Ch 9. Discrete Probability (2)

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Ch 7. Discrete Probability

7.1 An Introduction to Discrete Probability

7.2 Probability Theory

7.3 Bayes' Theorem

7.4 Expected Value and Variance

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3. Conditional probability, Bayes' theorem

Definition

The *conditional probability* of E1 given E2, written *P(E1 | E2)*, is the probability that E1 would be true, given that E2 is true.

"P(E1 | E2)" reads "probability of E1 given E2".

Which of the following is true?

(1) $P(E1 \cap E2) \le P(E1 \mid E2)$ Yes

(2) $P(E1 \cap E2) \ge P(E1 \mid E2)$ No

98 students were surveyed and ...

		objective social class position		
		working class	upper middle class	Total
subjective social class identity	poor	0	0	0
	working class	8	0	8
	middle class	32	13	45
	upper middle	8	37	45
	upper class	0	0	0
	Total	48	50	98

Source) Dr. Mine Çetinkaya-Rundel, Duke University

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What is the probability that
a student **is** objectively in the working class
and associates with (= "subjectively belongs to") the upper middle class?

Source) Dr. Mine Çetinkaya-Rundel, Duke University

98 students were surveyed and ...

		objective social class position		
		working class	upper middle class	Total
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	Total	48	50	98

What is the probability that
a student **who is** objectively in the working class
associates with (= "subjectively belongs to") the upper middle class?

Source) Dr. Mine Çetinkaya-Rundel, Duke University

conditional

		o[illegible] social class position		
		working class	upper middle class	Total
subjective social class identity	poor	0	0	0
	working class	8	0	8
	middle class	32	13	45
	upper middle	8	37	45
	upper class	0	0	0
	Total	48	50	98

What is the probability that a student who is objectively in the working class associates with upper middle class?

$P(\text{subj UMC} \mid \text{obj WC})$

$= 8 / 48 \approx 0.17$

counts

Source) Dr. Mine Çetinkaya-Rundel, Duke University

Bayes' Theorem

Theorem Let E1 and E2 be two events such that $P(E2) > 0$.

$$P(E1 \mid E2) = P(E1 \cap E2) \,/\, P(E2)$$

Bayes' theorem:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

		objective social class position		
		working class	upper middle class	Total
subjective social class identity	poor	0	0	0
	working class	8	0	8
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	upper class	0	0	0
	Total	48	50	98

$$P(\text{subj UMC} \mid \text{obj WC}) = \frac{P(\text{subj UMC \& obj WC})}{P(\text{obj WC})} = \frac{8/98}{48/98} = 8/48 \approx 0.17$$

probabilities

What is the probability that
a student who is objectively in the working class
associates with upper middle class?

Source) Dr. Mine Çetinkaya-Rundel, Duke University

General Product Rule

Theorem Let E1 and E2 be two events such that P(E2) > 0.

$$P(E1 \mid E2) = P(E1 \cap E2) \,/\, P(E2)$$

Corollary 1. Let E and F be two events such that P(E2) > 0. Then:

$$P(E1 \cap E2) = P(E1 \mid E2) \times P(E2)$$

		objective social class position		
		working class	upper middle class	Total
subjective social class identity	poor	0	0	0
	working class	8	0	8
	middle class	32	13	45
	upper middle	8	37	45
	upper class	0	0	0
	Total	48	50	98

P(E1 ∩ E2)
= 8/98

		objective social class position		
		working class	upper middle class	Total
subjective social class identity	poor	0	0	0
	working class	8	0	8
	middle class	32	13	45
	upper middle	8	37	45
	upper class	0	0	0
	Total	48	50	98

P(E1 | E2)
X
P(E2)

= 8/48
x
48/98

Corollary 1. Let E and F be two events such that P(E2) > 0. Then:

$$P(E1 \cap E2) = P(E1 \mid E2) \times P(E2)$$

☛ Called the *General Product Rule*

Cf 1)

The Multiplication Principle

Let A and B be finite sets. The number of elements (i.e., ordered pairs) in $A \times B$ is $|A| \cdot |B|$. So there are $|A| \cdot |B|$ ways to choose two items in sequence, with the first item coming from A and the second item from B.

