2020 Fall CS300 Homework # 4

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Due: 11/11(Wed), 18:00:00 KST

· Total 100 points

Problem 1. [3 × 10 = 30 Points]

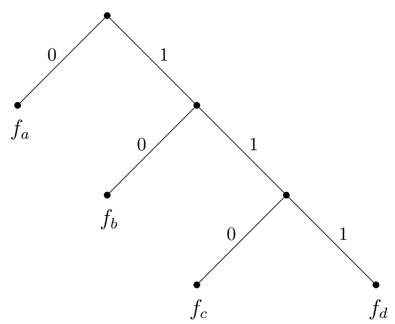
We use Huffman's algorithm to obtain an encoding of alphabet $\{a, b, c, d\}$ with frequencies f_a, f_b, f_c, f_d. In each of the following cases, either give an example of frequencies (f_a, f_b, f_c, f_d) that would yield the specified code, or explain why the code cannot possibly be obtained (no matter what the frequencies are).

(a) Code: $\{0, 10, 110, 111\}$

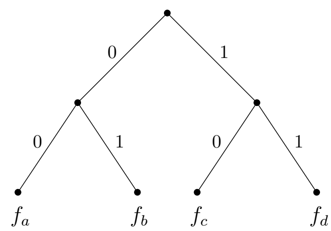
(b) Code: $\{1, 00, 01, 110\}$

(c) Code: $\{00, 01, 10, 11\}$

Solution: (a) Any example that satisfying $f_a > f_b > f_c > f_d$ and $f_a > f_c + f_d$.



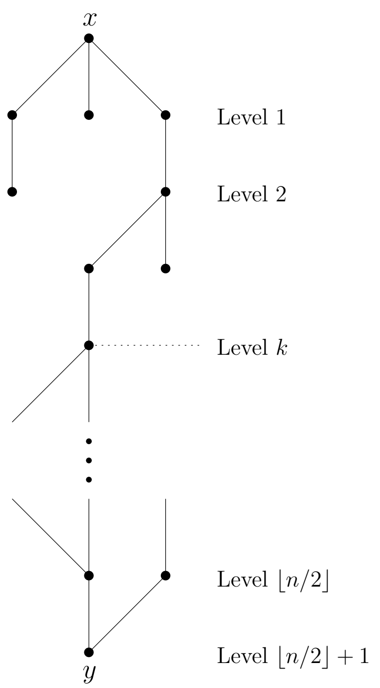
(b) Not possible. Only the leaf nodes can contain symbol in Huffman's tree or 1 is the prefix of 110.
(c) Any example that satisfying $f_a = f_b = f_c = f_d$.



Problem 2. [40 Points]

Let G be a graph with n vertices. Suppose there are two vertices $x, y \in V(G)$ such that the every simple path between x and y has length strictly greater than $n/2$. Prove that there exists some vertex $z(\neq x, y)$ such that deleting z from G destroys all x-y paths (in other words, no path from x to y).

Solution: For simplicity, we assume that n is even. Consider any breadth-first search tree T of G with x as the root. For each vertex v of T, we can define a *level* which means the length of a shortest path from x to v. Since every path between x and y has length at least $\lfloor n/2 \rfloor + 1$, the level of vertex y is at least $\lfloor n/2 \rfloor + 1$. And we have $n-2$ vertices except x and y so, by the pigeonhole principle, there exists some level $1 \leq k \leq \lfloor n/2 \rfloor$ which contains only one vertex. Therefore, removing the vertex at level k will destroy all x-y paths.



Problem 3. [2 × 15 = 30 Points]

For each of the following statements, decide whether it is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

(a) Suppose we are given an instance of the Minimum Spanning Tree Problem on a graph G, with edge costs that are all positive. Let T be a minimum spanning tree for this instance. Suppose we replace each edge cost c_e by adding 1, $c_e + 1$, thereby creating a new instance of the problem with the same graph but different costs.

True or false? "T must still be a minimum spanning tree for this new instance."

(b) Suppose we are given an instance of the Shortest s-t Path Problem on a directed graph G. We assume that all edge costs are positive. Let P be a minimum-cost s-t path for this instance. Suppose we replace each edge cost c_e by adding 1, $c_e + 1$, thereby creating a new instance of the problem with the same graph but different costs.

True or false? "P must still be a minimum-cost s-t path for this new instance.

Solution: (a) True. If G has n vertices, then any spanning tree has $n-1$ edges. So, incrementing each edge weight by 1 increases the cost of every spanning tree by $n-1$. Therefore, T is still minimum spanning tree.
(b) False, the counterexample is as follows: