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HW-7: Mathematical Induction

3) Base case: If $n=0 \Rightarrow C(0)=0$, but we know

$0 = \frac{3^{0+1}-2\cdot 0-3}{4} = \frac{3^1-3}{4} = \frac{0}{4} = 0$, $\boxed{C(0)=\frac{3^{0+1}-2\cdot 0-3}{4}}$

holds ✓ (we have $C(0)=0$ from recurrence relation)

Let $f(n)=\frac{3^{n+1}-2n-3}{4} \Rightarrow C(0)=f(0)$ (they match)

Inductive Hypothesis: Let $k>0$, suppose as inductive hypothesis that $C(k-1)=\frac{3^{k-1+1}-2(k-1)-3}{4} \Rightarrow$
for some $k>0$

$C(k-1)=\frac{3^k-2k+2-3}{4}=\frac{3^k-2k-1}{4} \Rightarrow C(k-1)=\frac{3^k-2k-1}{4}$

Inductive Step: Using the recurrence relation, (2nd part)

$C(k)=k+3C(k-1)=k+\frac{3^{k+1}-6k-3}{4}=\frac{3^{k+1}-6k-3+4k}{4}=$

$=\frac{3^{k+1}-2k-3}{4} \Rightarrow C(k)=\frac{3^{k+1}-2k-3}{4}=f(k)$ (since $k>0$) (inductive hypothesis)

So, by induction, $\boxed{C(n)=\frac{3^{n+1}-2n-3}{4} \text{ for all } n\geq 0}$

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2) We use firstly induction on n to prove that $H(2n)=H(2n-1)$ for all $n \geq 1$:

Base case: $H(2)=1$, from recurrence definition, and $H(1)=1$ from recurrence definition $\Rightarrow H(2\cdot 1)=H(2\cdot 1-1)$ holds ✓ True for $n=1$ ✓

Inductive Hypothesis: Suppose as inductive hypothesis that $H(2k)=H(2k-1)$, for some $k \geq 1$

Inductive Step: Since $k \geq 1 > 0 \Rightarrow 2k+2>2$, and from recurrence definition, $H(2k+2)=H(2k+1)+H(2k)-$ $-H(2k-1)$. Since from inductive hypothesis, $H(2k)=H(2k-1) \Rightarrow H(2k+2)=H(2k+1)$ becomes true

Or, $H(2(k+1))=H(2(k+1)-1)$ is satisfied ✓ ☒

Hence, from induction, $\boxed{H(2n)=H(2n-1) \text{ for } \forall n \geq 1}$

$H(2m+3)=H(2m+2)+H(2m+1)-H(2m)$, where $2m+3>2$ where $m \geq 0 \Rightarrow H(2m+4)=H(2m+3)$ and $H(2m+2)=H(2m+1) \Rightarrow H(2(m+2))=H(2(m+2)-1)$ is true for $m \geq 0$, $m+2 \geq 2 > 1$ and $m+1 \geq 1 \Rightarrow$ $H(2(m+1))=H(2(m+1)-1)$, $H(2m+3)=H(2m+4)$

$H(2m+4) = H(2m+3) = H(2m+2) + H(2m+1) - H(2m) =$
$= H(2m+2) + H(2m+2) - H(2m) = 2H(2m+2) - H(2m)$

$\boxed{H(2m+4) + H(2m) = 2H(2m+2), \text{ for } \forall m \geq 0}$ Now,

$\underline{H(0) = 0}$, from definition; $\underline{H(2) = 1}$, from definition

$m = 0 \Rightarrow H(4) = 2H(2) - H(0) = 2 - 0 = 2$, $\underline{H(4) = 2}$

Let's prove by strong induction on n that

$H(2n) = n$ for all $n \geq 1$ $\quad H(4) = 2$ ✓

Base Case: For $n = 1, 2$ we had previously (above, we proved it) $H(2) = 1$ and

Inductive Hypothesis: Let $K > 2$, suppose as inductive hypothesis that $H(2i) = i$ for all i such that $1 \leq i < K$

Inductive Step: Using the recurrence relation, we've

$K - 2 > 0 \Rightarrow H(2(K-2)+4) + H(2(K-2)) = 2H(2(K-2)+2)$

and by inductive hypothesis, $K - 2 > 0 \Rightarrow K - 2 \geq 1$ or

$\underline{1 \leq K-2 < K-1 < K}$, meaning $\underline{H(2(K-2)) = K-2}$ and

$\underline{H(2(K-1)) = K-1} \Rightarrow H(2K-4+4) + H(2(K-2)) =$

$= 2H(2K-4+2) = 2H(2K-2) = 2H(2(K-1)) = 2(K-1)$

$H(2K) + H(2(K-2)) = H(2K) + K - \cancel{2} = 2K - \cancel{2} \Rightarrow \underline{H(2K) = K}$

