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HW-9: Selections and Arrangements

1) The order of the heads/tails is not important (since we are interested in the # of heads, not the order of the heads), thus we need to use the definition of <u>combination</u>: Given variables n, defining # of flips of a coin, and r, defining # of heads, we get that # of ways obtaining r heads with n flips of a coin is $\boxed{C(n,r)=\binom{n}{r}=\frac{n!}{r!(n-r)!}}$ ✠ where order is not important

Thus, if we flip coin n=5 times, in $\binom{5}{1}=\frac{5!}{1!\,4!}=\boxed{5 \text{ ways}}$, we'll get exactly <u>1 head</u>. Similarly, if we flip coin for <u>n=5 times</u>, in $\binom{5}{2}=\frac{5!}{2!\,3!}=\frac{4\cdot 5}{2}=\boxed{10 \text{ ways}}$, we'll get exactly <u>2 heads</u>. ■

2) Using the formula $C(n,k)=\frac{n!}{k!(n-k)!}$ for $n \geq k \geq 0$,

considering that $n \geq r-1 \geq 0$, $\binom{n}{r-1}+\binom{n}{r}=\frac{n!}{(r-1)!(n-r+1)!}$ $+\frac{n!}{r!(n-r)!}=\frac{n!\cdot r+n!\cdot(n-r+1)}{r!(n-r+1)!}=\frac{n!(r+n-r+1)}{r!(n-r+1)!}$, hence

We found $\binom{n}{r-1}+\binom{n}{r}=\dfrac{n!\,(r+n-r+1)}{r!\,(n-r+1)!}=\dfrac{n!\,(n+1)}{r!\,(n+1-r)!}=$

$=\dfrac{(n+1)!}{r!\,(n+1-r)!}=\binom{n+1}{r}$, from the definition of combination. Thus,

$$\boxed{\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}, \text{ where } n\geq r-1\geq 0} \; \blacksquare$$

3) a) There are 5 possible grades (A, B, C, D, F) for each of the 7 students $\Rightarrow$
Here, each student from 1^{st} to 7^{th} has 5 possible ways. Therefore,

1^{st} student: 5 ways
2^{nd} student: 5 ways
$\vdots$
7^{th} student: 5 ways

using multiplication principle, (# of ways that events can occur in sequence)

There are $\underbrace{5\cdot 5\cdot\ldots\cdot 5}_{7}=5^7=\boxed{78125}$ ways

b) There are 4 possible grades (A, B, C, D) for each of the 7 students, as no student receives an F.
Moreover, only 1 of the students can receive an A.
- Student that receives an A: 7 ways (as there are 7 students)
- 1^{st} student: 3 ways (B, C, or D)
- 2^{nd} student: 3 ways (B, C, or D)

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- 3^{rd} student: 3 ways (B, C, or D)
- 4^{th} student: 3 ways (B, C, or D)
- 5^{th} student: 3 ways (B, C, or D)
- 6^{th} student: 3 ways (B, C, or D)

Combining these results, and rule of multiplication ⇒

So, using multiplication principle, (# of ways that events can occur in sequence)

we get $7 \cdot \underbrace{3 \cdot 3 \cdot \ldots \cdot 3}_{6} = 7 \cdot 3^6 = \boxed{5103}$ ways ■

4) a) Suppose we have to choose a total of 2 objects from 2 boxes, each containing n of them. This can be done in $\binom{2n}{2}$ # of ways. Now, we can choose both of them from 1 box or 1 from each box. For the 1^{st} case, there are $\binom{n}{2}$ # of ways to do it for either of the boxes, whereas in the 2^{nd} case, the product rule (multiplication rule) implies that # of ways $= \binom{n}{1} \cdot \binom{n}{1} = \left(\frac{n!}{1!\,(n-1)!}\right)^2 = n^2$. Combining both,

$$\binom{2n}{2} = 2 \cdot \binom{n}{2} + \binom{n}{1}^2 = 2\binom{n}{2} + n^2 \Rightarrow \boxed{\binom{2n}{2} = 2\binom{n}{2} + n^2}$$

(there are n objects in each box, choosing 1 from each is 2^{nd} case)

