Ch 2. Basic Structures: Sets, Functions
Ch 9. Relations

Sets

Sungwon Kang

Acknowledgement

- [Rosen 19] Kenneth H. Rosen, for Discrete Mathematics & Its Applications (8th Edition), Lecture slides
- [Hunter 11] David J. Hunter, Essentials of Discrete Mathematics, 2nd Edition, Jones & Bartlett Publishers, 2011, Lecture Slides

Ch 2. Basic Structures: Sets, Functions

Sets

1. Set Membership
2. Subset & Equality Relationship
3. Constructing New Sets form Existing Sets
4. Set Identities
5. Discussion on Set Comprehension

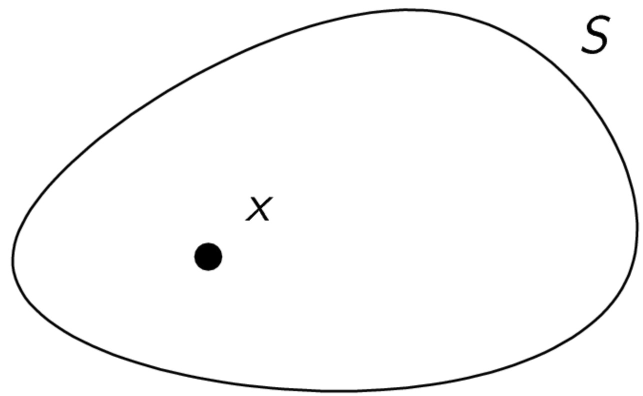
1. Set Membership

$x \in S$ denotes that x is contained in S.

or x is an element of S
or x is a member of S

$x \notin S$ denotes that x is not an element of S

$x \in S$ denotes that x is contained in S.



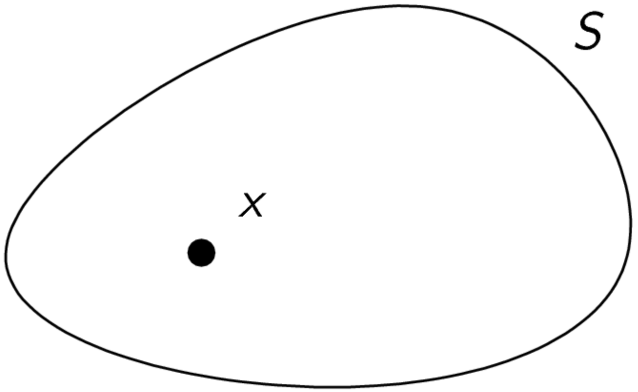
Set Examples

$\varnothing$, { }: the empty set, the null set
{Larry, Joe, Moe}

$\mathbb{N}$: the set of natural numbers including 0
$\mathbb{N}^+$: the set of natural numbers excluding 0
$\mathbb{Z}$: the set of integers
$\mathbb{Q}$: the set of rational numbers
$\mathbb{R}$: the set of real numbers

Defining Sets

$x \in S$ denotes that x is contained in S.



What is the limitation of this approach?

Two ways to define sets:

(1) Listing elements of a set: $S = \{x_1, x_2, \ldots, x_n\}$

[redacted]

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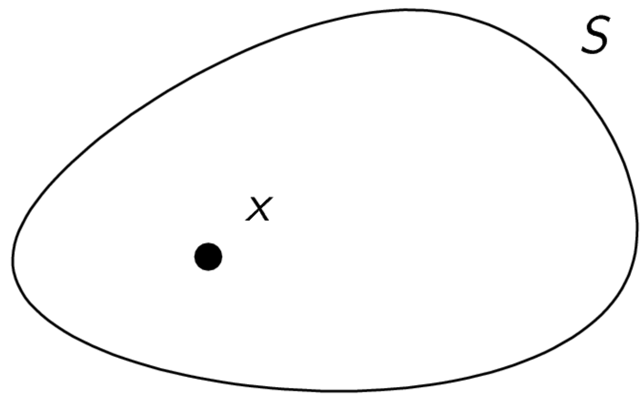


What is the limitation of this approach?

