# *Ch 6. Counting*

## Selections and Arrangements

**Sungwon Kang**

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# Ch 6. Counting

6.1 The Basics of Counting

6.2 The Pigeonhole Principle

6.3 Permutations and Combinations

6.4 Binomial Coefficients and Identities

# Selections and Arrangements

- Permutations: The Arrangement Principle
- Combinations: The Selection Principle
- The Binomial Theorem

# Permutations: counting ordered arrangements

## The Arrangement Principle

The number of ways to form an ordered list of $r$ distinct elements drawn from a set of $n$ elements is

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Read "permute n r".

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**Example:** A baseball team has a 24-man roster. How many different ways are there to choose a 9-man batting order?

## Solution

*A batting order is simply a list of nine players in order, so there are* $P(24, 9) = 24!/15! = 474,467,051,520 \approx 4.74 \times 10^{11}$ *ways to make such a list.*

# Permutations: putting a set in order

**Example:** How many different ways are there to rearrange the letters in the word GOURMAND?

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Solution

*The important thing to notice about the word GOURMAND is that all the letters are different. The number of ways to do put these letters in order is* $P(8,8) = 8! = 40,320$.

**Example:** A kitchen drawer contains ten different plastic food containers and ten different lids, but any lid will fit on any container. How many different ways are there to pair up containers with lids?

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**Solution**

$P(10,10) = 10! = 3,628,800$. *(Just arrange the lids.)*

## Example: Arrangements vs. Multiplication

An urn contains 10 ping pong balls, numbered 1 through 10. Four balls are drawn from the urn in sequence, and the numbers on the balls are recorded. How many ways are there to do this, if

(a) the balls are replaced before the next one is drawn.

(b) the balls are drawn and not replaced.

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(a) the balls are replaced before the next one is drawn.

(b) the balls are drawn and not replaced.

**Solution**

*In case (a), there are always 10 balls in the urn, so there are always 10 choices. By the multiplication principle, the number of ways to draw four balls is* $10^4 = 10,000$. *In case (b), the balls are not replaced, so the number of choices goes down by one each time a ball is drawn. Hence there are* $P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5,040$ *ways to draw four balls.*

# Combinations: counting unordered groupings

## The Selection Principle

The number of ways to choose a subset of $r$ elements from a set of $n$ elements is

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Read "choose n r".

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**Example:** A baseball team has a 24-man roster. How many different ways are there to choose an unordered group of 9 players?

## Solution

*The number of ways to choose this group is*

$$C(24,9) = \frac{24!}{15!\,9!} = 1,307,504.$$

# Example: an unordered urn problem

Suppose an urn contains 10 ping pong balls numbered 1 through 10. Instead of drawing four balls in sequence, reach in and grab a handful of four balls. How many different handfuls can you grab?

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**Solution**

*A "handful" of four ping pong balls is an unordered set, so there are* $C(10, 4) = 210$ *different possible outcomes.*

Compare this to the last "urn" problem.

# Example: counting strings using selections

How many different ways are there to rearrange the letters in the word PFFPPPPFFFF?

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### Solution

*Although this example resembles the GOURMAND example, the solution is quite different because the letters of PFFPPPFFFF are not all distinct. In order to view this as a selection problem, notice that we have to fill ten blanks*

$$\_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_$$

*using four P's and six F's. Once we choose where the P's go, there are no more choices to make, since the remaining blanks get filled up with the F's. The order of the blanks we choose doesn't matter, because we are putting P's in all of them. So the number of ways to fill in the blanks is* $C(10, 6) = 210$.

# Example: counting integer solutions

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13$$

if $x_1, \ldots, x_5$ must be nonnegative integers?

The answer is 2,380. How can we obtain this answer?

## Solution

*Such a solution corresponds to a distribution of 13 units among the five variables* $x_1, \ldots, x_5$. *For example, the solution*

$$x_1 = 4,\ x_2 = 0,\ x_3 = 5,\ x_4 = 1, x_5 = 3$$

*amounts to dividing up thirteen 1's into groups as follows.*

$$1\,1\,1\,1 \,|\, | \,1\,1\,1\,1\,1 \,|\, 1 \,|\, 1\,1\,1$$

*We can view this division into groups as a string containing four* $|$*'s and thirteen 1's; every such string defines a different solution to the equation, and all solutions can be represented this way. So the number of possible solutions is* $C(17, 4) = 2,380$.

**Binomial Expression**

A *binomial expression* is the sum of two terms, such as $x + y$.

Each term x and y may be a product of a constant and a variable.

A *power of a binomial expression* is an expression of the form $(x+y)^n$ where n is a natural number.

$$
\begin{aligned}
& (3x-5)^4 \\
= \quad & 81x^4 - 540x^3 + 1350x^2 - 1500x + 625
\end{aligned}
$$

How can we get these coefficients of this power of an binomial expression?

$$
\begin{aligned}
& (3x-5)^4 \\
= \;& 81x^4 - 540x^3 + 1350x^2 - 1500x + 625
\end{aligned}
$$

Hard way:

$$
\begin{aligned}
& (3x-5)^4 \\
= \;& (3x-5)(3x-5)(3x-5)(3x-5) \\
= \;& (9x^2 - 15x - 15x + 25)(3x-5)(3x-5) \\
= \;& (27x^3 - 45x^2 - 45x^2 + 75x - 45x^2 + 75x + 75x - 125)(3x-5) \\
= \;& (27x^3 - 135x^2 + 225x - 125)(3x-5) \\
= \;& 81x^4 - 405x^3 + 675x^2 - 375x - 135x^3 + 675x^2 - 1125x + 625 \\
= \;& 81x^4 - 540x^3 + 1350x^2 - 1500x + 625
\end{aligned}
$$

Is there an easier way?

# The Binomial Theorem

### Theorem

*Let $j$ and $k$ be nonnegative integers such that $j + k = n$. The coefficient of the $a^j b^k$ term in the expansion of $(a + b)^n$ is $C(n, j)$.*

### Corollary

*The Binomial Theorem.*

$$\begin{aligned}(a+b)^n &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots \\ &\quad + \binom{n}{j} a^{n-j} b^j + \cdots + \binom{n}{n} b^n\end{aligned}$$

$$(a + b)^n = \overbrace{(a + b)\,(a + b)\,(a + b)\ \ldots\ (a + b)\,(a + b)}^{\text{n occurrences of (a+b)}}$$

# Apply the Binomial Theorem

Use the binomial theorem to expand $(3x-5)^4$.
Apply the corollary with $a=3x$ and $b=-5$.

$$(3x-5)^4$$

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$$(3x-5)^4$$

$$= ?\,(3x)^4 + \; ?\,(3x)^3(-5) + \; ?\,(3x)^2(-5)^2 + \; ?\,(3x)(-5)^3 + ?(-5)^4$$

Use the binomial theorem to expand $(3x-5)^4$.
Apply the corollary with $a = 3x$ and $b = -5$.

$$\begin{aligned} &(3x-5)^4 \\ = &\underbrace{\phantom{x}}_{{}_4C_0}(3x)^4 + \underbrace{\binom{4}{1}}_{{}_4C_1}(3x)^3(-5) + \underbrace{\binom{4}{2}}_{{}_4C_2}(3x)^2(-5)^2 + \underbrace{\binom{4}{3}}_{{}_4C_3}(3x)(-5)^3 + \underbrace{\phantom{x}}_{{}_4C_4}(-5)^4 \end{aligned}$$

Use the binomial theorem to expand $(3x - 5)^4$.
Apply the corollary with $a = 3x$ and $b = -5$.

$$
\begin{aligned}
& (3x-5)^4 \\
= \;& (3x)^4 + \binom{4}{1}(3x)^3(-5) + \binom{4}{2}(3x)^2(-5)^2 + \binom{4}{3}(3x)(-5)^3 + (-5)^4 \\
= \;& 81x^4 + (4)(27x^3)(-5) + (6)(9x^2)(25) + (4)(3x)(-125) + (625) \\
= \;& 81x^4 - 540x^3 + 1350x^2 - 1500x + 625
\end{aligned}
$$

**Example**

Find the coefficient of $x^2y^3z^4$ in the expansion of $(x + y + z)^9$.

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Find the coefficient of $x^2y^3z^4$ in the expansion of $(x + y + z)^9$.

$$(x + (y+z))^9$$

$$= \underbrace{{}_9C_0x^9 + {}_9C_1x^8(y+z) + {}_9C_2x^7(y+z)^2 + \ldots + \underline{{}_9C_k\, x^{9-k}\, (y+z)^k} + \ldots + {}_9C_9\, (y+z)^9}_{\text{General Term: } {}_9C_k\, {}_kC_i\, x^{9-k}\, y^{k-i}\, z^i.}$$

$$(y+z)^k = \underbrace{{}_kC_0\, y^k + {}_kC_1\, y^{k-1}\, z^1 + {}_kC_2\, y^{k-2}\, z^2 + \ldots + {}_kC_k\, z^k}_{\text{General Term: } {}_kC_i\, y^{k-i}z^i}$$

**Example**

Find the coefficient of $x^2y^3z^4$ in the expansion of $(x + y + z)^9$.

$$(x + (y+z))^9$$
$$= {}_9C_0x^9 + {}_9C_1x^8(y+z) + {}_9C_2x^7(y+z)^2 + \dots + \underline{{}_9C_k\, x^{9-k}\, (y+z)^k} + \dots + {}_9C_9\, (y+z)^9$$

$$(y+z)^k = {}_kC_1\, y^k + {}_kC_1\, y^{k-1}\, z + {}_kC_2\, y^{k-2}\, z^2 + \dots + {}_kC_k\, z^k$$

General Term: ${}_kC_i\, y^{k-i}z^i$

General Term: ${}_9C_k\, {}_kC_i\, x^{9-k}\, y^{k-i}\, z^i$.

