Ch 1. The Foundations: Logic and Proofs

Predicate Logic-2

Natural Deduction

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- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Well-Formed Formulas of Proportional Logic

- Well-Formed Formulas(wffs): Grammatically (or syntactically) correct expressions of a language.

Well-Formed Formulas of Propositional Logic

1) Every proposition symbols such as p, q, r, ... and P, Q, R, ... are wffs.

2) If α and β are wffs, then so are $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $(\alpha \rightarrow \beta)$.

- Note that we do now consider $\leftrightarrow$ here because
  $\alpha \leftrightarrow \beta \equiv (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$
  and thus it is redundant.

Well-Formed Formulas of Predicate Logic

Term: A variable or an expressions that is built by prefixing a function symbol to a sequence of terms.

Examples f**,** f(*t*), f(*t1*,*t2*) ..., g, g(*t*), g(*t1*,*t2*), ... where *t*, *t1*, *t2*, ... are terms.

Atomic Formula: A proposition symbol or an expression that is built by prefixing a predicate symbol to a sequence of terms.

Examples p, q, r, ..., P, Q, R, ..., P(*t*), P(*t1*,*t2*), ..., Q(*t*), Q(*t1*,*t2*), ... where *t*, *t1*, *t2*, ... are terms.

Well-Formed Formulas of Predicate Logic

1) Atomic formulas are wffs.

2) If α and β are wffs, then so are $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $(\alpha \rightarrow \beta)$.

3) If *x* is a variable and α is a wff, then $(\forall x)\alpha$ and $(\exists x)\alpha$ are wffs.

Gentzen's Natural Deduction

Rules for Propositional Logic

$\wedge$-intro

$$\frac{A, B}{A \wedge B}$$

$\wedge$-elim

$$\frac{A \wedge B}{A}$$

$\wedge$-elim

$$\frac{A \wedge B}{B}$$

$\vee$-intro

$$\frac{A}{A \vee B}$$

$\vee$-intro

$$\frac{B}{A \vee B}$$

$\vee$-elim (proof by cases)

$$\frac{\Sigma, A \mid\!\!- C \qquad \Sigma, B \mid\!\!- C}{\Sigma, A \vee B \mid\!\!- C}$$

$\rightarrow$ -intro

$$\frac{\Sigma, A \mid\!\!- B}{\Sigma \mid\!\!- A \rightarrow B}$$

$\rightarrow$ -elim (modus ponens)

$$\frac{A, A \rightarrow B}{B}$$

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!\!- B, \qquad \Sigma, A \mid\!\!- \neg B}{\Sigma \mid\!\!- \neg A}$$

$\neg$-elim

$$\frac{\neg\neg A}{A}$$

"|-" is read "yield".

D1, . . ., Di |- E

means

“E is provable from D1, ..., Di”,

i.e.

“There is a formal proof
(using only the given rules of inference) of E
from D1, ... , Di.”

D1, ..., Di are called *assumptions*.
E is called the *conclusion*.

Σ is used to represent a set of assumptions.

The name for the symbol “|-” is “turnstile”.

Gentzen's Natural Deduction

Rules for Propositional Logic

$\wedge$-intro

$$\frac{A,\ B}{A \wedge B}$$

$\wedge$-elim

$$\frac{A \wedge B}{A}$$

$\wedge$-elim

$$\frac{A \wedge B}{B}$$

$\vee$-intro

$$\frac{A}{A \vee B}$$

$\vee$-intro

$$\frac{B}{A \vee B}$$

$\vee$-elim (proof by cases)

$$\frac{\Sigma, A \mid\!- C \qquad \Sigma, B \mid\!- C}{\Sigma, A \vee B \mid\!- C}$$

$\rightarrow$ -intro (Deduction Theorem)

$$\frac{\Sigma, A \mid\!- B}{\Sigma \mid\!- A \rightarrow B}$$

$\rightarrow$ -elim (modus ponens)

$$\frac{A,\ A \rightarrow B}{B}$$

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!- B,\ \neg B}{\Sigma \mid\!- \neg A}$$

$\neg$-elim

$$\frac{\neg\neg A}{A}$$

$\wedge$-intro rule and $\wedge$-elim rule

$\wedge$-intro

$$\frac{A, B}{A \wedge B}$$

$\wedge$-elim

$$\frac{A \wedge B}{A}$$

$\wedge$-elim

$$\frac{A \wedge B}{B}$$

$\wedge$-intro rule introduces $\wedge$ in the conclusion.

