sum of the degrees of the vertices. □

Exercise Explain why it is impossible for a group of 11 teams to each play five games against different opponents in the group.

Apply Explain why it is impossible for a group of 11 teams to each play five games against different opponents in the group.

Problem	Graph Model
team	vertex
game	edge
11 teams →	11 vertices
Each team play 5 games. →	The degree of each vertex is 5.
Number of games = 11 x 5 / 2 ←	Number of edges = 11 x 5 / 2

For each of 11 teams to play 5 games,
we need 55 playing teams and 55/2 games !!

5. Paths and Circuits

Definition

Let G be a graph. A *path* in G is a sequence

$$v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$$

of vertices v_i and edges e_i such that edge e_i joins vertex v_{i-1} to vertex v_i, where $n \geq 1$. A *circuit* is path with $v_0 = v_n$. A path or a circuit is called *simple* if $e_1, e_2, \ldots, e_n$ are all distinct.

Euler path: A path starting from any node, going through each edge of the graph exactly once.

Euler circuit: A path starting from any node, going through each edge of the graph exactly once and terminating at the start node.

More Theorems from Euler

Theorem

If all the vertices of a connected graph G have even degree, then G has an Euler circuit.

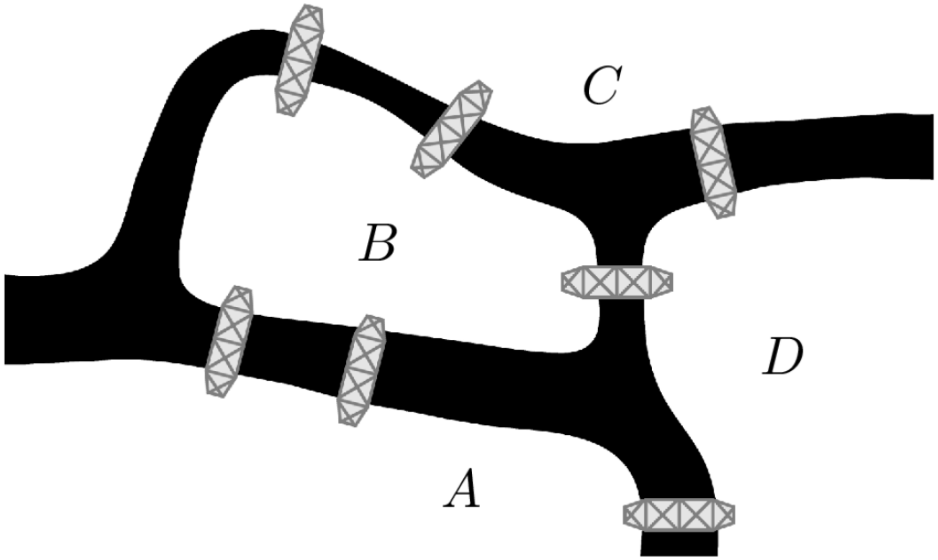
Theorem

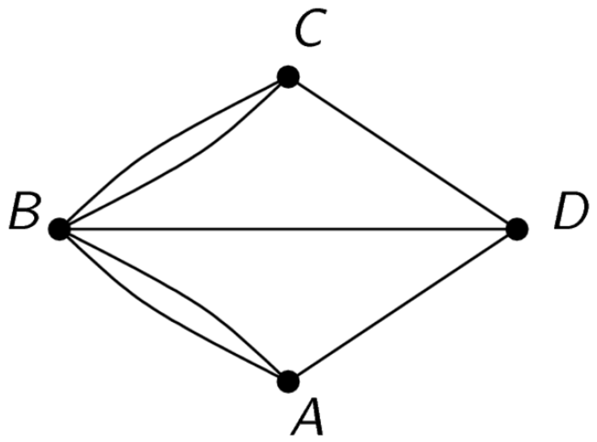
If a graph G has an Euler circuit, then all the vertices of G have even degree.

See the text for modern proofs.