Two ways to define sets:

(1) Listing elements of a set: $S = \{x_1, x_2, \ldots, x_n\}$

(2) Denoting defining property of a set: $\{x \in S \mid x \text{ has property } p\}$

(= set comprehension)

How can we describe properties and relationships of variables for set comprehension ?

☛ Use predicate logic

Example

The set of all even numbers

=

How can we describe properties and relations of variables for set comprehension ?

☛ Use predicate logic

Example

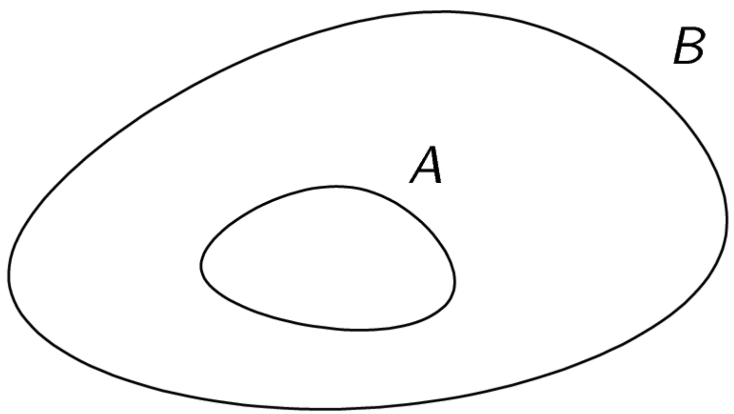
The set of all even numbers

$= \{x \in \mathbb{Z} \mid \exists y \; (y \in \mathbb{Z} \land x = 2 \times y) \}$

2. Subset Relationship

"$A \subseteq B$" means "$(\forall x)(x \in A \rightarrow x \in B)$."

definition



For example,

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}.$$

Set Equality

Definition 1. (Set Equality)
Two sets A and B are equal (written $A = B$) if and only if $A \subseteq B$ and $B \subseteq A$.

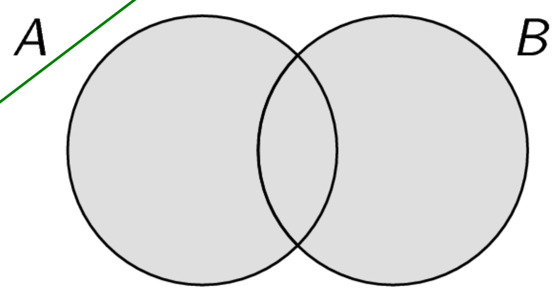
Definition 2. (Set Equality)
Two sets A and B are equal ($A = B$) if and only if $\forall x(x \in A \leftrightarrow x \in B)$ holds.

Union and Intersection

Definition of set union: $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$.

In predicate logic: $(\forall x)(x \in A \cup B \leftrightarrow (x \in A) \vee (x \in B))$

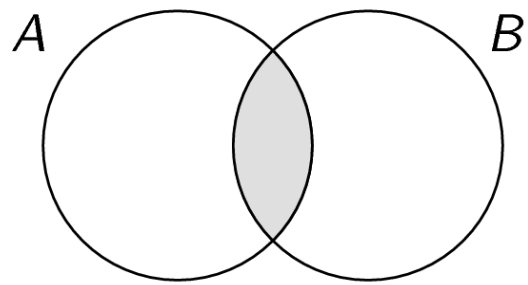
That is, rewriting the definition in predicate logic