Cf 2) $P(E1 \cap E2) = P(E1) \times P(E2)$ - Product Rule for Independent Events

Corollary 1. Let E and F be two events such that P(E2) > 0. Then:

$$P(E1 \cap E2) = P(E1 \mid E2) \times P(E2)$$

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The Multiplication Principle

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Cf 2) $P(E1 \cap E2) = P(E1) \times P(E2)$ - Product Rule for Independent Events

- What if E1 and E2 are dependent?

☛ E1 affects E2 or E2 affects E1,
although we may not know the exact causal relationship.

- Sometimes conditional probabilities are easier to find out.

Example

Assume the probability of getting a flu is 0.2

the probability of having a high fever given the flu is 0.9

Would it be easier to find out the probability of a person having a high fever?

What is the probability of getting a flu with fever?

$P(\text{flu} \cap \text{fever}) = P(\text{fever} \mid \text{flu}) \times P(\text{flu}) = 0.9 \times 0.2 = 0.18$

Switching the conditioning events

Corollary 2. Let E1 and E2 be two events such that $P(E2) > 0$. Then

$$P(E1 \mid E2) = P(E2 \mid E1) \times P(E1) / P(E2)$$

Bayes' Theorem

$$P(E1 \mid E2) = P(E1 \cap E2) \mathbin{\boldsymbol{/}} P(E2)$$

Proof

$$\begin{aligned} P(E1 \mid E2) &= P(E1 \cap E2) / P(E2) && \text{-- Bayes' Theorem} \\ &= \{P(E1 \cap E2) / P(E1)\} \times \{P(E1) / P(E2)\} \\ &= \{P(E2 \cap E1) / P(E1)\} \times \{P(E1) / P(E2)\} && \text{-- Commutativity of } \cap \\ &= P(E2 \mid E1) \times \{P(E1) / P(E2)\} && \text{-- Bayes' Theorem} \end{aligned}$$

$P(E1 \mid E2) = P(E2 \mid E1) \times P(E1) / P(E2)$

Example

Assume the probability of getting a flu is 0.2

the probability of getting a fever is 0.3

the probability of having a high fever given the flu is 0.9

What is the probability of having a flu given the fever?

$$P(E1 \mid E2) = P(E2 \mid E1) \times P(E1) / P(E2)$$

Example

Assume the probability of getting a flu is 0.2

the probability of getting a fever is 0.3

the probability of having a high fever given the flu is 0.9

What is the probability of having a flu given the fever?

$$
\begin{aligned}
P(\text{flu} \mid \text{fever}) &= P(\text{fever} \mid \text{flu}) \times P(\text{flu}) / P(\text{fever}) \\
&= 0.9 \times 0.2/0.3 \\
&= 0.18/0.3 \\
&= 0.6
\end{aligned}
$$

$$
\begin{aligned}
P(\text{flu} \cap \text{fever}) &= P(\text{fever} \mid \text{flu}) \times P(\text{flu}) \\
&= 0.9 \times 0.2 \\
&= 0.18
\end{aligned}
$$

Alternative Form of Bayes' Theorem

Corollary 2. $P(E1 \mid E2) = P(E2 \mid E1) \times P(E1) \,/\, P(E2)$

Corollary 3. Let E1 and E2 be two events such that P(E1), P(E2) > 0. Then

$$P(E1 \mid E2) = \frac{P(E2 \mid E1) \times P(E1)}{\underbrace{P(E2 \mid E1) \times P(E1)}_{\parallel \atop P(E1 \cap E2)} + \underbrace{P(E2 \mid \sim E1) \times P(\sim E1)}_{\parallel \atop P(\sim E1 \cap E2)}}$$

