Review $A = [a_{ij}]_{n\times n}$

- $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

- the minor of entry a_{ij} (or called the ijth entry of A)

  M_{ij} = the determinant of the submatrix obtained by deleting the ith row and the jth row of A.

- the cofactor of entry a_{ij} (or called the ijth cofactor of A)

  $C_{ij} = (-1)^{i+j} M_{ij}$

- $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$: the cofactor expansion along the ith row

  $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$: the cofactor expansion along the jth column.

- $\det(A) = \det(A^T)$

- $$\begin{cases} B = E_i(k)A & \Rightarrow \det(B) = k\det(A) \\ B = E_{ij}A & \Rightarrow \det(B) = -\det(A) \\ B = E_{ij}(k)A & \Rightarrow \det(B) = \det(A) \end{cases}$$

Note that $\det(E_{ij}) = -1$, $\det(E_i(k)) = k$, $\det(E_{ij}(k)) = 1$.

Hence for an elemetary matrix E, $\det(EA) = \det(E)\det(A)$.

Check the following.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B_2 = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{21}+a_{31} & ka_{22}+a_{32} & ka_{23}+a_{33} \end{bmatrix}$$

$$\Rightarrow \det(B_1) = ka_{21}C_{21} + ka_{22}C_{22} + ka_{23}C_{23}$$

$$= k(a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23})$$

$$= k\det(A)$$

$$\det(B_2) = a_{21}a_{12}a_{33} + a_{22}a_{13}a_{31} + a_{23}a_{11}a_{32} - a_{23}a_{12}a_{31} - a_{22}a_{11}a_{33} - a_{21}a_{13}a_{32}$$

$$= -\det(A)$$

$$\det(B_3) = (ka_{21}+a_{31})C_{31} + (ka_{22}+a_{32})C_{32} + (ka_{23}+a_{33})C_{33}$$

$$= ka_{21}C_{31} + ka_{22}C_{32} + ka_{23}C_{33} + a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} + \det(A) = \det(A)$$

Thm 4.2.3 $\quad A = [a_{ij}]_{n\times n}$

(a) If A has two identical rows and columns, then $\det(A)=0$.

(b) If A has two proportional rows or columns, then $\det(A)=0$.

(c) $\det(kA) = k^n \det(A)$.

(d) $$\left.\begin{cases} a_{i1}C_{i'1} + a_{i2}C_{i'2} + \cdots + a_{in}C_{i'n} = 0 & \text{if } i \neq i' \\ a_{1j}C_{1j'} + a_{2j}C_{2j'} + \cdots + a_{nj}C_{nj'} = 0 & \text{if } j \neq j' \end{cases}\right\}$$ Thm 4.3.1

Ex
$$\begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & 3 \\ 3 & 1 & 8 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18.$$

Thm 4.2.4 A square matrix A is invertible iff $\det(A) \neq 0$.

pf) Let R be the reduced row echelon form of A. Then

$\exists$ elementary matrices $E_1, E_2, \cdots, E_k$ s.t. $E_k \cdots E_2 E_1 A = R$.

By Thm 4.2.2, $\det(A) \neq 0 \iff \det(R) \neq 0$

Note that $R = I_n$ or the last row of R consists of all zeros.

$\therefore \det(R) \neq 0 \iff R = I_n$.

A: invertible $\iff R = I_n \iff \det(R) \neq 0 \iff \det(A) \neq 0$. ▨

Thm 4.2.5 Assume that A and B are $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B).$$

pf) ① Suppose that A or B is singular. Then AB is also singular.

$\therefore \det(A)\det(B) = 0 = \det(AB)$.

② Suppose that both A and B are invertible.

Then there exist elementary matrices $E_1, \cdots, E_r$ s.t. $A = E_1 \cdots E_r$.

$\therefore \det(AB) = \det(E_1 \cdots E_r B) = \det(E_1) \cdots \det(E_r)\det(B) = \det(A)\det(B)$.

▨

Cor. $\det(A^n) = \{\det(A)\}^n$.

$A = LU \Rightarrow \det(A) = \det(L)\det(U)$

$A = PLU \Rightarrow \det(A) = \underbrace{\det(P)}_{\pm 1}\det(L)\det(U)$

Thm 4.2.6 A : invertible $\Longrightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

Recall that we have :

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*

We can add one more equivalent statement.

