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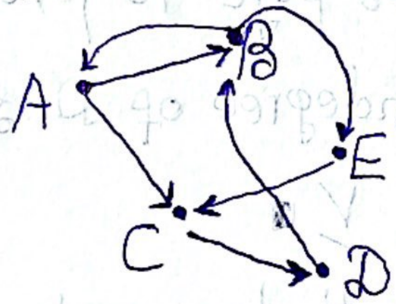
Homework-12: Graphs

1) Below is a complete graph on five vertices

This graph has 10 edges

$\frac{5 \cdot 4}{2} = 10 \rightarrow$ A complete graph on 5 vertices has 10 edges ✓

2) Let the graph below represent some web pages and link on Internet



a) Yes, it will be a directed graph as there'll be direction of path which it will follow to lead you to the web page for e.g. C have a link which will take you to D

b) Yes, it will be a connected graph as there is a path between all vertices i.e. we can get to a web page from another either via direct or indirect link

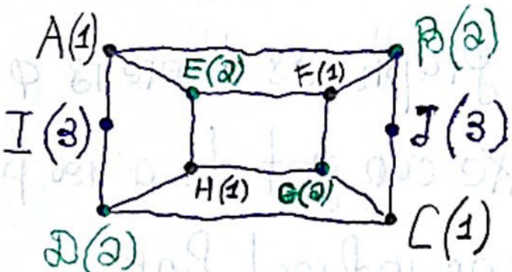
c) No, it need not to be complete as it's not possible to have link to all the web page directly on each web page

d) No, the graph is not simple as a graph is called simple if it has no multiple edges or loops, but in order to have a complete graph, we will have to create multiple edges to move back and forth from a web page as we can see between A and B in the graph

e) Outdegree of p represent that the # of links taking you to a different web page from p, e.g. as in the graph, we can see that outdegree of C is taking you from web page C to D

f) Indegree of p represent that the # of links taking you from a different web page to p, e.g. as in the graph, we can see that indegree of D is taking you from web page C to D ✓ ■

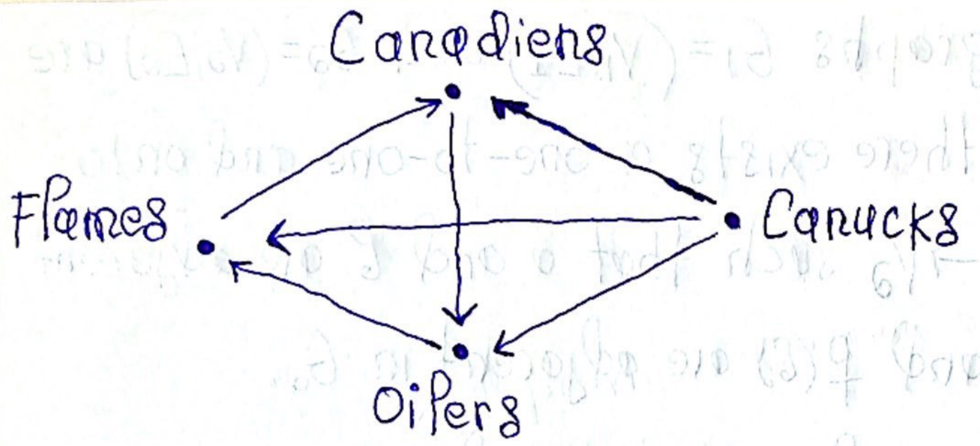
3) Below is the graph so that no vertices of the same color share an edge with as few colors as possible



Minimum 3 colors are needed in this graph, as for I and J, it already had both the colors on adjacent vertices because of which we had to introduce third color in the graph

Minimum 3 colors are needed for valid coloring of the graph ✓ ■

4) a) Below is the model, given results with a directed graph, where each vertex represents a team and each edge represents a game, pointing from the winner to the loser



b) Finding a circuit in this graph: Let's assume

vertex Canadiens = A
vertex Oilers = B
vertex Flames = C

then, we have a circuit ABCA

c) Explaining why the existence of a circuit in such a graph makes it hard to rank the teams from best to worst.

