Ch 1. The Foundations: Logic and Proofs

Predicate Logic-5

Proof Examples

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Example 1: Prove that the following inference is valid.

Socrates is a human.
All humans dies.
____
Socrates dies.

Example 2a: $\exists x \forall y\ P(x,y) \Rightarrow \forall y \exists x\ P(x,y)$

$$\overset{?}{\Rightarrow}$$

Example 2b: $\forall y \exists x\ P(x,y) \overset{?}{\Rightarrow} \exists x \forall y\ P(x,y)$

Example 1: Prove that the following inference is valid.

Socrates is a human.
All humans dies.
____
Socrates dies.

human(__): "___ is human"
dies(__): " __ dies"

1	human(Socrates)	- premise
2	$\forall$x (human(x) $\rightarrow$ dies(x))	- premise
3		
4	dies(Socrates)	

Example 1: Prove that the following inference is valid.

Socrates is a human.
All humans dies.
———————————
Socrates dies.

1	human(Socrates)	- premise
2	$\forall x\ (\text{human}(x) \rightarrow \text{dies}(x))$	- premise
3		
4	dies(Socrates)	- $\rightarrow$-elim, 3,1

Example 1: Prove that the following inference is valid.

Socrates is a human.
All humans dies.
Socrates dies.

1	human(Socrates)	- premise
2	$\forall$x (human(x) $\rightarrow$ dies(x))	- premise
3	human(Socrates) $\rightarrow$ dies(Socrates)	- $\forall$-elim, 2
4	dies(Socrates)	- $\rightarrow$-elim, 3,1

Example 2a: $\exists x \forall y\ P(x,y) \Rightarrow \forall y \exists x\ P(x,y)$

1	$\exists x \forall y\ P(x,y)$	- premise
2		
3		
4		
5		
6	$\forall y \exists x\ P(x,y)$	

Example 2a: $\exists x \forall y\ P(x,y) \Rightarrow \forall y \exists x\ P(x,y)$

1	$\exists x \forall y\ P(x,y)$	- premise
2		
3		
4		
5		
6	$\forall y \exists x\ P(x,y)$	- $\exists$-elim, 2-5

$\exists$-elim Rule

1	$\exists x\ A(x)$	- premise
2	$A(x)$	- premise
3	C	
4	C	- $\exists$-elim, 1, 2-3

C does not contain x free

$\forall y \exists x\ (x > y)$ does not contain x free

Example 2a: $\exists x \forall y\, P(x,y) \Rightarrow \forall y \exists x\, P(x,y)$

1	$\exists x \forall y\, P(x,y)$	- premise
2	$\quad \forall y\, P(w,y)$	- premise
3		
4		
5	$\quad \forall y \exists x\, P(x,y)$	
6	$\forall y \exists x\, P(x,y)$	- $\exists$-elim, 2-5

$\exists$-elim Rule

1	$\exists x\, A(x)$	- premise
2	$\quad A(x)$	- premise
3	$\quad C$	
4	C	- $\exists$-elim, 1, 2-3

C does not contain x free

Example 2a: ∃x∀y P(x,y) ⇒ ∀y∃x P(x,y)

1	∃x∀y P(x,y)	- premise
2	∀y P(w,y)	- premise
3	P(w,z)	- ∀-elim, 2
4		
5	∀y∃x P(x,y)	
6	∀y∃x P(x,y)	- ∃-elim, 2-5

∃-elim Rule

1	∃x A(x)	- premise
2	A(x)	- premise
3	C	
4	C	- ∃-elim, 1, 2-3

C does not contain x free

$\forall$-intro

$$\frac{P(x)}{\forall x\, P(x)}\ \text{x is free}$$

Example 2a: $\exists x \forall y\, P(x,y) \Rightarrow \forall y \exists x\, P(x,y)$

1	$\exists x \forall y\, P(x,y)$	- premise
2	$\quad \forall y\, P(w,y)$	- premise
3	$\quad P(w,z)$	- $\forall$-elim, 2
4	$\quad \exists x\, P(x,z)$	- $\exists$-intro, 3
5	$\quad \forall y \exists x\, P(x,y)$	
6	$\forall y \exists x\, P(x,y)$	- $\exists$-elim, 2-5