$\wedge$-elim rule eliminates $\wedge$ in the premise.

$\vee$ -intro rule and $\vee$-elim rule

$\vee$-intro	$\vee$-intro	$\vee$-elim (proof by cases)
$\dfrac{A}{A \vee B}$	$\dfrac{B}{A \vee B}$	$\dfrac{\Sigma, A \vdash C \quad \Sigma, B \vdash C}{\Sigma, A \vee B \vdash C}$

$\vee$-intro rule introduces $\vee$ in the conclusion.

$\vee$-elim rule eliminates $\vee$ in the premise $A \vee B$ so that it does not appear in the premises of the two subproofs.

Example (Proof by cases):

Let Σ : {r, s}

A: p

B: q

C: $(p \wedge r) \vee (q \wedge r)$

Show $\{r, s\}, p \vee q \vdash (p \wedge r) \vee (q \wedge r)$

$\vee$-elim (proof by cases)

$$\frac{\Sigma, A \mid\!- C \qquad \Sigma, B \mid\!- C}{\Sigma, A \vee B \mid\!- C}$$

Example (Proof by cases):

Let Σ : {r, s}

A: p

B: q

C: $(p \wedge r) \vee (q \wedge r)$

Show $\{r, s\}, p \vee q \mid\!- (p \wedge r) \vee (q \wedge r)$

$\boxed{\Sigma, A \vee B \mid\!- C}$ (1)

Proof Need to prove the following two cases

(2) $$\frac{\underbrace{r \quad s}_{\Sigma} \quad \underbrace{p}_{A}}{\underbrace{(p \wedge r) \vee (q \wedge r)}_{C}}$$

(3) $$\frac{\underbrace{r \quad s}_{\Sigma} \quad \underbrace{q}_{B}}{\underbrace{(p \wedge r) \vee (q \wedge r)}_{C}}$$

Then by $\vee$-elim Q.E.D.

$\rightarrow$-intro rule and $\rightarrow$-elim rule

$\rightarrow$ -intro (Deduction Theorem)

$$\frac{\Sigma, A \mid\!- B}{\Sigma \mid\!- A \rightarrow B}$$

$\rightarrow$ -elim (modus ponens)

$$\frac{A,\ A \rightarrow B}{B}$$

$\rightarrow$-intro rule introduces $\rightarrow$ in the conclusion.

$\rightarrow$-elim rule eliminates $\rightarrow$ in the premise $A \rightarrow B$.

$\neg$-intro rule and $\neg$-elim rule

$\neg$-intro (proof by contradiction)

$$\frac{\Sigma, A \mid\!- B, \neg B}{\Sigma \mid\!- \neg A}$$

$\neg$-elim

$$\frac{\neg\neg A}{A}$$

$\neg$-intro rule introduces $\neg$ in the conclusion.

$\neg$-elim rule eliminates $\neg$ in the premise $\neg\neg A$.

Quiz 06-1

[1] State whether the following statement is true or false.

"Any expression (for propositional logic) with more left parentheses than right parentheses is not a well-formed formula."

[2] Which of the following is NOT a well-formed formula of predicate logic?

(a) $((p \lor q) \to r)$

(b) $(\exists x)\ (P(f(x),x) \land q)$

(c) $(\exists x)\ P(f(x),x) \land q)$

(d) $(\exists x)\ P(f(x),x) \land q$