Theorem

Let a and b be scalars. Let A, B, and C be matrices with the same size.

1. $A + B = B + A$ — commutative
2. $A + (B + C) = (A + B) + C$ — associative
3. $(ab)A = a(bA)$ — commutative
4. $(a \pm b)A = aA \pm bB$
5. $a(A \pm B) = aA \pm bB$

} distributative.

Proof: Exercise.

Theorem

Assume that the sizes of the matrices A, B, and C are such that the indicated operations can be performed.

1. $A(BC) = (AB)C$ – associative
2. $A(B \pm C) = AB \pm AC$
3. $(B \pm C)A = BA \pm CA$

distributive

4. $a(BC) = (aB)C = B(aC)$ *for* $a \in \mathbb{R}$. – commutative for scalar multiplication.

Proof: Exercise.

Theorem

Assume that the sizes of the matrices A, B, and C are such that the indicated operations can be performed.

1. $A(BC) = (AB)C$
2. $A(B \pm C) = AB \pm AC$
3. $(B \pm C)A = BA \pm CA$
4. $a(BC) = (aB)C = B(aC)$ *for* $a \in \mathbb{R}$.

Proof: Exercise.

True or False?

1. If AB and BA are defined, then they have the same size. (False)
2. If A and B are square matrices of the same order, then $AB = BA$.

False!

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

$(AB)_{11} = 3$

$(BA)_{11} = [1 \ -1] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1-3 = -2$

A matrix whose entries are all zero is called a *zero matrix*, denoted by $\mathbf{0}$.

$[0]$, $[0 \quad 0]$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\cdots$

Theorem

Let c be a scalar. The sizes of matrices are such that the operations can be performed.

1. $A + \mathbf{0} = \mathbf{0} + A$
2. $A - \mathbf{0} = A$
3. $A - A = \mathbf{0}$
4. $0A = \mathbf{0}$
5. $cA = \mathbf{0}$ *implies that* $c = 0$ *or* $A = \mathbf{0}$.

Proof: Exercise.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

True or False?

1. $AB = AC$ implies $B = C$.
2. Nonzero matrices can have a zero product.

A square matrix with 1's on the main diagonal and zeros elsewhere is called an *identity matrix*. Denote by I_n the $n \times n$ identity matrix.

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cdots$$

Then for an $m \times n$ matrix, we have that $AI_n = A$ and $I_m A = A$.

Theorem

Let A be a square matrix of size n and let R be the reduced row echelon form of A. Then either R has a row of zeros or $R = I_n$.

pf) On the last row of R has a leading 1, then $(a_{n1}, a_{n2}, \cdots, a_{nn}) = (0, \cdots, 0, 1)$.

If the last colum of R is $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th} \ (i < n)$, then the last $(n-i)$ rows are 0.

Definition

For a square matrix A, if there is a matrix B with the same size as A such that $AB = BA = I$, then A is *invertible* (or *nonsingular*), and B is called an *inverse* matrix of A. If not, A is *singular*.

Theorem

An invertible matrix has a unique inverse.

Proof: Suppose that $AB = BA = I$ & $AC = CA = I$.

Then $B = BI = BAC = IC = C$.

We denote by A^{-1} the *inverse* of an invertible matrix A.

Theorem

A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible if and only if* $ad - bc \neq 0$*, in which case the inverse is given by* $A^{-1} = \underbrace{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\substack{\parallel \\ B}}$.

Proof: Exercise.

Check $AB = BA = I$.

- ad-bc is called the determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- We will study the notion of determinant for general square matrices in CH 4.

$\boxed{n\times n\ A:\ \text{Invertible} \iff \text{determinant of } A \neq 0.}$

Theorem

A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible if and only if* $ad - bc \neq 0$*, in which case the inverse is given by* $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof: Exercise.

Theorem

Let A *and* B *be square matrices of the same order. If* A *and* B *are invertible, then* AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Exercise. (Use the associativity.)
If $A_1, A_2, \ldots, A_n$ are invertible, then $A_1 A_2 \cdots A_n$ is also invertible and

$$(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_1^{-1}.$$

Hence, if $A_1 A_2 \cdots A_n$ is singular, then one of $A_1, A_2, \ldots, A_n$ is singular.

