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CS 204: Discrete Mathematics
Final Examination

December 15(Tuesday), 2020

1:00 pm ~ 3:00 pm

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(Total number of pages including cover: 13)

Part I. (Logic)

1. (12 pts) Let H stand for Homes (Sherlock Holmes) and let M stand for Moriarty. Let us abbreviate "x can catch y" by "C(x,y)". Give predicate logic expressions of the following.

(a) Homes can catch anyone whom Moriarty can catch.

(b) If anyone can catch Moriarty, then Holmes can.

(c) Anyone who can catch Holmes can catch Moriarty.

(d) Everyone can catch someone who cannot catch Moriarty.

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2. (12 pts) Consider the following inference.

$$\frac{F(a) \leftrightarrow G(x)}{\exists x\,(F(x) \leftrightarrow G(x))}$$

where x is a variable for a domain U and a is a constant designating some element in U.

(a) Is the inference valid? (Answer with Yes or No.)

(b) If your answer to (a) is Yes, then give a formal proof. If you answer to (a) is No, then give a model that establishes that your answer is correct.

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3. (12 pts) Prove the following theorems using natural deduction:

$$\frac{\exists x\, \exists y\, F(x,y) \lor \forall x \forall y\, G(y,x)}{\exists x\, \exists y\, (F(x,y) \lor G(x,y))}$$

Use only the inference rules of Gentzen's Natural Deduction.

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Part II. (Sets and Relations)

1. (12 pts) Let R1 and R2 be arbitrary relations on a set A. (So both the domain and the range of R1 and R2 are A.)

(a) Is the assertion "If R1 and R2 are reflexive, then R1°R2 is reflexive" true? Answer with True or False.

(b) Prove or disprove depending on your answer in (a).

(c) Is the assertion "If R1 and R2 are irreflexive, then R1°R2 is irreflexive" true? Answer with True or False.

(d) Prove or disprove depending on your answer in (c).

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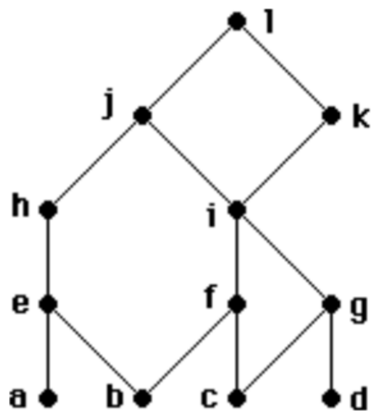
2. (12 pts) Consider the partial order with the following Hasse diagram.



(a) Find all maximal elements of the partial order.

(b) Find the least element of the partial order if it exists, or show that it does not exist.

(c) What is the greatest lower bound of the set {a, b, c}?

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Part III. (Induction and Recursion)

1. (10 pts) Let the following definitions be given, where s is a string.

Definition 1. Define the number $l(s)$ as follows.

B1. $l(s) = 0$ if s is the empty string.

B2. $l(s) = 1$ if s is a single symbol.

R. $l(s) = l(x) + l(y)$ if $s = xy$.

Definition 2. Let n be a natural number. Define the string ns as follows:

B. $1s = s$.

R. $ns = (n-1)s\ s$ if $n > 1$.

Use these definitions to prove that $l(ns) = n\ l(s)$ for all $n \geq 1$.

2. (12 pts)

a) Give a recursive definition of the function $m(s)$, which equals the smallest digit in a nonempty string of decimal digits.

b) Use structural induction to prove that $m(st) = \min(m(s), m(t))$.

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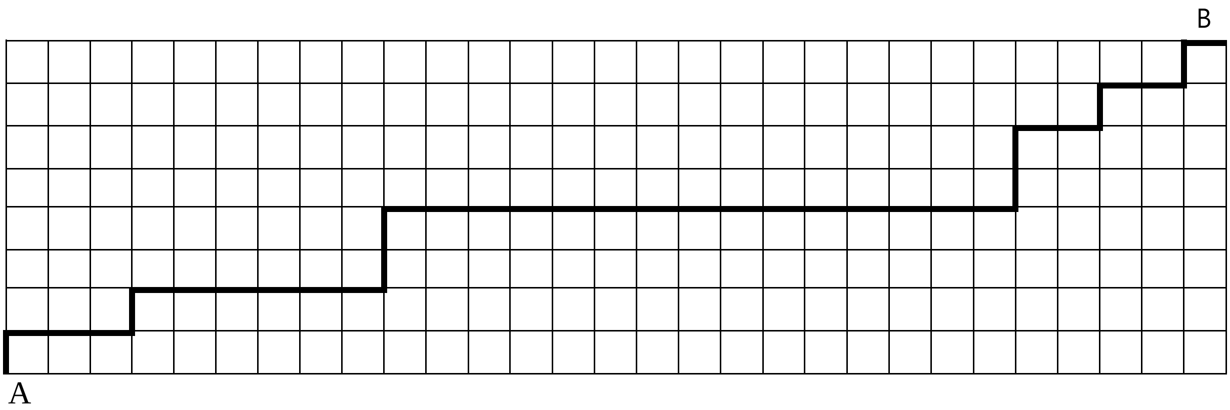
Part IV. (Counting and Probability)

1. (12 pts) The following 8 x 24 grid is divided into squares that are 1 unit by 1 unit.



The shortest possible path on this grid from A to B is 32 units long. One such path is shown in the figure. Let X be the set of all 32 unit long paths from A to B.

(a) There is a one-to-one correspondence between X and the set Y of all binary strings with 8 1's and 24 0's. Describe the function, and explain why it is a one-to-one correspondence.

(b) Compute |X|, the number of 32 unit long paths from A to B.

2. (12 pts) Two identical urns contain balls. One of the urns has 6 red balls and 3 blue balls. The other urn has 5 red balls and 8 blue balls. An urn is chosen at random and a ball is drawn at random from this urn. If the ball turns out to be red, what is the probability that this is the urn with 6 red balls?

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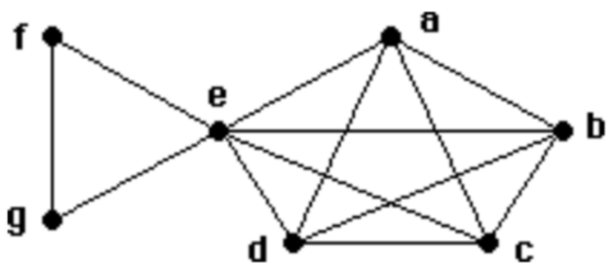
Part V. (Trees and Graphs)

1. (12 pts)

How many nonisomorphic undirected graphs with no self loops are there with three vertices? Draw examples of each of these.

2. (12 pts) Consider the following graph.



(a) Is there an Euler circuit in the graph above? If so, find such a circuit. If not, explain why no such circuit exists.

(b) Is there a Hamilton circuit in the graph above? If so, find such a circuit. If not, prove why no such circuit exists.

3. (10 pts) Form a binary search tree from the words of the sentence

"This test is not so difficult"

using alphabetical order, inserting words in the order they appear in the sentence.