What is the coefficient of $x^2y^3z^4$?

☛ ${}_9C_2\ {}_7C_3$

# Pascal's Identity

**Theorem** For $1 \le k \le n$,

$$C(n+1,k) = C(n,k) + C(n,k-1)$$

*Proof*

$$\frac{(n+1)!}{k!\,(n+1-k)!} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!}$$

**Another approach**

Recall the proof of

$$|\mathbb{P}(S)| = 2^{|S|}$$

where S is a finite set.

# Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$\cdots$$

# Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
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. . .

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

. . .

# Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

...

Has many usages:

# Pascal's Triangle

| | |
|---|---|
| 1 | |
| 1 1 | ← one coin |
| 1 2 1 | ← two coins |
| 1 3 3 1 | ← three coins |
| 1 4 6 4 1 | ← four coins |
| 1 5 10 10 5 1 | |
| 1 6 15 20 15 6 1 | |
| 1 7 21 35 35 21 7 1 | |
| 1 8 28 56 70 56 28 8 1 | |
| 1 9 36 84 126 126 84 36 9 1 | |

...

Has many usages:

1) Tells the numbers of heads and tails in coin tossing
2) Fibonacci sequence

. . .

# Pascal's Triangle

| | |
|---|---|
| 1 | 1 |
| 1 | 1 1 |
| 2 | 1 2 1 |
| 3 | 1 3 3 1 |
| 5 | 1 4 6 4 1 |
| 8 | 1 5 10 10 5 1 |
| 13 | 1 6 15 20 15 6 1 |
| | 1 7 21 35 35 21 7 1 |
| | 1 8 28 56 70 56 28 8 1 |
| | 1 9 36 84 126 126 84 36 9 1 |

...

Has many usages:

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# Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

5

1 5 10 10 5 1

8

1 6 15 20 15 6 1

13

...

# Another Theorem

If n and r are natural numbers and $k \leq n$, then

$$\sum_{i=k}^{n} C(i,k) = C(n+1,k+1)$$

If n=4 and k=2, then

$C(2,2) + C(3,2) + C(4,2) = C(5,3)$

# Example

$$\binom{0}{0}$$

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$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \underline{\binom{5}{3}}$$

If n and r are natural numbers and k ≤ n, then

$$\sum_{i=k}^{n} C(i,k) = C(n+1,k+1)$$

If n=4 and k=2, then

C(2,2) + C(3,2)+ C(4,2) = C(5,3)

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$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \underline{\binom{5}{3}} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
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$$\cdots$$

$$1$$
$$1 \quad 1$$
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$$1 \quad 5 \quad 10 \quad \underline{10} \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

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$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
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If n and r are natural numbers and k ≤ n, then

$$\sum_{i=k}^{n} C(i,k) = C(n+1,k+1)$$

If n=4 and k=2, then

C(2,2) + C(3,2)+ C(4,2) = C(5,3)

C(5,3) = C(4,3) + C(4,2)
= {C(3,3) + C(3,2)} + C(4,2)
= C(2,2) + C(3,2) + C(4,2)

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

…

# Proof

- Two ways to prove

1) Proof by induction

2) Combinatorial proof

- C(n+1,k+1): the number of bit strings of length n+1 containing k+1 ones.
- LHS: corresponds to case analyses according to the positions of the last one

Case location = n+1:C(n,k)

Case location = n:    C(n-1,k)

...

Case location = k+1: C(k,k) Is a location < k+1 possible ?

The sum of these counts is the LHS. (Note C(k+1,k+1) = C(k,k))

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1) Proof by induction

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   - C(n+1,k+1): the number of bit strings of length n+1 containing k+1 ones.
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| | | | | |
|---|---|---|---|---|
| | | | | |

Length: 4 + 1

How many bit strings with 2+1 ones?

| | | | | | | |
|---|---|---|---|---|---|---|
| case 1 | | | | | **1** | → C(4,2) |
| case 2 | | | | **1** | | → C(3,2) |
| case 3 | | | **1** | | | → C(2,2) |

Are they mutually exclusive?

# Quiz 18-1

Which of the following is NOT true ?

(a) $C(13, 6) = C(13,7)$

(b) $2^n = \sum_{i=0}^{n} C(n,i)$

(c) $C(13,6) = C(12,6) + C(12,5)$

(d) $C(7,3) = C(5,3) + C(5,2) + C(6,3)$

(e) $C(9,5) = C(7,5) + C(7,4) + C(8,4)$