# *Ch 5. Induction and Recursion*

## Recursive Definition and Structural Induction

Sungwon Kang

**Acknowledgement**

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

# 3. Structural Induction

- To prove a property of the elements of a recursively defined set, we can use structural induction.

**Structural induction**

**Basis Step**: Show that the result holds for all elements specified in the basis step of the recursive definition.

**Induction Step**: Show that, if the statement is true for <u>each of the ways</u> to construct new elements in the recursive step of the definition, the result holds for these new elements.

# Example 1

**Example (Recursive definition)**

Define a set $X \subseteq \mathbf{Z}$ recursively as

**B.** $4 \in X$.

**R$_1$.** If $x \in X$ then $x - 12 \in X$.

**R$_2$.** If $x \in X$ then $x^2 \in X$.

☛ X = {4, -8, 16, -20, 64, 256, . . . }

- Prove that every element of $X$ is divisible by 4.

Will mathematical induction work?

[redacted]

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Will mathematical induction work?

- When we use Mathematical Induction, we know the (n+1)-th object from the n-th object.
- Many reclusively generated sets do not have an obvious **well-ordering of objects**.

In such cases, we should use **Structural Induction**.

## Example (Recursive definition)

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Prove that every element of $X$ is divisible by 4.

**Structural induction: Induction on the recursive definition**

We prove this claim by showing that

1. the claim is true for the base case of the definition, and
2. that the recursive cases of the definition preserve the truth of the claim.

To prove that, for each $x \in X$, x is divisible by 4, let's proceed as follows:

**Basis Step** $4 \in X$ is divisible by 4.

**Induction Step**

Let $y \in X$.
Suppose y is divisible by 4. (☚ Induction Hypothesis)

Then
$y - 12 \in X$ and $y^2 \in X$.
Since y is divisible by 4 by induction hypothesis,
y-12 and $y^2$ are both divisible by 4.

Therefore, by **structural induction**, every element in X is divisible by 4.
■

# Proof: (Induction on the recursive definition.)

**Base Case:** Since $4 = 1 \cdot 4$, $4 \mid 4$, so the claim holds for the base case of the definition.

**Inductive Hypothesis:** Suppose as inductive hypothesis that some $x \in X$ is divisible by 4. Then $x = 4a$ for some integer $a$.

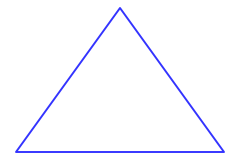
Part of inductive step

**Inductive Step:** Now $x - 12 = 4a - 12 = 4(a - 3)$, and $x^2 = (4a)^2 = 4(4a^2)$, so both $x - 12$ and $x^2$ are divisible by 4. Therefore cases $\mathbf{R_1}$ and $\mathbf{R_2}$ always produce integers that are divisible by 4 (given that $4 \mid x$), and the base case **B** gives an integer that is divisible by 4. So, by induction, all elements of $X$ are divisible by 4. $\square$

# Example 2

## Koch snowflake: definition

Define a sequence of shapes as follows.

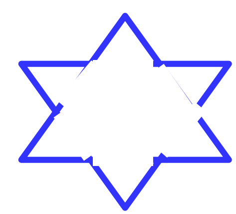


**B.** $K(1)$ is an equilateral triangle.

**R.** For $n > 1$, $K(n)$ is formed by replacing each line segment

of $K(n-1)$ with the shape

such that the central vertex points outwards.



# Example: Number of segments in $K(n)$

**Theorem**

*Let $K(1), K(2), K(3), \ldots$ be the sequence of shapes whose limit is the Koch Snowflake. Then $K(n)$, the nth term in this sequence, is composed of $4^{n-1} \cdot 3$ line segments.*

# Proof:

Show $K(n) = 4^{n-1} * 3$

**Base Case:** The base case of the definition states that $K(1)$ consists of 3 line segments, and $3 = 4^{1-1} \cdot 3$, so the theorem is true when $n = 1$.

# Proof:

**Base Case:** The base case of the definition states that $K(1)$ consists of 3 line segments, and $3 = 4^{1-1} \cdot 3$, so the theorem is true when $n = 1$.

**Inductive Hypothesis:** Suppose as inductive hypothesis that $K(k-1)$ is composed of $4^{k-1-1} \cdot 3 = 4^{k-2} \cdot 3$ line segments, for some $k > 1$.