The existence of a circuit in such a graph makes it hard to rank the teams from best to worst, because this is a never ending loop because of which we can't determine who is better than whom.

Example we can have a look at the circuit in part (b) A won against B, and B won against C, but C won against A. It is not possible to determine which one is best among these three. We have a circuit → Canadiens

Canadiens → Oilers → Flames →

5) Two simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are <u>isomorphic</u> if there exists a one-to-one and onto function $f: V_1 \to V_2$ such that a and b are adjacent in $G_1 \Leftrightarrow f(a)$ and $f(b)$ are adjacent in G_2.

Note that this implies that the degree of a vertex needs to be the same as the degree of the image of the vertex.

We notice that all vertices in the graph have degree 2, so it does not matter which vertex is chosen as the image of the first vertex. Let's first assign the image of a to be equal to $1 \Rightarrow \underline{f(a)=1}$

b and g are adjacent to a, while 3 and 6 are adjacent to 1. Let's then assign 3 as the image of b, and assign 6 as the image of $g \Rightarrow \underline{f(b)=3,\ f(g)=6}$

c is adjacent to b, while 5 is adjacent to $3 \Rightarrow$
$\underline{f(c)=5}$

d is adjacent to c, while 7 is adjacent to $5 \Rightarrow \underline{f(d)=7}$

e is adjacent to d, while 2 is adjacent to $7 \Rightarrow \underline{f(e)=2}$

f is adjacent to e, while 4 is adjacent to $2 \Rightarrow \underline{f(f)=4}$

Finally, we note that f and g are also adjacent, while $f(f)=4$ and $f(g)=6$ are adjacent, which implies that the function f makes the 2 graphs isomorphic ⇒

$\boxed{G_1 \text{ and } G_2 \text{ are isomorphic}}$ ✓

Note: I will provide an alternative solution for problem 2

a) Each web page can be directed from 1 page to another and can return back to the previous page, which means that the internet graph can be considered as a directed graph Yes ✓

b) To get the information from web pages, each webpage should be linked, indicating the graph is connected Yes ✓

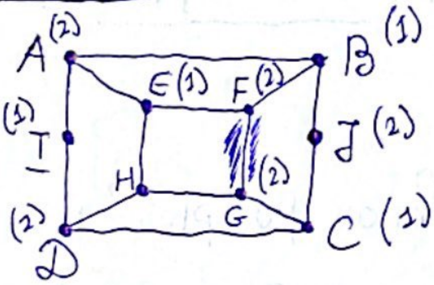
c) The graph is not complete, because a webpage cannot direct you to every webpage No ✓

d) This graph can't be simple. Every webpage contains a "Back" and a "Refresh" button. Every vertex or webpage can refresh itself and bring back the same information along with any page B can resend any information to any page A No ✓

e) The # of links that direct to a new page defines the outdegree of p ✓

f) The # of links from which web page P can be reached is considered the indegree of P ✓ ■

Note: I will provide some more details on problem 3



As we explained earlier, because of I and J,

I, A - different
I, D - different
} at least 2 colours

If we used exactly 2 colours, then

I → 1st color
A → 2nd color
D → 2nd color
E → 1st color
F → 2nd color
B → 1st color
J → 2nd color
C → 1st color
G → 2nd color

Since they (F and G) are neighbors (adjacent), ✱ they can't be same-colored $\boxed{\text{impossible}}$ ✓ So, 2 colours do not satisfy, and we need at least 3 colors. The required construction was done in previous pages ✓ ■

6) A circuit is a path with at least 1 edge and no repeated edges. An Euler circuit is a circuit that contains every edge of the graph. A connected, undirected graph has an Euler circuit $\Leftrightarrow$ each of the vertices has an even degree. An Euler path is a path that contains every edge of the graph. A connected, undirected graph has an Euler path $\Leftrightarrow$ there are exactly 2 vertices who have an odd degree.

a) The given graph has 45 vertices. We note that the graph contains four vertices of degree 2 (corners), two vertices of degree 3 (vertices between corners), and all remaining vertices have degree 4.

