§4.1. Determinants ; Cofactor expansion

A bijection from $\{1,2,\cdots,n\}$ to itself is called a <u>permutation</u>.

Let S_n be the set of all permutations on $\{1,2,\cdots,n\}$.

For two permutations $\sigma, \tau \in S_n$, $\sigma \circ \tau \in S_n$.

Each permutation $\sigma \in S_n$ can be written as an ordered arrangement

$$\sigma(1)\,\sigma(2)\,\cdots\,\sigma(n)$$

<u>Ex.</u> $S_3 = \{123, \overset{\sigma}{213}, \overset{\tau}{132}, 231, 312, 321\}$

- 123: identity
- 213: swapping 1 & 2
- 132: swapping 2 & 3

sgn = +	sgn = −
123	213
231	132
312	321

$$\sigma \circ \tau = (\sigma\circ\tau)(1)\,(\sigma\circ\tau)(2)\,(\sigma\circ\tau)(3)$$
$$= 231$$

$$\tau \circ \sigma = (\tau\circ\sigma)(1)\,(\tau\circ\sigma)(2)\,(\tau\circ\sigma)(3)$$
$$= 312$$

$$\sigma\circ\tau\circ\sigma = \tau\circ\sigma\circ\tau = 321 \quad \text{(check!)}$$

A <u>transposition</u> τ_{ij} is the permutation swapping i and j $(i \neq j)$.

<u>Fact</u> Every permutation σ can be written as the composition of transpositions. If σ is the composition of even (resp. odd) number of transpositions, then $\text{sgn}(\sigma) = +$ (resp. $\text{sgn}(\sigma) = -$)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad \sigma = 12, \ \tau = 21 \in S_2$$

$$= a_{1\sigma(1)}a_{2\sigma(2)} + \operatorname{sgn}(\tau)\, a_{1\tau(1)}a_{2\tau(2)}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$= \operatorname{sgn}(123)\, a_{11}a_{22}a_{33} + \operatorname{sgn}(231)\, a_{12}a_{23}a_{31} + \operatorname{sgn}(312)\, a_{13}a_{21}a_{32}$$
$$+ \operatorname{sgn}(321)\, a_{13}a_{22}a_{31} + \operatorname{sgn}(213)\, a_{12}a_{21}a_{33} + \operatorname{sgn}(132)\, a_{11}a_{23}a_{32}$$

For each permutation $\sigma \in S_n$ and an $n \times n$ matrix $A = [a_{ij}]$

$$a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

is called an elementary product and

$$\operatorname{sgn}(\sigma)\ a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

is called a signed elementary product.

__Def.__ For a square matrix A, the __determinant__ of A, denoted by $\det(A)$ or $|A|$ is the sum of all signed elementary products of A, i.e.,

$$\det(A) = |A| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

__Thm 4.1.2__ If a square matrix with a row or a column of zeros, then $\det(A) = 0$.

Thm 4.1.3 If A is triangular, then $\det(A)$ is the product of the entries on the main diagonal.

pf) A : lower triangular
(upper)

$\Rightarrow a_{ij} = 0$ when $i < j$ $(i > j)$

$\Rightarrow a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \neq 0$ iff $i \geq \sigma(i)$ $\forall i$

$(i \leq \sigma(i))$

$\Rightarrow \sigma = 12\cdots n$

$\therefore \det(A) = a_{11} a_{22} \cdots a_{nn}$ ▨

If the size of A is very large, then it is not easy to compute $\det(A)$ by only using the definition. We will use the cofactor expansion.

Def. Let $A = [a_{ij}]_{n\times n}$

- the minor of entry a_{ij}, denoted by M_{ij}, is the determinant of the submatrix of A obtained by deleting the ith row and the jth column of A.

i.e.,

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

- the cofactor of entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Ex) $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix} \implies M_{12} = \det\begin{bmatrix} 2 & 6 \\ 1 & 8 \end{bmatrix} = 10$

$C_{12} = (-1)^{1+2} M_{12} = -10.$

Note that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}C_{11} + a_{12}C_{22} + a_{13}C_{13}.$$

$$= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} \qquad (i = 2 \text{ or } 3)$$

$$= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} \qquad (j = 1, 2, 3).$$