Part of inductive step

**Inductive Step:** By the recursive part of the definition, $K(k)$ is formed by replacing each line segment with four others, so the number of line segments is multiplied by four. Therefore $K(k)$ is composed of $4 \cdot 4^{k-2} \cdot 3 = 4^{k-1} \cdot 3$ line segments, as required. $\square$

**For this proof, Mathematical Induction will work, too!**

# Example 3

## binary trees - A recursively defined data structure

**$B_1$.** The empty tree is a binary tree.

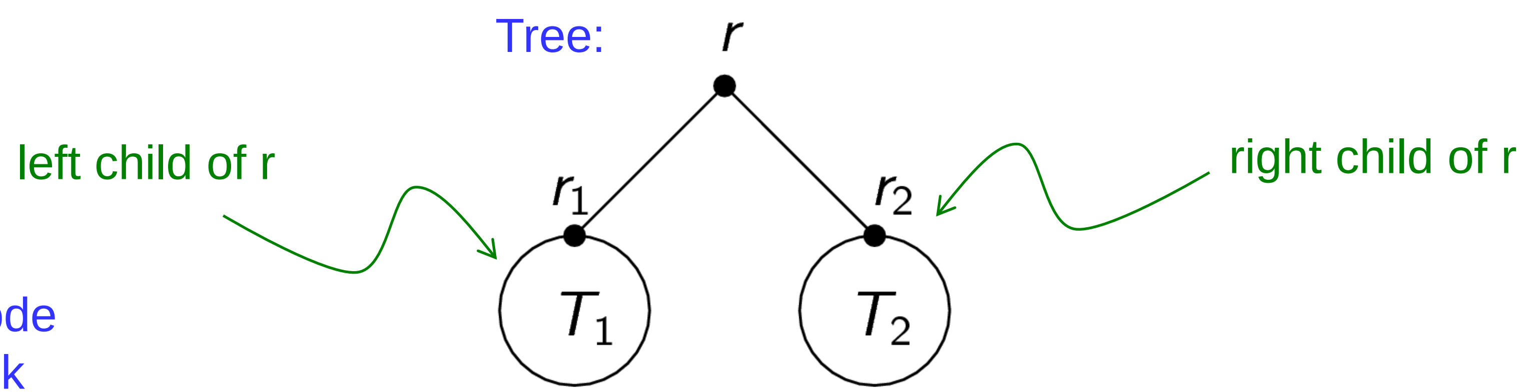
**Tree**: *empty tree* or *null tree*

**$B_2$.** A single vertex is a binary tree. In this case, the vertex is the root of the tree.

Tree: ● r

**R.** If $T_1$ and $T_2$ are binary trees with roots $r_1$ and $r_2$ respectively, then the tree

Tree:




vertex = node
edge = link

is a binary tree with root $r$. Here the circles represent the binary trees $T_1$ and $T_2$. If either of these trees $T_i$ $(i = 1, 2)$ is the empty tree, then there is no edge from $r$ to $T_i$.

- Various properties of binary trees can be proved using Structural Induction:

  - A binary tree with height h has at most $2^{h+1}$ -1 nodes.

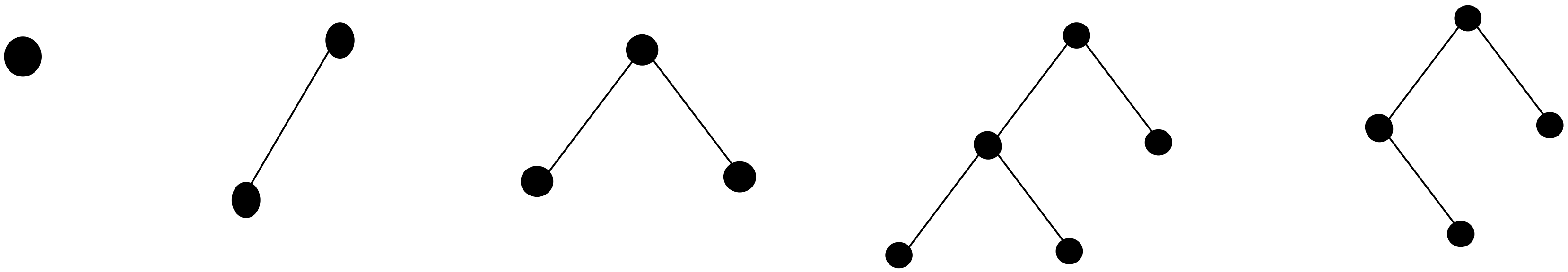
  - The number of leaf nodes of a binary tree is at most $2^h$.

  - . . .

**Definitions**:

A *full binary tree* is a binary tree in which each vertex has either two children or zero children.

Which of the following are full binary trees?

