Ch 1. The Foundations: Logic and Proofs

Predicate Logic-3

Formal Proof

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Acknowledgement
- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Example 1: $(P \rightarrow Q) \Rightarrow (\neg P \vee Q)$

Here we use a big turnstile notation.

1	$(P \rightarrow Q)$	- premise
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

Example 1: $(P \rightarrow Q) \Rightarrow (\neg P \vee Q)$

Hint:
Use proof by contradiction !

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!- B, \ \neg B}{\Sigma \mid\!- \neg A}$$

1	$(P \rightarrow Q)$	- premise
2	$\quad \neg(\neg P \vee Q)$	- premise
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13	$\neg\neg(\neg P \vee Q)$	- $\neg$-intro, 2-12, 8, 12
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

Example 1: $(P \rightarrow Q) \Rightarrow (\neg P \vee Q)$

1	$(P \rightarrow Q)$	- premise
2	$\quad \neg(\neg P \vee Q)$	- premise
3		
4		
5		
6		
7		
8	$\quad\quad Q$	
9		
10		
11		
12	$\quad\quad \neg Q$	
13	$\neg\neg(\neg P \vee Q)$	- $\neg$-intro, 2-12, 8, 12
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

Example 1: $(P \rightarrow Q) \Rightarrow (\neg P \vee Q)$

1	$(P \rightarrow Q)$	- premise
2	$\quad \neg(\neg P \vee Q)$	- premise
3		
4		
5		
6		
7	$\quad\quad P$	
8	$\quad\quad Q$	- $\rightarrow$-elim, 1, 7
9		
10		
11		
12	$\quad\quad \neg Q$	
13	$\neg \neg(\neg P \vee Q)$	- $\neg$-intro, 2-12, 8, 12
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!- B, \ \neg B}{\Sigma \mid\!- \neg A}$$

Example 1: $(P \to Q) \Rightarrow (\neg P \lor Q)$

¬-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!- B,\ \neg B}{\Sigma \mid\!- \neg A}$$

1	$(P \to Q)$	- premise
2	$\neg(\neg P \lor Q)$	- premise
3	$\neg P$	- premise
4		
5		
6	$\neg\neg P$	- ¬-intro, 3-5, 4, 5
7	P	- ¬-elim, 6
8	Q	- →-elim, 1, 7
9		
10		
11		
12	$\neg Q$	
13	$\neg\neg(\neg P \lor Q)$	- ¬-intro, 2-12, 8, 12
14	$(\neg P \lor Q)$	- ¬-elim, 13

Example 1: $(P \to Q) \Rightarrow (\neg P \vee Q)$

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \vdash B, \ \neg B}{\Sigma \vdash \neg A}$$

1	$(P \to Q)$	- premise
2	$\neg(\neg P \vee Q)$	- premise
3	$\neg P$	- premise
4	$\neg P \vee Q$	- $\vee$-intro, 3
5	$\neg(\neg P \vee Q)$	- 2
6	$\neg\neg P$	- $\neg$-intro, 3-5, 4, 5
7	P	- $\neg$-elim, 6
8	Q	- $\to$-elim, 1, 7
9		
10		
11		
12	$\neg Q$	
13	$\neg\neg(\neg P \vee Q)$	- $\neg$-intro, 2-12, 8, 12
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

Example 1: $(P \rightarrow Q) \Rightarrow (\neg P \vee Q)$

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!- B, \ \neg B}{\Sigma \mid\!- \neg A}$$

1	$(P \rightarrow Q)$	- premise
2	$\neg(\neg P \vee Q)$	- premise
3	$\neg P$	- premise
4	$\neg P \vee Q$	- $\vee$-intro, 3
5	$\neg(\neg P \vee Q)$	- 2
6	$\neg\neg P$	- $\neg$-intro, 3-5, 4, 5
7	P	- $\neg$-elim, 6
8	Q	- $\rightarrow$-elim, 1, 7
9	Q	- premise
10		
11		
12	$\neg Q$	- $\neg$-intro, 9-11, 10, 11
13	$\neg\neg(\neg P \vee Q)$	- $\neg$-intro, 2-12, 8, 12
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

Example 1: $(P \rightarrow Q) \Rightarrow (\neg P \vee Q)$

1	$(P \rightarrow Q)$	- premise
2	$\neg(\neg P \vee Q)$	- premise
3	$\neg P$	- premise
4	$\neg P \vee Q$	- $\vee$-intro, 3
5	$\neg(\neg P \vee Q)$	- 2
6	$\neg\neg P$	- $\neg$-intro, 3-5, 4, 5
7	P	- $\neg$-elim, 6
8	Q	- $\rightarrow$-elim, 1, 7
9	Q	- premise
10	$\neg P \vee Q$	- $\vee$-intro, 9
11	$\neg(\neg P \vee Q)$	- 2
12	$\neg Q$	- $\neg$-intro, 9-11, 10, 11
13	$\neg \neg(\neg P \vee Q)$	- $\neg$-intro, 2-12, 8, 12
14	$(\neg P \vee Q)$	- $\neg$-elim, 13

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \vdash B, \ \neg B}{\Sigma \vdash \neg A}$$

Quiz 06-2

Which of the following is NOT a good approach to constructing a formal proof?

(a) Use the main connective or the quantifier of the conclusion as the key to making an overall proof plan.

(b) Use the forward-backward technique if necessary.

(c) If the conclusion is a conjunction, it is a good idea to prove each conjunct one by one and apply the $\wedge$-Intro rule.

(d) If the conclusion is a disjunction and none of the disjuncts are easily proved, it is a good idea to try a proof by cases or a proof by contradiction.

(e) Try to derive as many formulas as possible from the formulas that have been already derived to determine the most appropriate one from which to move forward with the proof.