Ch 1. The Foundations: Logic and Proofs

Predicate Logic-4

Quantifiers

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Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Gentzen's Natural Deduction

Additional Proof Rules for Predicate Logic

Proof Rules for Propositional Logic

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$\forall$-intro

$$\frac{P(x)}{\forall x\, P(x)}\ \text{x is free}$$

$\forall$-elim

$$\frac{\forall x\, P(x)}{P(t)}$$

$\exists$-intro

$$\frac{P(t)}{\exists x\, P(x)}$$

$\exists$-elim

$$\frac{\Sigma(x),\ P(x) \mid\!- C}{\Sigma(x),\ \exists x\, P(x) \mid\!- C}\quad C \text{ does not contain x free}$$

t is a term which is free for x in A(x) and
where term is defined to be a variable, a constant or an expression of the form $f(t_1,\ldots t_n)$ where $t_1,\ldots, t_n$ are terms

∀-intro rule (1/2)

P(x)

———————— x is free

∀x P(x)

x is free. So it is any x.
If any x satisfies the property P, then it means every object in the given domain satisfies P.

$\forall$-intro rule (2/2)

Consider the following proof:

Prove the following.

For all integers a, b, and c, if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Proof.

Let integers a, b, and c be given, and suppose $a \mid b$ and $a \mid c$. Then, by the definition of $\mid$, there is some integer k_1 such that $b = k_1 a$ and there is some integer k_2 such that $c = k_2 a$. Therefore,

$$b + c = k_1 a + k_2 a = (k_1 + k_2)a.$$

By the closure axiom, $k_1 + k_2$ is an integer, so $a \mid (b + c)$, again by the definition of $\mid$. $\square$

It was shown that, for any integers a, b and c, **-- a, b, c are free**

$(a \mid b) \wedge (a \mid c) \rightarrow a \mid (b+c)$ -- (1)

From this, we can assert that

$\forall a\ \forall b\ \forall c\ [(a \mid b) \wedge (a \mid c) \rightarrow a \mid (b+c)]$ -- (2)

$\forall$-elim rule (1/2)

$\forall x\ P(x)$

$P(t)$

t is a term which is free for x in A(x) and where term is defined to be a variable, a constant or an expression of the form $f(t_1,...t_n)$ where $t_1,...,t_n$ are terms

Example 1

From $\forall x.\ (Q(y) \vee \exists y.R(x,y))$, by applying $\forall$-elim rule, we can deduce

$Q(y) \vee \exists y.R(z,y)$ – (1)

but we can NOT deduce

$Q(y) \vee \exists y.R(y,y)$ – (2)

Reason

In (1), z is substituted for x but z does not get bound by $\exists y$.

But in (2), y is substituted for x but y gets bound by $\exists y$.

$\forall$-elim rule (2/2)

Example 2

"For any x, if x is even then (x+1)(x-1) is not even" translates to

$$\forall x[\ \exists y(x = 2y) \rightarrow \neg\exists z((x-1)(x+1) = 2z)\ ]$$

We can substitute w for x but we cannot substitute y or z for x.

$\exists$-intro rule

P(t)

$\exists$x P(x)

t is a term which is free for x in A(x) and where term is defined to be a variable, a constant or an expression of the form $f(t_1,...t_n)$ where $t_1,...$, t_n are terms

Example

From $\forall$x.Q(y,x), by applying $\exists$-intro, we can deduce

$\exists$z $\forall$x. Q(z,x) – (1)

but we can NOT deduce,

$\exists$x $\forall$x.Q(x,x) -- (2)

Reason

In (1), z does not become bound by $\forall$x.

In (2), x gets bound by $\forall$x.

$\exists$-elim rule (1/5)

$$\frac{\Sigma(x),\ P(x)\ \vdash\ C}{\Sigma(x),\ \exists x\ P(x)\ \vdash\ C}\qquad \textbf{C does not contain x free}$$

Given $\Sigma(x)$ and $\exists x$ P(x), we are asked to prove C.

Due to $\exists$-elim rule, it is sufficient to prove C from $\Sigma(x)$ and P(x), provided that C does not contain x free.

If C contains x free, then x that represented only a specific object in C within the proof $\Sigma(x)$, P(x) |- C would stand for any object in C within the proof $\Sigma(x)$, $\exists x$ P(x) |- C.

∃-elim rule (2/5)

Assume that in the following the variables range over the set of integers.

Prove the following

If $\exists y\ (x = 2y)$ is true, then $\neg\exists z((x+1)(x-1) = 2z)$.

How do you translate this into a natural language sentence?

I am going to proceed as follows:

∃-elim rule (3/5)

Assume that in the following the variables range over the set of integers.

Prove the following

If $\exists y\ (x = 2y)$ is true, then $\underbrace{\neg\exists z((x + 1)(x - 1) = 2z)}_{(*)}$.

I am going to proceed as follows:

Let $x = 2k$. Then try to prove

$\exists$-elim rule (4/5)

Assume that in the following the variables range over the set of integers.

Prove the following

If $\exists y\,(x = 2y)$ is true, then $\underbrace{\neg\exists z((x+1)(x-1) = 2z)}_{(*)}$.

I am going to proceed as follows:

Let $x = 2k$. Then try to prove

$$\neg\exists z\big((x+1)(x-1) = 2z\big) \qquad \text{-- (1)}$$

that is,

$$\neg\exists z\big((2k+1)(2k-1) = 2z\big) \qquad \text{-- (2)}$$

∃-elim rule (5/5)

Assume that in the following the variables range over the set of integers.

Prove the following

$$\left.\text{If } \exists y\,(x = 2y) \text{ is true, then } \underbrace{\neg\exists z((x+1)(x-1) = 2z)}_{C}.\right\}\ \text{(A)}$$

$$\boxed{\exists y\,(x = 2y)\ \vdash C}$$

I am going to proceed as follows:

Let $x = 2k$. Then try to prove

$$\neg\exists z\big((x+1)(x-1) = 2z\big)$$

that is ,

$$\neg\exists z\big((2k+1)(2k-1) = 2z\big)$$

(B)

$$\boxed{x = 2\,k\ \vdash C,\quad C \text{ does not contain k free.}}$$

Quiz 07-1 (Due April 7, 14:30)

A *hasty generalization* is a fallacy in which a conclusion reached is not logically justified by sufficient evidence. Examples are:

- A person's grandparents do not know how to use a computer. That person thinks that all older people must be computer illiterate.
- A driver with a New York license plate cuts you off in traffic. You decide that all New York drivers are terrible drivers.

Assume that the Natural Deduction inference rules reflect most of the ways of reasoning that people commonly use. Which of the Natural Deduction inference rules do you think is being misapplied in hasty generalization? (People would do fallacious reasoning because the reasoning is similar to a really logical reasoning. Consider the case of *sophistry*, which is subtly deceptive reasoning.)

a) $\forall$-intro rule

b) $\forall$-elim rule

c) $\exists$-intro rule

d) $\exists$-elim rule