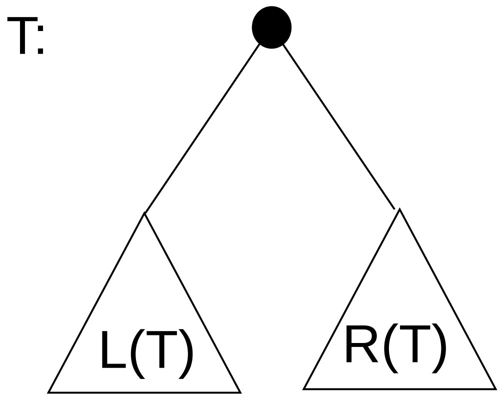


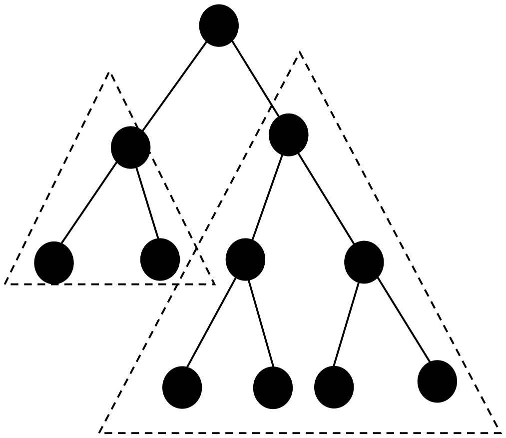
(1) (2) (3) (4) (5)

**Theorem** A full binary tree of height h has at most $2^{h+1} - 1$ vertices.

**Definition** (Height of a tree)
The *height* of a rooted tree is the maximum length of the path from the root to a leaf.

General form of a full binary tree



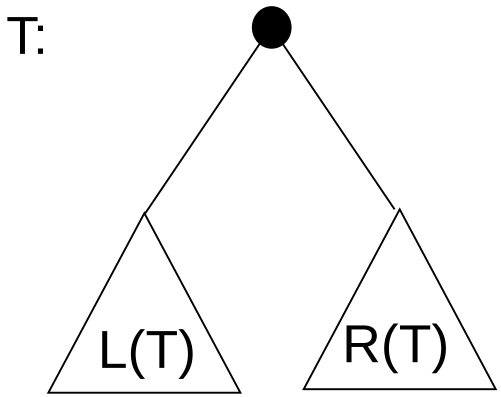



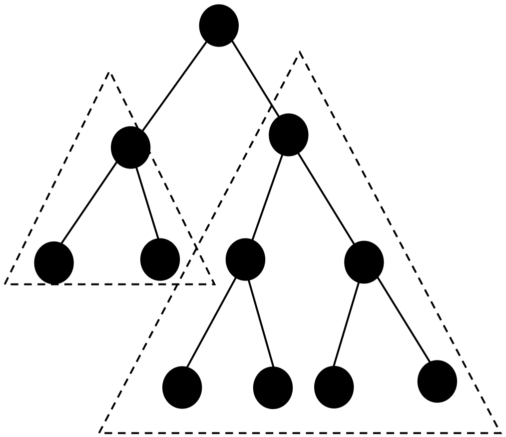
**Notation**
v(T): The number of vertices in T
h(T): The height of T
R(T): The right subtree of T
L(T): The left subtree of T
2**x: $2^x$

- Need to prove v(T) ≤ 2**(h(T)+1) -1

Basis Step) ?

General form of a full binary tree



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Induction Step) ?

# Prove $v(T) \leq 2^{**}(h(T)+1) -1$

**Basis Case)** Need to prove $v(T) \leq 2^{**}(h(T)+1) -1$

But $h(T) = 0$, $v(T) = 1$ So the claim is true.

**Inductive Case)**

Let T be a general form full binary tree with $h(T) = k$.

We want to prove that the claim for T.

Assume that for $h(T) < k$, the claim is true.

Then $v(L(T)) \leq 2^{**}(h(L(T)) + 1) – 1$ -- by induction hypothesis

and $v(R(T)) \leq 2^{**}(h(R(T)) + 1) – 1$ -- by induction hypothesis

So $v(T) = v(L(T)) + v(R(T)) + 1$

$\leq \{2^{**}(h(L(T)) + 1) – 1\} + \{2^{**}(h(R(T)) + 1) – 1\} +1$ -- from induction hypothesis

assuming without loss of generality $h(L(T)) \geq h(R(T))$

$\leq 2^*\{2^{**}(h(L(T)) + 1) – 1\} +1$

since $h(T) = h(L(T)) + 1$

$\leq 2^*\{2^{**}h(T) – 1+1$

$= 2^*2^{**}h(T) - 1$

$\leq 2^{**}(h(T)+1) -1$

By structural induction, the claim has been proved.

Notation

v(T): The number of vertices in T

h(T): The height of T

R(T): The right subtree of T

L(T): The left subtree of T

2**x: $2^x$

# Quiz 15-1

For which of the following proof problems the first and the second principles of mathematical induction are not suitable and therefore structural induction need be used?

(a) To prove that every nonprime numbers $>$ 2 is a product of prime numbers.

(b) To prove that the string reverse operation is correct.

(c) To prove that the maximum number of nodes of a binary tree is $2^h$ -1 where h is the height of the tree.

(d) To prove that, in Example 1 of this lecture, every element in X is divisible by 4.