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Assignment # 1: 2020 Fall CS300

1) a) False

b) False

c) True

d) True

2) a) Using Master's Theorem, $a=b=3$ and $f(n)=\frac{n}{2}$

$n^{\log_b a}=n^{\log_3 3}=n^1$ with $f(n)=\frac{n}{2}=\Theta(n)$, then applying Case 2 of Master's Theorem yields $\boxed{T(n)=\Theta(n\lg n)}$

b) $a=4$, $b=2$, $f(n)=\frac{n}{\lg n} \Rightarrow n^{\log_b a}=n^{\log_2 4}=n^2$ where taking $\varepsilon=1 \Rightarrow f(n)=\frac{n}{\lg n}=O(n^{2-\varepsilon})=O(n^1)$, since $\frac{n}{\lg n}\le c\cdot n$ or plugging $c=1$ gives $\frac{n}{\lg n}\le n$ for all $n\geq 2$

Using the first case from Master's Theorem, we find $\boxed{T(n)=\Theta(n^2)}$

c) We'll implement method of "changing variables"

For convenience, we shall not worry about rounding off values, such as $\sqrt{n}$, to be integers. Renaming $m=\lg n$, $n=2^m$ or $T(2^m)=2T(2^{\frac{m}{2}})+m$. We can now rename $S(m)=T(2^m)$ to produce the new recurrence

$S(m) = 2S\left(\frac{m}{2}\right) + m$. Using master's method, $a = b = 2$ and $f(m) = m \Rightarrow m^{\log_b a} = m^{\log_2 2} = m$, $f(m) = m = \Theta(m) = \Theta(m^{\log_b a})$, then applying case 2 yields that

$S(m) = \Theta(m^{\log_b a} \lg m) = \Theta(m \lg m)$. Changing back from $S(m)$ to $T(n)$, $S(m) = T(2^m) = T(n) = \Theta(m \lg m) = \Theta(\lg n \cdot \lg(\lg n)) \Rightarrow \boxed{T(n) = \Theta(\lg n \cdot \lg(\lg n))}$ ☆

d) We'll use induction on n to find asymptotic tight bound for this recurrence. We assume $T(1) = 1$ as a base case for our inductive hypothesis: $T(n) = 2^n - 1$ (we could assume any values, since this method applies to large n, but it's important to handle base cases)

$n = 1 \Rightarrow$ holds ✓ Assume $n = k \Rightarrow$ holds ✓ where $k \geq 1$

$T(k+1) = 2T(k) + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1 \Rightarrow$ Therefore, $n = k+1 \Rightarrow$ holds ✓ meaning $T(n) = 2^n - 1$, given $T(1) = 1 \Rightarrow$

Transforming it into Θ-notation, $\boxed{T(n) = \Theta(2^n)}$ ☆

e) We'll again use "changing variables" method as in part (c): $m = \lg n$ or $n = 2^m \Rightarrow T(2^m) = T\left(2^{\frac{m}{2}}\right) + 1$

Let $S(m) = T(2^m) \Rightarrow S(m) = S\left(\frac{m}{2}\right) + 1$, where applying Master's theorem

$a=1$, $b=2$, $f(m)=1 \Rightarrow m^{\log_b a}=m^{\lg 1}=m^0=1$ and

$f(m)=1=\Theta(1)=\Theta(m^{\log_b a})$ gives us second case of

Master's Theorem: $S(m)=\Theta(m^{\log_b a}\lg m)=\Theta(\lg m)$

$S(m)=T(2^m)=\Theta(\lg m)$ or plugging $m=\lg n$, $n=2^m \Rightarrow$

$\boxed{T(n)=\Theta(\lg(\lg n))}$

g) $T(n)=\sqrt{n}\,T(\sqrt{n})+n$, implementing "changing variables" method, let $n=2^k \Rightarrow k=\lg n$, $T(2^k)=2^{\frac{k}{2}}T(2^{\frac{k}{2}})+2^k$

dividing by 2^k, $\frac{T(2^k)}{2^k}=\frac{T(2^{\frac{k}{2}})}{2^{\frac{k}{2}}}+1$, Let $\boxed{S(k)=\frac{T(2^k)}{2^k}}$

