7.6.6

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives the first, the third, and the fourth columns form a basis of the column space. Hence, we have a basis $\{(1,-2,3,-1,1,4),(2,0,1,-3,2,1),(1,-6,8,-6,1,11)\}$ for the column space of A . You may note that the second, the third and the fourth columns also form a basis. Similarily, if we make reduced column echelon form, which is equivalent to the reduced row echelon form of A^T, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{7}{4} & 0 & 0 \end{bmatrix}$$

which gives the first, the second, and the fourth rows form a basis of the row space. Hence we have a basis $\{(1,3,2,1),(-2,-6,0,-6),(-1,-3,-3,-6)\}$ for the row space of A. You may note that this is not the unique choice for the basis.

7.6.10

The reduced row echelon form of $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This shows that if we choose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$, then it form a basis for the space spanned by the vectors, and $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_5 = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_4$.

7.6.14

From the reduced row echelon form we may find bases for $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$, and especially, our basis for $\text{col}(A)$ will be chosen from columns of A. To find a basis for $\text{null}(A^T)$, we have to find the reduced row echelon form of A for $\left[A \mid I\right]$, which is

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{8} & -1 & \frac{3}{8} & 0 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{7}{16} & 1 & -\frac{5}{16} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & -2 & 1 \end{array}\right]$$

Hence, $\{(2, 0, -1, 0), (0, 4, 5, 0), (0, 0, 0, 1)\}$, which are the scalar multiple of the first, the second, and the third rows of the reduced row echelon form of A, is a basis of $\text{row}(A)$. The solutions of A$\mathbf{x}$=0 form a basis for null(A), so $\{(2, -5, 4, 0)\}$ is a basis of $\text{null}(A)$. Also, $\{(3, -1, 1, 7), (2, 2, 6, 2), (1, -2, -5, 0)\}$ is a basis of $\text{col}(A)$, which are the first, the second, and the fourth columns of A, and $\{(0, 5, -2, 1)\}$ is a basis of $\text{null}(A^T)$.

7.6.15

(a) From the reduced row echelon form, we may choose the first, the third, and the fourth columns. Hence, $\{(4,0,0,0,1),(8,5,0,3,2),(8,15,4,9,2)\}$ is a basis for $\mathrm{col}(A)$ which consisting of column vectors.

(b) From the reduced row echelon form of the transpose, we may choose the first, the second and the third row. Hence, $\{(4,16,8,8),(0,0,5,15),(0,0,0,4)\}$ is a basis for $\mathrm{row}(A)$ which consisting of row vectors.

(c) From the second column of the row echelon form, $\{(-4,1,0,0)\}$ is a basis for $\mathrm{null}(A)$.

(d) From the fourth and the fifth column of the reduced row echelon form of the transpose, $\{(0,-3,0,5,0),(-1,0,0,0,4)\}$ is a basis for $\mathrm{null}(A^T)$.

7.6.D1

(a) Since the number of leading 1's is the rank, so the first blank is 3, which is minimum among the number of rows and the number of columns. The number of parameters in a general solution of $A\mathbf{x} = \mathbf{0}$ is the number of column minus the rank, so the second blank is 5. The rank of A and the rank of A^T is same, and so the third and the fourth blanks are both 3. Lastly, the nullity of A^T is the number of row minus the rank, so is at most 3.

(b) As similarily as (a), the answer is 3,3,3,3,5.

(c) As similarily as (a), the answer is 4,4,4,4,4.

(d) As similarily as (a), the answer is $\min\{m, n\}, n, \min\{m, n\}, \min\{m, n\}, m$.

If we consider only nonzero matrices, then the answer will be $\min\{m, n\}, n-1, \min\{m, n\}, \min\{m, n\}, m-1$.

7.6.D4

(a) From the reduced row echelon form, we may choose $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$. There are many other answers such as $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$,$\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$, etc.