<u>Thm 4.1.5</u> For an $n\times n$ matrix $A = [a_{ij}]_{n\times n}$,

$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

: the cofactor expansion along the ith row

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

: the cofactor expansion along the jth column.

proof) Note that the monomials appearing in $a_{ij} M_{ij}$ are the elementary products of A containing a_{ij} as a factor. Hence every elementary product of A can appear in the cofact expansion

$$a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$
$$\left(= a_{i1}(-1)^{i+1} M_{i1} + a_{i2}(-1)^{i+2} M_{i2} + \cdots + a_{in}(-1)^{i+n} M_{in} \right)$$

We only need to check that the sign of the elementary product

$$(a_{ij})(a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{nj_n}) = a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{ij} a_{i+1,j_{i+1}} \cdots a_{nj_n} \quad (\text{in } \det(A))$$

equals the sign of

$$(-1)^{i+j} \times \left(\text{the sign of } a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{nj_n} \ (\text{in } M_{ij}) \right)$$

$$\boxed{j_1, \cdots, j_{i-1}, j, j_{i+1}, \cdots, j_n}$$

interchanges from the top to the down determines the sign of $a_{1j_1} \cdots a_{ij} \cdots a_{nj_n}$ in $\det A$)

$\}$ # interchanges $= i-1$.

the integers $1, \cdots, j-1, j+1, \cdots, n$

$$\boxed{j, \underbrace{j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_n}}$$

they are the integers $1, \cdots, j-1, j+1, \cdots, n$

$\}$ # interchanges from the above to the below determines the sign of the elementary product $a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{nj_n}$ in M_{ij}.

$$\boxed{j, 1, \cdots, j-1, j+1, \cdots, n}$$

$\}$ # interchanges $= j-1$.

$$\boxed{1, \cdots, j-1, j, j+1, \cdots, n}$$

$\therefore$ the sign of $(a_{ij})(a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{nj_n})$ in $\det(A)$ is

$$(-1)^{i-1} \times (-1)^{j-1} \times (\text{the sign of } a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{nj_n} \text{ in } M_{ij})$$
$$= (-1)^{i+j} \times (\text{the sign of } a_{1j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{nj_n} \text{ in } M_{ij})$$

$\blacksquare$

<u>Ex</u>

$$\det\begin{bmatrix} 3 & -1 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & 3 \\ 3 & 1 & 8 & 0 \end{bmatrix} = (-1)^{2+1}\det\begin{bmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{bmatrix}$$

$$= -\left\{ (-1)\begin{vmatrix} 3 & 3 \\ 8 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 3 \end{vmatrix} \right\}$$

$$= -(24 + 3 - 9) = -18$$

§4.2. Properties of determinants

<u>Thm 4.2.1</u> For a square matrix A, $\det(A) = \det(A^T)$

pf) The cofactor expansion along the ith row of A

$=$ The cofactor expansion along the ith column of A^T.

$\therefore \det(A) = \det(A^T)$. □

<u>Thm 4.2.2</u> Let A be an $n \times n$ matrix.

(a) If B is the matrix obtained by multiplying a constant k to row i (or column j) of A, then $\det(B) = k\det(A)$.

(b) If B is the matrix obtained from A by interchanging row k and row l (or column k and column l) of A, then $\det(B) = -\det(A)$.

(c) If B is the matrix obtained by adding c-multiple of row k (resp. column k) to row l (resp. column l), then $\det(B) = \det(A)$.

pf) Set $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$.

(1)

$$B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{i-1,1} & \cdots & a_{i-1,n} \\ ka_{i1} & \cdots & ka_{in} \\ a_{i+1,1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Note that for each $j=1,\cdots,n$, the minor of entry b_{ij} in B equals the minor of entry a_{ij} in A.

$$\det(B) = b_{i1}C_{i1} + b_{i2}C_{i2} + \cdots + b_{in}C_{in}$$

$$= ka_{i1}C_{i1} + ka_{i2}C_{i2} + \cdots + ka_{in}C_{in}$$

$$= k\det(A)$$

(2)

$$\begin{cases} b_{ij} = a_{ij} & \text{if } i \neq k, l \\ b_{kj} = a_{lj} \\ b_{lj} = a_{kj} \end{cases}$$

We may assume that $k < l$.