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b) From the definition, $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{(n-1)n}{2}$ and

$$2\binom{n}{2} + n^2 = \cancel{2}\cdot\frac{(n-1)n}{\cancel{2}} + n^2 = n^2 - n + n^2 = 2n^2 - n = n(2n-1)$$

$$= \frac{2n}{2}\cdot(2n-1) = \frac{(2n-1)2n}{2} = \frac{(2n-2)!\,(2n-1)2n}{2!\,(2n-2)!} =$$

$$= \frac{(2n)!}{2!\,(2n-2)!} = \binom{2n}{2} \Rightarrow \boxed{2\binom{n}{2} + n^2 = \binom{2n}{2} \text{ Becomes true}}$$

from using algebraic manipulations and definition of "combination" formula ■

Counting & Counting with Functions

1) There are 250 different classes, while there are 12 different time slots $\Rightarrow$ $N = 250, K = 12$

Generalized pigeonhole principle: When N objects are placed into K boxes, then there exists a box with at least $\lceil \frac{N}{K} \rceil$ objects

$\lceil \frac{N}{K} \rceil = \lceil \frac{250}{12} \rceil = \lceil 20.833... \rceil = 21$, By the generalized pigeonhole principle, there are, thus, at least 21 classes that need to occur during the same time slot, and thus, the minimum # of classrooms needed to accommodate all the classes is 21 $\Rightarrow$ $\boxed{21 \text{ classrooms}}$ ■

a) There are 100 lottery tickets of which 12 are winning tickets, which implies that there are $100-12 = 88$ losing tickets.

We then want to divide the 88 tickets between the 13 groups, between consecutive winning tickets (including the group before all winning tickets and the group after all winning tickets)

$N=88, K=13 \Rightarrow$ Generalized pigeonhole principle:

When N objects are placed into K boxes, then there exists a box with at least $\lceil \frac{N}{K} \rceil$ objects.

$\lceil \frac{N}{K} \rceil = \lceil \frac{88}{13} \rceil = \lceil 6.769... \rceil = \boxed{7} \Rightarrow$ From this principle, there are, thus, at least 7 losing tickets in the same group $\Rightarrow$ Hence, there must be at least $l=7$ losing tickets in a row $\boxed{l=7}$ ∎

b) We want to select 4 of the 4 distinct squares of stained glass $\Rightarrow n=4, r=4$. The order of the squares matters, because a different order results in a different window, and thus we should use the definition of permutation: $P(4,4) = \frac{4!}{0!} = 24$

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However, some of these 24 windows will be the same when flipping or rotating the window

For each window, there are 2 windows that are identical when rotating one of the windows by 180°; there are 2 windows identical when flipping one of the windows vertically, and there are 2 windows identical when flipping one of the windows horizontally

Moreover, there are then $2 \cdot 2 \cdot 2 = 8$ identical windows for each window.

This then implies that there are $\frac{24}{8} = 3$ unique windows $\Rightarrow$ $\boxed{3 \text{ different windows}}$ ■

5) From Selection Principle, we know that # of ways to choose a subset of r elements from a set of n elements is $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

a) Thus, any set with n elements have exactly $\binom{n}{k}$ # of subsets with k elements. Since $n=10$, and we have to assign k to 0, 1, 2, 3, 4 to get fewer than 5 elements $\Rightarrow$ # of subsets with fewer than 5 elements

$= \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} = \frac{10!}{0!10!} + \frac{10!}{1!9!} + \dots + \binom{10}{4}$

$\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4}=\frac{10!}{0!10!}+\frac{10!}{1!9!}+\frac{10!}{2!8!}+$

$+\frac{10!}{3!7!}+\frac{10!}{4!6!}=1+10+\frac{9\cdot10}{2}+\frac{8\cdot9\cdot10}{6}+\frac{7\cdot8\cdot9\cdot10}{24}=$

$=11+45+10\cdot12+70\cdot3=11+45+120+210=56+$

$+330=\boxed{386}$ ⊠

b) Since any set with n elements have exactly $\binom{n}{k}$ # of subsets with k elements, we take $n=10$, and $k>7$ ⇒ # of subsets with more than 7 elements $=$

$=\binom{10}{8}+\binom{10}{9}+\binom{10}{10}=\frac{10!}{8!2!}+\frac{10!}{9!1!}+\frac{10!}{10!0!}=\frac{9\cdot10}{2}+$