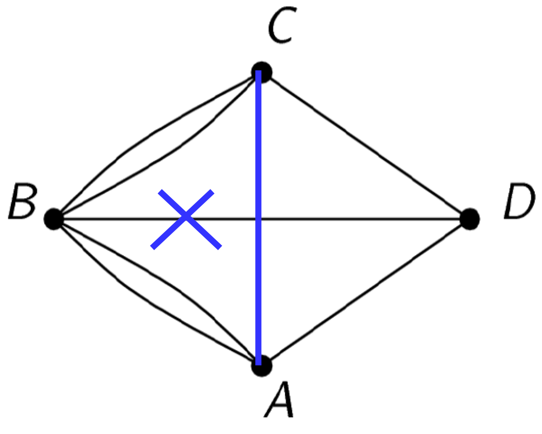
Königsberg Bridges Problem: Can you start from any island and walk across all the bridges exactly once in returning to the starting island?

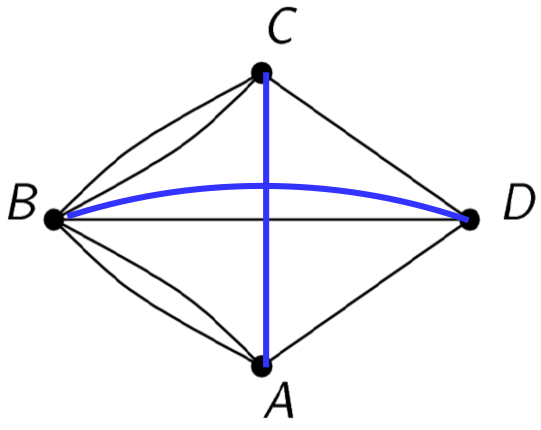
Does the Königsberg bridges problem have an Euler cycle solution?





How about the following graphs?





Hamilton paths and circuits

Definition

A *Hamilton path* in a graph G is a path

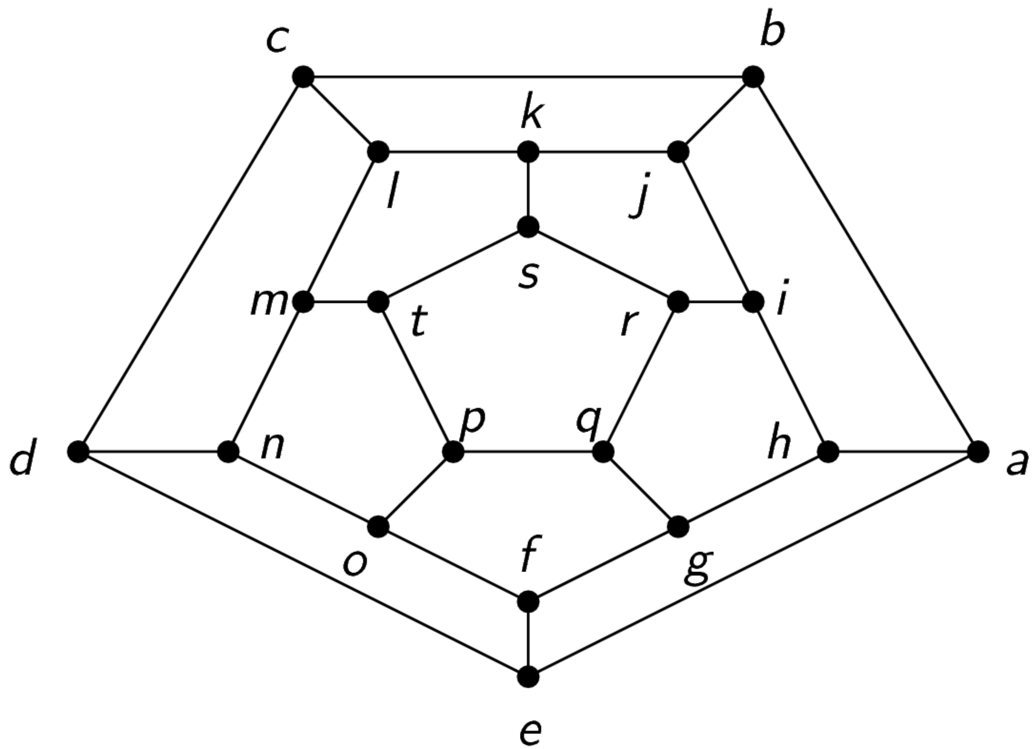
$$v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