Powers of a square matrix

For a nonnegative integer n, we define the nth power of a square matrix A:

$$(i) \quad A^0 = I$$
$$(ii) \quad A^n = \underbrace{AA\cdots A}_{n \text{ factors}} \text{ if } n > 0.$$

Hence for nonnegative integers r and s, we have $A^r A^s = A^{r+s}$.If A is invertible, then we can define a negative integer power of A such that

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}}.$$

Theorem

Let A be an invertible matrix and n a nonnegative integer. Then

1. *A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.*
2. *kA is invertible for any nonzero scalar k and $(kA)^{-1} = \frac{1}{k}A^{-1}$.*

For a square matrix A and a polynomial $p(x) = a_0 + \underset{a_1x}{\cancel{ax}} + a_2x^2 + \cdots + a_mx^m$, we define a *matrix polynomial*

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m.$$

For a square matrix A and a polynomial $p(x) = a_0 + a_x + a_2x^2 + \cdots + a_mx^m$, we define a *matrix polynomial*

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots a_mA^m.$$

Since $\boxed{A^rA^s = A^{r+s}}$, if $p(x) = p_1(x)p_2(x)$,

For a square matrix A and a polynomial $p(x) = a_0 + a_x + a_2x^2 + \cdots + a_mx^m$, we define a *matrix polynomial*

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots a_mA^m.$$

Since $A^rA^s = A^{r+s}$, if $p(x) = p_1(x)p_2(x)$, then $p(A) = p_1(A)p_2(A)$.

For a square matrix A and a polynomial $p(x) = a_0 + a_x + a_2x^2 + \cdots + a_mx^m$, we define a *matrix polynomial*

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots a_mA^m.$$

Since $A^rA^s = A^{r+s}$, if $p(x) = p_1(x)p_2(x)$, then $p(A) = p_1(A)p_2(A)$. For example, $(A+I)(A-I) = A^2 - I$.

$x^2 - 1 = (x+1)(x-1)$

For a square matrix A and a polynomial
$p(x) = a_0 + \cancel{a_x}^{a_1 x} + a_2 x^2 + \cdots + a_m x^m$, we define a *matrix polynomial*

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m.$$

Since $A^r A^s = A^{r+s}$, if $p(x) = p_1(x)p_2(x)$, then $p(A) = p_1(A)p_2(A)$. For example, $(A+I)(A-I) = A^2 - I$.

Note that $AB \neq BA$ in general.

$A^2 + BA - AB - B^2$

$\parallel$

Note that $(A+B)^2 \neq A^2 + 2AB + B^2$ and $(A+B)(A-B) \neq A^2 - B^2$ in general. // (even though $(x+y)^2 = x^2 + 2xy + y^2$ & $(x+y)(x-y) = x^2 - y^2$)

$A^2 + AB + BA + B^2$

Theorem

If the sizes of the matrices are such that the stated operations can be performed, then

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$ $\neq A^T B^T$

Proof of (4):

1. Check the sizes of $(AB)^T$ and $B^T A^T$.
2. Show $((AB)^T)_{ji} = (B^T A^T)_{ji} \quad \forall\, i,\, j$.

$(AB)^T_{ji} = (AB)_{ij}$

$= \mathbb{r}_i(A) \cdot \mathbb{c}_j(B)$

$= \mathbb{c}_j(B) \cdot \mathbb{r}_i(A)$

$= \mathbb{r}_j(B^T) \cdot \mathbb{c}_i(A)$

$= (B^T A)_{ji}$ $\square$

In general, $(A_1 A_2 \cdots A_n)^T = A_n^T A_{n-1}^T \cdots A_1^T$.

Theorem

If A is invertible, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof: $A^T(A^{-1})^T = (A^{-1}A)^T = I$ & $(A^{-1})^T A^T = (AA^{-1})^T = I$. ☑

Theorem

Let A and B be square matrices with the same size.

1. $\operatorname{tr}(A^T) = \operatorname{tr}(A)$
2. $\operatorname{tr}(cA) = c\operatorname{tr}(A)$
3. $\operatorname{tr}(A \pm B) = \operatorname{tr}(A) \pm \operatorname{tr}(B)$
4. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof: $\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} (A)_{ik}(B)_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} (B)_{ki}(A)_{ik}$

$= \sum_{k=1}^{n} (BA)_{kk} = \operatorname{tr}(BA)$. ☑

True or False?

$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$

$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 3 & * \\ * & 11 \end{bmatrix}$

Theorem

Let $\mathbf{r}$ *be a* $1 \times n$ *row vector and* $\mathbf{c}$ *an* $n \times 1$ *column vector. Then* $\mathbf{rc} = \mathrm{tr}(\mathbf{cr})$.

Proof: Exercise.