Since there are some vertices of odd degree, the graph has no Euler circuit ✓

b) The given graph has 45 vertices. We note that the graph contains four vertices of degree 2 (corners), two vertices of degree 3 (vertices between corners), and all remaining vertices have degree 4. Since there are exactly two vertices of odd degree (while the graph is a connected, undirected graph), the graph has an Euler path ✓

c) The given graph has 45 vertices, while we have also been given that each vertex has degree 4. Let's first determine the total degree of the graph:

$$\text{Total degree of } G = \underbrace{4+4+\dots+4}_{45} = 45 \cdot 4 = 180$$

Handshake theorem: A graph G with vertices $v_1, v_2, \dots, v_n$ and m edges has the property that

$$2m = \sum_{i=1}^{n} \deg(v_i) = \text{Total degree of } G$$

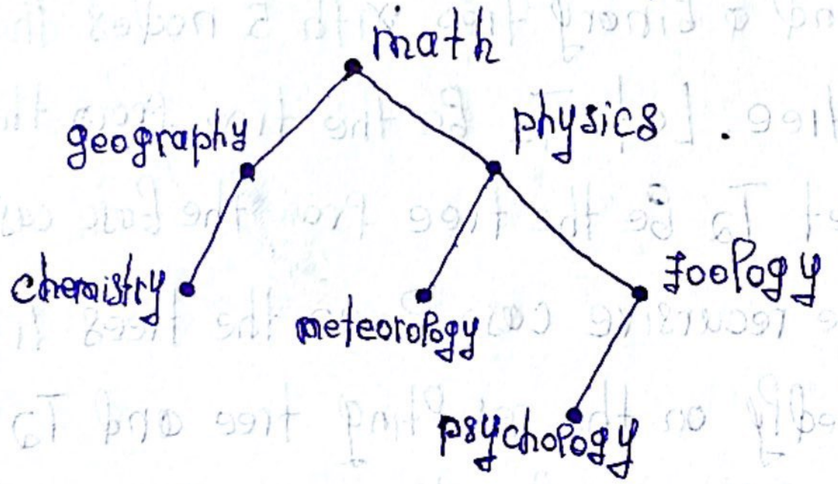
(also called $\boxed{\text{Euler}}$ 2.6 on David's book Degree-counting Theorem)

Note: This's in fact Euler's 1^{st} observation in the 23-1 slides (page 11)

By the handshake theorem, # of edges is then total degree of G divided by $2 \Rightarrow m = \frac{\text{Total degree of } G}{2} = \frac{180}{2} = 90$, thus $\boxed{\text{the graph has 90 edges}}$ (90 edges) ✓ ■

7) Using alphabetical order, the following figure displays the steps used to construct this BST. The word "math" is the key of the root. Because "physics" comes after "math" (in alphabetical order), add a right child of the root with key "physics". Because "geography" comes before "math", add a left child of the root with key "geography". Next, add a right child of the vertex with key "physics", and assign it the key "zoology", because "zoology" comes after "math" and after "physics". Similarly, add a left child

of the vertex with key "physics" and assign this new vertex the key "meteorology". Add a left child of the vertex with key "zoology", and assign it the key "psychology". Add a left child of the vertex with key "geography", and assign it the key "chemistry" (The required comparisons are done at each of these steps)



physics > math
geography < math
zoology > math
zoology > physics

meteorology > math
meteorology < physics
psychology > math
psychology > physics
psychology < zoology

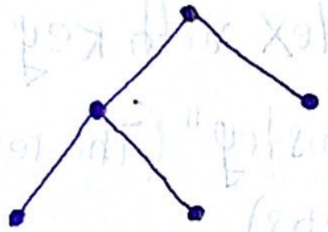
chemistry < math
Chemistry < geo-
-graphy

V ■

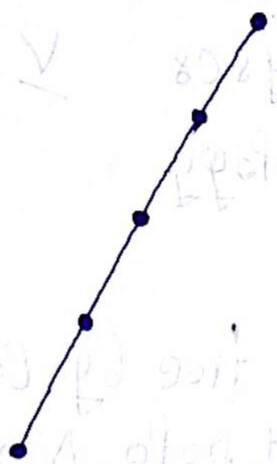
Trees

1) a) The single vertex is a binary tree by base case B_2, which is a T binary tree, with 1 node. Next, we obtain a T binary tree with 3 nodes by using the recursive case R with T_1 and T_2 the tree in the base case.