$+10+1=45+11=\boxed{56}$ ⊠ where we took $k=8,9,10$

c) A subset of r elements from a set with n elements is an r-combinations (subset that contains r of the n elements $\longrightarrow C(n,r)=\binom{n}{r}$) and thus, we are interested in the # of r-combinations of a set with $n=10$ elements that contain an odd # of elements ⇒ $n=10$; $r=1,3,5,7,$ or 9. Then, # of such subsets $=\sum_{r\in\{1,3,5,7,9\}} C(10,r)=\binom{10}{1}+\binom{10}{3}+\binom{10}{5}$ $+\binom{10}{7}+\binom{10}{9}$

$$\binom{10}{1}+\binom{10}{3}+\binom{10}{5}+\binom{10}{7}+\binom{10}{9} = \frac{10!}{1!\,9!}+\frac{10!}{3!\,7!}+\frac{10!}{5!\,5!}+\frac{10!}{7!\,3!}+\frac{10!}{9!\,1!} = 10+\frac{8\cdot 9\cdot 10}{6}+\frac{6\cdot 7\cdot 8\cdot 9\cdot 10}{120}+\frac{8\cdot 9\cdot 10}{6}$$

$$+10 = 10+10\cdot 12+42\cdot\frac{720}{120}+10\cdot 12+10 = 10+120+42\cdot 6+120+10 = 20+240+252 = 260+252 = 512$$

Thus, there are $\boxed{512}$ subsets with an odd # of elements of a set with 10 elements ∎

3) a) For any function from a set of 3 elements to a set of 4 elements, each element of the 3-set is assigned with an element of the 4-set uniquely, i.e. for that particular function, no other element of the 4-set can be assigned to that specific element of the 3-set. For each of these assigning, other elements of the 3-set can also be assigned with any element from the second set w/o any restriction. So, the task of assigning a value to an element of the 3-set can be done in 4 different ways, for each element of the 1st set.

Hence, according to the product rule, # of functions, i.e. # of different tasks of assigning values is the product $4 \cdot 4 \cdot 4 = 64$ times $\Rightarrow$ $\boxed{\# \text{ of functions} = 64}$

b) If the function is one-to-one, then any element of the 4-set can be assigned to only 1 element from the 3-set. This decreases the # of functions drastically. Suppose, the values are assigned to the elements of the 3-set one by one, then the first element can have 4 options, the second can have 3 options, and so on... in general, the i^{th} element can have only $5-i$ options, or the i^{th} task can be performed in $5-i$ # of different ways. Hence, according to the product rule, the total # of one-to-one functions is $\prod_{i=1}^{3}(5-i) = (5-1)(5-2)(5-3)$
$= 4 \cdot 3 \cdot 2 = 24$ $\Rightarrow$ $\boxed{\# \text{ of one-to-one functions} = 24}$

c) A function is onto means when its image and co-domain are equal, and thus the co-domain can't be larger than the domain. Hence, not a single onto function is possible in this case as $4 > 3$
(we map each element to a single element of the co-domain)

(there's always some element that's left over in the co-domain since $4>3$) $\Rightarrow$ $\boxed{\#\text{ of onto functions} = 0}$ $\blacksquare$

4) Take the equation $x_1 + x_2 + \ldots + x_n = m$, where $n \geq 1$ and $m \geq 0$. Then, there are $\binom{m+n-1}{n-1}$ number of solutions to this equation, where $x_1, x_2, \ldots, x_n$ must be non-negative integers

- Such a solution corresponds to a distribution of m units among the n variables $x_1, x_2, \ldots, x_n$. For example, the solution $x_1 = m-1$, $x_2 = 1$, $x_3 = 0, \ldots, x_n = 0$ amounts to dividing up m 1's into groups as follows

$$\underbrace{1\,1 \ldots 1}_{m-1} \,|\, 1 \,|\, | \ldots \,|\,|$$

We can view this division into groups as a string containing $(n-1)$ $|$'s and m 1's; every such string defines a different solution to the equation, and all solutions can be represented this way. So, we just need to count the strings of this type. By the method of "combinations", and selection principle the # of possible strings (and also the # of possible sol's) is $C(m+n-1, n-1)$ ✓