Hence, we proved $\underline{H(2K) = K}$ as required

Therefore, by strong induction on n, we concluded $\boxed{H(a_n)=n \text{ for all } n\geq 1}$ Now, combining this result $\boxed{V}$

with previous one, we find $\boxed{H(a_n)=H(a_{n-1})=n, \forall n\geq 1}$

4) We can define an ordering on $N^+ \times N^+$, by specifying that (x_1, y_1) is less than or equal to (x_2, y_2) iff either $x_1 < x_2$, or $(x_1 = x_2$ and $y_1 < y_2)$; this is called the lexicographic ordering. From this, we can recursively define the terms $f(m,n)$ with $m, n \in N^+$, and prove results about them using mathematical induction

Using $(m,n)=(1,1)$ as the basis case, we use strong induction: Basis step $\rightarrow$ Let $(m,n)=(1,1)$, and from recursive definition, $f(1,1)=5=2\cdot 2+1=2(1+1)+1$ So, $f(1,1)=2(1+1)+1$ and this completes the basis step

Inductive step $\rightarrow$ Suppose that $f(m', n')=2(m'+n')+1$ whenever (m', n') is less than (m,n) in the lexicographic ordering of $N^+ \times N^+$. By the recursive definition, if $n=1$, then $f(m,n)=f(m-1,n)+2$ with $m>1$. Because $(m-1,n)$ is smaller than (m,n), inductive hypothesis tells that $f(m-1,n)=2(m+n-1)+1=2m+2n-1$

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So, $f(m,n)=2m+2n-1+2=2(m+n)+1$, giving us the desired equality. Now, suppose $n>1$, then from the definition $\Rightarrow f(m,n)=f(m,n-1)+2$ and since $(m,n-1)$ is smaller than (m,n), inductive hypothesis tells us $f(m,n-1)=2(m+n-1)+1=2(m+n)-1$, so $f(m,n)=$ $=2(m+n)-1+2=2(m+n)+1$. This finishes the inductive step $\Rightarrow \boxed{f(a,b)=2(a+b)+1,\ \forall a,b\in N^+}$ $\boxed{V}$ ■

Note: In inductive step, when we assumed result was true for all $(m',n')<(m,n)$ in the lexicographic ordering of $N^+\times N^+$, it's easy to see that $(m,n)\neq$ $\neq(1,1)$, as this case was already verified. Therefore, if $n=1$, we had $m>1 \Rightarrow$ generalized induction proved the problem $\boxed{V}$ ■

1) From recursive definition, $f(0,n)=n+1$ and choosing $n>0 \Rightarrow f(1,n)=f(0,f(1,n-1))$, then

Claim: $f(1,n)=n+2$ for all $n\geq 0$ (*)

Proof: We apply induction on n. The base case: For $n=0$, $f(1,0)=f(0,1)=1+1=2$, since we used only recursive definitions, and $f(1,0)=0+2$ becomes satisfied V basic step is proved

Inductive hypothesis: Let $k>0$, and suppose as true inductive hypothesis that $f(1, k-1) = k-1+2 = k+1$ is for some $k>0$

Inductive Step: Using the recurrence relation, $1>0$ and $k>0 \Rightarrow f(1,k) = f(0, f(1,k-1))$ and since $f(1,k-1) = k+1$ from inductive hypothesis, we get $f(1,k) = f(0, k+1) = k+1+1 = k+2$ from given recursive definition. Hence, $f(1,k) = k+2$ and inductive step is proved

Hence, by induction on $n \Rightarrow \boxed{f(1,n) = n+2, \ \forall n \geq 0}$ ☒

Claim: $f(2,m) = 2m+3$, for all $m \geq 0$

Proof: We apply induction on m. The base step:
$m=0 \Rightarrow f(2,0) = f(1,1)$ from recursive definition, and from previous claim, $f(1,1) = 1+2 = 3 \Rightarrow f(2,0) = 2 \cdot 0 + 3$
The basis step is satisfied ✓

Inductive Hypothesis: Let $k>0$, and assume as inductive hypothesis that $f(2, k-1) = 2(k-1)+3 = 2k-2+3 = 2k+1$ is true for some $k>0$

Inductive Step: Using the recurrence relation, $2>0$ and $k>0 \Rightarrow$

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$f(2,k)=f(1,f(2,k-1))=f(1,2k+1)$, from recurrence relation and inductive hypothesis, respectively. Using previous claim that $f(1,n)=n+2$ for $\forall n\geq 0 \Rightarrow$