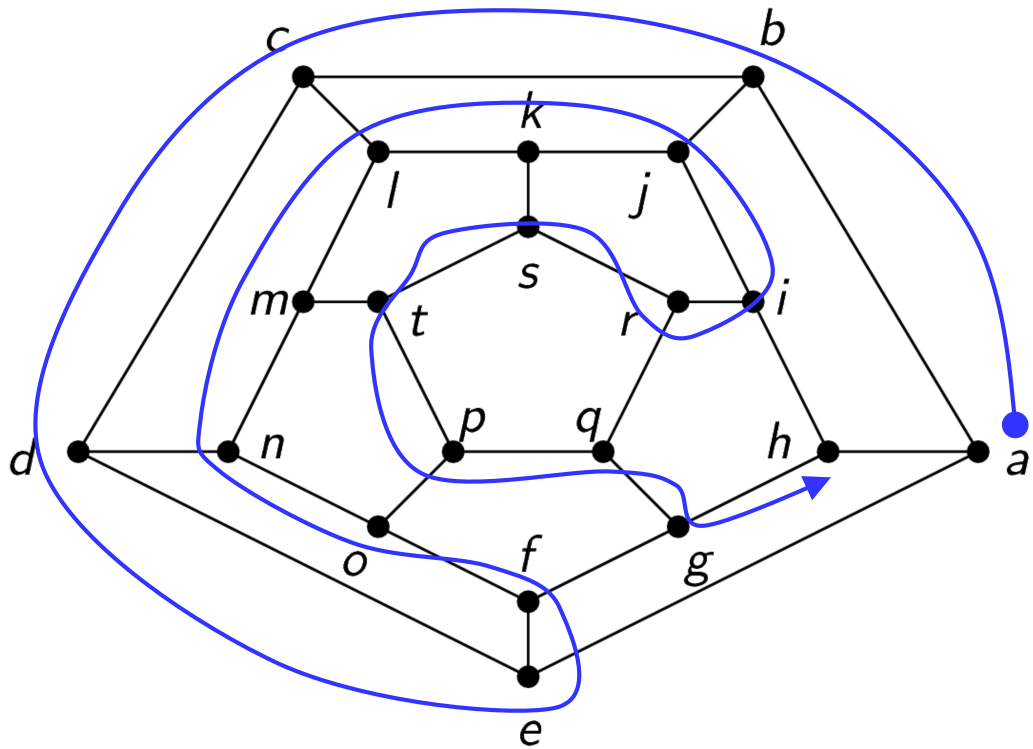
such that $v_0, v_1, \ldots, v_n$ is a duplicate-free list of all the vertices in G. A Hamilton circuit is a circuit $v_0, e_1, v_1, e_2, \ldots, e_n, v_n, e_{n+1}, v_0$, where $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ is a Hamilton path.

An Euler path passes through (= traverses) **each edge** exactly once.

A Hamiltonian path visits **each node** exactly once (except the start node).

Can you find a Hamilton circuit in this graph?





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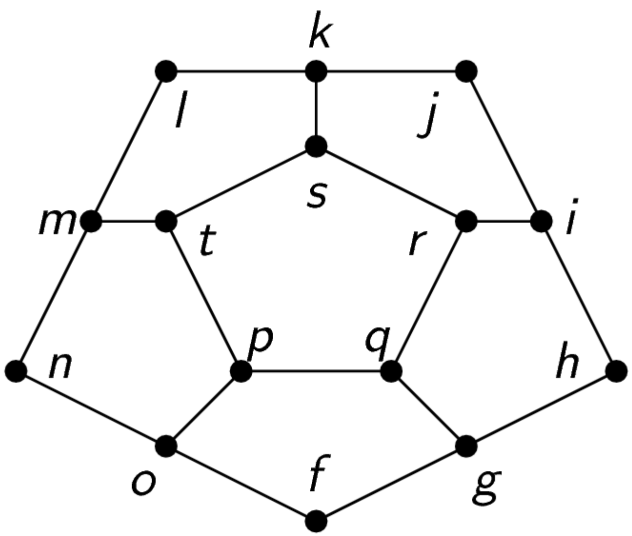
Why is there no Hamilton circuit in this graph?