9. $\det(A) \neq 0$. (Thm 4.2.7)

§4.3. Cramer's rule ; Formula for A^{-1} ; Applications of determinants.

Def. Let $C = [C_{ij}]_{n\times n}$ be the matrix of cofactors from A.

The adjoint of A, denoted by $\mathrm{adj}(A)$, is

$$\mathrm{adj}(A) = C^T.$$

Thm 4.3.3 A : invertible $\Rightarrow$ $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

pf) ETS : $A\,\mathrm{adj}(A) = \det(A)\,I$.

The (i,j)-entry of $A\,\mathrm{adj}(A)$ is

$$\mathbf{r}_i(A)\,\mathbf{c}_j(\mathrm{adj}(A)) = \mathbf{r}_i(A)\,\mathbf{r}_j(C)$$

$$= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

$$= \begin{cases} \det(A) & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$$

$\therefore$ $A\,\mathrm{adj}(A) = \det(A)\,I$. ☒

Thm 4.3.4 (Cramer's rule)

Let A be an $n\times n$ matrix. Then $A\mathbf{x} = \mathbf{b}$ has a unique solution iff $\det(A)\neq 0$, in which case the solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \ \ldots, \ x_n = \frac{\det(A_n)}{\det(A)},$$

where $A_j = [\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \mathbf{b}\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n]$.

There was a mistake when I wrote this matrix. This is right.

pf) By Thm 4.3.3, $\mathbf{x} = \frac{1}{\det(A)}\mathrm{adj}(A)\,\mathbf{b}$

For each $j=1,\ldots,n$,

$$x_j = \frac{1}{\det(A)}\begin{bmatrix} C_{1j} & C_{2j} & \cdots & C_{nj}\end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n\end{bmatrix}$$

$$= \frac{1}{\det(A)}(b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}) = \frac{\det(A_j)}{\det(A)}$$ ☒

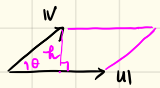
* Geometric interpretation of determinants

Thm 4.3.5 (just as a numerical value ; we do not consider the unit of area or volume)

(a) For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $|\det(A)|$, where $A = [\mathbf{u} \ \ \mathbf{v}]$.

(b) For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, the volume of the parallepiped determined by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is $|\det(A)|$, where $A = [\mathbf{u} \ \ \mathbf{v} \ \ \mathbf{w}]$.

pf) (a) Assume that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.



$h = \|\mathbf{v}\| \sin\theta$

$$(\text{Area})^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2\theta = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

$$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1 x_2 + y_1 y_2)^2$$

$$= (x_1 y_2 - x_2 y_1)^2 = (\det(A))^2.$$

$$\therefore \ \text{Area} = |\det(A)|.$$

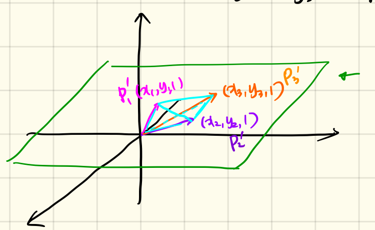
(b) Exercise. (Hint. $\det[\mathbf{u} \ \ \mathbf{v} \ \ \mathbf{w}] = \det\begin{bmatrix} - \mathbf{u} - \\ - \mathbf{v} - \\ - \mathbf{w} - \end{bmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$)

Thm 4.3.6 Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$ be lie on $\mathbb{R}^2$.

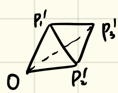
The area of the triangle $\triangle P_1 P_2 P_3$ is

$$\frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

pf)



The volume of the trigonal pyramid



is

$$\frac{1}{3} \triangle P_1 P_2 P_3 = \frac{1}{6} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

$\square$

If n points in the xy-plane have distinct x-coordinates, then there exists a unique polynomial of degree $\leq n-1$ whose graph passes through those points.

pf) Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n) \in \mathbb{R}^2$.

Suppose that

$\exists\ y = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ s.t. $y_i = a_0 + a_1 x_i + \cdots + a_{n-1} x_i^{n-1}$ $(i = 1, \cdots, n)$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad (*)$$

The determinant of this matrix is $\prod_{1 \leq i < j \leq n} (x_i - x_j)$, called the Vandermonde determinant.

$\therefore$ If x_i's are distinct, then we can find the solution of the linear system $(*)$. $\blacksquare$