$$P(E1 \cap E2) + P(\sim E1 \cap E2) = P(E2)$$

Alternative Form of Bayes' Theorem

Corollary 2. $P(E1 \mid E2) = P(E2 \mid E1) \times P(E1) \,/\, P(E2)$

Corollary 3. Let E1 and E2 be two events such that P(E1), P(E2) > 0. Then

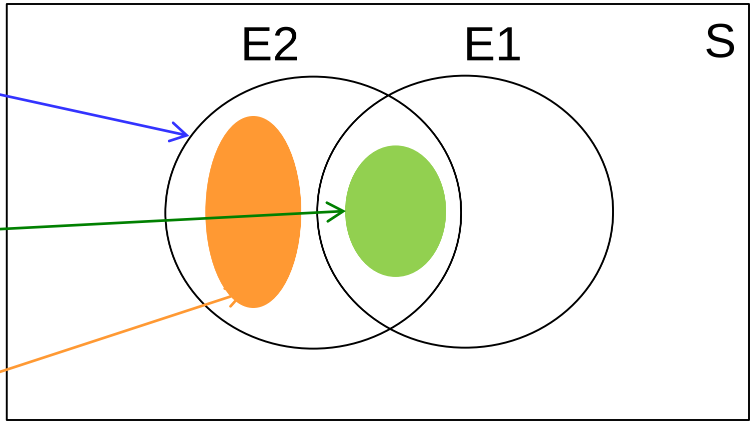
$$P(E1 \mid E2) = \frac{P(E2 \mid E1) \times P(E1)}{\underbrace{P(E2 \mid E1) \times P(E1)}_{= P(E1 \cap E2)} + \underbrace{P(E2 \mid \sim E1) \times P(\sim E1)}_{= P(\sim E1 \cap E2)}}$$

$$P(E1 \cap E2) + P(\sim E1 \cap E2) = P(E2)$$

P(E2)

$P(E1 \cap E2)$

$P(\sim E1 \cap E2)$



$$P(E1 \mid E2) = \frac{P(E2 \mid E1) \times P(E1)}{P(E2 \mid E1) \times P(E1) + P(E2 \mid \sim E1) \times P(\sim E1)}$$

Example

One person in 100,000 has a rare disease for which there is an accurate diagnostic test.

This test is correct 99.0% when given to a person selected at random who has disease;

it is correct (i.e. tests negative) 99.5% when given to a person selected at random who does not have disease.

What is the probability that a person who tests positive for the disease has the disease?

$$P(E1 \mid E2) = \frac{P(E2 \mid E1) \times P(E1)}{P(E2 \mid E1) \times P(E1) + P(E2 \mid \sim E1) \times P(\sim E1)}$$

Example

One person in 100,000 has a rare disease for which there is an accurate diagnostic test.

This test is correct 99.0% when given to a person selected at random who has disease;

it is correct (i.e. tests negative) 99.5% when given to a person selected at random who does not have disease.

What is the probability that a person who tests positive for the disease has the disease?

E1: event that a person selected at random has the disease
E2: event that a person selected at random tests positive for the disease
P(E1) = 1/100,000 = 0.00001
P(E2|E1) = 0.99;
P(~E2|~E1) = 0.995
P(E1|E2) = ?
Need P(E2|E1), P(E2|~E1), P(E1), P(~E1).
P(~E1) = 1 - 0.00001 = 0.99999
P(E2|~E1) = 1 - P(~E2|~E1) = 1 - 0.995 = 0.005

$$P(E1 \mid E2) = \frac{0.99 \times 0.00001}{0.99 \times 0.00001 + 0.005 \times 0.99999} \approx 0.002$$