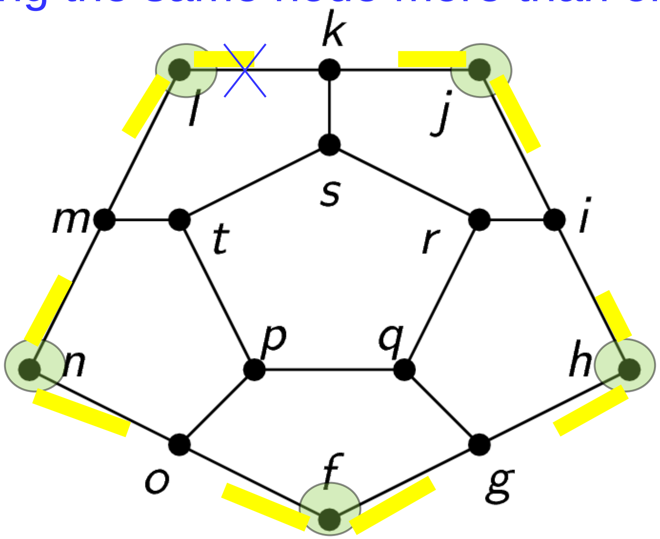


A promising line of thinking:

1) Nodes f, h, j, l, n have to be traversed.

2) Therefore all the edges marked with yellow must be traversed.
If not, we will be visiting the same node more than once as follows:

3) Suppose, for example, that edge l-k is not traversed. (Then l-m must be traversed twice)



. . .

4) If all yellow edges are traversed,
then some node on them would be visited twice.

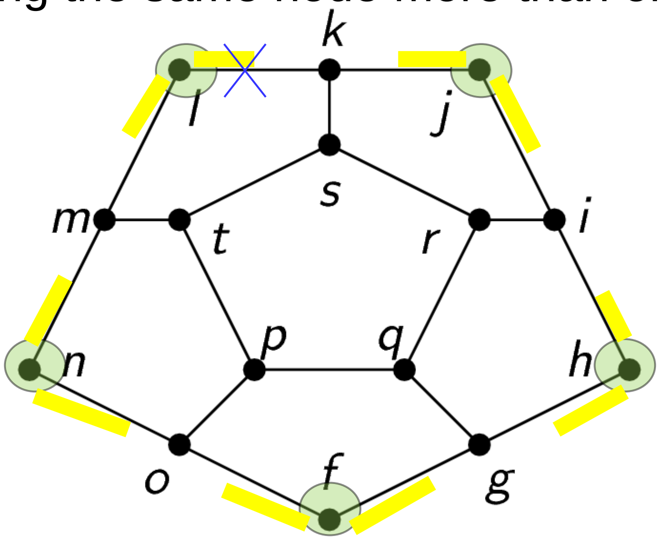
A promising line of thinking:

1) Nodes f, h, j, l, n have to be traversed.

2) Therefore all the edges marked with yellow must be traversed.
If not, we will be visiting the same node more than once as follows:

3) Suppose, for example, that edge l-k is not traversed.

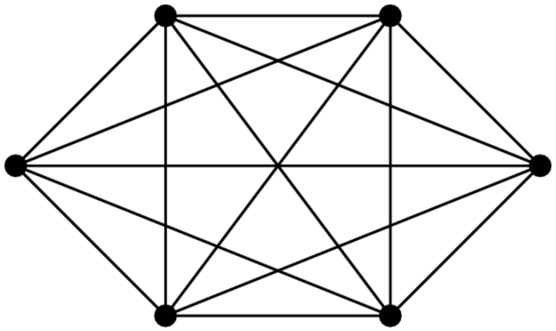
3A) If l is the start node, then we visit m twice.
3B) If l is not the start node, m must be the start node since otherwise m would be visited twice.
If m is the start node, then after visiting l we cannot come back to m.



4) If all yellow edges are traversed,
then some node on them would be visited twice.

Example: edge coloring

Let G be the complete graph on six vertices. This graph has 15 edges. Suppose that some edges are colored red and the rest green. Show that there must be some triangular circuit whose edges are the same color.