Subtraction rule: If an event can occur either in m ways or in n ways, (overlapping), the # of ways the event can occur is then (m+n) decreased by the # of ways that the event can occur commonly to the 2 different ways

- Definition permutation (order is important)

No repetition allowed: $P(n,r) = \frac{n!}{(n-r)!}$

Repetition allowed: $n^r \rightarrow$ # of r-permutations of a set of n objects with repetition allowed

- Definition combination (order is not important)

No repetition allowed: $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Repetition allowed: $C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!} \rightarrow$ there are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed (proof is similar to what we did in previous page, where we make a list of n-1 bars and r stars. The (n-1) bars are used to mark off n different cells, with the i^{th} cell containing a star for each time the i^{th} element of the set occurs in the combination)

(As we see, each different list containing $(n-1)$ bars and r stars correspond to an r-combination of the set with n elements, when repetition is allowed. The # of such lists is $C(n-1+r, r)$, because each list corresponds to a choice of the r positions to place the r stars from the $n-1+r$ positions that contain r stars and $(n-1)$ bars)

a) $x_1+x_2+x_3=17$, where $x_1>1$, $x_2>3$, and $x_3>3$

Let's assume $\underline{y_1=x_1-2,\ y_2=x_2-4,\ y_3=x_3-4}$ (Note that we then get $\underline{y_1, y_2, y_3 \geq 0}$) $\Rightarrow y_1+y_2+y_3=$

$= x_1+x_2+x_3-10=17-10=7$, the order of the sol'n does not matter (since different order leads to the same sol'ns), thus we need to use definition of "combination"

$\underline{y_1+y_2+y_3=7,\ y_1, y_2, y_3 \geq 0}$ $\Rightarrow$ We need to select 7 values (ones, such that we obtain sum of 7) from 3 boxes

$\underline{n=3, r=7}$ $\Rightarrow$ repetition's allowed, $\binom{n+r-1}{r}=\binom{9}{7}=$

$=\frac{9!}{7!2!}=\frac{\cancel{8}^{4}\cdot 9}{\cancel{2}}=36 \Rightarrow \boxed{36}$ solutions are there

b) $x_1+x_2+x_3=17$, where $x_1<6$; $x_3>5$

of solutions with $x_3 > 5$

Let's then redefine $x_3' = x_3 - 6$, which will then result in non-negative variables $\Rightarrow x_1 + x_2 + x_3' = x_1 + x_2 + x_3 - 6 =$
$= 17 - 6 = 11$, $x_1 + x_2 + x_3' = 11$ and $x_1, x_2, x_3' \geq 0$

We want to select 11 indistinguishable objects from 3 distinguishable boxes (variables) $\Rightarrow n = 3, r = 11$

Since repetition is allowed, $C(3 + 11 - 1, 11) = \binom{13}{11} =$
$= \frac{13!}{11!\,2!} = \frac{12 \cdot 13}{2} = 78$

of solutions with $x_1 \geq 6$ and $x_3 > 5$

Let's redefine $x_1' = x_1 - 6$, and $x_3' = x_3 - 6$, which will then result in non-negative variables $\Rightarrow x_1' + x_2 + x_3' =$
$= x_1 + x_2 + x_3 - 12 = 17 - 12 = 5$, $x_1' + x_2 + x_3' = 5$ with $x_1', x_2, x_3' \geq 0$

We want to select 5 indistinguishable objects from 3 distinguishable boxes (variables) $\Rightarrow n = 3, r = 5$. Since repetition is allowed, $C(5 + 3 - 1, 5) = \binom{7}{5} = \frac{7!}{5!\,2!} =$
$= \frac{6 \cdot 7}{2} = 21$

# of solutions with $x_1<6$ and $x_3>5$

We note that there are 78 solutions with $x_3>5$, and there are 21 solutions with $x_1 \geq 6$ and $x_3>5$, thus there are $78-21=57$ solutions with $x_1<6$ and $x_3>5$