4. Independence

Independent Events

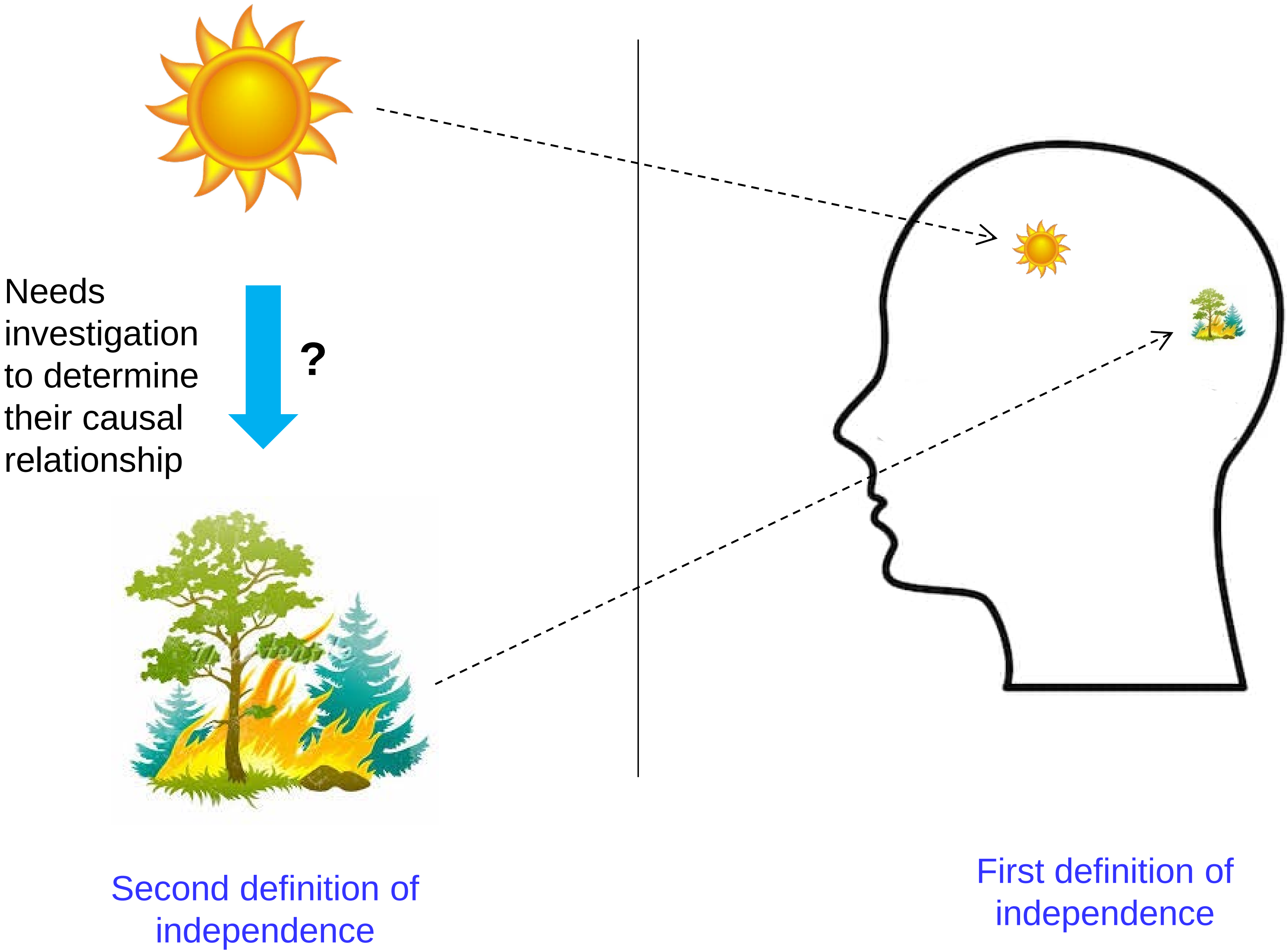
- Two events are *independent* if knowing the outcome of one provides no useful information about the outcome of the other. (Definition in terms of our knowledge)

Is this a good definition?

Compare the definition above with the following:

- Two events are *independent* if the outcome of one event does not depend on the outcome of the other.

  (Definition in terms of the real world.)


Needs investigation to determine their causal relationship
?
Second definition of independence
First definition of independence

Test for independence (Test 1)

Two events E1 and E2 are independent if and only if

$$P(E1 \cap E2) = P(E1) \times P(E2)$$

<= Compare this with the General Product Rule

$$P(E1 \cap E2) = P(E1 \mid E2) \times P(E2)$$

Test for independence (Test 1)

Two events E1 and E2 are independent if and only if

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<= Compare this with the General Product Rule

$$P(E1 \cap E2) = P(E1 \mid E2) \times P(E2)$$

Example 1.

E1: Flip a coin

E2: Roll a die.

$$P(\text{heads} \cap \text{"1"}) = P(\text{heads}) \times P(\text{"1"})$$

☛ E1 and E2 are independent events.

Test 1 : $P(E1 \cap E2) = P(E1) \times P(E2)$

Example 2.

Sample Space: The families with three children.
E1: The family has both boy and girl.
E2: The family has at most one boy.
Are E1 and E2 independent?

Sample Space = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} => size = 8
E1 = {BBG, BGB, GBB, BGG, GBG, GGB} => size = 6
E2 = {GGG, GGB, GBG, BGG} => size = 4
E1 ∩ E2 = {GGB, GBG, BGG} => size = 3

$P(E1 \cap E2) = 3/8$ and $P(E1) \times P(E2) = 6/8 \times 4/8 = 3/8$

☛ E1 and E2 are independent.

