# Other probability problems

1) If $X$ is the number of defective items in the sample, then $X$ is a binomial random variable with (Theorem on slide 12) parameters $(3, 0.1) \Rightarrow$ Thus, the desired probability is given by $P\{X=0\} + P\{X=1\} = \binom{3}{0}(0.1)^0(0.9)^3 + \binom{3}{1}(0.1)^1(0.9)^2 = (0.9)^3 + 3\cdot(0.1)(0.9)^2 = 0.729 + 0.243$
$= \boxed{0.972}$ ✓

2) Let $W$ be the event that a white ball is drawn, and let $H$ be the event that the coin comes up heads. The desired probability $P(H|W)$ may be calculated as follows:

$$P(H|W) = \frac{P(H \text{ and } W)}{P(W)} = \frac{P(W|H)\,P(H)}{P(W)} =$$

$$= \frac{P(W|H)\,P(H)}{P(W|H)P(H) + P(W|\sim H)P(\sim H)} = \frac{\frac{2}{9}\cdot\frac{1}{2}}{\frac{2}{9}\cdot\frac{1}{2} + \frac{5}{11}\cdot\frac{1}{2}} =$$

$$= \frac{\frac{2}{9}}{\frac{2}{9}+\frac{5}{11}} = \frac{\frac{2}{9}}{\frac{22+45}{99}} = \frac{2}{9}\cdot\frac{\overset{11}{\cancel{99}}}{67} = \boxed{\frac{22}{67}}$$ ✓

where we had that $P(H) = P(\sim H) = \frac{1}{2}$

$P(W|H) = P(\text{draw a white ball from } 1^{st} \text{ bag}) = \frac{2}{9}$, $P(W|\sim H) = P(\text{white ball from } 2^{nd} \text{ bag}) = \frac{5}{11}$

3) We will prove for the general $k$; namely, what is the probability that I will get infected when I meet with $k$ people, where $k \geq 1$?

Leaving out myself, there are 10 infected in the 99 remaining people. Since $P\left(\begin{matrix}\text{get infected when} \\ \text{meet with } k \text{ people}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{not get infected when} \\ \text{meet with } k \text{ people}\end{matrix}\right)$ and if any of these $k$ people have caught the virus, then I will get infected as well. Thus, in order to not getting infected, all of these $k$ people should be healthy $\Rightarrow$ there are 10 infected, and 89 healthy people (excluding myself, since I meet with these people) $\Rightarrow$ Choosing $k$ healthy people out of 89 such people is equivalent to $\binom{89}{k}$ number of possible combinations, meaning that $\underbrace{|\text{desired cases}| = \binom{89}{k}}$

and there are total of $\binom{99}{k}$ possible choosings of $k$ people out of 99 remaining ones $\Rightarrow \underbrace{|\text{all cases}| = \binom{99}{k}}$

Hence, $P\left(\begin{matrix}\text{not get infected} \\ \text{when meet with } k \text{ people}\end{matrix}\right) = \frac{|\text{desired cases}|}{|\text{all cases}|} = \frac{\binom{89}{k}}{\binom{99}{k}}$, and using the "complementary" property, we find that

$$\boxed{P(\text{get infected when meet with } k \text{ people}) = 1 - \frac{C(89,k)}{C(99,k)} \quad \text{for each } k \geq 1}$$

Now, plugging the correspondent values for k in our problem

a) $\underline{k=1} \Rightarrow \text{probability} = 1 - \dfrac{\binom{89}{1}}{\binom{99}{1}} = 1 - \dfrac{\frac{89!}{88!\,1!}}{\frac{99!}{1!\,98!}} = 1 - \dfrac{89}{99} =$

$= \dfrac{99-89}{99} = \boxed{\dfrac{10}{99}}$

b) $\underline{k=2} \Rightarrow \text{probability} = 1 - \dfrac{\binom{89}{2}}{\binom{99}{2}} = 1 - \dfrac{\frac{89!}{2!\,87!}}{\frac{99!}{2!\,97!}} = 1 - \dfrac{89 \cdot 88}{99 \cdot 98}$

$= 1 - \dfrac{7832}{9702} = \dfrac{9702-7832}{9702} = \dfrac{1870}{9702} = \boxed{\dfrac{935}{4851}}$