True or False?

$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$

Theorem

Let $\mathbf{r}$ *be a* $1 \times n$ *row vector and* $\mathbf{c}$ *an* $n \times 1$ *column vector. Then* $\mathbf{rc} = \mathrm{tr}(\mathbf{cr})$.

Proof: Exercise.

Definition

A square matrix A is *idempotent* if $A^2 = A$. A square matrix A is *nilpotent* if $A^k = \mathbf{0}$ for some positive integer k. The smallest value of k for which this equation holds is called the *index of nilpotency*.

Let $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Check $A^2 = A$ & $B^2 \neq \mathbf{0}$, $B^3 = \mathbf{0}$.

Example (Exercise)

Let $\mathbf{u}$ and $\mathbf{v}$ be column vectors in $\mathbb{R}^n$. Set $A = I + \mathbf{u}\mathbf{v}^T$. Show that if $\mathbf{u}^T\mathbf{v} \neq -1$, then A is invertible and $A^{-1} = I - \frac{1}{1+\mathbf{u}^T\mathbf{v}}\mathbf{u}\mathbf{v}^T$.

Proof: Since $A - I = \mathbf{u}\mathbf{v}^T$,

$$(A-I)^2 = (\mathbf{u}\mathbf{v}^T)(\mathbf{u}\mathbf{v}^T) = \mathbf{u}(\mathbf{v}^T\mathbf{u})\mathbf{v}^T = (\mathbf{v}^T\mathbf{u})(\mathbf{u}\mathbf{v}^T) = (\mathbf{v}^T\mathbf{u})(A-I).$$

Since $(A-I)^2 = A^2 - 2A + I = (A^2 - A) - (A - I)$, we get

$$A^2 - A = (1 + \mathbf{v}^T\mathbf{u})(A - I).$$

Hence if $1 + \mathbf{v}^T\mathbf{u} \neq 0$, then we have

$$\begin{aligned} I = A - \frac{1}{1+\mathbf{v}^T\mathbf{u}}(A^2 - A) &= A\left\{I - \frac{1}{1+\mathbf{v}^T\mathbf{u}}(A-I)\right\} \\ &= \left\{I - \frac{1}{1+\mathbf{v}^T\mathbf{u}}(A-I)\right\}A. \end{aligned}$$

Therefore, A is invertible and $A^{-1} = I - \frac{1}{1+\mathbf{v}^T\mathbf{u}}(A-I) = I - \frac{1}{1+\mathbf{v}^T\mathbf{u}}\mathbf{u}\mathbf{v}^T$.

Recall that there are three elementary row operations:

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another.

Definition

An *elementary matrix* is a matrix that results from applying a ~~simple~~ single elementary row operation to an identity matrix.

Examples of elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$\shortparallel$

$3(0,1,0) + (0,0,1)$

Theorem

Let A be an $m \times n$ matrix. If E is the elementary matrix that results by performing a certain elementary row operation on I_m, then EA (AE) is the matrix that results when the same row operation (column) is performed on A.

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Then we have

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

$AE_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

obtained from A by interchanging the first two columns.

Check AE_2 & AE_3.

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

The matrices E_1, E_2, and E_3 are invertible and their inverse are

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix},$$

respectively. Furthermore, E_1^{-1}, E_2^{-1}, and E_3^{-1} are also elementary matrices.

Theorem

Every elementary matrix is invertible and the inverse is also an elementary matrix.

Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \| \; E_1 A \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \| \; E_2 E_1 A$$

$$\xrightarrow{E_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \| \; E_6E_5E_4E_3E_2E_1A$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \| \; E_3E_2E_1A \qquad E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \| \; E_4E_3E_2E_1A$$

$$E_5 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \| \; E_5E_4E_3E_2E_1A \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = E_6E_5E_4E_3E_2E_1 \quad \& \quad A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$$

Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent.

1. *The reduced row echelon form of A is I_n.* ✓
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*

Proof.

$1 \Rightarrow 2$: Assume $E_k \cdots E_2 E_1 A = I_n$. Then

$$A = (E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

$2 \Rightarrow 3$: Assume $A = E_1 E_2 \cdots E_k$. Since every elementary matrix is invertible, A is also invertible.

$3 \Rightarrow 1$: Suppose that the reduced row echelon form of A is R. i.e, $E_k \cdots E_2 E_1 A = R$.Since A is invertible, R is also invertible. Since the reduced echelon form of a square matrix ~~is either $\mathbf{0}$ or I_n~~ either contains the rows consisting of zeros or is I_n, R must be I_n. □

For two matrices A and B, we say that A and B are *row equivalent* if $B = E_k \cdots E_1 A$, where $E_1, \ldots, E_k$ are elementary matrices.

