Ch 6. Counting

Counting with Functions

Sungwon Kang

Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 6. Counting

Counting with Functions

How can we **count** by **relating** a given problem to something familiar?

1. One-to-One Correspondence
2. The Pigeonhole Principle
3. The Generalized Pigeonhole Principle

1. One-to-One Correspondence

Definition A total function $f: X \rightarrow Y$ that is surjective and injective is called a *one-to-one correspondence* ($\neq$ one-to-one).

Theorem

Let $|X| = m$ and $|Y| = n$. If there is some $f : X \longrightarrow Y$ that is one-to-one, then $m \leq n$. Assume that f is a total function.

Will this theorem be true even if f is a partial function?

No.

Theorem

Let $|X| = m$ *and* $|Y| = n$. *If there is some* $f : X \longrightarrow Y$ *that is onto, then* $m \geq n$.

Will this be true if f is a total function?
Yes.

Will this be true even if f is a partial function?
Yes.

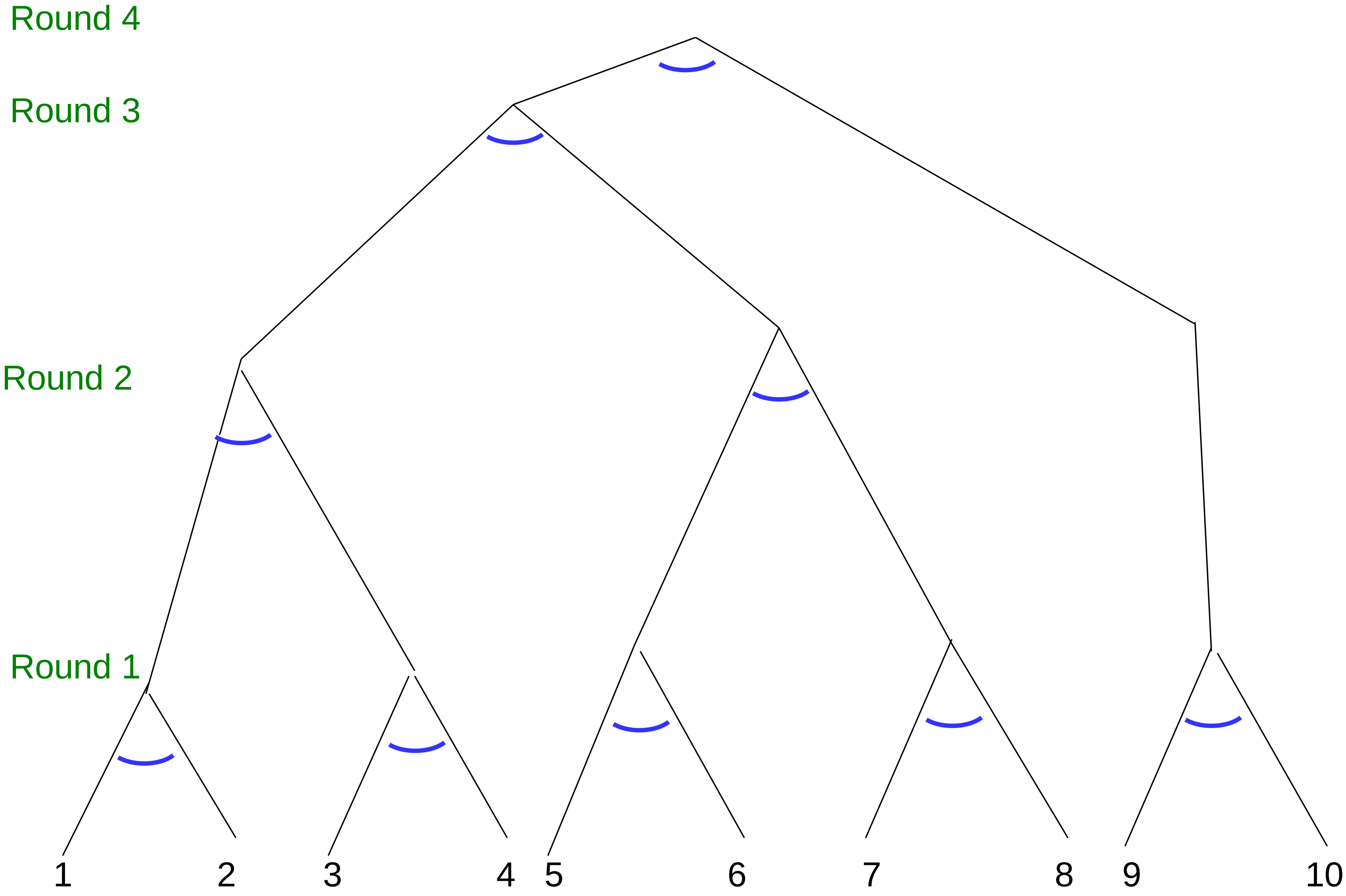
Corollary

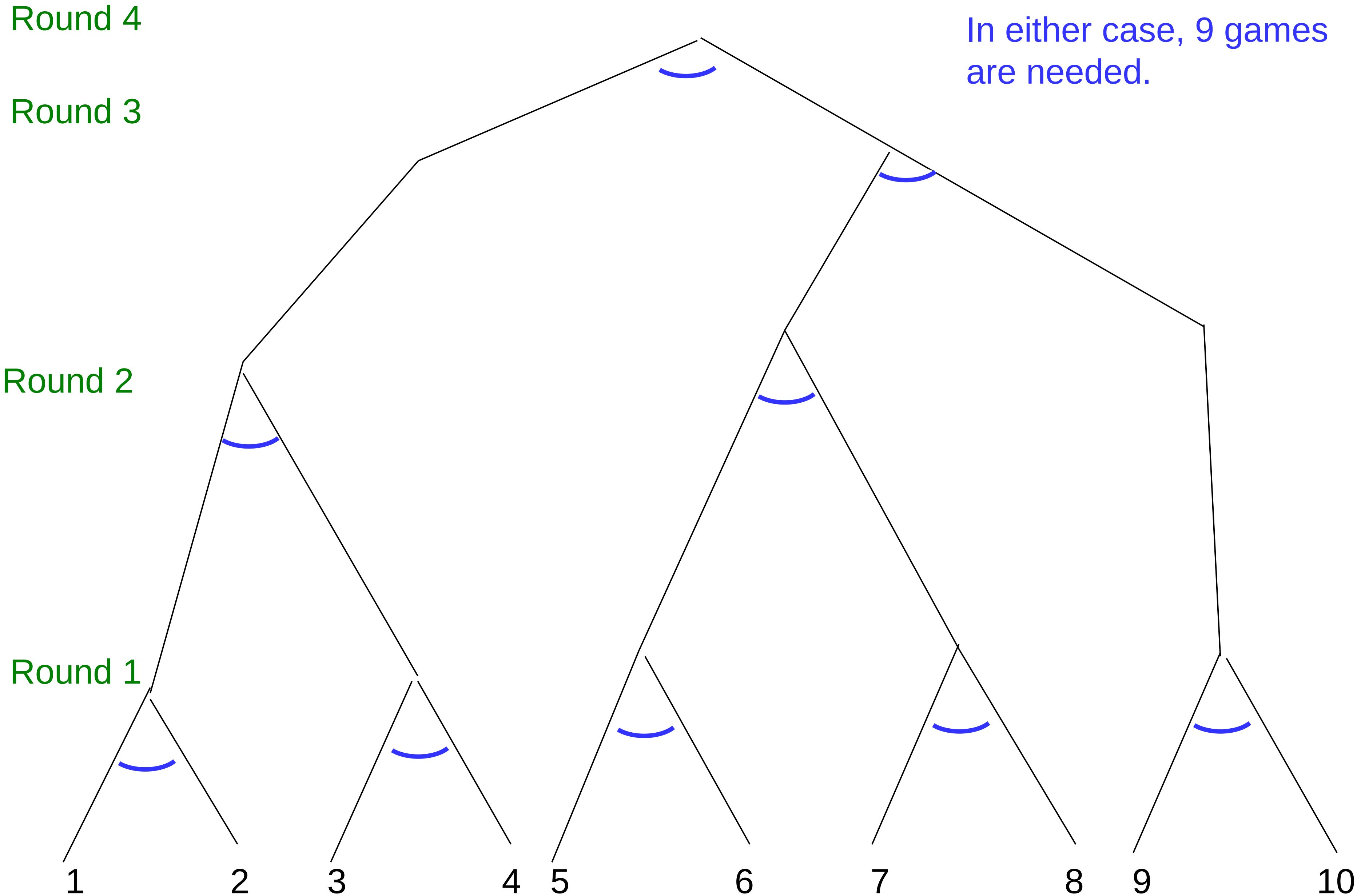
Let $|X| = m$ *and* $|Y| = n$. *If there is a one-to-one correspondence* $f : X \longrightarrow Y$, *then* $m = n$. Assume that X and Y are finite sets.