$\dfrac{935}{4851} = \boxed{\dfrac{85}{441}}$

c) $\underline{k=3} \Rightarrow \text{probability} = 1 - \dfrac{\binom{89}{3}}{\binom{99}{3}} = 1 - \dfrac{\frac{89!}{3!\,86!}}{\frac{99!}{3!\,96!}} =$

$= 1 - \dfrac{89 \cdot \cancel{88}^{\,4} \cdot \cancel{87}^{\,29}}{\cancel{99}_{\,\cancel{33}\,3} \cdot \cancel{98}_{\,49} \cdot 97} = 1 - \dfrac{89 \cdot 4 \cdot 29}{3 \cdot 49 \cdot 97} = 1 - \dfrac{10324}{14259} =$

$= \dfrac{3935}{14259} = \dfrac{5 \cdot 787}{3 \cdot 7^2 \cdot 97} = \boxed{\dfrac{3935}{14259}}$ ■

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## Homework-11: Integer random variables, Bernoulli trial; Expected value, Linearity of Expectation

1) 9 of the 100 numbers from 1 to 100 consist of 1 digit (that's, from 1 to 9), while 90 of the 100 numbers from 1 to 100 consist of 2 digits (that is, from 10 to 99), and 1 of the 100 numbers from 1 to 100 consist of 3 digits

| # of digits | 1 | 2 | 3 |
|---|---|---|---|
| Probability $(P(X=x))$ | $\frac{9}{100}$ | $\frac{90}{100}$ | $\frac{1}{100}$ |

The expected value (or mean) is the sum of the product of each possibility $x$ with its probability $P(X=x)$:

$$\mu = \sum x P(X=x) = 1 \times \frac{9}{100} + 2 \times \frac{90}{100} + 3 \times \frac{1}{100} = \frac{9+180+3}{100} =$$

$$= \frac{192}{100} = \boxed{1.92}$$ ✓

2) 50 true/false questions $n=50$ and $p=0.9$

The expected # of successes (heads) among a fixed number $n$ of mutually independent Bernoulli trials is $np$, with $p$ being the constant probability of success $\Rightarrow E(X) = np = 50 \times 0.9 = 45$ (expected # of correct answers in true/false)

25 multiple-choice questions $n=25$ and $p=0.8$

The expected # of successes (heads) among a fixed number $n$ of mutually independent Bernoulli trials is $np$, where $p$ - constant probability of success

$E(Y)=np=25\times0.8=20$ (expected # of correct answers in multiple-choice questions)

Total Properties of expected value:

$E(X_1+X_2+...+X_n)=E(X_1)+E(X_2)+...+E(X_n)$ and

$E(aX+b)=aE(X)+b$ $\boxed{\text{Linearity of Expectation}}$

We have to determine the expected total score consisting of the 50 true/false questions (worth 2 points) and 25 multiple-choice questions (worth 4 points):

$E(2X+4Y)=E(2X)+E(4Y)=2E(X)+4E(Y)=$

$=2\times45+4\times20=90+80=170 \Rightarrow$ Thus, $\boxed{\text{the expected score on the final is 170}}$

Proof of "$np$": Let $X_i$ be the random variable with $X_i((t_1,t_2,...,t_n))=1$ if $t_i$ is a success, and $X_i((t_1,...,t_n))=0$ if $t_i$ is a failure. The expected value of $X_i$ is $E(X_i)=1\times p+0\times(1-p)=p$ for $i=1,2,...,n$. Let $X=X_1+X_2+...+X_n$, so that $X$ counts # of successes when these $n$ Bernoulli trials are performed; then $E(X)=E(X_1)+...+E(X_n)=np$ $\boxed{Y}$

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Note: Here, we denoted X to be the random variable giving the score on the true/false questions, and Y to be the random variable giving the score on the multiple-choice questions. In both true/false, as well as multiple-choice questions, the expected number of questions answered correctly is the expectation of # of successes in n Bernoulli trials. If probability of success is p, and number of trials n, the expected number of successes is np ✓ (computing overall expectation by summing over the expectations of individual questions)

4) Let X be the number on first die, and let W be the number on 2nd die ⇒ Thus, $Y = X + W$

(probability of each outcome is $\frac{1}{6}$)

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} =$$

(it's because of fair die)

$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{\cancel{6} \cdot 7}{2 \cdot \cancel{6}} = \frac{7}{2} \Rightarrow \boxed{E(X) = \frac{7}{2}}$$