Operations being defined

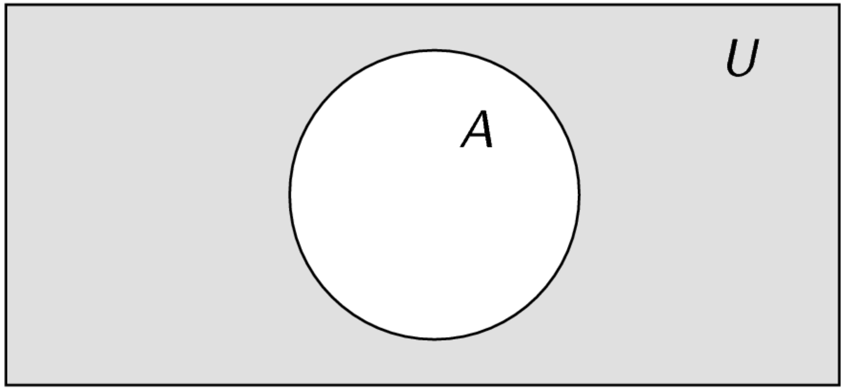
Definition of set intersection: $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$.

In predicate logic: $(\forall x)(x \in A \cap B \leftrightarrow (x \in A) \wedge (x \in B))$



Complement

Define: $A' = \{x \in U \mid x \notin A\}$.

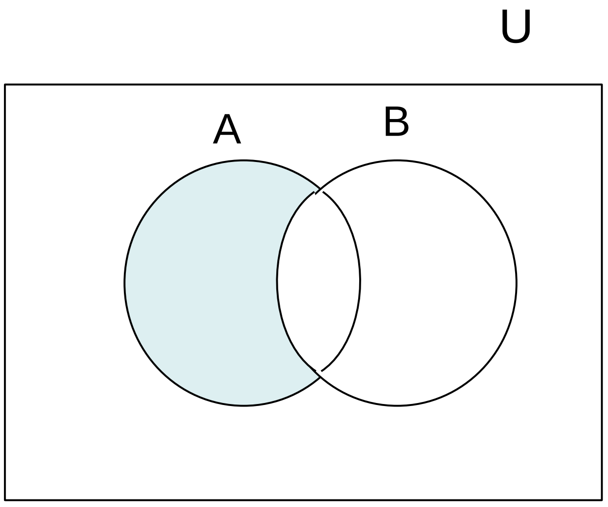


Note that we could also write $A' = \{x \in U \mid \neg(x \in A)\}$ to make the use of the $\neg$ connective explicit.

Set Difference

A - B = $\{x \in U \mid x \in A \wedge \neg(x \in B)\}$

"A - B" is also written "A \ B"



Examples [redacted]

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and suppose the universal set is $U = \{1, 2, \ldots 10\}$. Then

$$
\begin{aligned}
A \cup B &= \\
A \cap B &= \\
B' &= \\
\text{A - B} &=
\end{aligned}
$$

Examples

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and suppose the universal set is $U = \{1, 2, \ldots 10\}$. Then

$$
\begin{aligned}
A \cup B &= \{1, 2, \ldots, 8\} \\
A \cap B &= \{4, 5\} \\
B' &= \{1, 2, 3, 9, 10\} \\
A - B &= \{1, 2, 3\}
\end{aligned}
$$

Cartesian products and power sets

The *Cartesian product* $A \times B$ of two sets A and B is the set of all ordered pairs where the first item comes from the first set and the second item comes from the second set. Formally,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

The *power set* $\mathcal{P}(S)$ of the set S is the set of all subsets of S:

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}.$$

Examples [redacted]

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and suppose the universal set is $U = \{1, 2, \ldots 10\}$. Then

$$(A \cap B) \times A \quad =$$

$$\mathcal{P}(A \cap B) \quad =$$

Examples [redacted]

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and suppose the universal set is $U = \{1, 2, \ldots 10\}$. Then

$$
\begin{aligned}
(A \cap B) \times A &= \{(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),\\
&\qquad (5,2),(5,3),(5,4),(5,5)\}\\
\mathcal{P}(A \cap B) &= \{\emptyset, \{4\}, \{5\}, \{4,5\}\}
\end{aligned}
$$

Set of Strings over an Alphabet

Let Σ be a set of symbols $\{a_1, a_2, ..., a_n\}$, called an *alphabet*.