$\boxed{57 \text{ solutions when } x_1<6 \text{ and } x_3>5}$ #

c) $x_1+x_2+x_3=17$, with $x_1<4$, $x_2<3$, $x_3>5$

# of solutions with $x_3>5$

Assume $x_3'=x_3-6$, which will then result in non-negative variables. $x_1+x_2+x_3'=x_1+x_2+x_3-6=17-6=11 \Rightarrow$

$x_1+x_2+x_3'=11$ with $x_1, x_2, x_3' \geq 0$. We want to select 11 indistinguishable objects from 3 distinguishable boxes (variables)

$n=3$, $r=11 \Rightarrow$ repetition is allowed, $C(3+11-1, 11)=$

$$=\binom{13}{11}=\frac{13!}{11!2!}=\frac{12 \cdot 13}{2}=78$$

# of solutions with $x_1 \geq 4$ and $x_3>5$

Let's redefine $x_1'=x_1-4$ and $x_3'=x_3-6$, which will result in non-negative $\Rightarrow x_1'+x_2+x_3'=x_1+x_2+x_3-10=17-10=7$

$x_1'+x_2+x_3'=7$, with $x_1', x_2, x_3' \geq 0$. We want to select 7 indistinguishable objects from 3 distinguishable boxes (variables) $\Rightarrow$

$n=3$, $r=7$, repetition is allowed $\Rightarrow C(3+7-1, 7)=\binom{9}{7}=\frac{9!}{7!2!}=\frac{8 \cdot 9}{2}=36$

# of solutions with $x_2 \geq 3$ and $x_3 > 5$

Let's redefine $x_2' = x_2 - 3$ and $x_3' = x_3 - 6 \Rightarrow x_1 + x_2' + x_3' =$
$= x_1 + x_2 + x_3 - 9 = 17 - 9 = 8$, where $x_1 + x_2' + x_3' = 8$ and
$x_1, x_2', x_3' \geq 0 \Rightarrow$ We want to select 8 indistinguishable objects from 3 distinguishable boxes (variables) $\boxed{\begin{matrix} n=3 \\ r=8 \end{matrix}}$

Repetition is allowed $\Rightarrow \binom{3+8-1}{8} = \binom{10}{8} = \frac{10!}{8!2!} = \frac{9 \cdot 10}{2} = 45$

# of solutions with $x_1 \geq 4$, $x_2 \geq 3$, $x_3 > 5$

Let's redefine $x_1' = x_1 - 4$, $x_2' = x_2 - 3$, $x_3' = x_3 - 6$, then
$x_1' + x_2' + x_3' = x_1 + x_2 + x_3 - 13 = 17 - 13 = 4 \Rightarrow x_1' + x_2' + x_3' = 4$
with $x_1', x_2', x_3' \geq 0 \Rightarrow$ We want to select 4 indistinguishable objects from 3 distinguishable boxes (variables) $\boxed{\begin{matrix} n=3 \\ r=4 \end{matrix}}$

Repetition is allowed $\Rightarrow \binom{3+4-1}{4} = \binom{6}{4} = \frac{6!}{4!2!} = \frac{5 \cdot 6}{2} = 15$

# of solutions with ($x_1 \geq 4$ or $x_2 \geq 3$) and $x_3 > 5$

There are 36 solutions with $x_1 \geq 4$ and $x_3 > 5$,
45 solutions with $x_2 \geq 3$ and $x_3 > 5$,
15 solutions with $x_1 \geq 4$, and $x_2 \geq 3$, and $x_3 > 5$.
By the subtraction rule (inclusion-exclusion), there're then $36 + 45 - 15 = 36 + 30 = \boxed{66 \text{ solutions with } (x_1 \geq 4 \text{ or } x_2 \geq 3) \text{ and } x_3 > 5}$

# of solutions with $x_1 < 4$ and $x_2 < 3$ and $x_3 \geq 5$

There are 78 solutions with $x_3 \geq 5$, while 66 of the 78 solutions have $x_1 \geq 4$ or $x_2 \geq 3$, and thus there are $78 - 66 = 12$ solutions with $x_1 < 4$ and $x_2 < 3$ and $x_3 \geq 5$

12 such solutions ■

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Homework 9 - Basic Counting Techniques

5) We want to select 8 of the 8 distinct squares of stained glass => $n=8, r=8$ The order of the squares matters, because a different order results in a different window, and thus we should use the definition of permutation => $P(8,8) = \frac{8!}{0!} = 8! = 720 \cdot 56 = 40320$

