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Fundamental spaces of a matrix A : $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$, $\text{null}(A^T)$

$$\begin{cases} \text{row}(A)^\perp = \text{null}(A) \\ \text{col}(A)^\perp = \text{null}(A^T) \end{cases}$$

$$\begin{cases} \text{rank}(A) = \dim(\text{row}(A)) \\ \text{nullity}(A) = \dim(\text{null}(A)) \end{cases}$$

For an $m\times n$ matrix A, $\quad \text{rank}(A) + \text{nullity}(A) = n$

For a subspace W of $\mathbb{R}^n$, $\quad \dim(W) + \dim(W^\perp) = n$.

$\therefore \quad \dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m$

Today, we will show that $\text{rank}(A) = \text{rank}(A^T)$, rank theorem.

Recall that if W is a subspace of $\mathbb{R}^n$ with dim. 1, then it is a line through the origin, i.e., $W = \text{span}\{a\}$. Then $W^\perp = \text{span}\{a\}^\perp = a^\perp$, and $\dim(W^\perp) = n-1$.

<u>Thm 7.4.5</u> If W is a subspace of $\mathbb{R}^n$ with dim $n-1$, then $\exists\, a \in \mathbb{R}^n - \{0\}$ s.t. $W = a^\perp$.

pf) Since $W^\perp$ is a 1-dim'l subspace of $\mathbb{R}^n$, $\exists\, a \in \mathbb{R}^n - \{0\}$ s.t.

$$W^\perp = \text{span}\{a\}.$$

Hence $\quad W = (W^\perp)^\perp = (\text{span}\{a\})^\perp = a^\perp \quad \blacksquare$

Let A be an $m\times n$ matrix. If $\text{rank}(A) = 1$, then there exists $a \in \mathbb{R}^n - \{0\}$ s.t. all the rows of A are scalar multiple of a.

<u>Thm 7.4.7</u> If u is a nonzero $m\times 1$ matrix and v is a nonzero $n\times 1$ matrix, then $\text{rank}(uv^T) = 1$. Conversely, if A is an $m\times n$ matrix with $\text{rank}(A) = 1$, then A can be factored into a product of the form uv^T.

pf) $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ & $\mathbf{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\Rightarrow$ $\mathbf{u}\mathbf{w}^T = \begin{bmatrix} - & u_1 \mathbf{w}^T & - \\ - & u_2\mathbf{w}^T & - \\ & \vdots & \\ - & u_m \mathbf{w}^T & - \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1\mathbf{u} & v_2\mathbf{u} & \cdots & v_n\mathbf{u} \\ | & | & & | \end{bmatrix}$

If rank$(A)=1$, then the row vectors of A are scalar multiples of a nonzero vector $\mathbf{w}^T = [v_1 \cdots v_n]$. i.e.,

$$A = \begin{bmatrix} u_1 \mathbf{w}^T \\ u_2 \mathbf{w}^T \\ \vdots \\ u_m \mathbf{w}^T \end{bmatrix} = \mathbf{u}\mathbf{w}^T.$$

Since rank$(A)=1$, $\mathbf{u} \neq \mathbf{0}$.

$\blacksquare$

Thm 7.4.8

If $\mathbf{u}$ is a nonzero column vector, then $\mathbf{u}\mathbf{u}^T$ is a symmetric matrix of rank 1. Conversely, if A is a symmetric matrix of rank 1, then it can be factored as $A = \mathbf{u}\mathbf{u}^T$ or $A = -\mathbf{u}\mathbf{u}^T$ for some nonzero $n\times 1$ column vector $\mathbf{u}$.

Pf) By Thm 7.4.7, $A = \mathbf{x}\mathbf{y}^T$ for some nonzero column vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

Since A is symmetric, $(\mathbf{x}\mathbf{y}^T)^T = \mathbf{x}\mathbf{y}^T$

Then $\mathbf{x}(\mathbf{y}^T\mathbf{y}) = (\mathbf{x}\mathbf{y}^T)\mathbf{y} = (\mathbf{x}\mathbf{y}^T)^T\mathbf{y} = \mathbf{y}(\mathbf{x}^T\mathbf{y})$

Hence $\mathbf{x} = k\mathbf{y}$, where $k = \dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}$.

Then $A = k\mathbf{y}\mathbf{y}^T$.

If $\mathbf{x}^T\mathbf{y} > 0$, then let $\mathbf{u} := \sqrt{\dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}}\,\mathbf{y}$.

Then $\mathbf{u}\mathbf{u}^T = \dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}\mathbf{y}^T = \mathbf{x}\mathbf{y}^T = A$.