Test for independence (Test 2)

The two events E1 and E2 are independent if and only if

$$P(E1 \mid E2) = P(E1)$$

Product rule for independent events:

If A and B are independent, $P(A \text{ and } B) = P(A) \times P(B)$

2nd Test

Bayes' theorem:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

General product rule:

$$P(A \text{ and } B) = P(A \mid B) \times P(B)$$

Example

Test 2: $P(E1 \mid E2) = P(E1)$

E1: getting a fever

E2: getting a flu

Assume $P(E1) = 0.3$

$P(E2) = 0.2$

The probability of having a fever given the flu, i.e. $P(E1|E2) = 0.9$.

Are flu and fever independent ?

Example

Test 2: $P(E1 \mid E2) = P(E1)$

E1: getting a fever

E2: getting a flu

Assume $P(E1) = 0.3$

$P(E2) = 0.2$

The probability of having a fever given the flu, i.e. $P(E1|E2) = 0.9$.

Are flu and fever independent ?

1st Test

$P(\text{flu} \cap \text{fever}) = P(\text{fever} \mid \text{flu}) \times P(\text{flu}) = 0.9 \times 0.2 = 0.18$

$P(\text{flu}) \times P(\text{fever}) = 0.2 \times 0.3 = 0.06$

Independent or not?

2nd Test

Disjoint vs. Independent

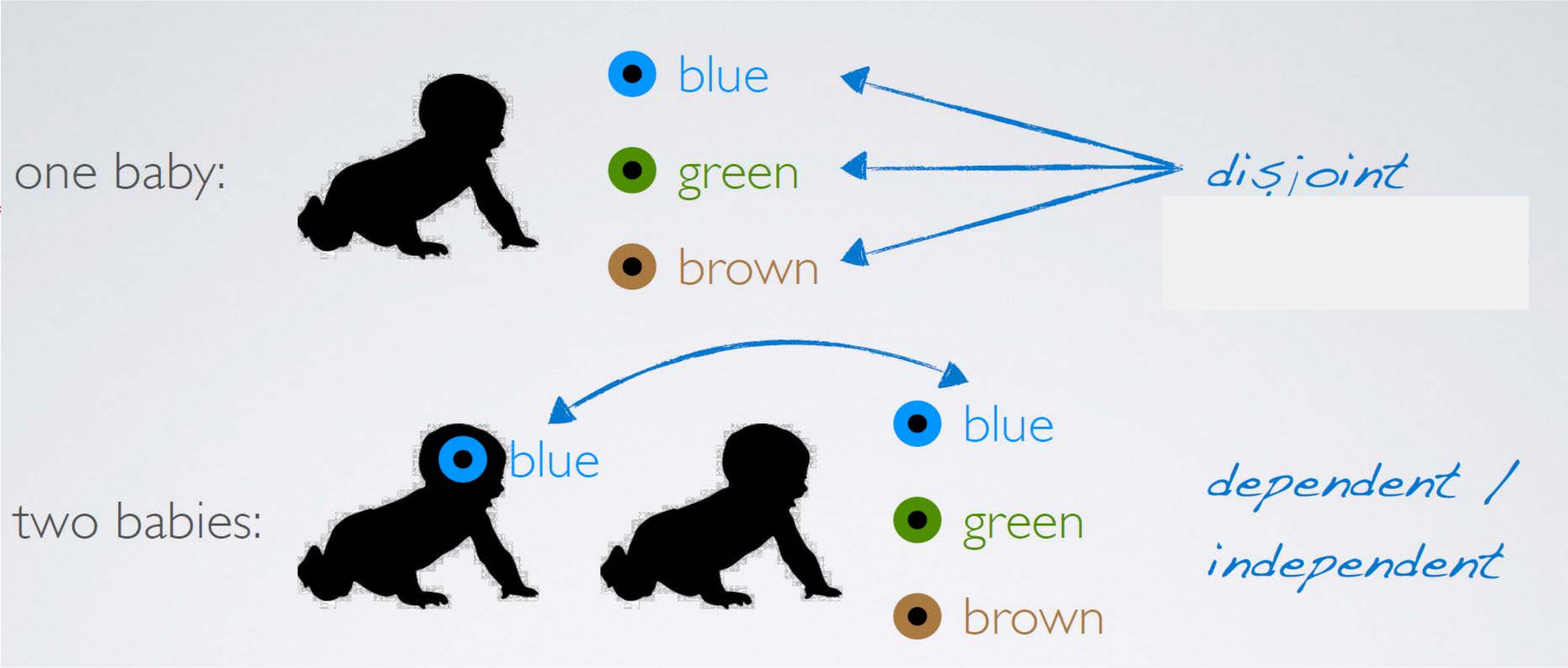
Source)
Dr. Mine Çetinkaya-Rundel
Duke University

