CS300 Homework #6

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Total 100 points | Due: December 4 at 18:00:00 KST

1 True / False (30 points)

Give true or false for each of the following statements. Briefly justify your answer in less than four sentences.

1) If 3-CNF-SAT $\in$ **P**, then CLIQUE $\in$ **P**. [10 points]

2) For decision problems $L_1, L_2 \in$ **NP**, if **P** $\neq$ **NP**, $L_1 \leq_p L_2$ and $L_2 \leq_p L_1$, then L_1 and L_2 are **NP-complete**. [10 points]

3) For decision problems $L_1, L_2 \in$ **NP-complete**, if $L_1 \notin$ **P**, then $L_2 \notin$ **P**. [10 points]

2 Closest string (70 points)

Given two binary strings $x = x_1 \cdots x_n$, $y = y_1 \cdots y_n$, let $d(x, y)$ denote the number of different bit pairs. That is,

$$d(x,y) = \sum_{i=1}^{n} (x_i \oplus y_i) \quad \text{with } x_i \oplus y_i = \begin{cases} 1 & \text{if } x_i \neq y_i \\ 0 & \text{otherwise} \end{cases}$$

For example, $d(0011, 0001) = 1$ and $d(0011, 1100) = 4$. We define the Closest String Problem as follow.

Definition Closest String Problem(CSP)
Given a natural number k and m strings $s_1, s_2, \cdots, s_m$ where $s_i \in \{0,1\}^n$ for $i = 1, \cdots, m$, is there a string $t \in \{0,1\}^n$ such that $d(t, s_i) \leq k$ for all i?

1) Assuming $m = O(n^c)$ for some constant c, prove that CSP $\in$ **NP**. [10 points]

We'll show that CSP $\in$ **NP-hard** by proving 3-CNF-SAT $\leq_p$ CSP, thus CSP $\in$ **NP-complete**.

Given a boolean formula $\phi = C_1 \wedge \cdots \wedge C_m$ of variables $x_1, \cdots, x_n$ in 3-CNF, generate the strings $s_1, \cdots, s_{6n+m}$ each of length $2n$-bit as follow.

For $i = 1, \cdots, 2n$

$$s_i = b_1 b_2 \cdots b_{2n} \quad \text{with } b_j = \begin{cases} 0 & j = i \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

For $i = 2n+1, \cdots, 4n$, $s_i = \overline{s_{i-2n}}$ where $\overline{s}$ is bit-wise negation of s.

For $i = 4n+1, \cdots, 5n$

$$s_i = b_1 b_2 \cdots b_n c_1 c_2 \cdots c_n \quad \text{with } b_j = c_j = \begin{cases} 0 & j = i - 4n \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

For $i = 5n+1, \cdots, 6n$, $s_i = \overline{s_{i-n}}$ where $\overline{s}$ is bit-wise negation of s.

For $i = 6n+1, \cdots, 6n+m$,

$$s_i = b_1 b_2 \cdots b_n c_1 c_2 \cdots c_n \quad \text{with}$$

$$b_j = \begin{cases} 1 & \text{if } C_{i-6n} \text{ includes the literal } x_j \\ 0 & \text{otherwise} \end{cases} \qquad c_j = \begin{cases} 1 & \text{if } C_{i-6n} \text{ includes the literal } \neg x_j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

For each string $s = b_1 \cdots b_n c_1 \cdots c_n$, b_j corresponds to x_j, and c_j does $\neg x_j$.

For example, given $\phi = (x_1 \vee x_2 \vee \neg x_4) \wedge (x_1 \vee \neg x_3 \vee x_4)$, we generate

$s_1 = 01111111$	$s_9 = 10000000$	$s_{17} = 01110111$	$s_{21} = 10001000$	
$s_2 = 10111111$	$s_{10} = 01000000$	$s_{18} = 10111011$	$s_{22} = 01000100$	$s_{25} = 11000001$
$\vdots$	$\vdots$	$s_{19} = 11011101$	$s_{23} = 00100010$	$s_{26} = 10010010$
$s_8 = 11111110$	$s_{16} = 00000001$	$s_{20} = 11101110$	$s_{24} = 00010001$	

Note that this process runs in polynomial time.

At first, we show that if 3-CNF-SAT produces the positive answer, then so does CSP.

2) Prove that if ϕ has a satisfying assignment, then there exists $t \in \{0,1\}^{2n}$ such that $d(t, s_i) \leq n+1$ for all i. [15 points]
(Hint: Convert the satisfying assignment $b_1, \cdots, b_n$(1 for `TRUE`, 0 for `FALSE`) into $t = b_1 \cdots b_n c_1 \cdots c_n$ with $c_j = \neg b_j$)

Next, we show that if CSP produces the positive answer, then so does 3-CNF-SAT.

For the problems 3) - 5), assume that there exists $t \in \{0,1\}^{2n}$ such that $d(t, s_i) \leq n+1$ for all i.

3) Prove that t has exactly n 1s and n 0s. [15 points]
(Hint: Use (1) to show that t contains at least n 1s)

4) Prove that if $t = b_1 \cdots b_n c_1 \cdots c_n$(binary string), then $b_j \neq c_j$ for j. [15 points]
(Hint: Use (2) to show that there does not exist k such that $b_k = c_k = 1$)

5) Prove that ϕ has a satisfying assignment. [15 points]
(Hint: Use (3) to show that t and s_{6n+i} where $i = 1, \cdots, m$ have at least a 1 in a common position.)