If $\mathbf{x}^T\mathbf{y} < 0$, then let $\mathbf{u} := \sqrt{-\dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}}\,\mathbf{y}$

Then $\mathbf{u}\mathbf{u}^T = -\dfrac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}\mathbf{y}^T = -A$.

$\blacksquare$

§ 7.5. Rank theorem

Rank theorem : The row space and the column space of a matrix have the same dimension.

Thm 7.5.2 For an $m\times n$ matrix A, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Pf) Let $A = \begin{bmatrix} - \mathbf{a}_1 - \\ - \mathbf{a}_2 - \\ \vdots \\ - \mathbf{a}_m \end{bmatrix}$ with rank k.

If R is the reduced row echelon form of A, then

$$R = \begin{bmatrix} - \mathbf{r}_1 - \\ - \mathbf{r}_2 - \\ \vdots \\ - \mathbf{r}_k - \\ - \mathbf{0} - \\ - \mathbf{0} - \end{bmatrix},$$ where $\mathbf{r}_1, \dots, \mathbf{r}_k$ are nonzero vectors.

Since $\mathrm{row}(A) = \mathrm{row}(R)$,
$$\mathbf{a}_1 = c_{11}\mathbf{r}_1 + \cdots + c_{1k}\mathbf{r}_k$$
$$\vdots$$
$$\mathbf{a}_m = c_{m1}\mathbf{r}_1 + \cdots + c_{mk}\mathbf{r}_k$$

Hence
$$\underbrace{\begin{bmatrix} - \mathbf{a}_1 - \\ \vdots \\ - \mathbf{a}_m - \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mk} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} - \mathbf{r}_1 - \\ \vdots \\ - \mathbf{r}_k - \end{bmatrix}}_{R}$$

Then $\mathbf{c}_j(A) = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = C \cdot \mathbf{c}_j(R) = C \begin{bmatrix} r_{1j} \\ \vdots \\ r_{kj} \end{bmatrix} = r_{1j}\mathbf{c}_1(C) + \cdots r_{kj}\mathbf{c}_k(C)$

$\therefore$ $\mathrm{col}(A) \subset \mathrm{col}(C)$.

$\therefore$ $\dim(\mathrm{col}(A)) \leq \dim(\mathrm{col}(C)) \leq \underbrace{k}_{\#\text{ colums of } C} = \mathrm{rank}(A)$.

$\therefore$ $\dim(\mathrm{col}(A)) \leq \dim(\mathrm{row}(A))$.

Replacing A with A^T in the above, we can see that

$$\underbrace{\dim(\mathrm{col}(A^T))}_{\mathrm{rank}(A)} \leq \underbrace{\dim(\mathrm{row}(A^T))}_{\mathrm{rank}(A^T)}$$

$\therefore$ $\mathrm{rank}(A) = \mathrm{rank}(A^T)$. $\square$

Remark Let A be an $m\times n$ matrix with rank k.

Then $\begin{cases} \operatorname{rank}(A^T) = k \\ \operatorname{nullity}(A) = n-k \\ \operatorname{nullity}(A^T) = m-k \end{cases}$

Q: For an $m\times n$ matrix A, what is the largest possible value for the rank of A?

A: $\min\{m,n\}$.

Thm 7.5.3 (Consistency theorem)

$A\mathbf{x} = \mathbf{b}$ is consistent $\Longleftrightarrow$ $\mathbf{b} \in \operatorname{col}(A)$ $\Longleftrightarrow$ A & $[A \vdots \mathbf{b}]$ have the same rank.

$\Longleftrightarrow$ $\operatorname{col}(A) = \operatorname{col}([A \vdots \mathbf{b}])$

Def. An $m\times n$ matrix A has full column rank (full row rank) if its column (row) vectors are linearly independent.

Thm 7.5.5 $A: m\times n$

- A has full column rank $\Longleftrightarrow$ $\operatorname{rank}(A) = n$
- A has full row rank $\Longleftrightarrow$ $\operatorname{rank}(A) = m$.

Rmk If an $m\times n$ matrix A has full column (row) rank, then $n \le m$. ($n \ge m$)

Thm 7.5.6 $A: m\times n$

(a) $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

(b) $A\mathbf{x} = \mathbf{b}$ has at most one solution.

(c) A has full column rank.

Pf) (a) $\Leftrightarrow$ (c)

Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \cdots \mathbf{a}_n]$. Then $A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$.

(a) $\Leftrightarrow$ $\mathbf{a}_1, \cdots, \mathbf{a}_n$ are lin. indep. $\Longleftrightarrow$ A has full col. rank. $\blacksquare$