(b) From the choise $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$, we have $\mathbf{a}_3 = 4\mathbf{a}_1 - 3\mathbf{a}_2$ and $\mathbf{a}_5 = 6\mathbf{a}_1 + 7\mathbf{a}_2 + 2\mathbf{a}_4$. If you choose $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ as a basis, then from $(0, 1, 0, 0, 0) = \frac{4}{3}(1, 0, 0, 0, 0) - \frac{1}{3}(4, -3, 0, 0, 0)$ and $(6, 7, 2, 0, 0) = \frac{46}{3}(1, 0, 0, 0, 0) - \frac{7}{3}(4, -3, 0, 0, 0) + 2(0, 0, 1, 0, 0)$, we have $\mathbf{a}_2 = \frac{4}{3}\mathbf{a}_1 - \frac{1}{3}\mathbf{a}_3$ and $\mathbf{a}_5 = \frac{46}{3}\mathbf{a}_1 - \frac{7}{3}\mathbf{a}_3 + 2\mathbf{a}_4$

7.7.8

The vecor component of $\mathbf{x}$ along $\mathbf{a}$ is

$$\frac{\mathbf{x} \bullet \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{10+0+3-7}{4+1+1+1}\mathbf{a} = \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7}\right)$$

and the vector componenet of $\mathbf{x}$ orthogonal to $\mathbf{a}$ is

$$(5, 0, -3, 7) - \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7}\right) = \left(\frac{23}{7}, -\frac{6}{7}, -\frac{15}{7}, \frac{55}{7}\right)$$

7.7.16

Construct the $M = \begin{bmatrix} 1 & 4 \\ -2 & -2 \\ 5 & 3 \end{bmatrix}$. Then, the standard matrix of the orthogonal projection, is $P = M(M^TM)^{-1}M^T$. Since

$$M^TM = \begin{bmatrix} 1 & -2 & 5 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & -2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 30 & 23 \\ 23 & 29 \end{bmatrix},$$

we have

$$P = \frac{1}{341} \begin{bmatrix} 1 & 4 \\ -2 & -2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 29 & -23 \\ -23 & 30 \end{bmatrix} \begin{bmatrix} 1 & -2 & 5 \\ 4 & -2 & 3 \end{bmatrix} = \frac{1}{341} \begin{bmatrix} 325 & -68 & -24 \\ -68 & 52 & -102 \\ -24 & -102 & 305 \end{bmatrix}$$

Easily, this matrix is symmetric. Since first row and second row are linearly independent, it has the rank at least 2. Now, $4(325, -68, -24) + 17(-68, 52, -102) + 6(-24, -102, 305) = (0, 0, 0)$. Hence, the rows of this matrix is not linearly independent, so it has the rank exactly 2. Lastly, to check whether this is an idempotent, by just computation, you may check $P^2 = P$.

7.7.20

Since the vector $(2,-1,3)$ spans the line orthogonal to given plane, which is orthogonal compoenent, we may compute orthogonal projection by finding a projection to direction $(2,-1,3)$ and then, subtract it from the identity matrix. Since the standard matrix of the projection along the direction $(2,-1,3)$ is $\frac{1}{4+1+9}\begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{bmatrix}$, the answer is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 10 & 2 & -6 \\ 2 & 13 & 3 \\ -6 & 3 & 5 \end{bmatrix}$$

Then, the projection of $\mathbf{v}$ is

$$\frac{1}{14}\begin{bmatrix} 10 & 2 & -6 \\ 2 & 13 & 3 \\ -6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 34 \\ 53 \\ -5 \end{bmatrix}$$

7.7.24

If you use the reduced row echelon form, the solution space of the linear space has a basis $\{(-3,0,1,2)\}$. Hence, the orthogonal projection of $\mathbf{v}$ on the solution space is

$$\frac{(-3,0,1,2)\bullet(1,1,2,3))}{\|(-3,0,1,2)\|^2}(-3,0,1,2)=\frac{5}{14}(-3,0,1,2)=\frac{1}{14}(-15,0,5,10)$$

7.7.28 Let $A = \begin{bmatrix} 3 & 0 & 2 & -1 \\ 1 & 4 & 1 & 2 \\ 4 & -8 & 2 & -6 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$. Show that $A\mathbf{x} = \mathbf{b}$ is consistent and find the solution $\mathbf{x}_{\text{row}}$ that lies in the row space of A.

Solution. The reduced row echelon form of the matrix $[A|b]$ is

$$[R|c] = \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{1}{3} & | & \frac{4}{3} \\ 0 & 1 & \frac{1}{12} & \frac{7}{12} & | & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