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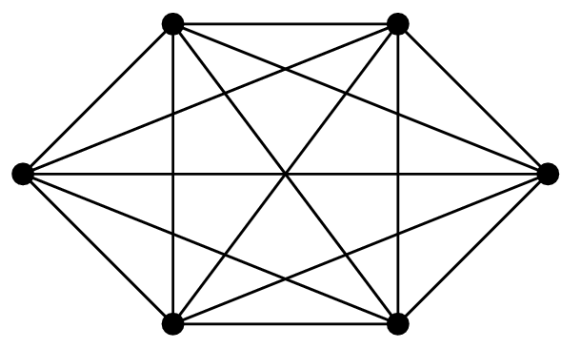


Let's call a triangle "suitable" if not all its sides have the same color.

Should show that there must be an unsuitable triangle.

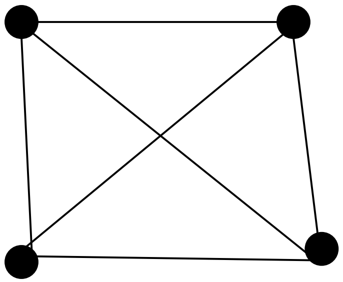
How many different triangles are there?

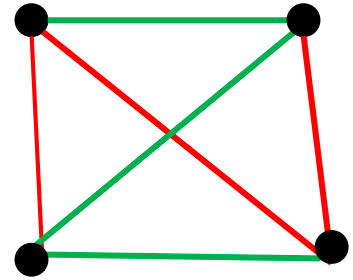
How many triangles are there?

${}_4C_3 = 4$

How many sides are there?

${}_4C_2 = 6$





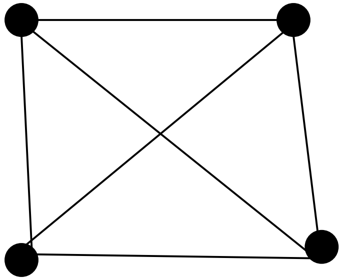
All triangles are suitable.

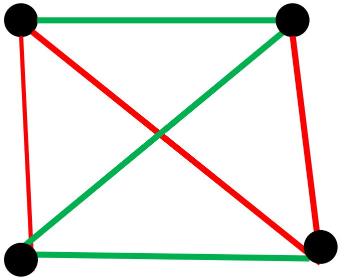
How many triangles are there?

${}_4C_3 = 4$

How many sides are there?

${}_4C_2 = 6$





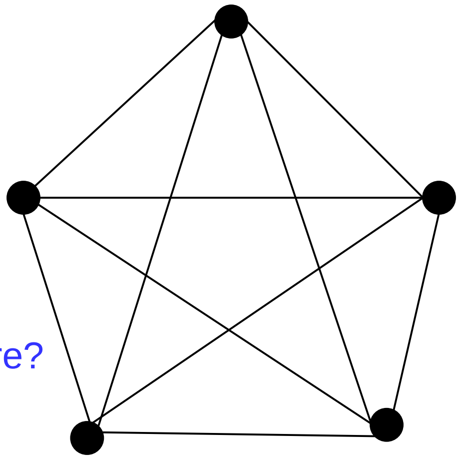
All triangles are suitable.

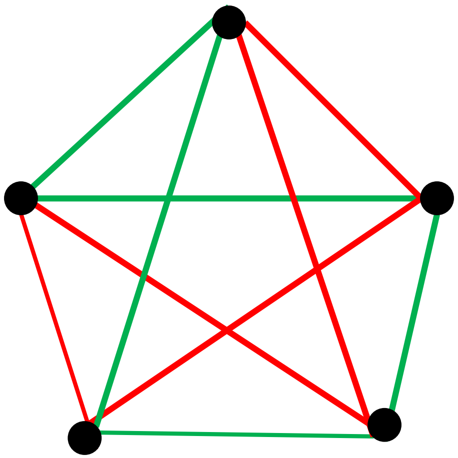
How many triangles are there?

${}_5C_3 = 10$

How many sides are there?