$f(1,2k+1)=2k+1+2=2k+3$ because $2k+1>1>0 \Rightarrow$

$f(2,k)=2k+3$ and by this, inductive step is done

Hence, induction on m concludes $\boxed{f(2,m)=2m+3, \forall m\geq 0}$

Claim: $f(3,t)=2^{t+3}-3$ for all $t\geq 0$

Proof: We apply induction on t. Base step: For $t=0$, $f(3,0)=f(2,1)$ from definition, and from the previous claim $\Rightarrow f(2,1)=2\cdot 1+3=5$, $f(3,0)=5$ or $f(3,0)=8-3=2^3-3=2^{0+3}-3$, $f(3,0)=2^{0+3}-3$ is true

Basis step is satisfied ✓

Inductive Hypothesis: Let $k>0$, and suppose as inductive hypothesis that $f(3,k-1)=2^{k-1+3}-3=2^{k+2}-3$ is satisfied for some $k>0$ ✓

Inductive Step: Using the recurrence definition for $3>0$ and $k>0 \Rightarrow f(3,k)=f(2,f(3,k-1))$ and since $f(3,k-1)=2^{k+2}-3$ from inductive hypothesis, we get $f(3,k)=f(2,2^{k+2}-3)$ and $2^{k+2}-3>2^{0+2}-3=4-3=1>0 \Rightarrow$

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from previous claim, $f(2, 2^{k+2}-3) = 2(2^{k+2}-3)+3 =$
$= 2^{k+3}-6+3 = 2^{k+3}-3$, hence $f(3,k) = 2^{k+3}-3$ is true

By this, inductive step is done ✓

Hence, by induction on $t \Rightarrow \boxed{f(3,t) = 2^{t+3}-3, \forall t \geq 0}$ ☒

Henceforth, using mathematical induction iteratively, we conclude $\boxed{f(3,n) = 2^{n+3}-3 \text{ for all } n \geq 0}$ ☒ and in particularly, $f(3,4) = 2^{4+3}-3 = 2^7-3 = 128-3 = 125$,

$\boxed{f(3,4) = 125}$ ☒ $\boxed{\checkmark}$ ■

5) Let a_n represent the # of ways to climb n stairs if the person climbing the stairs can take 1, 2, or 3 stairs at a time

<u>First case:</u> The last step is taken by 1 stair, then there are a_{n-1} ways to climb the first $(n-1)$ steps

<u>Second case:</u> The last step is taken by 2 stairs, then there are a_{n-2} ways to climb the first $(n-2)$ steps

<u>Third case:</u> The final step is taken by 3 stairs, then there are a_{n-3} ways to climb the first $(n-3)$ steps.

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Since all these 3 cases are mutually distinct, we can add the # of sequences of all three cases, and get

$\boxed{a_n = a_{n-1} + a_{n-2} + a_{n-3}}$ ⊠

For $\underline{n=0}$, there's exactly 1 way to climb the stairs (taking 0 steps) $\Rightarrow \boxed{a_0 = 1}$

For $\underline{n=1}$, there exists exactly 1 way to climb the stairs (just take 1 step) $\Rightarrow \boxed{a_1 = 1}$

For $\underline{n=2}$, there are exactly 2 ways to climb the stairs (just take 2 steps, or (1 step & 1 step)) $\Rightarrow \boxed{a_2 = 2}$

For $\underline{n=3}$, there are exactly 4 ways to climb the stairs (1 stair & 1 stair & 1 stair, 2 stairs & 1 stair, 1 stair & 2 stairs, or just 3 stairs) $\Rightarrow \boxed{a_3 = 4}$

Notice that $a_3 = 4 = 2+1+1 = a_2 + a_1 + a_0$, hence the recurrence relation holds for $n=3$, and it's not necessary to be an initial condition. Henceforth,

⊠ ⊠

$\boxed{a_0 = 1,\ a_1 = 1,\ a_2 = 2, \text{ and } a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for all } n \geq 3}$ $\boxed{\checkmark}$ ■

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6) a) First application: Apply the recursive step on $(0,0)$:

$(0,0+1)=(0,1)\in S$; $(0+1,0+1)=(1,1)\in S$; $(0+2,0+1)=$
$=(2,1)\in S \Rightarrow \boxed{(0,1),\ (1,1),\ (2,1)\in S}$

Second application: Apply the recursive step on $(0,1)$, $(1,1)$, and $(2,1)$:

