§ 3.5. The geometry of linear systems.

In the last class, we showed that the solution set of the following homogeneous linear system

$$\underbrace{\begin{bmatrix} 0 & 0 & -2 & 0 & 7 \\ 2 & 4 & -10 & 6 & 12 \\ 2 & 4 & -5 & 6 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is spanned by the vectors

$(-2, 1, 0, 0, 0)$ and $(-3, 0, 0, 1, 0)$.

Now, we consider the following (non-homogeneous) linear system

$$\underbrace{\begin{bmatrix} 0 & 0 & -2 & 0 & 7 \\ 2 & 4 & -10 & 6 & 12 \\ 2 & 4 & -5 & 6 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 12 \\ 28 \\ -1 \end{bmatrix}.$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 1 & 0 & -3 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$x_2 = s$, $x_4 = t$, $\quad x_1 = -2s - 3t + 7$, $x_3 = 1$, $x_5 = 2$

$$\therefore\ \mathbf{x} = \underbrace{s(-2, 1, 0, 0, 0) + t(-3, 0, 0, 1, 0)}_{\mathbf{x}_h} + \underbrace{(7, 0, 1, 0, 2)}_{\mathbf{x}_0}$$

$$A\mathbf{x} = A\mathbf{x}_h + A\mathbf{x}_0 = \mathbf{0} + A\mathbf{x}_0 = \begin{bmatrix} 12 \\ 28 \\ -1 \end{bmatrix}$$

<u>Thm 3.5.1</u> If the solution space of $A\mathbf{x}=\mathbf{0}$ is spanned by the vectors $\mathbf{v}_1,\cdots,\mathbf{v}_s$ and if $\mathbf{x}_0$ is a solution of $A\mathbf{x}=\mathbf{b}$, then the solution set of $A\mathbf{x}=\mathbf{b}$ is the set

$$\{\underbrace{\mathbf{x}_0+t_1\mathbf{v}_1+\cdots+t_s\mathbf{v}_s}_{(*)} \mid t_1,\cdots,t_s\in\mathbb{R}\}$$

$$\left(\begin{array}{l} =\mathbf{x}_0+\mathrm{span}\{\mathbf{v}_1,\cdots,\mathbf{v}_s\} \\ :\text{ the } \underline{\text{translation}} \text{ of } \mathrm{span}\{\mathbf{v}_1,\cdots,\mathbf{v}_s\} \text{ by } \mathbf{x}_0. \end{array}\right)$$

pf) Set $W:=\mathrm{span}\{\mathbf{v}_1,\cdots,\mathbf{v}_s\}$ & let S be the sol. set of $A\mathbf{x}=\mathbf{b}$.

① $\mathbf{x}_0+W\subseteq S$ (trivial)

② $\mathbf{x}_0+W\supseteq S$

For $\mathbf{x}\in S$, let $\mathbf{w}=\mathbf{x}-\mathbf{x}_0$. Then $A\mathbf{w}=A\mathbf{x}-A\mathbf{x}_0=\mathbf{0}$

$\therefore \mathbf{w}\in W$ & $\mathbf{x}=\mathbf{x}_0+\mathbf{w}\in\mathbf{x}_0+W$. ▨

We call $\mathbf{x}_0$ a <u>particular solution</u> of $A\mathbf{x}=\mathbf{b}$.

A solution of the form (*) is called a <u>general solution</u> of $A\mathbf{x}=\mathbf{b}$.

<u>Thm 3.5.2</u>

A general solution of a consistent linear system $A\mathbf{x}=\mathbf{b}$ can be obtained by adding a particular solution $\mathbf{x}_0$ of $A\mathbf{x}=\mathbf{b}$ to a <u>general solution of $A\mathbf{x}=\mathbf{0}$</u>. ▨

Say $\mathbf{x}_h$.

$\mathbf{x}_0+\mathbf{x}_h$ is a general sol. of $A\mathbf{x}=\mathbf{b}$.

Thm 3.5.3 Let A be an $m \times n$ matrix. TFAE

(a) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$.

(i.e., $A\mathbf{x} = \mathbf{b}$ is inconsistent or has a unique sol.)

$\therefore$ Set $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$. $\exists \mathbf{x}_0 \Rightarrow \mathbf{x} = \mathbf{x}_0 + \mathbf{0} = \mathbf{x}_0$

$\nexists \mathbf{x}_0 \Rightarrow \nexists \mathbf{x}$.

Thm 3.5.4 A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.

$\#$ free variables $\geq n - m > 0$.