Let's call this T binary tree T_3. Next, we find a T binary tree with 5 nodes by using the recursive case R with T_3 and the tree in the base case for T_2



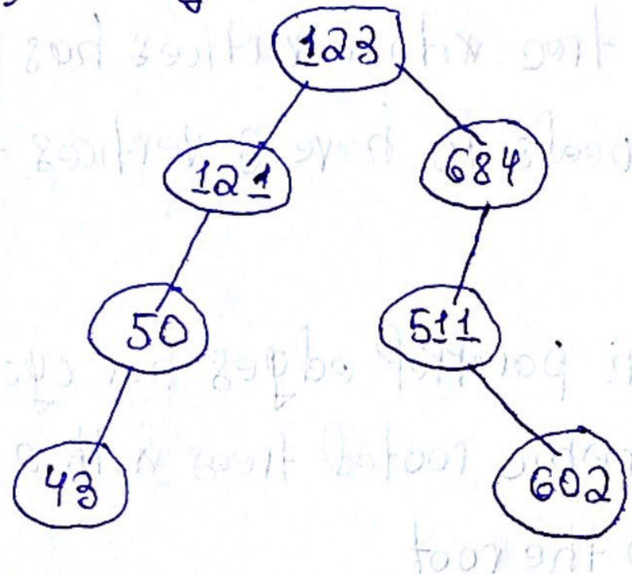
b) We need to find a binary tree with 5 nodes that is not a T binary tree. Let T_1 be the tree from the base case B_1, and let T_2 be the tree from the base case B_2. When we use the recursive case R on the trees T_1 and T_2, and then repeatedly on the resulting tree and T_2 until we obtain a tree with 5 nodes, then we obtain a binary tree with five nodes that is not a T binary tree (as each node with a child does not have 2 children)



a simple path of length 4

$\blacksquare$

2) a) Binary search tree of given order of numbers is

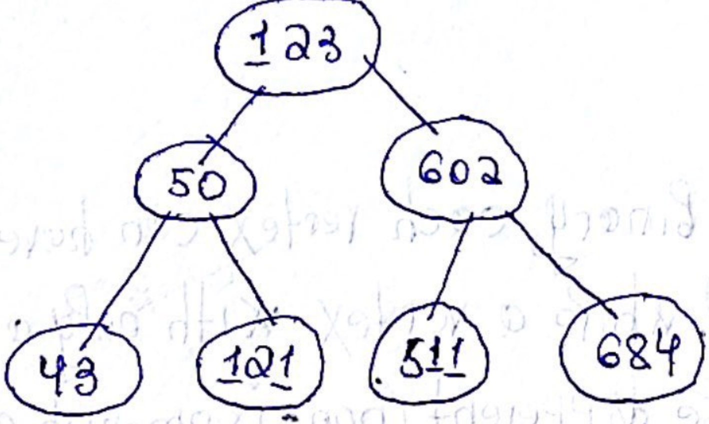


$123<684$, $50<123$
$121<123$, $50<121$
$511>123$, $43<123$
$511<684$, $43<121$
$602>123$, $43<50$
$602<684$
$602>511$

b) The height of tree is 3

c) After reordering, we get: 123, 602, 50, 684, 511, 121, 43

Binary tree for new arrangement with height of 2 is



$123<602$, $511>123$
$50<123$, $511<602$
$123<684$, $121<123$
$602<684$, $121>50$
$43<123$
$43<50$

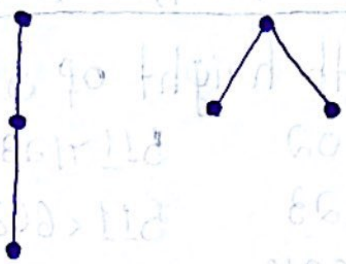
3) a) Two simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there exists a one-to-one and onto function $f: V_1 \to V_2$ such that a and b are adjacent in $G_1 \Leftrightarrow f(a)$ and $f(b)$ are adjacent in G_2.