So $x_0 = (\frac{4}{3}, -\frac{1}{3}, 0, 0)^T$ is one solution, and that $A\mathbf{x} = \mathbf{b}$ is consistent. Let B be the 4×2 matrix having the first two rows of R as its columns, and let $C = B^T B = \begin{bmatrix} \frac{14}{9} & -\frac{5}{36} \\ -\frac{5}{36} & \frac{97}{72} \end{bmatrix}$. Then the standard matrix for the orthogonal projection of R^4 onto $\text{row}(R) = \text{row}(A)$ is $P = BC^{-1}B^T$, and the solution of $A\mathbf{x} = \mathbf{b}$ which lies in $\text{row}(A)$ is given by

$$\mathbf{x}_{row(A)} = Px_0 = \begin{bmatrix} \frac{194}{299} & \frac{20}{299} & \frac{131}{299} & -\frac{53}{299} \\ \frac{20}{299} & \frac{224}{299} & \frac{32}{299} & \frac{124}{299} \\ \frac{131}{299} & \frac{32}{299} & \frac{90}{299} & -\frac{25}{299} \\ -\frac{53}{299} & \frac{124}{299} & -\frac{25}{299} & \frac{90}{299} \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{252}{299} \\ -\frac{48}{299} \\ \frac{164}{299} \\ -\frac{112}{299} \end{bmatrix}$$

$\square$

7.7.31 Find the orthogonal projection of the vector $\mathbf{v} = (1,1,1,1)$ on the orthogonal complement of the subspace of R^4 spanned by $\mathbf{v}_1 = (1,-2,3,0)$ and $\mathbf{v}_2 = (3,4,-1,2)$.

Solution. Let A be the 4×2 matrix having the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as its columns. Then

$$A^T A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 3 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -8 \\ -8 & 30 \end{bmatrix}$$

and the standard matrix for the orthogonal projection of R^4 onto the subspace $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 3 & -1 \\ 0 & 2 \end{bmatrix} \frac{1}{178} \begin{bmatrix} 15 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & 4 & -1 & 2 \end{bmatrix} = \frac{1}{89} \begin{bmatrix} 51 & 23 & 28 & 25 \\ 23 & 54 & -31 & 20 \\ 28 & -31 & 59 & 5 \\ 25 & 20 & 5 & 14 \end{bmatrix}$$

Thus the orthogonal projection of the vector $\mathbf{v} = (1,1,1,1)$ onto $W^\perp$ is given by

$$\mathbf{v} - P\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{89} \begin{bmatrix} 51 & 23 & 28 & 25 \\ 23 & 54 & -31 & 20 \\ 28 & -31 & 59 & 5 \\ 25 & 20 & 5 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{89} \begin{bmatrix} 127 \\ 66 \\ 61 \\ 64 \end{bmatrix} = \frac{1}{89} \begin{bmatrix} -38 \\ 23 \\ 28 \\ 25 \end{bmatrix}$$

□

7.7.D5 Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) If W is a subspace of R^n, then $\text{proj}_W \mathbf{u}$ is orthogonal to $\text{proj}_{W^\perp} \mathbf{u}$ for every vector $\mathbf{u}$ in R^n.

(b) An $n \times n$ matrix P that satisfies the equation $P^2 = P$ is the standard matrix for an orthogonal projection onto some subspace of R^n.

(c) If $\mathbf{x}$ is a solution of a linear system $A\mathbf{x} = \mathbf{b}$, then $\text{proj}_{row(A)} \mathbf{x}$ is also a solution.

(d) If P is the standard matrix for the orthogonal projection of R^n onto a subspace W, then $I - P$ is idempotent.

(e) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent linear system, then so is $A\mathbf{x} = \text{proj}_{col(A)} \mathbf{b}$.

Solution. (a) True. Since $\text{proj}_W \mathbf{u}$ belongs to W and $\text{proj}_{W^\perp}$ belongs to $W^\perp$, the two vectors are orthogonal

(b) False. For example, the matrix $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ satisfies $P^2 = P$ but is not symmetric and therefore does not correspond to an orthogonal projection.

(c) True. See the proof of Theorem 7.7.7.

(d) True. Since $P^2 = P$, we also have $(I - P)^2 = I - 2P + P^2 = I - P$; thus $I - P$ is idempotent.

(e) False. In fact, since $\text{proj}_{col(A)} \mathbf{b}$ belongs to col(A), the system $A\mathbf{x} = \text{proj}_{col(A)} \mathbf{b}$ is always consistent.

□

7.7.D6 If W is a subspace of R^n, what can you say about $((W^\perp)^\perp)^\perp$?

Solution. Since $(W^\perp)^\perp = W$ (Theorem 7.7.8), it follows that $((W^\perp)^\perp)^\perp = W^\perp$. □

7.7.P3 Let P be a symmetric $n \times n$ matrix that is idempotent and has rank k. Prove that P is the standard matrix for an orthogonal projection onto some k-dimensional subspace W of R^n.