However, some of these 40320 windows will be the same when flipping or rotating the window

For each window, there are 2 windows that are identical when rotating one of the windows by 180°, there are 2 windows identical when flipping one of the windows vertically, and there are 2 windows identical when flipping one of the windows horizontally

Moreover, there are then $2 \cdot 2 \cdot 2 = 8$ identical windows for each window

This then implies that there are $\frac{40320}{8} = 5040$ unique windows => $\boxed{5040 \text{ ways}}$

6) There are 3 natural numbers, while there are 2 possible parities for each natural number (even or odd)

$N=3, K=2 \Rightarrow$ Generalized pigeonhole principle: When N objects are placed into K boxes, then there exists a box with at least $\lceil \frac{N}{K} \rceil$ objects

$\lceil \frac{N}{K} \rceil = \lceil \frac{3}{2} \rceil = \lceil 1.5 \rceil = 2$, from this principle, there are thus, at least 2 natural numbers with same parity

However, sum of 2 even natural numbers = even, and sum of 2 odd natural numbers = even, which implies that 2 natural numbers with same parity have an even sum $\Rightarrow$ At least 2 of the numbers have same parity, and their sum will be even

2) We want to rearrange the 9 letters in the word INANENESS, which contains 1 I, 3 N's, 1 A, 2 E's, and 2 S's $\Rightarrow$ $n=9$, $n_1=1$, $n_2=3$, $n_3=1$, $n_4=2$, $n_5=2$

where $9=n=1+3+1+2+2$

$=n_1+n_2+n_3+n_4+n_5$

$\Rightarrow n=n_1+n_2+n_3+n_4+n_5$

Permutations with sets of Indistinguishable Objects:

If there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., n_k indistinguishable objects of type k and $n = n_1 + n_2 + ... + n_k$, then # of distinguishable permutations of n objects is $= \dfrac{n!}{n_1!\, n_2! \ldots n_k!}$

Plugging the values, we get $\dfrac{9!}{1!\,3!\,1!\,2!\,2!} =$

$= \dfrac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{\cancel{2} \cdot \cancel{2}} = 30 \cdot 56 \cdot 9 = 1680 \cdot 9 = 15120$

Thus, there are $\boxed{15120 \text{ ways}}$ to rearrange the letters in INANENESS ∎

Note: Because some of the letters of INANENESS are the same, the answer's not given by the # of permutations of 9 letters. This word contains 1 I, 3 Ns, 1 A, 2 Es, and 2 Ss. To determine the # of different strings that can be made by reordering the letters, first note that the three Ns can be placed among the 9 positions in $C(9,3)$ different ways, leaving 6 positions free. Then, the two Es can be placed in $C(6,2)$ ways, leaving 4 free positions. The two Ss can be placed in $C(4,2)$ ways, leaving 2 free positions. The I can be placed in $C(2,1)$ ways, leaving just 1 position free. Hence, A can be placed in $C(1,1)$ way. Consequently, from the product rule, # of different strings that can be made is =

$$= C(9,3)\cdot C(6,2)\cdot C(4,2)\cdot C(2,1)\cdot C(1,1) =$$

$$= \frac{9!}{3!\,6!}\cdot\frac{6!}{2!\,4!}\cdot\frac{4!}{2!\,2!}\cdot\frac{2!}{1!\,1!}\cdot 1 = \frac{7\cdot 8\cdot 9}{6}\cdot\frac{5\cdot 6}{2}\cdot\frac{3\cdot 4}{2}\cdot 2 =$$

$$= 7\cdot 12\cdot 15\cdot 6\cdot 2 = 84\cdot 90\cdot 2 = 84\cdot 180 = 15120 \text{ ways}$$

3) There are $2\cdot 2\cdot 2\cdot 2 = 2^4 = 16$ binary strings of length 4, ranging from 0000 to 1111 (we used product rule here). A code that represents each of the 10 decimal digits 0-9 with a different binary sequence of length 4 can therefore be concocted in any of

$$16\cdot 15\cdot\ldots\cdot 7 = \frac{16!}{(16-10)!} = P(16,10) = \frac{16!}{6!} = 7\cdot 8\cdot\ldots\cdot 15\cdot 16 \text{ ways.}$$

