CS300 Homework 2

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Due : 4/16(Thu) 23:59:00 / **Total 100 points**

No late submission will be accepted

Problem 1 [Total 50pts]

Let A and B be $n \times n$ matrices where n is an exact power of 2. Let $C = A \cdot B$. Suppose that we partition each of A, B and C into four $n/2 \times n/2$ matrices.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Suppose we create following matrices:

$S_1 = A_{21} + A_{22}$

$S_2 = S_1 - A_{11}$

$S_3 = B_{12} - B_{11}$

$S_4 = B_{22} - S_3$

$P_1 = S_2 S_4$

$P_2 = A_{11} B_{11}$

$P_3 = A_{12} B_{21}$

$P_4 = (A_{11} - A_{21})(B_{22} - B_{12})$

$P_5 = S_1 S_3$

$P_6 = (A_{12} - S_2) B_{22}$

$P_7 = A_{22}(S_4 - B_{21})$

$T_1 = P_1 + P_2$

$T_2 = T_1 + P_4$

(a) [18 pts] Prove $C = \begin{pmatrix} P_2 + P_3 & T_1 + P_5 + P_6 \\ T_2 - P_7 & T_2 + P_5 \end{pmatrix}$

(b) [10 pts] Use the above method to compute the matrix product $\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$.

(c) [9 pts] Is this way more efficient than Strassen's method? Explain in terms of the numbers of $n/2 \times n/2$ matrix multiplications and additions(subtractions).

(d) [6 pts] Construct a recurrence equation for this method.

(e) [7 pts] Calculate the asymptotic running time of this method in Θ-notation.

Problem 2 [Total 50pts]

Recall RANDOMIZED-QUICKSORT in Chapter 7.3.

(a) [7 pts] Explain the Randomized Quick Sort and why we should use it.

(b) [1 pts] Argue that, given an array of size n, the probability that any particular element is chosen as the pivot is $1/n$. Use this to define indicator random variables $X_i = \{1\ if\ ith\ smallest\ element\ is\ chosen\ as\ the\ pivot; 0\ otherwise\ \}$. What is $E[X_i]$?

(c) [6 pts] Let $T(n)$ be a random variable denoting the running time of quicksort on an array of size n. Argue that

$$E[T(n)] = E\left[\sum_{q=1}^{n} X_q\big(T(q-1) + T(n-q) + \Theta(n)\big)\right]. \qquad (1)$$

(d) [12 pts] Show that we can rewrite equation (1) as

$$E[T(n)] = \frac{2}{n}\sum_{q=2}^{n-1} E[T(q)] + \Theta(n). \qquad (2)$$

(e) [10 pts] Show that

$$\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2. \tag{3}$$

(Hint: Split the summation into two parts, one for $k = 2,3,\ldots,\lceil n/2 \rceil - 1$ and one for $k = \lceil n/2 \rceil, \ldots, n-1$)

(f) [14pts] Using the bound from equation (3), show that the recurrence in equation (2) has the solution $E[T(n)] = \Theta(n \lg n)$.
(Hint: Show, by substitution, that $E[T(n)] \leq an \lg n$ for sufficiently large n and for some positive constant a.)

※ lg means logarithm with base 2