Solution. Let P be a symmetric $n \times n$ matrix that is idempotent and has rank k. Then $W = \operatorname{col}(P)$ is a k-dimensional subspace of R^n. We will show that P is the standard matrix for the orthogonal projection of R^n onto W; i.e., that $P\mathbf{x} = \operatorname{proj}_W \mathbf{x}$ for all $\mathbf{x}$ in R^n. To this end, we first note that $P\mathbf{x}$ belongs to W and that

$$\mathbf{x} = P\mathbf{x} + (\mathbf{x} - P\mathbf{x}) = P\mathbf{x} + (I - P)\mathbf{x}.$$

To show that $P\mathbf{x} = \operatorname{proj}_W \mathbf{x}$ it suffices (from Theorem 7.7.4) to show that $(I - P)\mathbf{x}$ belongs to $W^\perp$, and since $W = \operatorname{col}(P)$, this is equivalent to showing that $P\mathbf{y} \cdot (I - P)\mathbf{x} = 0$ for all $\mathbf{y}$ in R^n. Finally, since $P^T = P = P^2$ (P is symmetric and idempotent), we have $P(I - P) = P - P^2 = P - P = O$ and so

$$P\mathbf{y} \cdot (I - P)\mathbf{x} = (P\mathbf{y})^T (I - P)\mathbf{x} = \mathbf{y}^T P^T (I - P)\mathbf{x} = \mathbf{y}^T P (I - P)\mathbf{x} = \mathbf{y}^T O \mathbf{x} = 0$$

for every $\mathbf{x}$ and $\mathbf{y}$ in R^n. This completes the proof.

□

7.8.2 Let $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Find the least square solution of $A\mathbf{x} = \mathbf{b}$ by solving the associated normal system, and show that the least squares error vector is orthogonal to the column space of A (as guaranteed by Theorem 7.8.4).

Solution. The columns of A are linearly independent and so A has full column rank. Thus the system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution given by

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 9 \\ -14 \end{bmatrix}$$

The least squares error vector is

$$\mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \frac{1}{21} \begin{bmatrix} 9 \\ -14 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} -4 \\ -16 \\ 8 \end{bmatrix}$$

and it is easy to check that this vector is orthogonal to each of the columns of A.

□

7.8.4 For the matrices in Exercise 7.8.2, find $\text{proj}_{col(A)}\mathbf{b}$, and confirm that the least squares solution satisfies the equation $A\mathbf{x} = \text{proj}_{col(A)}\mathbf{b}$

Solution. From Exercise 2, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \frac{1}{21}\begin{bmatrix} 9 \\ -14 \end{bmatrix}$; thus

$$A\mathbf{x} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \frac{1}{21} \begin{bmatrix} 9 \\ -14 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 46 \\ -5 \\ 13 \end{bmatrix}$$

On the other hand, the standard matrix of the orthogonal projection onto col(A) is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 20 & -4 & 2 \\ -4 & 5 & 8 \\ 2 & 8 & 17 \end{bmatrix}$$

and so we have

$$\text{proj}_{col(A)}\mathbf{b} = P\mathbf{b} = \frac{1}{21} \begin{bmatrix} 20 & -4 & 2 \\ -4 & 5 & 8 \\ 2 & 8 & 17 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 46 \\ -5 \\ 13 \end{bmatrix} = A\mathbf{x}.$$

□

7.8.8 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -5 \\ 1 & 3 & 4 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Find all least squares solutions of $A\mathbf{x} = \mathbf{b}$, and confirm that all of the solutions have the same error vector (and hence the same least squares error). Compute the least squares error.

Solution. The least squares solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by solving $A^T A\mathbf{x} = A^T\mathbf{b}$ which is

$$\begin{bmatrix} 6 & 11 & 17 \\ 11 & 22 & 33 \\ 17 & 33 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}.$$

The augmented matrix of this system reduces to $\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & \frac{19}{11} \\ 0 & 0 & 0 & 0 \end{bmatrix}$; thus there are infinitely many solutions given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 - t \\ \frac{19}{11} - t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{19}{11} \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The error vector is

$$\mathbf{b} - A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -5 \\ 1 & 3 & 4 \end{bmatrix} \left(\begin{bmatrix} -3 \\ \frac{19}{11} \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} \frac{5}{11} \\ \frac{9}{11} \\ \frac{24}{11} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ -\frac{2}{11} \end{bmatrix}$$

and the least squares error is $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{(\frac{6}{11})^2 + (\frac{2}{11})^2 + (-\frac{2}{11})^2} = \sqrt{\frac{44}{121}} = \frac{2\sqrt{11}}{11}$. □

7.8.12 Find the least squares quadratic fit $y = a_0 + a_1x + a_2x^2$ to the points (1,-2), (0,-1), (1,0), (2,4). Show that the result is reasonable by graphing the curve and plotting the data in the same coordinate system.

