1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let A be a square matrix with $\text{rank}(A) = 1$. For vectors u_1, u_2, v_1, v_2 such that $A = u_1 v_1^T = u_2 v_2^T$, $u_1^T v_1$ is equal to $u_2^T v_2$.

(b) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then,

$$\text{nullity}(AB) = \max\{\text{nullity}(A), \text{nullity}(B)\}.$$

(c) If A and B have full column rank, then AB has full column rank.

Solution.

(a) True. Since $A = uv^T$, the column space of A is the same as the space spanned by u. Thus, the space spanned by u_1 is equal to the space spanned by u_2. It implies that $u_1 = cu_2$ with some nonzero constant c. Similarly, we consider the row space of A. Then, $v_1 = dv_2$ with some nonzero constant d. Since $A = u_1 v_1^T = cd(u_2 v_2^T) = u_2 v_2^T$, $cd = 1$. Therefore,

$$u_1^T v_1 = cd(u_2^T v_2) = u_2^T v_2.$$

(b) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then, $\text{nullity}(A) = \text{nullity}(B) = 1$. However, $\text{nullity}(AB) = 2$ since AB is zero matrix.

(c) True. By Theorem 7.5.6, the matrix C has full column rank if and only if $Cx = 0$ has only the trivial solution. If $ABx = 0$, then $Bx = 0$ since $A(Bx) = 0$ and A has full column rank. Also, B has full column rank, $x = 0$. Thus, by Theorem 7.5.6, the matrix AB has full column rank.

2 Let A be a square matrix. Assume that the rank of A is not equal to the rank of
10 points A^2. Prove that

$$\mathrm{null}(A) \cap \mathrm{col}(A) \neq \{0\}.$$

Solution.

If $v \in \mathrm{null}(A)$, then $A^2 v = 0$. Thus,

$$\mathrm{null}(A) \subset \mathrm{null}(A^2).$$

By the dimension theorem,

$$\mathrm{rank}(A) \geq \mathrm{rank}(A^2).$$

By assumption, we get

$$\mathrm{rank}(A) > \mathrm{rank}(A^2).$$

It implies that

$$\mathrm{null}(A) \neq \mathrm{null}(A^2).$$

Thus, there exists vector v such that $v \in \mathrm{null}(A^2)$ and $v \notin \mathrm{null}(A)$. Since $A^2 v = A(Av) = 0$, $Av \in \mathrm{null}(A) \cap \mathrm{col}(A)$. However, $v \notin \mathrm{null}(A)$, thus Av is a nonzero vector. Therefore, $\mathrm{null}(A) \cap \mathrm{col}(A) \neq \{0\}$.