☛ In this chapter, we consider only total functions as we are going to use them for the purpose of counting.

Example: Counting tournament games

In a single elimination tournament, players are paired up in each round, and the winner of each match advances to the next round. If the number of players in a round is odd, one player gets a bye to the next round. The tournament continues until only two players are left; these two players play the championship game to determine the winner of the tournament. In a tournament with 270 players, how many games must be played?


Round 4
Round 3
Round 2
Round 1
1
2
3
4
5
6
7
8
9
10



Round 4
Round 3
Round 2
Round 1
In either case, 9 games are needed.
1
2
3
4
5
6
7
8
9
10

Round	Players left	Number of games played
0	270	0
1	135	135
2	68	67
3	34	34
4	17	17
5	9	8
6	5	4
7	3	2
8	2	1
9	1	1
Total		269

Is there an easier way?

Solution

This problem is easy if you realize that there is a one-to-one correspondence

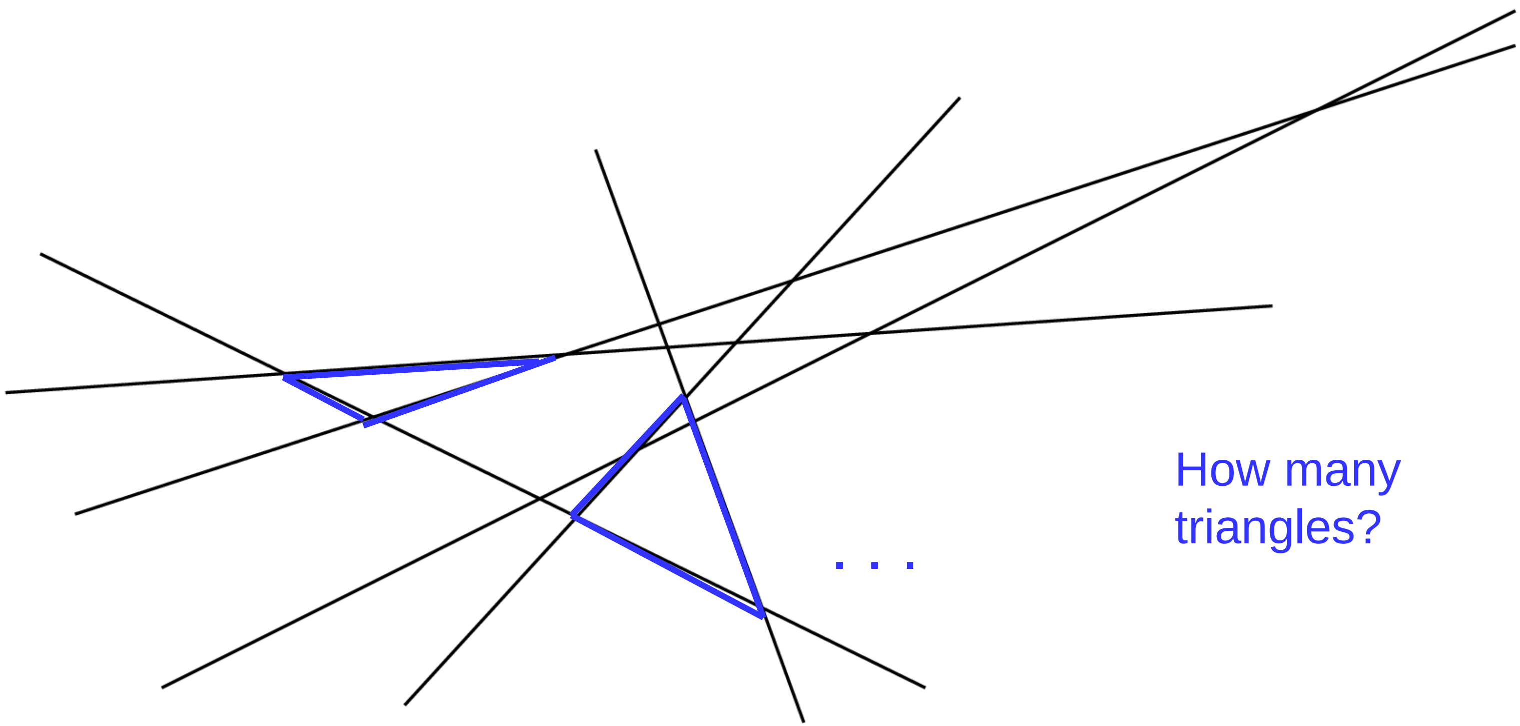
$$f : G \longrightarrow L$$

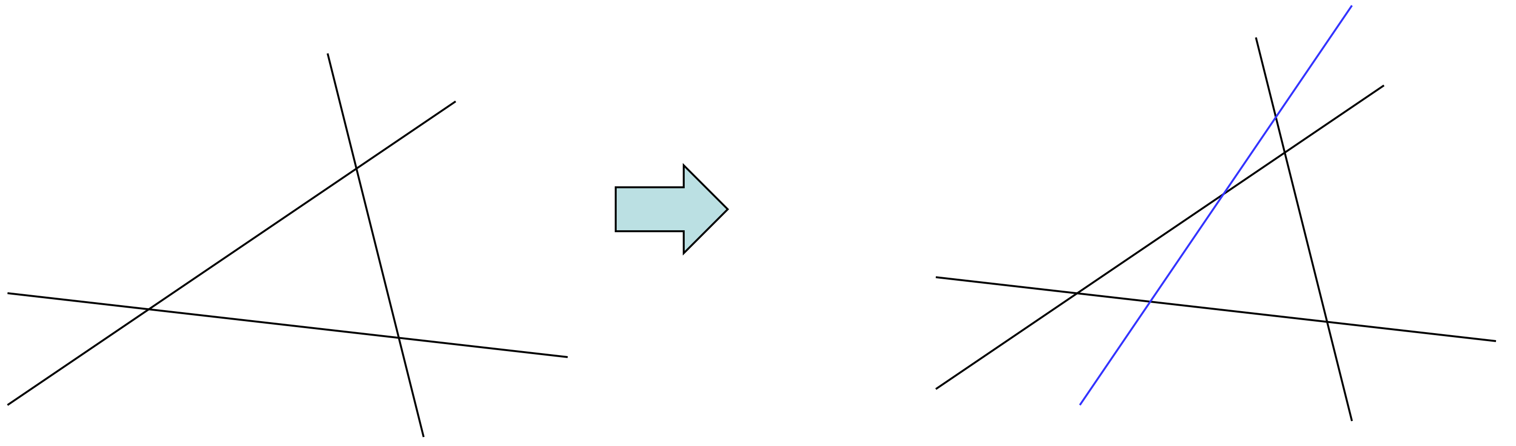
where G is the set of all games played, and L is the set of players who lose a game. The function is defined for any game x as $f(x) = l$, *where l is the loser of game x. Since every game has a single loser, the function is well-defined. Since this is a single elimination tournament, no player can lose two different games, so f is one-to-one. And since every loser lost some game, f is onto. So* **the number of games equals the number of losers.** *The winner of the tournament is the only non-loser, so there are 269 losers, hence 269 games.*