${}_5C_2 = 10$





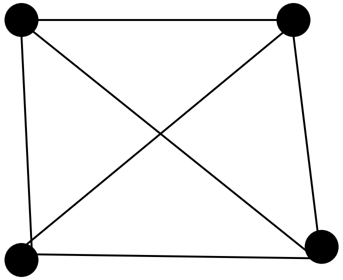
All triangles are suitable.

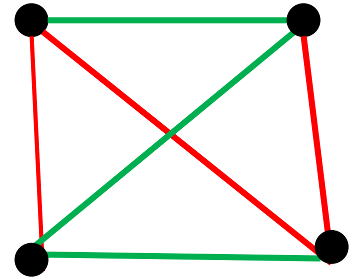
How many triangles are there?

${}_4C_3 = 4$

How many sides are there?

${}_4C_2 = 6$





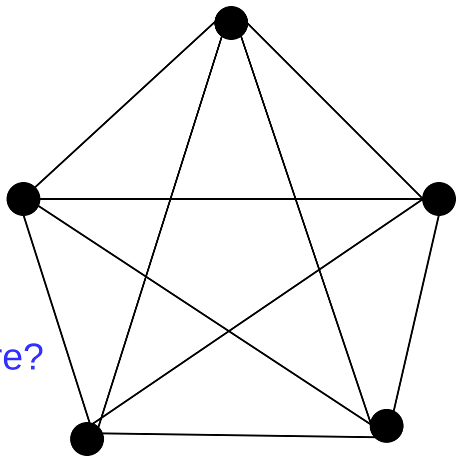
All triangles are suitable.

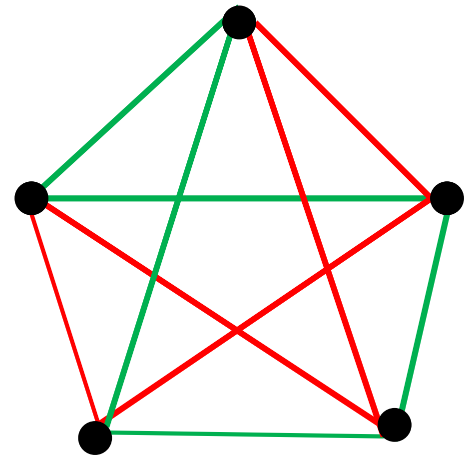
How many triangles are there?

${}_5C_3 = 10$

How many sides are there?

${}_5C_2 = 10$





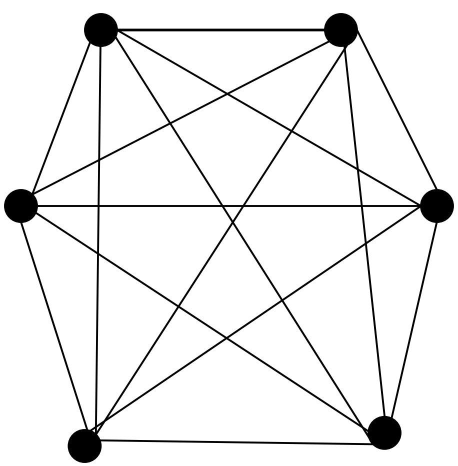
All triangles are suitable.

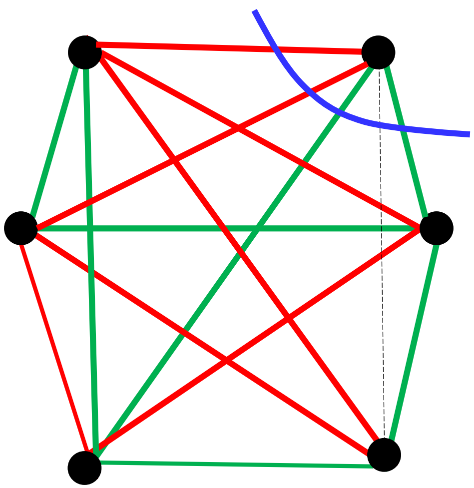
How many triangles are there?