Similarly, we get that $\boxed{E(W) = \frac{7}{2}}$ Since $Y = X + W$ is true, we get from Linearity of Expectation, $E(Y) = E(X+W) = E(X) + E(W) = \frac{7}{2} + \frac{7}{2} = 7 \Rightarrow \boxed{E(Y) = 7}$ From the definition of expectation,

$E(X^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} =$

$= \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{\frac{6\cdot 7\cdot 13}{6}}{6} = \frac{91}{6} \Rightarrow \boxed{E(X^2) = \frac{91}{6}}$

We know that $W$ and $X$ are independent (they are different dice) which means $E(WX) = E(W)\cdot E(X) = \frac{7}{2}\cdot\frac{7}{2} = \frac{49}{4}$ (famous property)

$\boxed{E(WX) = \frac{49}{4}}$ From the linearity of expectation,

$E(XY) = E(X(X+W)) = E(X^2+XW) = E(X^2) + E(XW) =$

$= \frac{91}{6} + \frac{49}{4} = \frac{182}{12} + \frac{147}{12} = \frac{329}{12}$, thus $\boxed{E(XY) = \frac{329}{12}}$

As we had $E(X) = \frac{7}{2}$ and $E(Y) = 7 \Rightarrow \boxed{E(X)E(Y) = \frac{49}{2}}$

Clearly, $\frac{329}{12} = E(XY) > \frac{294}{12} = \frac{49}{2} = E(X)E(Y)$, meaning

$\boxed{E(XY) > E(X)E(Y)}$, or just $\boxed{E(XY) \neq E(X)E(Y)}$ ■

<u>Note:</u> Since $X$ has non-zero variance, $Var(X) = E(X^2) - E(X)^2 > 0$, or just $E(X^2) > E(X)^2 \Rightarrow E(XY) = E(X(X+W))$
$= E(X^2+XW) = E(X^2) + E(XW) = E(X^2) + E(X)E(W) >$
$> E(X)^2 + E(X)E(W) = E(X)(E(X)+E(W)) = E(X)E(X+W)$
$= E(X)E(Y)$, where we used "linearity of expectation" and $W, X$-inde.

2) The number $X$ of Heads in 4 independent tosses of fair coins has $X \sim \text{Binom}(n=4, p=\frac{1}{2})$. We seek

$$P(X=2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = 6 \cdot \frac{1}{16} = 0.375 \quad \checkmark \; \blacksquare$$

By "brute-force" listing,

- TTTT
- TTTH TTHT THTT HTTT
- TTHH THTH THHT HTTH HTHT HHTT
- HHHT HHTH HTHH THHH
- HHHH

We want 3rd row, thus $P = \dfrac{\text{good cases}}{\text{total cases}} = \dfrac{\binom{4}{2}}{\sum_{i=0}^{4}\binom{4}{i}} =$

$$= \frac{\frac{4 \cdot 3}{2}}{2^4} = \frac{6}{16} \quad \checkmark \; \blacksquare$$

We can also see that # of total cases $= 16 = 1+4+6+4+1$ (from listing).

Two results for each of 4 coin flips. When ways to perform tasks in series, we multiply. So that, in our case, there are $2 \times 2 \times 2 \times 2 = 16$ results in total.

For the favorable cases, we need to count the ways to get 2 heads and 2 tails. The count of permutations of 2 pairs of symbols is $\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$. This was also easily confirmed by just counting third row $\Rightarrow$ Thus, probability is $\boxed{\frac{6}{16} = 0.375}$ $\checkmark$ $\blacksquare$

In general, for n fair coins and obtaining k heads and $(n-k)$ tails, the probability is $\boxed{\dfrac{\binom{n}{k}}{2^n}}$ $\left(\begin{array}{c}\text{our case,}\\ n=4, k=2\end{array}\right)$

5) Suppose that the probability that a coin comes up tails is $p$ (we'll consider our case at the end). We first note that the sample space consists of all sequences that begin with any number of heads, with denoted by H, followed by a tail, denoted by T ⇒ Thus, sample space is the set $\{T, HT, HHT, HHHT, \ldots\}$. Note, this is an infinite sample space. We can determine the probability of an element of the sample space by noting that coin flips are independent and that the probability of a head is $(1-p) \Rightarrow P(T)=p$, $P(HT)= (1-p)p$, $P(HHT)=(1-p)^2 p$, and in general, the probability that the coin is flipped n times before a tail comes up, that's, that $\underline{(n-1) \text{ heads come up followed by a tail, is } (1-p)^{n-1}p}$. In fact, we can see

$$1+(1-p)+(1-p)^2+\ldots=\sum_{k=0}^{\infty}(1-p)^k=\frac{1}{1-(1-p)}=\frac{1}{p}, \text{ since } |1-p|<1$$