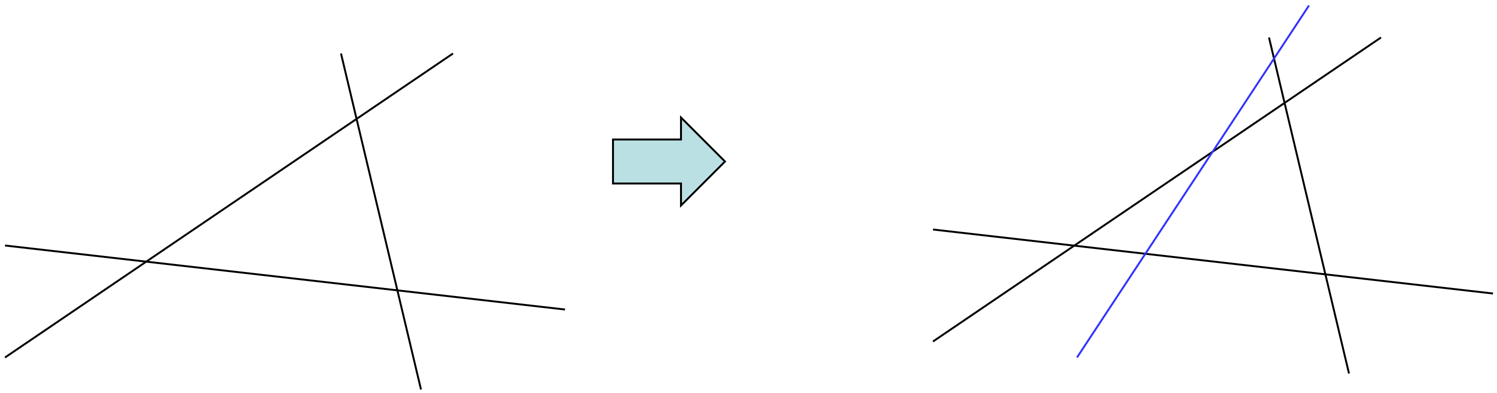
How many triangles are there?

The 6 lines below satisfy the following conditions.

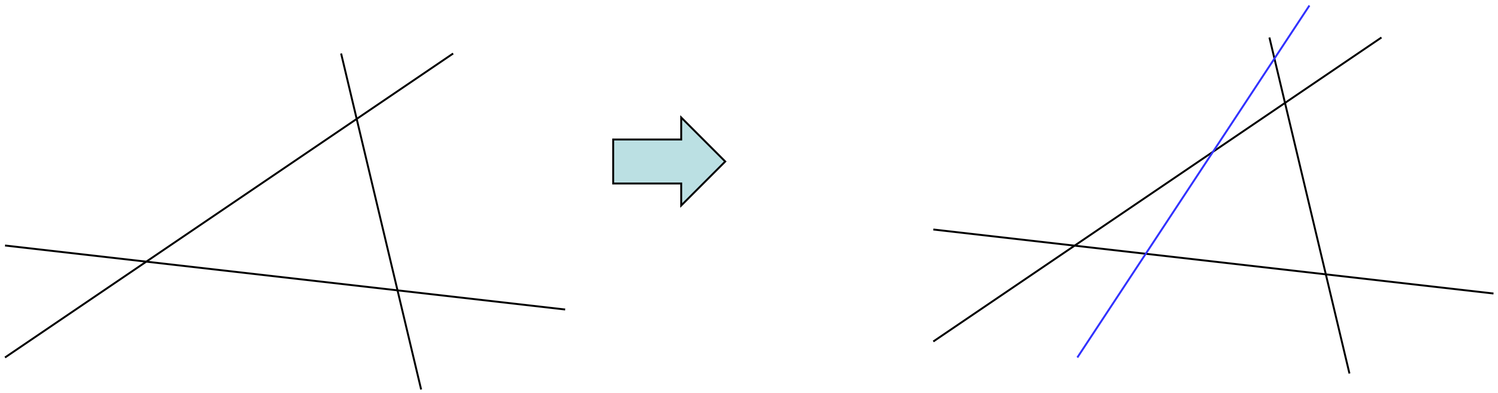
- Every line intersects every other line.
- No three lines intersect in a single point.