$\Downarrow$

$\because$ $A\mathbf{x} = \mathbf{b}$, $A : m \times n \Rightarrow m < n. \Rightarrow A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. $\therefore$ $\#$ sol. of $A\mathbf{x} = \mathbf{b}$ depends on the existence of a particular system.

Def. (column space)

$\mathrm{col}(A) = \mathrm{span}\{\mathbf{c}_1(A), \mathbf{c}_2(A), \cdots, \mathbf{c}_n(A)\}$ for an $m \times n$ matrix A

Thm 3.5.5

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is in the column space of A.

$\therefore$ $A\mathbf{x} = \mathbf{b}$, $A : m \times n$

$\shortparallel$

$x_1 \mathbf{c}_1(A) + \cdots + x_n \mathbf{c}_n(A)$

For a nonzero vector $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \ : \text{hyperplane in } \mathbb{R}^n.$$

$$\Updownarrow$$

$$\mathbf{a} \cdot \mathbf{x} = b$$

Notation For $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a}^\perp = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} = 0 \}$, a perp.

: the orthogonal complement of $\mathbf{a}$.

Thm. 3.5.6 $A : m \times n$ matrix

The sol. space of $A\mathbf{x} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.

∵ $$A\mathbf{x} = \begin{bmatrix} - & \mathbf{r}_1(A) & - \\ & \vdots & \\ - & \mathbf{r}_m(A) & - \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1(A) \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m(A) \cdot \mathbf{x} \end{bmatrix}$$

$$\therefore \ A\mathbf{x} = \mathbf{0} \iff \mathbf{r}_i(A) \cdot \mathbf{x} = 0 \quad \forall i.$$

※ $\operatorname{span}\{ \mathbf{r}_1(A), \cdots, \mathbf{r}_m(A) \} =: \operatorname{row}(A)$.

$\{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \} = \operatorname{null}(A)$ & $\operatorname{row}(A)^\perp = \operatorname{null}(A)$.

§ 3.6. Matrices with special forms

Diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}_{n\times n}$$

$$(D)_{ij} = 0 \quad \text{for } i \neq j$$

D is invertible $\iff$ $d_i \neq 0 \quad \forall i = 1, \cdots, n$

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

For $k \in \mathbb{N} \cup \{0\}$, $\quad D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$

If D is invertible, then $\exists D^k \quad \forall k \in \mathbb{Z}$.

Triangular matrix

Let $A = [a_{ij}]_{n\times n}$.

We say that A is lower (resp. upper) triangular if

$(A)_{ij} = 0$ for $i < j$ (resp. $i > j$).

We say that A is strictly lower (resp. upper) triangular if

$(A)_{ij} = 0$ for $i \leq j$ (resp. $i \geq j$).

Thm 3.6.1

(a) The transpose of a lower (resp. upper) triangular matrix is upper (resp. lower) triangular.

(b) A product of lower (resp. upper) triangular matrices is lower (resp. upper) triangular.

(c) A triangular matrix is invertible iff its diagonal entries are all nonzero.

(d) The inverse of an invertible lower (resp. upper) triangular matrix is lower (resp. upper) triangular.

pf) We will prove (c) and (d) in CH.4.

(a) : trivial.

(b) Assume A and B are lower triangular.

Then $\mathbb{r}_i(A) = [a_{i1} \ a_{i2} \cdots a_{ii} \ 0 \ 0 \cdots 0] \quad (i=1,\cdots,n)$

and $\mathbb{c}_j(B) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{jj} \\ \vdots \\ b_{nj} \end{bmatrix} \quad (j=1,\cdots,n)$

If $i<j$, then $(AB)_{ij} = \mathbb{r}_i(A)\,\mathbb{c}_j(B) = a_{i1}\times 0 + \cdots + a_{ii}\times 0$
$+ 0\times 0 + \cdots + 0\times 0$
$+ 0\times b_{jj} + \cdots + 0\times b_{nj}$
$= 0$

Def. A square matrix A is symmetric (resp. skew-symmetric)

if $A = A^T$ (resp. $A = -A^T$).

One can check that every diagonal entry of a skew-symmetric matrix is zero!

Thm 3.6.2 A, B : symmetric matrices with the same size

k : any scalar

$\Rightarrow$ (a) A^T is symmetric.

(b) $A \pm B$ is symmetric

(c) kA is symmetric

Thm 3.6.3 A, B : symmetric matrices with the same size

AB is symmetric iff $AB = BA$.

pf) It follows from the fact that $(AB)^T = B^T A^T = BA$.

$\square$

Thm 3.6.4 If A is an invertible symmetric matrix, then A^{-1} is symmetric.

($\because$ $(A^{-1})^T = (A^T)^{-1} = A^{-1}$.)