Let a string be a sequence of symbols. (We will defined this precisely later.)

The string of length 0 is called the *empty string* or the null string and is denoted λ (= ε = “”).

Σ^0 : set of strings of length 0, i.e. $\{\lambda\}$

Σ^1 : set of strings of length 1, i.e. $\{a_1, a_2, ..., a_n\} = \Sigma$

. . .

Σ^2 : set of strings of length 2, i.e. $\{a_1a_1, a_2a_1, a_2a_1, a_2a_2, ..., a_na_n\}$

. . .

Σ^k : set of strings of length k

Σ^*: the set of all strings of finite length with symbols from Σ, i.e.

$\cup \{\Sigma^k \mid k \in \mathbb{N}\} = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup . . . \cup \Sigma^k \cup . . .$

Cardinality of Set

Cardinality of a set S is the number of elements in S and written $|S|$.

Example

(1) If $|A| = a$ and $|B| = b$, what is $|A \times B|$?

(2) What is $|\mathcal{P}(S)|$?

Cardinality of Set

Cardinality of a set S is the number of elements in S and written |S|.

Example

(1) If |A| = a and |B| = b, what is |A x B|? a x b

(2) What is | $\mathcal{P}$(S) | ? $2^{|S|}$

Identities from propositional logic

The inference rules for propositional logic give identities for set theory. For example, the addition inference rule

$$q \vee p \Leftrightarrow p \vee q$$

allows us to prove the identity

$$B \cup A = A \cup B$$

in set theory.

2nd Definition of Set Equality
When A and B are sets,
$A = B$ iff $\forall x(x \in A \leftrightarrow x \in B)$.

Proof that $B \cup A = A \cup B$

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Proof that $B \cup A = A \cup B$

Let $x \in B \cup A$. Then, by definition of $\cup$, $x \in B \vee x \in A$, which is equivalent to $x \in A \vee x \in B$, by commutativity of $\vee$. Then the last formula implies $x \in A \cup B$, by definition of $\cup$. So $B \cup A \subseteq A \cup B$. Similarly, $A \cup B \subseteq B \cup A$. Thus $B \cup A = A \cup B$.

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws

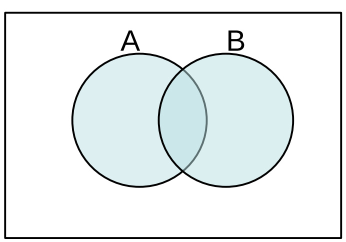
Source:
[Rosen 19] p.136

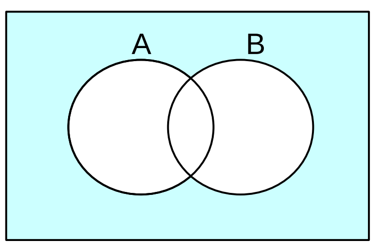
Identity	*Name*
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

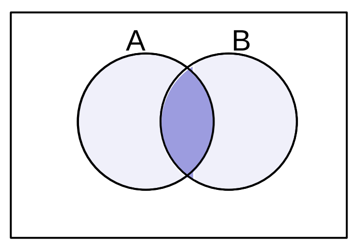
Theorem (Set Theoretic version of De Morgan's laws).

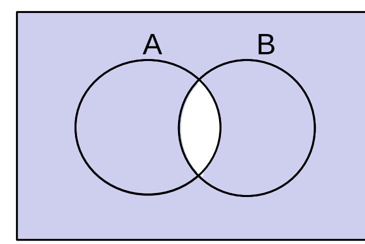
Let A and B be sets. Then

1. $(A \cup B)' = A' \cap B'$
2. $(A \cap B)' = A' \cup B'$

[1a]


[1b]


[2a]


[2b]


Prove Part [1] of the theorem.

De Morgan's Laws: proof

Proof.

We will prove part 1. (Part 2 is similar.) Let $x \in (A \cup B)'$. In other words, if $P(x)$ is the statement "$x \in A$" and $Q(x)$ is the statement "$x \in B$," we are starting with the assumption $\neg(P(x) \vee Q(x))$. By De Morgan's laws for propositional logic, this is equivalent to $\neg P(x) \wedge \neg Q(x)$, which, in the language of set theory, is the same as $x \in A' \cap B'$. We have shown that

$$x \in (A \cup B)' \quad \Leftrightarrow \quad x \in A' \cap B',$$

hence the sets $(A \cup B)'$ and $A' \cap B'$ are equal. $\square$

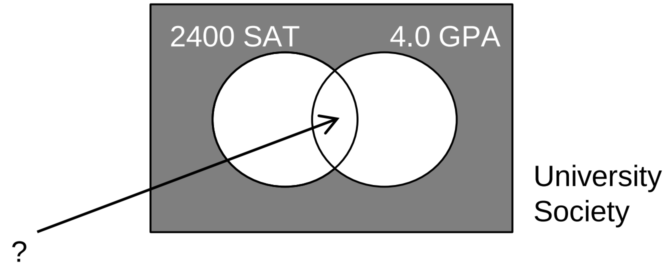
Inclusion-exclusion principle

For finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$

Example

The Master of the Universe Society at a certain college accepts members who have 2400 SAT's or 4.0 GPA's in high school. Of the 11 members of the society, 8 had 2400 SAT's and 5 had 4.0 GPA's. How many members had both 2400 SAT's and 4.0 GPA's?

Solution



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Solution

$11 = 8 + 5 - |A \cap B|$, *so there are two.*

5. Discussion on Set Comprehension

Consider the following:

The barber is the “one who shaves all those, and those only, who do not shave themselves,” The question is, does the barber shave himself?

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Can you describe it with set comprehension?

Let A be the set of such barbers. So

$$A = \{x \mid x \text{ shaves } y \text{ iff } y \text{ does not shave } y\}$$
$$= \{x \mid \forall y(x \text{ shaves } y \leftrightarrow \neg(y \text{ shaves } y))\}$$

If $z \in A$, does z shave z?

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Let $z \in A$. Then

$$\forall y(x \text{ shaves } y \leftrightarrow \neg(y \text{ shaves } y))$$

and by the $\forall$-elim rule

$$z \text{ shaves } z \leftrightarrow \neg(z \text{ shaves } z)$$

which is always false. Therefore $z \notin A$.

What does this mean?

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which is always false. Therefore $z \notin A$.

What does this mean?

$$A = \emptyset.$$

☛ Called *the Barber's paradox* (Not a genuine paradox. It is only paradoxical.)

Is $\{1\} \notin \{1\}$ true? Or is the set $\{1\}$ an element of itself?

Is it a “meaningful” statement?

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Is it a “meaningful” statement?

Consider the following set comprehension.

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Is $A \in A$ true?

☛ If true, then $A \in A$, hence $A \notin A$ must be true.

If false, then $A \notin A$, hence $A \in A$ must be true.

Neither true nor false.

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Is it a "meaningful" statement?

Consider the following set comprehension.

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Is $A \in A$ true?

If true, then $A \in A$, hence $A \notin A$ must be true.

If false, then $A \notin A$, hence $A \in A$ must be true.

Neither true nor false.

☛ Russell's paradox

- ☛ Should not allow all possible descriptions for set comprehension.
- ☛ Restrict the underlying domains of variables.

Quiz 10-1

Let S be a set.

[1] What is the smallest subset of S?

[2] What is the largest subset of S?

[3] What is the smallest element of P(S)?

[4] What is the largest element of P(S)?

[5] What is the smallest subset of P(S)?

[6] What is the largest subset of P(S)?