${}_3C_3$ ${}_4C_3$



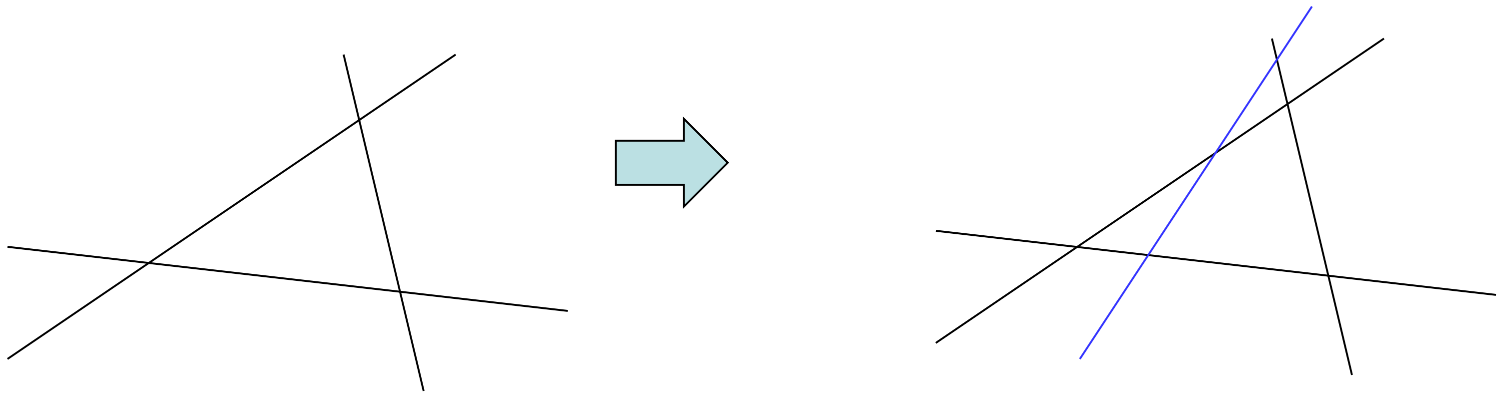
$_3C_3$ $_4C_3$

One-to-one correspondence?

Domain: The set of combinations of three different lines.

Range: The set of triangles.

Call this mapping f.



$_3C_3$ $_4C_3$

- Is it possible for a combination of the same three lines to make two different triangles?
  NO => f is a function
- Is it possible for a combination of three lines not to make a triangle?
  NO (by Constraints 1 and 2) => f is total
- Is it possible for two different combinations of three lines to make the same triangle?
  NO => f is one-to-one
- Is any triangle made from a combination of three lines?
  Yes => f is onto.

■ Every line intersects every other line.	- Constraint 1
■ No three lines intersect in a single point.	- Constraint 2

Solution

Observe that every triangle is formed from three lines, and any set of three lines forms a triangle. Thus there is a one-to-one correspondence

$$\{\textit{triangles in figure}\} \longleftrightarrow \{\textit{sets } \{l_1, l_2, l_3\} \mid l_i \textit{ is a line}\}$$

So to count the number of triangles, we can just count the number of sets of three lines. There are $C(6,3) = 20$ *of these.*

n-to-one functions

An *image* of a function f:X→Y is the set of all values in Y that f can take.

Definition

A function $f : X \longrightarrow Y$ is called *n-to-one* if every y in the image of the function has exactly n different elements of X that map to it. In other words, f is n-to-one if

$$|\{x \in X \mid f(x) = y\}| = n$$

for all $y \in f(X)$.

Theorem

Let $|X| = p$ and $|Y| = q$. If there is an n-to-one function $f : X \longrightarrow Y$ that maps X onto Y, then $p = qn$.

f is total.

Example: Rearranging letters with repeats

How many different strings can you form by rearranging the letters in the word ENUMERATE?