${}_6C_3 = 20$

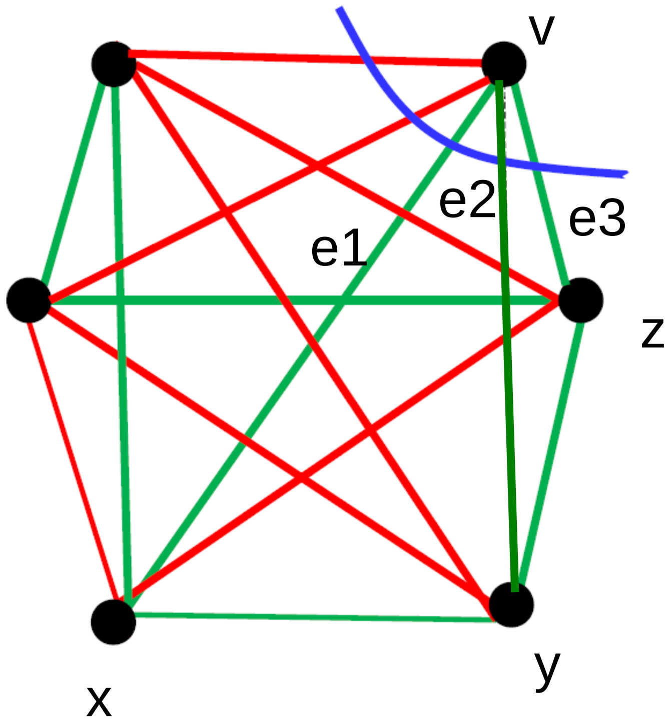
How many sides are there?

${}_6C_2 = 15$

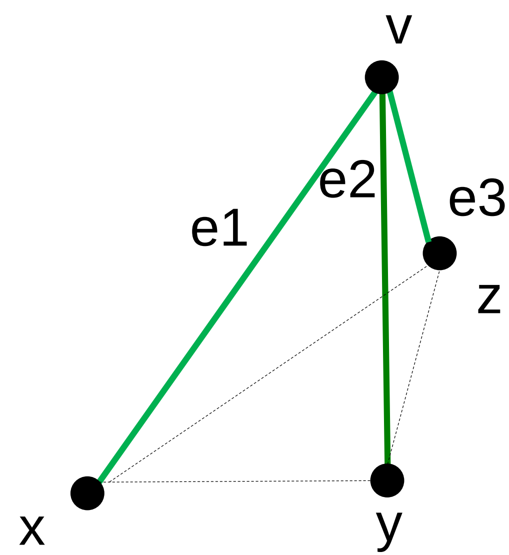




Not all triangles can be suitable.



Lemma If a vertex has three incident edges with the same color, then there is an unsuitable triangle.



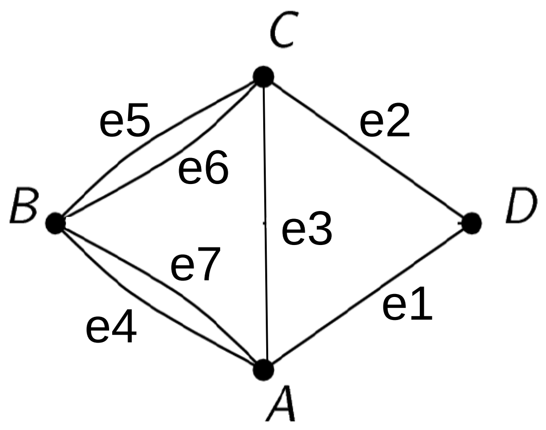
Is it possible for all triangles of this **tetrahedral** to be suitable?

Fact 1. In a complete graph with 6 vertices, each vertex has five incident edges.
Fact 2. When coloring five edges incident on a vertex with two colors, three edges will have the same.
(By Generalized Pigeonhole Principle)

Solution

Pick a vertex v. There are 5 edges on v, so, by the Extended Pigeonhole Principle, three of these edges e_1, e_2, e_3 must be the same color (say green, without loss of generality). Let the other vertices of edges e_1, e_2, e_3 be x, y, z, respectively. If the triangular circuit formed by x, y, z has all red edges, we are done. But if one of the edges is green, then it forms a green triangular circuit with v.

Quiz 24-1



How many different Euler circuits are there in the graph that begins at A and then visits D as the immediate next vertex?