Two events are *disjoint (mutually exclusive)* if they cannot happen at the same time

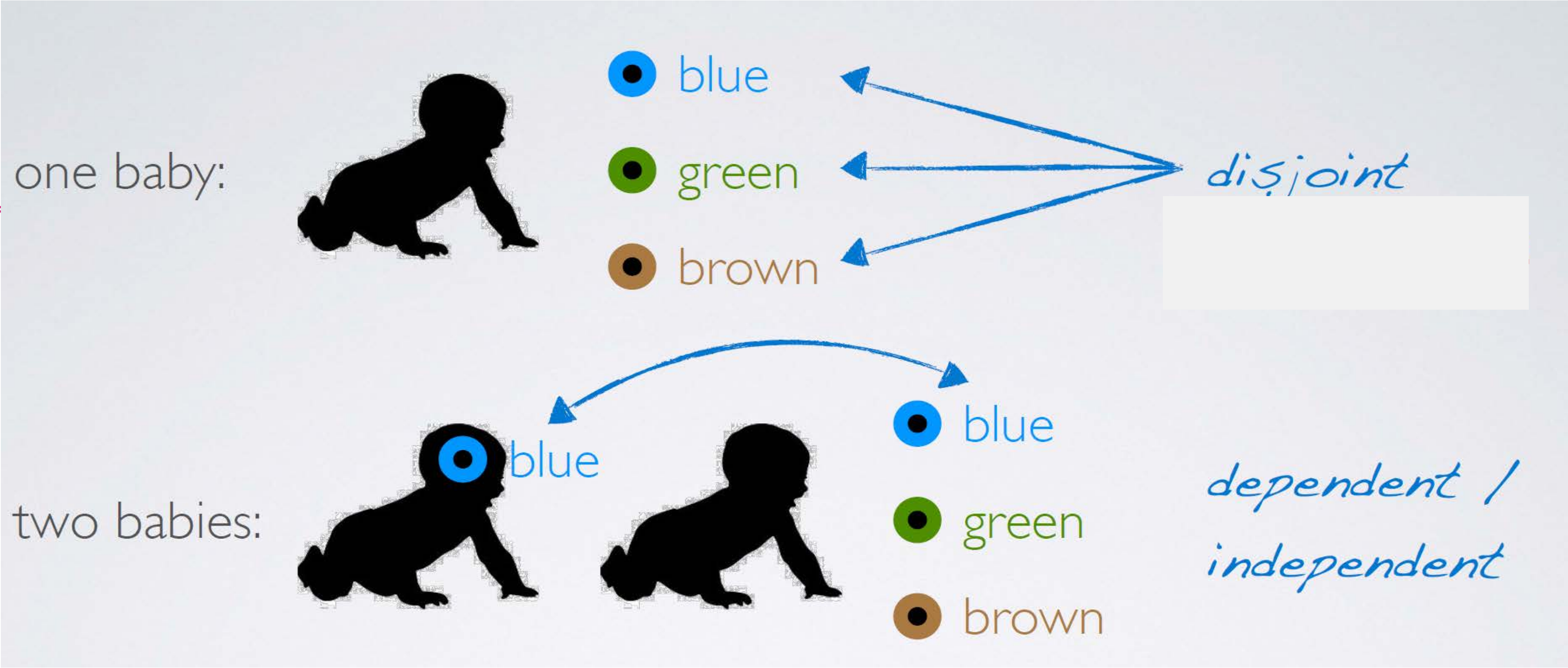
$P(A \cap B) = 0$

Two events are *independent* if the outcome of one event does not depend on the outcome of the other.

$P(A \mid B) = P(A)$



one baby:
blue
green
brown
disjoint
two babies:
blue
blue
green
brown
dependent /
independent



Questions:

- Can two disjoint events be dependent?
  Yes!
- Can two dependent events be disjoint?
  Yes!

Quiz 21-1

In the student survey example discussed at the beginning of this lecture, what is the probability that a student who is subjectively in the middle class belongs to the upper middle class?