How many different strings can you form by rearranging the letters in the word ENUMERATE?

Solution

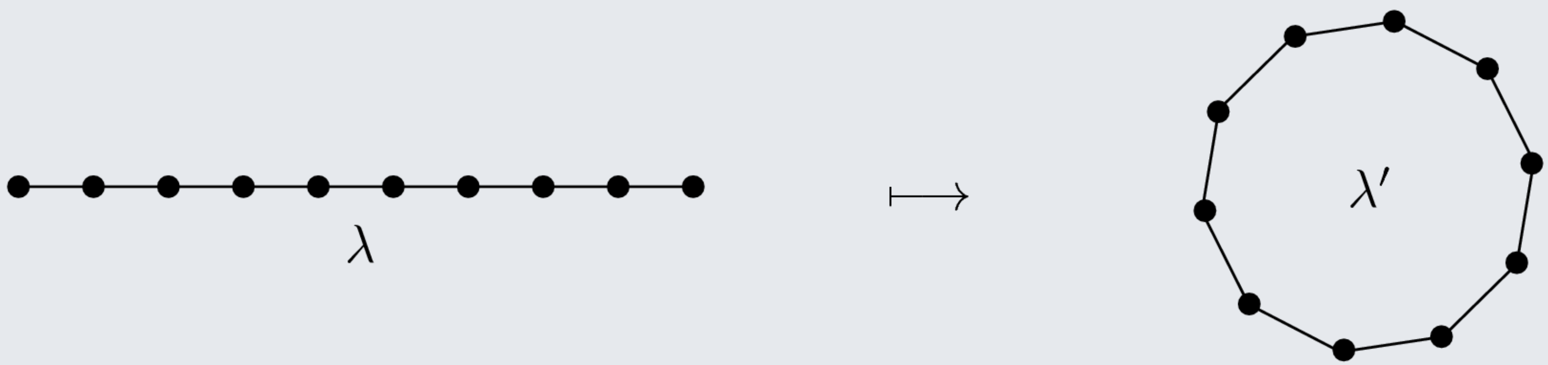
Pretend that the three E's are different for a moment: call them E_1*,* E_2*, and* E_3*. Let* X *be the set of all strings you can form by rearranging the letters in the word* $E_1NUME_2RATE_3$*. Since the elements of* X *are just permutations of 9 distinct symbols,* $|X| = 9!$*. Now let* Y *be the number of ways to rearrange ENUMERATE, and define a function* $f : X \longrightarrow Y$ *by* $f(\lambda) = \lambda'$*, where* λ' *is the string* λ *with the subscripts on the E's removed. This function is onto, because you can always take a string in* Y *and put the subscripts 1, 2, and 3 on the E's. Moreover, there are exactly* $3! = 6$ *ways to do this, so* f *is 6-to-one. Therefore,* $|X| = 6 \cdot |Y|$*, so there are* $|Y| = 9!/6 = 60,480$ *arrangements.*

Example: sitting around a campfire

A group of ten people sits in a circle around a campfire. How many different seating arrangements are there? In this situation, a seating arrangement is determined by who sits next to whom, not by where on the ground they sit. Let's also agree not to distinguish between clockwise and counterclockwise; all that matters is who your two neighbors are, not who is on your left and who is on your right.

Solution

Use the following map:



This is a 20-to-one function. Therefore $|Y| = 10!/20 = 181,440$.

Why 20 ?

2. The Pigeonhole Principle

Theorem

Let $|X| = n$ and $|C| = r$, and let $f : X \longrightarrow C$. If $n > r$, then there are distinct elements $x, y \in X$ with $f(x) = f(y)$. Assume f is total.

That is, there are only r pigeonholes.
If the number of pigeons is larger than the number of pigeonholes, then some pigeonhole should be occupied by more than one pigeons.

Theorem

Let $|X| = n$ *and* $|C| = r$, *and let* $f : X \longrightarrow C$. *If* $n > r$, *then there are distinct elements* $x, y \in X$ *with* $f(x) = f(y)$.

