1. (7pts x 6 = 42 pts) Give True or False for each of the following statements. Justify your answers. Incorrect justification will earn 0 pts even with correct answer.

   (a) If a problem satisfies the optimal substructure property, then a locally optimal solution is globally optimal.
   (FALSE) Even if a certain problem satisfies the optimal substructure property, a locally optimal solution may not be globally optimal. Consider the 0-1 knapsack problem. It has the optimal substructure property but cannot always be solved with greedy algorithm (i.e., each locally optimal choices does not guarantee the globally optimal solution).

   (b) If a problem A is polynomial time reducible to a problem B and A has a polynomial time algorithm, then B has a polynomial time algorithm.
   (FALSE) If $\mathrm{A} \leq_{\mathrm{P}} B$, B is at least as hard as A. Consider $\mathrm{A} \in \mathrm{P}(\subseteq \mathrm{NP})$ and $\mathrm{B} \in \mathrm{NP}-\mathrm{hard}$. By definition of NP-hard, A can be reduced to B in polynomial time and A is P, but B is not P.

   (c) $\mathrm{P} \subseteq \mathrm{co}-\mathrm{NP}$ is an open question.
   (FALSE) Since P is closed under complement, for a language L in P, $\bar{\mathrm{L}}$ is also in P and therefore in NP ($\because$ P $\subseteq$ NP). This implies L is in co-NP, thus $\mathrm{P} \subseteq \mathrm{co}-\mathrm{NP}$ is true statement.

   (d) If $\mathrm{NP} \neq \mathrm{co}-\mathrm{NP}$, then $\mathrm{P} \neq \mathrm{NP}$.
   (TRUE) Proof by contrapositive. If $\mathrm{P} = \mathrm{NP}$, since P is closed under complement, $\mathrm{NP} = \mathrm{P} = \mathrm{co}-\mathrm{P} = \mathrm{co}-\mathrm{NP}$ and therefore $\mathrm{NP} = \mathrm{co}-\mathrm{NP}$.

   (e) We can decide any problem in NP in exponential time.
   (TRUE) SAT problem is NP-complete and can be decided in exponential time. By definition, any of NP problem can be reduced to SAT and therefore decided also in exponential time.

   (f) If we can decide whether a graph has a Hamiltonian cycle in polynomial time, then we can list the vertices of a Hamilonian cycle, in order, in polynomial time.
   (TRUE) Suppose that $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ is hamiltonian. Pick any one vertex v in the graph, and consider all the possibilities of deleting all but two of the edges

passing through that vertex. For some pair of edges to save, the resulting graph must still be hamiltonian because the hamiltonian cycle that existed originally only used two edges. Since the degree of a vertex is bounded by |V|-1, we are only less than squaring that number by looking at all pairs $\left(\binom{n-1}{2} \in O(n^2)\right)$. Once we have some pair of vertices where deleting all the others coming off of v still results in a hamiltonian graph, we will remember those as special, and ones that we will never again try to delete. We repeat the process with both of the vertices that are now adjacent to v, testing hamiltonicity of each way of picking a new vertex to save. We continue in this process until we are left with only |V| edge, and so, we have just constructed a hamiltonian cycle. We are only running the polynomial tester polynomially many independent times, so the runtime is polynomial.

< Grading policy > For each subproblem,
Wrong answer (T/F) : 0
Correct answer but no justification : 0
Correct answer but completely error (or not related solution) : 2
Correct answer and some logic error : 4
Correct answer and correct justification : 7

2. (15 pts) We are given a list of *n* TV shows to watch. Each show *i* is represented by a channel number c_i, a start time s_i, an end time e_i, and a pleasure rating r_i. We can record only one TV show at a time and we do not record only part of the show (We should record any show from the start time till the end time of the show.) Design an efficient algorithm to select a subset of TV shows that maximizes the total "pleasure" – the sum of the pleasure ratings of recorded shows. Prove the correctness of your algorithm and analyze the running time.

Sol)

Greedy algorithm fails when we consider the pleasure rating, so we use DP here. First sort the shows in ascending order of end time. After sorting we can assume $\mathrm{e}_1 \le e_2 \le \cdots \le e_n$. For each show i, compute $\mathrm{p(i) = show\ with\ largest\ index\ j < i\ s.t.\,i\ and\ j\ are\ compatible.}$ Now make a list R with $\mathrm{n+1}$ elements. Set $\mathrm{R[0]}$ as 0 and then for each j from 1 to n, set $\mathrm{R[j] = max}\left(\mathrm{r_j} + R[p(j)], R[j-1]\right)$. Now we have the maximum total pleasure in $\mathrm{R[n]}$ and with proper backtracking, we can find which combination of the shows produce the maximum total pleasure. This algorithm has the running time of $\mathrm{O(nlogn)}$ due to the sorting.

If you did not give a full algorithm, you get 0 point.

If you suggested brute force, you get 0 point.

If your solution has a major error, you get 5 points.

Major errors : algorithms with running time of worse than $\mathrm{O(n^2)}$, wrong algorithms.

If your solution has a minor error, you get 10 points.

Minor errors : wrong running time analysis, assumption that the shows are sorted.

Other than the above cases, you get 15 points.

3.

criteria

If you solve it with a Dynamic programming or other algorithm, you'll lose all the pts.

If you express time complexity as linear time, you’ll get 3 pts.

Solution

```
VERTEX-COVER-TREES(G)
C = ∅
while ∃ leaves in G
    Add all parents to C
    Remove all leaves and their parents from G
return C
```

Running time is O(V), and the returned solution is a vertex cover.

4.

This problem is so called metric-TSP. To prove metric-TSP is NP-complete, we have to prove it is NP and NP-hard.

(1) If there's given Hamilton path which can be solution of metric-TSP, we can check it is solution by adding every cost of edge and compare it to given cost bound. And it takes not more than polynomial time. So metric TSP is NP.

(2) We'll prove that HAMILTONIAN PATH <= metric-TSP. Let G=(V,E) be graph in HAMILTONIAN PATH. (I mean it has Hamiltonian path.) Then we can construct complete weighted graph H from G by followings. Vertex of H is same as G, and the weight of edge is 1 if such edge is also in G, and 2 otherwise. It takes at most $O(V^2)$ times, i.e. polynomial. Note that weight(or cost) of edge of H satisfy triangle identity.

    If G has Hamiltonian path, then H must have Hamiltonian path whose total weight is lower than |V|. Because we can construct Hamiltonian path of H by only using edges already in G.

    Conversely, if H has Hamiltonian path whose total weight is greater than |V|, there must be edge of weight 2 in such path. Because number of edge in Hamiltonian path is fixed as |V|.

    In conclusion, HAMILTON PATH can be reduced to metric-TSP. However, HAMILTONIAN PATH is NP-hard problem, so metric-TSP is also NP-hard problem.

Criteria:
2 for NP, 8 for NP-hard.