That's, we pick one of the 16 binary strings to represent the decimal digit 0, one of the 15 remaining

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strings to represent the decimal digit 1, one of the 14 remaining strings for the decimal digit 2, and so on up to 7 choices for the decimal digit 9 (Note, there will be six binary strings left over). Therefore, we get $\boxed{16\cdot 15\cdot 14\cdot\ldots\cdot 7 = P(16,10)}$ possible encodings ■
$\boxed{29059430400}$

1) We prove the following property:

<u>Counting Subsets of a Finite Set:</u> Number of different subsets of a finite set S is $2^{|S|}$

- Let S be a finite set. List the elements of S in arbitrary order. Recall that there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. Namely, a subset of S is associated with the bit string with a "1" in the i^{th} position if the i^{th} element in the list is in the subset, and a "0" in this position otherwise. By the product rule, there are $2^{|S|}$ bit strings of length $|S| = n$. Hence, we get that $\boxed{|P(S)| = 2^{|S|}}$ ✓

Moreover, we know that if $A_1, A_2, \ldots, A_n$ - finite sets, then $|A_1 \times A_2 \times \ldots \times A_n| = |A_1|\cdot|A_2|\cdot\ldots\cdot|A_n|$ (Theorem 2 on §4.1.2)

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From the definition, a binary relation over sets X and Y is a subset of the Cartesian product $X \times Y$; that is, it is a set of ordered pairs (x, y) consisting of elements x in X and y in Y.

Given $|A| = m$, $|B| = n$, with A, B - finite sets, we know that, from aforementioned properties, that every subset of $A \times B$ is a binary relation. From the given formula, $|A \times B| = |A| \cdot |B| = mn$, then $\boxed{A \times B \text{ has } mn \text{ elements}}$

Considering first property, we get that there are exactly 2^{mn} subsets of $A \times B \Rightarrow \boxed{|P(A \times B)| = 2^{|A \times B|} = 2^{mn}}$

Thus, $\boxed{\text{there are } 2^{mn} \text{ binary relations from } A \text{ to } B}$

■

4) From the selection principle and definition of "combination", the # of words of length n with exactly k ones is $\binom{n}{k}$. Moreover, there are 2^n binary strings of length n since there're 2 possible choices for each of the n entries ($\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{n} = 2^n$) According to Binomial Theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ is true for arbitrary integers x and y

Choosing $x=y=1 \Rightarrow (1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$

So, we found $\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k}}$ ① ★ became true. Similarly, if we pick $x=1, y=-1$

$(1+(-1))^n = 0 = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$, hence

$\boxed{0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}}$ ② ★ also become true, where we found them from plugging specific x, y values to famous Binomial identity.

Notice that adding the positive terms in the 2nd summation counts all the binary strings of length n with an even number of ones $\left(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots \right)$ while adding the negative terms gives the additive inverse of the # of strings of length n with an odd number of ones $\left(\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots \right)$. Since this sum is equal to zero, the # of binary strings of length n with an even number of ones is equal to the # of binary strings of length n with an odd number of ones. Because every binary string of length n must have an even # of ones, or an odd # of ones, exactly half the strings of length n have an even number of ones.

As, there are 2^n such strings, the # of binary strings of length n with an even number of ones is $\frac{1}{2}\cdot 2^n = 2^{n-1}$, as is the number of binary strings of length n with an odd number of ones

-This can also be seen algebraically. If we add the 2 equations mentioned in previous page (eq'n 1 & 2) we find $2^n + 0 = 2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} =$

$= 2\sum_{k=0}^{n} \binom{n}{2k}$, since the terms with an odd number of ones cancel. Hence, the number of binary strings of length n with an even number of ones is

$2^{n-1} = \sum_{k=0}^{n} \binom{n}{2k}$ where $\binom{n}{2k} = 0$ if $2k > n$.

The number of binary strings of length n with an odd number of ones is found by subtracting 2^{n-1} from 2^n, which yields $2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$

Henceforth, 2^{n-1} number of words of length n will have an even parity and

2^{n-1} number of words of length n will have an odd parity