- No late submission will be accepted. Please submit before the submission deadline.
- Consider lg as logarithm with base 2.

1 Problem 1 (20 points)

Decide whether these statements are TRUE or FALSE. Consider $f(\cdot)$ and $g(\cdot)$ as positive functions.

(a) (5 points) For the functions n^k and c^n, when $k \geq 1$ and $c > 1$, n^k is $\Omega(c^n)$.

(b) (5 points) If $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = g(n)$.

(c) (5 points) $f(n) + g(n) = \theta(max\{f(n), g(n)\})$.

(d) (5 points) $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Answer : F, F, T, T

2 Problem 2 (60 points)

Solve the following recurrence relations to determine a good asymptotic tight bound for each of them. Include steps to your answer. (Specify case it is when applying Master Theorem.)

(a) (5 points) $T(n) = 3T(n/3) + n/2$

Solve by Master method (Case 2). $T(n) = \Theta(nlgn)$
Grading policy : 3pts(logic) + 2pts(answer), -2 pts if the answer is big-O or big-Ω

(b) (5 points) $T(n) = 4T(n/2) + n/lgn$

Solve by Master method (Case 1). $T(n) = \Theta(n^2)$
Grading policy : 3pts(logic) + 2pts(answer), -2 pts if the answer is big-O or big-Ω

(c) (10 points) $T(n) = 2T(\sqrt{n}) + lg(n)$

Let $S(m) := T(2^m)$, then the given formula is equivalent to $S(m) = T(2^m) = 2T(\sqrt{2^m}) + lg(2^m) = S(m/2) + m$. We can solve this equation by applying Master method (Case 2). It says $S(m) = T(2^m) = \Theta(mlgm)$ and when substituting $m = lgn$, $T(n) = \Theta(lgn \cdot lglgn)$.
Grading policy : 5pts(logic) + 5pts(answer), -5 pts if the answer is big-O or big-Ω

(d) (10 points) $T(n) = 2T(n-1) + 1$

Let $S(m) := T(n) + 1$, then the given formula is equivalent to $S(m) = 2S(m-1)$. Then by solving the recurrence, it gives $S(m) = 2^{n-1}S(1)$ and then $T(n) = 2^{n-1}T(1) - 1 = \Theta(2^n)$.
Grading policy : 5pts(logic) + 5pts(answer), -5 pts if the answer is big-O or big-Ω

(e) (10 points) $T(n) = T(\sqrt{n}) + 1$

Let $S(m) := T(2^m)$, then the given formula is equivalent to $S(m) = T(2^m) = T(2^{m/2}) + 1 = S(m/2) + 1$. Then the recurrence can be solved by Master method (Case 2), and it gives $S(m) = T(2^m) = \Theta(lgm)$. By substituting $m = lgn$, $T(n) = \Theta(lglgn)$.
Grading policy : 5pts(logic) + 5pts(answer), -5 pts if the answer is big-O or big-Ω

(f) (10 points) $T(n) = T(n/3) + T(2n/3) + n$

Use substitution method.

i) $T(n) \le cnlgn + cn$: For the base case $n < n_0$, it is true for sufficiently large c' since $T(1) = \Theta(1)$. Suppose it is true for all $k < n$. Then $T(n) \le c(\frac{n}{3})lg(\frac{n}{3}) + c(\frac{2n}{3})lg(\frac{2n}{3}) + cn + n = cnlgn + cn + n(1 - c(lg3 - \frac{2}{3})) \le cnlgn + cn$, for $c \ge max(c', 1/(lg3 - \frac{2}{3}))$. Therefore $T(n) \le cnlgn + cn$ for all n and gives $T(n) = O(nlgn)$.

ii) $T(n) \ge dnlgn$: For the base case $n < n_0$, it is true for sufficiently small positive d' since $T(1) = \Theta(1)$. Suppose it is true for all $k < n$. Then $T(n) \ge d(\frac{n}{3})lg(\frac{n}{3}) + d(\frac{2n}{3})lg(\frac{2n}{3}) + n = dnlgn + n(1 - d(lg3 - \frac{2}{3})) \ge dnlgn$ for $0 < d \le min(d', 1/(lg3 - \frac{2}{3}))$. Therefore $T(n) \ge dnlgn$ for all n and gives $T(n) = \Omega(nlgn)$

By i), ii), $T(n) = \Theta(nlgn)$
Grading policy : 5pts(logic) + 5pts(answer), -5 pts if the answer is big-O or big-Ω

(g) (10 points) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

Let $S(m) := T(2^k)/2^k$, then the given formula is equivalent to $S(m) = \frac{T(2^k)}{2^k} = \frac{2^{k/2}T(2^{k/2})}{2^k} + 1$. Then $S(m) = S(\frac{m}{2}) + 1$, and by Master method (Case 2), we get $S(m) = lgk$, $T(2^k) = 2^k \log k \implies T(n) = nlglgn$.
Grading policy : 5pts(logic) + 5pts(answer), -5 pts if the answer is big-O or big-Ω

3 Problem 3 (20 points)

Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω or Θ of B. Assume that $k \ge 1$ and $\epsilon > 0$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box. You don't need to prove each of them. Note that the statement A is O of B means $A = O(B)$.

(A,B)	O	o	Ω	ω	Θ
$(lg^k n, n^\epsilon)$	Yes	Yes	No	No	No
$(n, lg^{lgn} n)$	Yes	Yes	No	No	No
$(n!, n^{n/2})$	No	No	Yes	Yes	No
$(\sqrt{n}, n^{cosn}$	No	No	No	No	No

Grading policy : 1pt per box