$(0,1+1)=(0,2)\in S$; $(0+1,1+1)=(1,2)\in S$; $(0+2,1+1)=$
$=(2,2)\in S$; $(1,1+1)=(1,2)\in S$; $(1+1,1+1)=(2,2)\in S$;
$(1+2,1+1)=(3,2)\in S$; $(2,1+1)=(2,2)\in S$; $(2+1,1+1)=$
$=(3,2)\in S$; $(2+2,1+1)=(4,2)\in S \Rightarrow$ Henceforth,
$\Rightarrow \boxed{(0,2),\ (1,2),\ (2,2),\ (3,2),\ (4,2)\in S}$

Third application: Apply the recursive step on $(0,2)$, $(1,2)$, $(2,2)$, $(3,2)$, $(4,2)$:

$(0,2+1)=(0,3)\in S$, $(0+1,2+1)=(1,3)\in S$, $(0+2,2+1)=$
$=(2,3)\in S$, $(1,2+1)=(1,3)\checkmark$ $(1+1,2+1)=(2,3)\checkmark$
$(1+2,2+1)=(3,3)\in S$, $(2,2+1)=(2,3)\checkmark$ $(2+1,2+1)=(3,3)\checkmark$
$(2+2,2+1)=(4,3)\in S$, $(3,2+1)=(3,3)\checkmark$ $(3+1,2+1)=(4,3)\checkmark$
$(3+2,2+1)=(5,3)\in S$

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$(4,2+1)=(4,3)$ ✓ $(4+1,2+1)=(5,3)$ ✓ $(4+2,2+1)=\underline{(6,3)\in S}$

Hence, $\boxed{(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\in S}$

<u>Fourth application:</u> Apply the recursive step on $(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)$:

$(0,3+1)=\underline{(0,4)\in S}$, $(0+1,3+1)=\underline{(1,4)\in S}$, $(0+2,3+1)=$

$=\underline{(2,4)\in S}$; $(1,3+1)=(1,4)$ ✓ $(1+1,3+1)=(2,4)$ ✓

$(1+2,3+1)=\underline{(3,4)\in S}$; $(2,3+1)=(2,4)$ ✓ $(2+1,3+1)=(3,4)$ ✓

$(2+2,3+1)=\underline{(4,4)\in S}$; $(3,3+1)=(3,4)$ ✓ $(3+1,3+1)=(4,4)$ ✓

$(3+2,3+1)=\underline{(5,4)\in S}$; $(4,3+1)=(4,4)$ ✓ $(4+1,3+1)=(5,4)$ ✓

$(4+2,3+1)=\underline{(6,4)\in S}$; $(5,3+1)=(5,4)$ ✓ $(5+1,3+1)=(6,4)$ ✓

$(5+2,3+1)=\underline{(7,4)\in S}$; $(6,3+1)=(6,4)$ ✓ $(6+1,3+1)=(7,4)$ ✓

$(6+2,3+1)=\underline{(8,4)\in S}\Rightarrow$ Hence, from 4^{th} application,

$\boxed{(0,4),(1,4),(2,4),(3,4),(4,4),(5,4),(6,4),(7,4),(8,4)\in S}$

B) <u>Proof by induction:</u> Let $P(n)$ be "$a\leq 2b$ is true in the n^{th} application of the recursive step"

<u>Basis step:</u> For $n=0$, notice $(0,0)\in S$ and $0\leq 2\cdot 0=0$

Thus, $\boxed{P(0)=\text{true}}$

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Inductive step: We assume that $a \leq 2b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step and consider an element obtained with $(k+1)$ applications of recursive steps. We have to show that $P(k+1)$ is also true. Let $(a, b) \in S$ in the k^{th} application of the recursive step with $a \leq 2b$, then we get $(a, b+1) \in S$, and $(a+1, b+1) \in S$, and $(a+2, b+1) \in S$ in the $(k+1)^{th}$ application $\Rightarrow$ $a \leq 2b \leq 2(b+1)$, $\boxed{a \leq 2(b+1)}$

$a \leq 2b \leq 2b+1$ or $a+1 \leq 2b+2 = 2(b+1)$, $\boxed{a+1 \leq 2(b+1)}$

$a \leq 2b \Rightarrow a+2 \leq 2b+2 = 2(b+1)$; $\boxed{a+2 \leq 2(b+1)}$ implying that $P(k+1)$ is true ✓

From the strong induction, $\boxed{P(n) \text{ is true for all } n \geq 0}$ which states $\boxed{a \leq 2b \text{ whenever } (a, b) \in S}$ $\boxed{V}$ ■

Note: In inductive step, because the final application of the recursive step to an element (a, b) needed to be applied to an element obtained with fewer applications of the recursive step, we knew that $a \leq 2b$ was true. The rest was to use inequalities ■ ⊕

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