Example: In a club with 400 members, must there be some pair of members who share the same birthday?

Theorem

Let $|X| = n$ and $|C| = r$, and let $f : X \longrightarrow C$. If $n > r$, then there are distinct elements $x, y \in X$ with $f(x) = f(y)$.

Example: In a club with 400 members, must there be some pair of members who share the same birthday?

Solution

Yes. Let X be the set of all people, and let C be the set of all possible birthdays. Let $f : X \longrightarrow C$ be the defined so that $f(x)$ is the birthday of person x. Since $|X| > |C|$, there must be two people x and y with the same birthday, that is, with $f(x) = f(y)$.

More pigeonhole examples

Example: Chandra has a drawer full of 12 red and 14 green socks. How many socks must he grab (without looking) in order to be assured of having a matching pair?

2 red socks or 2 greens socks will make a matching pair.

Example: Chandra has a drawer full of 12 red and 14 green socks. How many socks must he grab (without looking) in order to be assured of having a matching pair?

Solution

Let $C = \{red, green\}$ and let X be the set of socks Chandra selects. Let $f : X \longrightarrow C$ be the function that assigns a color to each sock. There are two colors, so he needs $|X| > 2$ socks. Three is enough.

Example: In a round-robin tournament, every player plays every other player exactly once. Prove that, if no player goes undefeated, at the end of the tournament there must be two players with the same number of wins.

Example: In a round-robin tournament, every player plays every other player exactly once. Prove that, if no player goes undefeated, at the end of the tournament there must be two players with the same number of wins.

The set of all possible numbers of wins is $\underbrace{\{0, 1, 2, \ldots, n-3, n-2\}}_{\text{n-1 different numbers of wins}}$.

There are n players and
there are n-1 different number of wins.

There must be the same number of wins
because . . .

Try to assign all different numbers of wins to n players

☛ Impossible !

Example: In a round-robin tournament, every player plays every other player exactly once. Prove that, if no player goes undefeated, at the end of the tournament there must be two players with the same number of wins.

Solution

Apply Theorem 26 with X being the set of players, and let $|X| = n$. Each player plays $n - 1$ games, and no player wins every game, so the set of all possible numbers of wins is $C = \{0, 1, 2, \ldots, n - 2\}$. Define $f : X \longrightarrow C$ so that $f(x)$ is the number times player x wins. Since $|C| < |X|$, there exists a pair of players with the same number of wins.

2. The Generalized Pigeonhole Principle

Theorem

Let $|X| = n$ and $|C| = r$, and let $f : X \longrightarrow C$. If $n >$ $r \times l$
then there is some subset $U \subseteq X$ such that $|U| =$ $l + 1$
$f(x) = f(y)$ for any $x, y \in U$. Assume f is total.

Corollary

Let $|X| = n$ and $|C| = r$, and let $f : X \longrightarrow C$. Then there is some subset $U \subseteq X$ such that

$$|U| = \left\lceil \frac{n}{r} \right\rceil$$

and $f(x) = f(y)$ for any $x, y \in U$.

Instead of one-to-one correspondence, n-to-one function is used for pigeonhole mapping.

Example: an image bank

A website displays an image each day from a bank of 30 images. In any given 100 day period, show that some image must be displayed four times.

Suppose that no image is displayed four times.

If every image is displayed three times, then after 90 days, then there are no more images left to display.

A website displays an image each day from a bank of 30 images. In any given 100 day period, show that some image must be displayed four times.

Solution

Apply the theorem with X being the set of days and C being the set of images. Let $f : X \longrightarrow C$ be the function that returns the image $f(x)$ that gets displayed on day x. Since $100 > 30(4-1)$, there is some image that will be displayed four times.

<– Example of an application of the Generalized Pigeonhole Principle.

Quiz 19-1

Concerning the round-robin tournament that appeared in this lecture slides, answer the following questions, supposing that there are n players.

1) What is the sum of wins and losses for each player?

2) If there is a player who is undefeated,

then how many wins does the player have?

3) If there is no player who is undefeated, then what is the most wins of a player?