Theorem

Let A and B are square matrices of the same size. The following are equivalent.

1. *A and B are row equivalent.*
2. *There is an invertible matrix E such that $B = EA$.*
3. *There is an invertible matrix F such that $A = FB$.*

♣ A and B are *column equivalent* if $B = AE_1E_2 \cdots E_k$, where $E_1, \ldots, E_k$ are elementary matrices.

The inversion algorithm

1. Find a sequence of elementary row operations that reduces A to I.
2. Perform the same sequence of operations on I to obtain A^{-1}.

$$[A|I]=\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_1} \overset{E_1A \qquad\quad E_1}{\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]} \xrightarrow{E_2} \overset{E_2E_1A \qquad\quad E_2E_1}{\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right]}$$

$$E_1=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{E_3} \overset{E_3E_2E_1A \qquad E_3E_2E_1}{\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]} \xrightarrow{E_4} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{E_5} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$E_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad E_4=\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_5=\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{E_6} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] = \boxed{[I \mid A^{-1}]} \qquad E_6=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 0 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = I$$

$$I = E_6 \cdots E_2E_1A \qquad \boxed{E_6E_5E_4E_3E_2E_1} = A^{-1}$$

Theorem

Let $A\mathbf{x} = \mathbf{b}$ *be a linear system of* n *equations in* n *unknowns. If* A *is invertible, then the system has a unique solution, namely* $\mathbf{x} = A^{-1}\mathbf{b}$*.*

Theorem

Let $A\mathbf{x} = \mathbf{0}$ *be a homogeneous linear system of* n *equations in* n *unknowns. Then the system has only the trivial solution if and only if* A *is invertible.*

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Theorem

1. *Let A and B be square matrices such that $AB = I$ or $BA = I$. Then A and B are invertible and $A^{-1} = B$ and $B^{-1} = A$.*
2. *Let A and B be square matrices such that AB is invertible. Then A and B are invertible.*

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.*

Theorem 1.1.5 on page 9

$\mathbb{R}^n$ is closed under (vector) addition and scalar multiplication, and they satisfy the following properties, see Theorem 1.1.5 on page 9.

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $(k+\ell)\mathbf{u}=k\mathbf{u}+\ell\mathbf{u}$
6. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
7. $k(\ell\mathbf{u})=(k\ell)\mathbf{u}$
8. $1\mathbf{u}=\mathbf{u}$

Definition

A nonempty set of vectors in $\mathbb{R}^n$ is called a *subspace* of $\mathbb{R}^n$ if it is closed under scalar multiplication and addition.

i.e., A subset $W \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

$$\begin{cases} ① \ \forall \mathbf{v} \ \& \ \mathbf{w} \in W, \quad \mathbf{v}+\mathbf{w} \in W \\ ② \ \forall c \in \mathbb{R}, \quad c\mathbf{v} \in W. \end{cases}$$

Definition

A nonempty set of vectors in $\mathbb{R}^n$ is called a *subspace* of $\mathbb{R}^n$ if it is closed under scalar multiplication and addition.

1. $\{\mathbf{0}\}$
2. $\mathbb{R}^n$

trivial subspaces

3. $\{(x, y) \in \mathbb{R}^2 \mid ax + by = 0\}$ is a subspace of $\mathbb{R}^2$.
4. $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0,\ 2x - y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

Note that every subspace W of $\mathbb{R}^n$ contains $\mathbf{0}$, and it satisfies the properties (1)$\sim$(8) in Theorem 1.1.5 on page 9.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be vectors in $\mathbb{R}^n$. Then the set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ is a subspace of $\mathbb{R}^n$.

Let $W = \{\mathbf{x} \mid \mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_s\mathbf{v}_s, \forall t_i \in \mathbb{R}\}$. The subspace W is called the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_s$ and is denoted by

$$W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}.$$

We also say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ *span* W.

We will study the following subspaces

$$\begin{cases} * \ \operatorname{span}\{\mathbb{c}_1(A), \mathbb{c}_2(A), \cdots, \mathbb{c}_n(A)\} \\ * \ \operatorname{span}\{\mathbb{r}_1(A), \mathbb{r}_2(A), \cdots, \mathbb{r}_m(A)\} \end{cases}$$

for an $m \times n$ matrix A.