CS300 Homework #2

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Total 100 points Due: 2020-10-05 18:00:00 KST

- Write in English or Korean.
- Make sure your writing is readable before upload it (especially for scanned image).

1. Matrix Multiplication (50 pts)

Let A and B be $n \times n$ matrices where n is an exact power of 2. Let $C = A \cdot B$. Suppose that we partition each of A, B and C into four $n/2 \times n/2$ matrices.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Suppose we create following matrices:

$$\begin{aligned}
S_1 &= A_{21} + A_{22} & P_1 &= S_2 S_4 & T_1 &= P_1 + P_2 \\
S_2 &= S_1 - A_{11} & P_2 &= A_{11}B_{11} & T_2 &= T_1 + P_4 \\
S_3 &= B_{12} - B_{11} & P_3 &= A_{12}B_{21} & & \\
S_4 &= B_{22} - S_3 & P_4 &= (A_{11} - A_{21})(B_{22} - B_{12}) & & \\
& & P_5 &= S_1 S_3 & & \\
& & P_6 &= (A_{12} - S_2)B_{22} & & \\
& & P_7 &= A_{22}(S_4 - B_{21}) & &
\end{aligned}$$

(a) (14 pts) Show that $C = \begin{pmatrix} P_2 + P_3 & T_1 + P_5 + P_6 \\ T_2 - P_7 & T_2 + P_5 \end{pmatrix}$

(b) (10 pts) Use the above method to compute the matrix product $\begin{pmatrix} 4 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 1 & 3 \end{pmatrix}$

(c) (6 pts) Is this way more efficient that Strassen's method? Explain in terms of the numbers of $n/2 \times n/2$ matrix multiplications and additions(subtractions).

(d) (6 pts) Constuct a recurrence equation for this method.

(e) (6 pts) Calculate the asymptotic running time of this method in Θ-notation.

(f) (8 pts) Even if n is not an exact power of 2, you can use the above method by finding the smallest power of 2 such that $n \leq 2^m$, and padding your matrices to $2^m \times 2^m$. Use it to prove that the asymptotic running time of this method is the same as the answer to (e) for any positive integer n.

2. Randomized Quicksort (50 pts)

Recall RANDOMIZED-QUICKSORT in Chapter 7.3 in CLRS.

(a) (7 pts) Explain the Randomized Quicksort and why we should use it.

(b) (1 pts) Argue that, given an array of size n, the probability that any particular element is chosen as the pivot is $1/n$. Use this to define indicator random variables

$$X_i = \begin{cases} 1 & \text{if } i \text{ th smallest element is chosen as the pivot;} \\ 0 & \text{otherwise} \end{cases}$$

which is slightly different from the lecture slides. What is $E[X_i]$?

(c) (6 pts) Let $T(n)$ be a random variable denoting the running time of quicksort on an array of size n. Show that

$$E[T(n)] = E\left[\sum_{q=1}^{n} X_q(T(q-1) + T(n-q) + \Theta(n))\right]. \tag{1}$$

(d) (12 pts) Show that we can rewrite equation (1) as

$$E[T(n)] = \frac{2}{n}\sum_{q=2}^{n-1} E[T(q)] + \Theta(n). \tag{2}$$

(e) (10 pts) Show that

$$\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2. \tag{3}$$

where $n \geq 3$. (Hint: Split the summation into two parts, one for $k = 2, 3, ..., \lceil n/2 \rceil - 1$ and the other for $k = \lceil n/2 \rceil, ..., n-1$)

(f) (14 pts) Using the bound from equation (3), show that the recurrence in equation (2) has the solution $E[T(n)] = O(n \lg n)$.

(Hint: Show, by substitution, that $E[T(n)] \leq an \lg n$ for sufficiently large n and for some positive constant a.)

※ lg means logarithm with base 2.