Solution. The quadratic least squares model for the given data is $M\mathbf{v} = \mathbf{y}$ where $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 4 \end{bmatrix}$. The least squares solution is obtained by solving the normal system $M^T M\mathbf{v} = M^T\mathbf{y}$ which is

$$\begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 14 \end{bmatrix}.$$

Since the matrix on the left is nonsingular, this system has a unique solution given by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 6 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{5}{2} \\ \frac{5}{2} \end{bmatrix}.$$

□

7.8.13 Set up but do not solve the normal system for finding the stated least squares fit: the least squares cubic fit $y = a_0 + a_1x + a_2x^2 + a_3x^3$ to the data points (1, 4.9), (2, 10.8), (3, 27.9), (4, 60.2), (5, 113).

Solution. The model for the least squares cubic fit to the given data is $M\mathbf{v} = \mathbf{y}$ where

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 4.9 \\ 10.8 \\ 27.9 \\ 60.2 \\ 113.0 \end{bmatrix}.$$

The associated normal system $M^TM\mathbf{v} = M^T\mathbf{y}$ is

$$\begin{bmatrix} 5 & 15 & 55 & 225 \\ 15 & 55 & 225 & 979 \\ 55 & 225 & 979 & 4425 \\ 225 & 979 & 4425 & 20515 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 216.8 \\ 916.0 \\ 4087.4 \\ 18822.4 \end{bmatrix}$$

and the solution is approximately $(a_0, a_1, a_2, a_3) \approx (5.160, -1.864, 0.811, 0.775)$. $\square$

7.8.15 Show that if

$$M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

then the normal system $M^T M\mathbf{v} = M^T\mathbf{y}$ can be written as

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Solution. It is just a calculation. □

7.8.D1 Fill in the blanks:

(a) The distance in R^3 from the point $P_0 = (1, -2, 1)$ to the plane $x + y - z = 0$ is _______, and the point in the plane that is closest to P_0 is _______.

(b) The distance in R^4 form the point $\mathbf{b} = (1, 2, 0, -1)$ to the hyperplane $x_1 - x_2 + 2x_3 - 2x_4 = 0$ is _______, and the point in the hyperplane that is closest to $\mathbf{b}$ is _______.

Solution. (a) The distance d in R^3 from the point $P_0 = (1, -2, 1)$ to the plane $x + y - z = 0$ is

$$d = \frac{|(1)(1) + (1)(-2) + (-1)(1)|}{\sqrt{(1)^2 + (1)^2 + (-1)^2}} = \frac{2}{\sqrt{3}}.$$

The latter one can be found by computing the orthogonal projection of the vector $\mathbf{b} = \overrightarrow{OP_0}$ onto the plane: The column vectors of the matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the plane W with equation $x + y - z = 0$, and so the orthogonal projection of $\mathbf{b}$ onto W is given by

$$\text{proj}_W \mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}.$$

So the answer is $d = \frac{2}{\sqrt{3}}$ and the point $Q = (\frac{5}{3}, -\frac{4}{3}, \frac{1}{3})$.

(b) By exactly the same method, we can obtain the solution as:

$$\text{distance: } \frac{1}{\sqrt{10}}, \quad \text{point: } Q = (\tfrac{9}{10}, \tfrac{21}{10}, -\tfrac{2}{10}, -\tfrac{8}{10}).$$

□

7.8.P2 If A has linearly independent column vectors, and if $\mathbf{b}$ is orthogonal to the column space of A, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = 0$.

Solution. By assumption, $\text{proj}_{col(A)}\mathbf{b} = 0$. The solutions of $\mathrm{A}\mathbf{x} = \text{proj}_{\mathrm{col(A)}}\mathbf{b}$ are the least squares solutions of $\mathrm{A}\mathbf{x} = \mathbf{b}$. Thus, since the columns of A are linearly independent, we have $A\mathbf{x} = \text{proj}_{col(A)}\mathbf{b} = 0$ if and only if $\mathbf{x} = 0$. $\square$