CS 204 Discrete Mathematics

Homework 4 – Predicate Logic - Part II

Sample Solutions

1. Let the following predicates be given in the domain of triangles.

R(x) = "x is a right triangle."

B(x) = "x has an obtuse angle."

Consider the following statements.

$S1 = \neg(\exists x)\ (R(x) \wedge B(x))$

$S2 = (\forall x)\ (R(x) \rightarrow \neg B(x))$

(a) Write a proof sequence to show that $S1 \Leftrightarrow S2$.

(b) Write S1 in ordinary English.

(c) Write S2 in ordinary English.

Solution)

(a)

$$
\begin{aligned}
S_1 &= \neg(\exists x)(R(x) \wedge B(x)) \\
&\Leftrightarrow (\forall x)\neg(R(x) \wedge B(x)), \text{ exist. neg.} \\
&\Leftrightarrow (\forall x)(\neg R(x) \vee \neg B(x)), \text{ De Morgan} \\
&\Leftrightarrow (\forall x)(R(x) \rightarrow \neg B(x)), \text{ implication} \\
&= S_2
\end{aligned}
$$

(b) There are no triangles that are both right and have obtuse angles.

(c) All right triangles have no obtuse angles.

2.

(a) Give an example interpretation of a pair of predicates P(x) and Q(x) in some domain to show that the $\exists$ quantifier does not distribute over the $\wedge$ connective. That is, give an example to show that the statements

$$(\exists x)(P(x) \wedge Q(x)) \quad \text{and} \quad (\exists x)P(x) \wedge (\exists x)Q(x)$$

are not logically equivalent.

(b) It is true, however, that $\exists$ distributes over $\vee$. That is,

$$(\exists x)(P(x) \vee Q(x)) \quad \Leftrightarrow \quad (\exists x)P(x) \vee (\exists x)Q(x)$$

is an equivalence rule for predicate logic. Verify that your example from part (a) satisfies this equivalence.

Solution)

Let P(x) = “x is even number.”

Q(x) = “x is odd number.”

Domain : all natural numbers

(a) In ($\exists$ x)(P(x) $\wedge$ Q(x)), there isn’t such a number which is even and odd at the same time.

In ($\exists$ x)P(x) $\wedge$ ($\exists$ x)Q(x), there are even number, odd number respectively.

Therefore, former one is false and latter one is true.

(b) In ($\exists$ x)(P(x) $\vee$ Q(x)), there exist even or odd number : True

In ($\exists$ x)P(x) $\vee$ ($\exists$ x)Q(x), either there exist even number or there exist odd number : True

3.

Any equation or inequality with variables in it is a predicate in the domain of real numbers. For each of the following statements, tell whether the statement is true or false.

(a) $(\forall x)(x^2 > x)$

(b) $(\exists x)(x^2 - 2 = 1)$

(c) $(\exists x)(x^2 + 2 = 1)$

(d) $(\forall x)(\exists y)(x^2 + y = 4)$

(e) $(\exists y)(\forall x)(x^2 + y = 4)$

Solution)

(a) false (for example, $x = 0.5$.)

(b) true

(c) false

(d) true

(e) false

4. The domain of the following predicates is all integers greater than 1.

P(x) = "x is prime"

Q(x,y) = "x divides y"

Consider the following statement.

For every x that is not prime, there is some prime y that divides it.

(a) Write the statement in predicate logic.

(b) Formally negate the statement

(c) Write the English translation of your negated statement

Solution)

(a) $\forall x\, [\neg P(x) \rightarrow \exists y(P(y) \wedge Q(y,x))\,]$

(b) $\exists x\, [\neg P(x) \wedge \forall y(P(y) \rightarrow \neg Q(y,x))\,]$

(c) There is a nonprime x such that no prime y divides it.

Since there are very few students who answered both questions 9 (b) and (d) correctly, answer for 9 (b) and 9 (d) are shown below.

Solution

(b)

$\forall x(H(x) \rightarrow A(x))$

$\forall x\ [\exists y(H(y) \wedge IsTailOf(x,y)) \rightarrow \exists y(A(y) \wedge IsTailOf(x,y))\ ]$

(d)

1.	$\forall x(H(x) \rightarrow A(x))$	- premise 1
2.	$\exists y(H(y) \wedge IsTailOf(w,y))$	- premise 2
3.	$H(a) \wedge IsTailOf(w,a))$	- $\exists$-elim, 2
4.	$H(a)$	- $\wedge$-elim, 3
5.	$H(a) \rightarrow A(a)$	- $\forall$-elim 1
6.	$A(a)$	- $\rightarrow$-elim, 4, 3
7.	$IsTailOf(w,a))$	- $\wedge$-elim, 3
8.	$A(a) \wedge IsTailOf(w,a))$	- $\wedge$-intro, 6, 7
9.	$\exists y(A(y) \wedge IsTailOf(w,y))$	- $\exists$-intro, 8
10.	$\exists y(H(y) \wedge IsTailOf(w,y)) \rightarrow \exists y(A(y) \wedge IsTailOf(w,y))$	- $\rightarrow$-intro, 2-9
11.	$\forall x\ [\exists y(H(y) \wedge IsTailOf(x,y)) \rightarrow \exists y(A(y) \wedge IsTailOf(x,y))\ ]$	- $\forall$-intro, 11