Then, $$p+p(1-p)+p(1-p)^2+\ldots=\sum_{k=0}^{\infty}p(1-p)^k=\boxed{\sum_{n=1}^{\infty}p(1-p)^{n-1}=1}$$
meaning, sum of probabilities of the points in the sample space is 1

Now, let $X$ be the random variable equal to the # of flips in an element in the sample space. That is, $X(T)=1$, $X(HT)=2$, $X(HHT)=3$, and so on. Note that $\boxed{p(X=j)=(1-p)^{j-1}p}$ The expected # of flips until the coin comes up tails $= E(X)$. From the definition, we know $E(X)=\sum_{j=1}^{\infty} j\cdot p(X=j)=\sum_{j=1}^{\infty} j(1-p)^{j-1}p=$

$$=p\cdot\sum_{j=1}^{\infty}\frac{j(1-p)^j}{1-p}=\frac{p}{1-p}\sum_{j=1}^{\infty} j(1-p)^j$$

We have from power rule for derivatives that $\frac{d}{dx}\sum_{n\geq1} x^n=\sum_{n\geq1} nx^{n-1}$, and from the sum of infinite geometric sequence, $\sum_{n\geq1} x^n = x+x^2+\ldots=\frac{1}{1-x}-1=\frac{1-1+x}{1-x}=$

$=\frac{x}{1-x}$, where $|x|<1 \Rightarrow \frac{d}{dx}\sum_{n\geq1} x^n=\frac{d}{dx}\left(\frac{x}{1-x}\right)=\frac{x'(1-x)-x(1-x)'}{(1-x)^2}$

$$=\frac{1-x-x(-1)}{(1-x)^2}=\frac{1-x+x}{(1-x)^2}=\frac{1}{(1-x)^2}, \text{ thus}$$

$\boxed{\sum_{n\geq1} nx^{n-1}=\frac{1}{(1-x)^2}, \text{ with } |x|<1}$ Here, we used power rule for derivatives: $\frac{d}{dx}(x^n)=nx^{n-1}$, $n\geq1$

Sum of infinite geometric sequence: $\sum_{n=0}^{\infty} z^n=\frac{1}{1-z}$, where $|z|<1$

Chain Rule for derivatives: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, where $u, v$ - functions of $x$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{d}{dx}(u)v - u \cdot \frac{d}{dx}(v)}{v^2}.$$

Hence, we got $\sum_{n \geq 1} n x^{n-1} = \frac{1}{(1-x)^2}$, with $|x| < 1 \Rightarrow$

plugging $x = 1 - p$ where $p \neq 1 \Rightarrow 1 > 1 - p > 0 > -1$, $|1-p| < 1$

and then, $\sum_{n \geq 1} n(1-p)^{n-1} = \frac{1}{(1-(1-p))^2} = \frac{1}{(1-1+p)^2} = \frac{1}{p^2}$

$\boxed{\sum_{n \geq 1} n(1-p)^{n-1} = \frac{1}{p^2}}$ Since we had $E(X) = \sum_{j \geq 1} j(1-p)^{j-1} p$
$= p \sum_{j \geq 1} j(1-p)^{j-1} \Rightarrow$

$E(X) = p \sum_{j \geq 1} j (1-p)^{j-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}$, thus $\boxed{E(X) = \frac{1}{p}}$

It follows that the expected # of times the coin is flipped until tails comes up is $\frac{1}{p}$. Now, let's solve the given problem via these rules. Note that when the coin is fair, we have $\boxed{p = \frac{1}{2}}$

a) From the above, $p(X=5) = (1-p)^4 p = \left(\frac{1}{2}\right)^4 \frac{1}{2} = \frac{1}{32}$

Thus, $\boxed{p(X=5) = \frac{1}{32}}$ - probability that exactly 5 flips are made

b) From above, $E(X) = \frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$, $\boxed{E(X) = 2}$ - expected # of flips