1 Problem 1 (50 points)

Complexity classes are for decision problems. For example, the well-known **NP**-Complete problem 3SAT only asks, for a given Boolean formula, if there is an assignment that satisfies the formula. It is a little disappointing that in practice, we are usually interested in function problems: for example, we wish to know specific assignments that make a Boolean formula satisfied, if there exists. Hence, we often define a decision problem that corresponds to a function problem that we are interested in, and see in which complexity class that the decision variant belongs to. Our intuition says that function problems and their corresponding decision problems have to be related somehow; i.e., defining a decision version of a function problem will only make sense if the complexity class that the decision version belongs to implies something meaningful for the original function problem. In this set of problems, we will see how they are related for some cases.

- (10 points) Consider the following function problem: for a 3CNF formula, compute a satisfying assignment, if there is one. Prove that the function problem is solvable in polynomial time if and only if 3SAT is in **P**.
- (10 points) Recall that TSP is a decision problem where you are given a distance matrix D and a value k, and required to decide if there is a route visiting each city exactly once such that the total distance of the route is less than or equal to k.

  Now, consider the following function problem: given a distance matrix D, compute the the shortest distance that a TSP route can have. Show that the function problem can be solved in polynomial time if and only if TSP is in **P**.
- (10 points) Suppose the modified decision problem TSP': given a distance matrix D, a value k, and a non-empty list of cities l, decide if there is a route that visits each city exactly once, the initial part of the route is l, and the overall distance of the route is less than or equal to k. Show that TSP' is **NP**-Complete.
- (10 points) Prove that TSP is in **P** if and only if TSP' is in **P**.
- (10 points) Consider the following function problem: for a distance matrix D and a value k, compute a route that visits each city exactly once such that the total distance of the route is less than or equal to k.

  Prove that the function problem is solvable in polynomial time if and only if TSP' is in **P** (hence, if and only if TSP is in **P**).

2 Problem 2 (50 points)

To show a problem X is **NP**-Complete, we need to show X belongs to **NP**, and X is **NP**-Hard. Prove the problems shown below are **NP**-Complete.

- (25 points) Consider SUBSET-SUM problem. Given n non-negative integers $w_1, ..., w_n$ and a target sum S, the problem is to decide whether such a subset of the items exists with total weight of S.

  (1) (10 points) Show that SUBSET-SUM is in **NP**.

  (2) (15 points) Show SUBSET-SUM is **NP**-Complete by reduction from Independent Set.

- (25 points) 0-1 KNAPSACK problem is as follows: We have a set of n items $x_1, ..., x_n$, each with a weight w_i and a cost c_i, given a maximum weight W, and a budget C. The problem is to decide whether a set of n variables $x_1, ..., x_n$ with $x_i \in \{0, 1\}$ exists such that

$$\sum_{i=1}^{n} x_i c_i \geq C, \sum_{i=1}^{n} x_i w_i \leq W.$$

  (1) (10 points) Show that 0-1 KNAPSACK belongs to **NP**.

  (2) (15 points) Show 0-1 KNAPSACK is **NP**-Complete by reduction from SUBSET-SUM.