$\exists$-elim Rule

1	$\exists x\, A(x)$	- premise
2	$\quad A(x)$	- premise
3	$\quad C$	
4	C	- $\exists$-elim, 1, 2-3

C does not contain x free

$\forall$-intro	
$P(x)$	
———————	x is free
$\forall x\ P(x)$	

Example 2a: $\exists x \forall y\ P(x,y) \Rightarrow \forall y \exists x\ P(x,y)$

1	$\exists x \forall y\ P(x,y)$	- premise
2	$\forall y\ P(w,y)$	- premise
3	$P(w,z)$	- $\forall$-elim, 2
4	$\exists x\ P(x,z)$	- $\exists$-intro, 3
5	$\forall y \exists x\ P(x,y)$	- $\forall$-intro, 4
6	$\forall y \exists x\ P(x,y)$	- $\exists$-elim, 2-5

$\exists$-elim Rule

1	$\exists x\ A(x)$	- premise
2	$A(x)$	- premise
3	C	
4	C	- $\exists$-elim, 1, 2-3

C does not contain x free

Example 2a(A concrete example): $\exists x \forall y\ (x > y) \Rightarrow \forall y \exists x\ (x > y)$

1	$\exists x \forall y\ (x > y)$	- premise
2	$\quad \forall y\ (w > y)$	- premise
3	$\quad (w > z)$	- $\forall$-elim, 2, z is arbitrary
4	$\quad \exists x\ (x > z)$	- $\exists$-intro, 3
5	$\quad \forall y \exists x\ (x > y)$	- $\forall$-intro, 4, z is arbitrary
6	$\forall y \exists x\ (x > y)$	- $\exists$-elim, 2-5, $\forall y \exists x\ (x > y)$ does not contain x free

If w > any number, there is some number (i.e. w) > any number.

Therefore, for any number y, some number (for example, w) will be > y.

Recall that $\exists x \forall y\ P(x,y) \not\equiv \forall y \exists x\ P(x,y)$.

One failed approach

Example 2b1: $\forall y \exists x\ P(x,y) \overset{?}{\Rightarrow} \exists x \forall y\ P(x,y)$

1	$\forall y\ \exists x\ P(x,y)$	- premise
2	$\exists x\ P(x,b)$	- $\forall$-elim, 1
3		
4		
5	$\forall y\ P(x,y)$	
6	$\exists x \forall y\ P(x,y)$	- $\exists$-intro, 5

$\exists$-elim Rule

1	$\exists x\ A(x)$	- premise
2	$A(x)$	- premise
3	C	
4	C	- $\exists$-elim, 1, 2-3

C **does not contain x free**

Example 2b1: $\forall y \exists x\, P(x,y) \overset{?}{\Rightarrow} \exists x \forall y\, P(x,y)$

1	$\forall y\, \exists x\, P(x,y)$	- premise
2	$\exists x\, P(x,b)$	- $\forall$-elim, 1
3	$\quad P(x,b)$	- premise
4	$\quad \forall y\, P(x,y)$	- $\forall$-intro, 3
5	$\forall y\, P(x,y)$	
6	$\exists x \forall y\, P(x,y)$	- $\exists$-intro, 5

$\exists$-elim Rule

1	$\exists x\, A(x)$	- premise
2	$\quad A(x)$	- premise
3	$\quad C$	
4	C	- $\exists$-elim, 1, 2-3

C **does not contain x free**

Example 2b1: $\forall y \exists x\, P(x,y) \overset{?}{\Rightarrow} \exists x \forall y\, P(x,y)$

1	$\forall y\, \exists x\, P(x,y)$	- premise
2	$\exists x\, P(x,b)$	- $\forall$-elim, 1
3	$P(x,b)$	- premise
4	$\forall y\, P(x,y)$	- $\forall$-intro, 3
5	$\forall y\, P(x,y)$	- ~~∃-elim~~, 3-4
6	$\exists x \forall y\, P(x,y)$	- $\exists$-intro, 5

∃-elim Rule

1	$\exists x\, A(x)$	- premise
2	$A(x)$	- premise
3	C	
4	C	- $\exists$-elim, 1, 2-3

C does not contain x free

Line 5 cannot be justified by ∃-elim because $\forall y\, P(x,y)$ contains x free.

In line 4, x was only a constrained object of the domain.