We need to determine all non-isomorphic rooted trees having three vertices

Since the trees are rooted, we need to take the root of the tree into account. A tree with n vertices has (n-1) edges. Since the tree needs to have 3 vertices, the tree has two edges.

As a tree can't contain parallel edges nor cycles, there are 2 non-isomorphic rooted trees with 2 vertices:

- two edges incident to the root
- one edge incident to the root, and the remaining edge incident to the child of the root

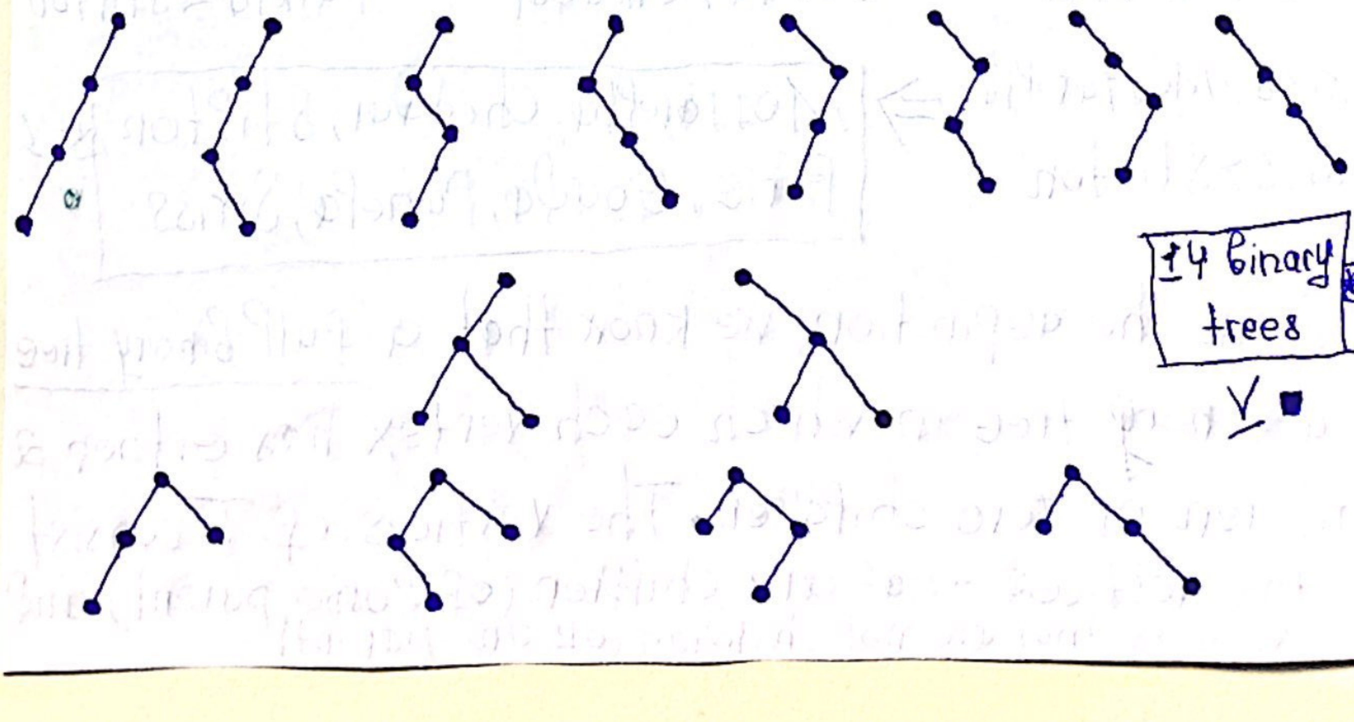


$\boxed{\text{2 rooted trees}}$ ✓

b) Since the trees are binary, each vertex can have a left and/or right child, while a vertex with only a left child is considered to be different (non-isomorphic) from a vertex with only a right child.