$S(n)=S\left(\frac{n}{2}\right)+1$. Using Master's Method for

$a=1$, $b=2$, $f(n)=1 \Rightarrow n^{\log_b a}=n^{\lg 1}=n^0=1=f(n) \Rightarrow$

$f(n)=1=\Theta(1)=\Theta(n^{\log_b a})$, meaning case 2 should be applied and we get

$S(n)=\Theta(n^{\log_b a}\lg n)=\Theta(\lg n)$; Since $S(n)=\frac{T(2^n)}{2^n} \Rightarrow$

$T(2^n)=2^n\,\Theta(\lg n)$ or $2^n=m \Rightarrow T(m)=m\,\Theta(\lg(\lg m))$

$\boxed{T(m)=\Theta(m\lg(\lg m))}$

f) $T(n)=T\left(\frac{n}{3}\right)+T\left(\frac{2n}{3}\right)+n$

We claim that $T(n)=\Theta(n\lg n)$ satisfies given recurrence relation

We can show that $T(n) \le d \cdot n \lg n$ for suitable constant d and considering that we handled base cases $\Rightarrow$

$$T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n \le d \cdot \frac{n}{3} \cdot \lg \frac{n}{3} + d \cdot \frac{2n}{3} \cdot \lg\left(\frac{2n}{3}\right) + n =$$

$$= \frac{dn}{3} \lg n - \frac{dn}{3} \lg 3 + \frac{2nd}{3} \lg 2 + \frac{2nd}{3} \lg n - \frac{2nd}{3} \lg 3 + n =$$

$$= \cancel{nd \lg n} + \frac{2nd}{3} + n - nd \lg 3 \overset{?}{\le} \cancel{n \lg n \cdot d} \Rightarrow n\left(\frac{2d}{3} + 1\right) \overset{?}{\le}$$

$$\overset{?}{\le} n \cdot d \lg 3 \Rightarrow \frac{2d}{3} + 1 \overset{?}{\le} d \lg 3, \quad d\left(\lg 3 - \frac{2}{3}\right) \overset{?}{\ge} 1 \Rightarrow$$

choosing $\boxed{d \ge \frac{1}{\lg 3 - \frac{2}{3}}} \Rightarrow \boxed{T(n) \le d \cdot n \lg n}$ considering that we have already handled base cases in constant time $\Theta(1)$

Hence, $\boxed{T(n) = O(n \lg n)}$ In the similar approach, we can prove that $\underline{T(n) \ge a \cdot n \lg n - b}$, for some a, b

Assuming we handled base cases and applying inductive hypothesis, we should be able to find suitable a and b

$$T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n \ge a \cdot \frac{n}{3} \cdot (\lg n - \lg 3) + a \cdot \frac{2n}{3}(\lg 2 + \lg n - \lg 3)$$

$$- 2b + n = \cancel{\lg n \cdot an} - \lg 3 \cdot an \overbrace{+ a \cdot \frac{2n}{3}}\, \cancel{-2b} + n \overset{?}{\ge} \cancel{a \cdot n \lg n} + b$$

$$n\left(\frac{2a}{3} + 1 - a \lg 3\right) \overset{?}{\ge} b \quad \text{choosing } \frac{2a}{3} + 1 - a \lg 3 > 0 \text{ or}$$

$$\frac{2a}{3} + 1 > a \lg 3, \quad 2a + 3 > 3a \lg 3, \quad 2a + 3 > a \lg 3^3 = a \lg 27$$

$$3 > a(\lg 27 - 2) = a \lg \frac{27}{4} \Rightarrow \boxed{\frac{3}{\lg\left(\frac{27}{4}\right)} > a} \text{ and } \boxed{n \ge \frac{b}{\frac{2a}{3} + 1 - a \lg 3}}$$

we find $T(n) \ge a \cdot n \lg n - b$ holds, considering base cases hold

Hence, Choosing such a and b with base cases to be true yields

$\boxed{T(n)=\Omega(n\lg n)}$ From previous result, we conclude that

$\boxed{T(n)=\Theta(n\lg n)}$

3)

(A, B)	O	o	Ω	ω	Θ
$(\lg^k n, n^\varepsilon)$	Yes	Yes	No	No	No
$(n, \lg^{\lg n} n)$	Yes	Yes	No	No	No
$(n!, n^{n/2})$	No	No	Yes	Yes	No
$(\sqrt{n}, n^{\cos n})$	No	No	No	No	No