In line 5, it would mean any object of the domain.

Another failed approach

Example 2b2: $\forall y \exists x\, P(x,y) \overset{?}{\Rightarrow} \exists x \forall y\, P(x,y)$

$$
\begin{array}{l|l l}
1 & \forall y\, \exists x\, P(x,y) & \text{- premise} \\
\hline
2 & \exists x\, P(x,b) & \text{- } \forall\text{-elim, 1} \\
3 & \quad P(x_b,b) & \text{- premise} \\
4 & \quad \forall y\, P(x_b,y) & \text{- } \cancel{\forall\text{-intro}}\text{, 3} \\
5 & \quad \exists x \forall y\, P(x,y) & \text{- } \exists\text{-intro, 4} \\
6 & \exists x \forall y\, P(x,y) & \text{- } \exists\text{-elim, 2, 3-5}
\end{array}
$$

∃-elim Rule

$$
\begin{array}{l|l l}
1 & \exists x\, A(x) & \text{- premise} \\
\hline
2 & \quad A(x) & \text{- premise} \\
3 & \quad C & \\
4 & C & \text{- } \exists\text{-elim, 1, 2-3}
\end{array}
$$

C does not contain x free

b in line 2 is arbitrary b.
But x in the premise of line 3 may depend on b.
So it is written x_b in line 3 to syntactically indicate its dependency on b.
Although b in line 3 is arbitrary, we cannot infer line 4 because y in line 4 would be any y that has no dependency on b.
It is too strong an assertion that is not supported by line 3.

Example 2b2(A concrete example): $\forall y \exists x\,(x > y) \overset{?}{\Rightarrow} \exists x \forall y\,(x > y)$

1	$\forall y\, \exists x\,(x > y)$	- premise
2	$\exists x\,(\,x > b\,)$	- $\forall$-elim, 1
3	$x > b$	- premise
4	$\forall y\,(x > y)$	- ~~$\forall$-intro~~, 3
5	$\exists x \forall y\,(x > y)$	- $\exists$-intro, 4
6	$\exists x \forall y\,(x > y)$	- $\exists$-elim, 2, 3-5

$\exists$-elim Rule

$\exists x\, A(x)$	- premise
$A(x)$	- premise
C	
C	- $\exists$-elim, 1, 2-3

C **does not contain x free**

Let's consider what went wrong in line 4 with a concrete example where $P(x,y)$ represents $x > y$. $\exists x \forall y\,(x > y)$ does not contain x free. So it is okay to apply $\exists$-elim at line 6.

But line 3 means, "Let x be > b." Then x > any number (line 4) or existence of such x (line 5) can NOT be asserted as in the previous slide.

Soundness and Completeness of the Natural Deduction System

Natural Deduction is a “complete” predicate logic system in the sense that any valid inference can be proved using its 12 inference rules.

Soundness (= Consistency) Theorem

If there is a proof of a conclusion φ from a set of premises Σ using Natural Deduction (i.e. $\Sigma \vdash \varphi$), then φ is true whenever Σ is true (i.e. $\Sigma \models \varphi$).

Proof Idea) Each inference rule is valid. (In the case of propositional logic inference rules, we can check their validity with truth tables.) Then a sequence of applications of valid inference rules results in a conclusion that is true whenever the given assumptions are true.

Completeness Theorem [Gödel 1930]

If φ is true whenever a set of premises Σ is true (i.e. $\Sigma \models \varphi$), then there is a proof of a conclusion φ from Σ using Natural Deduction (i.e. $\Sigma \vdash \varphi$).

Quiz 07-2

Answer with “Yes” or “No”.

[1] Does the set of inference rules of Gentzen’s Natural Deduction have redundancy in the sense that without some rule of the system it can still be complete?

[2] Is “$\forall x \forall y\ P(x,y) \Rightarrow \forall y \forall x\ P(x,y)$” valid?

[3] Is “$\exists x \exists y\ P(x,y) \Rightarrow \exists y \exists x\ P(x,y)$” valid?

[4] Is “$\exists x \forall y\ P(x,y) \Rightarrow \forall y \exists x\ P(x,y)$” valid?

[5] Is “$\forall y \exists x\ P(x,y) \Rightarrow \exists x \forall y\ P(x,y)$” valid?