A tree with n vertices has (n-1) edges. Since the tree needs to have 4 vertices, the tree has three edges. Let L represent left child and R right child. As a tree can't contain parallel edges nor cycles, there are <u>14 non-isomorphic binary trees with 4 vertices:</u>

- simple path of length 3: LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR
- root degree 1 with left child and left child has 2 children
- root degree 1 with right child and right child has two children
- root degree 2 and fourth vertex is the left child of the left child of the root
- root degree 2 and 4^{th} vertex is the right child of the left child of the root
- root degree 2 and 4^{th} vertex is the left child of the right child of the root
- root degree 2 and 4^{th} vertex is the right child of the right child of the root



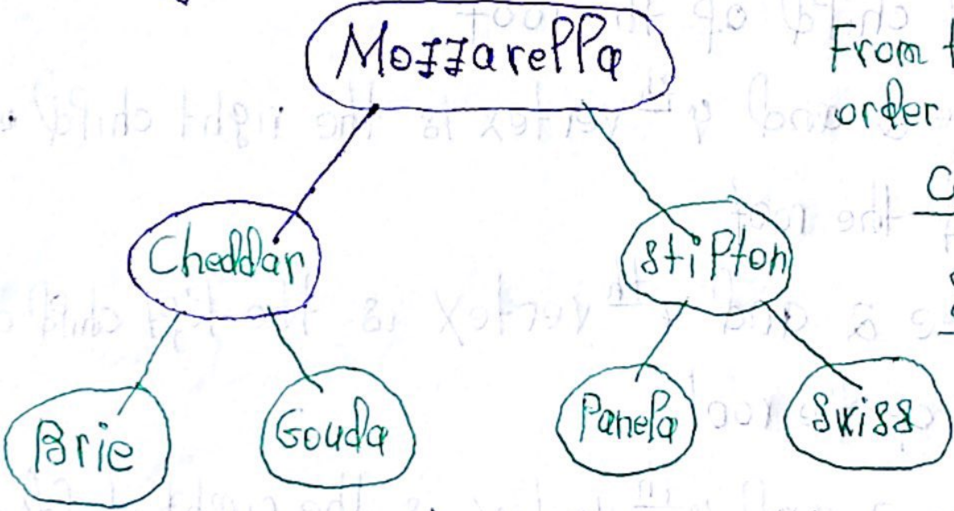
14 binary trees

V

5) Putting the word below in the order to have smallest binary tree height:

Mozzarella, Cheddar, Stilton, Brie, Gouda, Panela, Swiss

As the minimum height(h) of a binary tree is $h = \log_2(n+1) - 1$ $\left(h = \lfloor \log_2(n) \rfloor\right)$, where n is $7 \Rightarrow$ Thus, $h = \log_2(8) - 1 = 2$ $\left(h = \lfloor \log_2 7 \rfloor = 2\right)$, and the following is the binary tree for the above order of words:



From the alphabetical order of comparisons, Cheddar < Mozzarella, Stilton > Mozzarella,

Brie < Mozzarella
Brie < Cheddar

Gouda < Mozzarella
Gouda > Cheddar

Panela > Mozzarella
Panela < Stilton

Swiss > Mozzarella
Swiss > Stilton

$\Rightarrow$ $\boxed{\text{Mozzarella, Cheddar, Stilton, Brie, Gouda, Panela, Swiss}}$

4) From the definition, we know that a full binary tree is a binary tree in which each vertex has either 2 children or zero children. The vertices of T consist of the vertices that are children (of some parent) and the vertices that are not children (of any parent)

There is 1 nonchild - the root. Since there are i internal vertices, each having two children, (from the Definition) there are $2i$ children $\Rightarrow$ Thus, the total # of vertices of T is $1+2i$, and the # of terminal vertices is $(2i+1)-i = i+1$

Note: # of vertices of T = # of internal vertices + + # of terminal vertices, since each vertex is either a leaf or an internal vertex. As we proved there are a total of $(2i+1)$ vertices where # of internal vertices $= i \Rightarrow$

# of leaf vertices $= (2i+1)-i = i+1$

Thus, $\boxed{\text{T has } (i+1) \text{ terminal vertices and } (2i+1) \text{ total vertices}}$

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