$$\det(B) = \sum \pm 1\, b_{1j_1} \cdots b_{kj_k} \cdots b_{lj_l} \cdots b_{nj_n}$$

$$= \sum \pm 1\, a_{1j_1} \cdots a_{k-1,j_{k-1}}\, a_{lj_k}\, a_{k+1,j_{k+1}} \cdots a_{l-1,j_{l-1}}\, a_{k,j_l}\, a_{l+1,j_{l+1}} \cdots a_{nj_n}$$

$\therefore$ The elementary product $b_{1j_1} \cdots b_{kj_k} \cdots b_{lj_l} \cdots b_{nj_n}$ is the elementary product $a_{1j_1} \cdots a_{k-1,j_{k-1}}\, a_{k,j_l}\, a_{k+1,j_{k+1}} \cdots a_{l-1,j_{l-1}}\, a_{lj_k}\, a_{l+1,j_{l+1}} \cdots a_{nj_n}$.

Then the sign of $b_{1j_1} \cdots b_{kj_k} \cdots b_{lj_l} \cdots b_{nj_n}$ is determined by the number of interchanges from $(j_1, \cdots, j_k, \cdots, j_l, \cdots, j_n)$ to $(1, 2, \cdots, n)$. However, the sign of $a_{1j_1} \cdots a_{k,j_l} \cdots a_{lj_k} \cdots a_{nj_n}$ is determined by the number of interchanges from $(j_1, \cdots, j_l, \cdots, j_k, \cdots, j_n)$.

Since the minimal number of interchanges

from $(j_1, \cdots, j_k, \cdots, j_\ell, \cdots, j_n)$ to $(1, 2, \cdots, n)$

is equal to

$1 +$ the minimal # of interchanges from $(j_1, \cdots, j_\ell, \cdots, j_k, \cdots, j_n)$ to $(1, \cdots, n)$,

The sign of $b_{1\bar{j}_1} \cdots b_{k\bar{j}_k} \cdots b_{\ell \bar{j}_\ell} \cdots b_{n\bar{j}_n}$ in $\det(B)$ is different

from the sign of $a_{1\bar{j}_1} \cdots a_{k,\bar{j}_\ell} \cdots a_{\ell \bar{j}_k} \cdots a_{n\bar{j}_n}$ in $\det(A)$.

$\therefore \det(B) = -\det(A)$

(c)

$$B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ ca_{k1} + a_{\ell 1} & \cdots & ca_{kn} + a_{\ell n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Note that for each $j = 1, \cdots, n$, the minor of entry b_{kj} in B equals the minor of entry a_{kj} in A.

$$\det(B) = b_{\ell 1} C_{\ell 1} + b_{\ell 2} C_{\ell 2} + \cdots + b_{\ell n} C_{\ell n}.$$

$$= (ca_{k1} + a_{\ell 1}) C_{\ell 1} + (ca_{k2} + a_{\ell 2}) C_{\ell 2} + \cdots + (ca_{kn} + a_{\ell n}) C_{\ell n}$$

$$= c[a_{k1} C_{\ell 1} + a_{k2} C_{\ell 2} + \cdots + a_{kn} C_{\ell n}]$$

$$+ \underbrace{a_{\ell 1} C_{\ell 1} + a_{\ell 2} C_{\ell 2} + \cdots + a_{\ell n} C_{\ell n}}_{\det(A)}$$

Note that $a_{k1} C_{\ell 1} + a_{k2} C_{\ell 2} + \cdots + a_{kn} C_{\ell n}$ is the determinant

of the matrix

Note that $a_{k1}C_{l1} + a_{k2}C_{l2} + \cdots + a_{kn}C_{ln}$ is the cofactor expansion along the lth row of the matrix

$$\begin{array}{r c}
 & \\
k\text{th row} \rightarrow & \\
 & \\
l\text{th row} \rightarrow & \\
 &
\end{array}
\begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \vdots \\
a_{k1} & \cdots & a_{kn} \\
\vdots & & \vdots \\
a_{l-1,1} & \cdots & a_{l-1,1} \\
a_{k,1} & \cdots & a_{k,n} \\
a_{l+1,1} & \cdots & a_{l+1,1} \\
\vdots & & \vdots \\
a_{n1} & \cdots & a_{n1}
\end{bmatrix} =: C$$

Let C' be the matrix obtained by interchanging row k and row l of C. Then $C = C'$. However, it follows from (b) that $\det(C') = -\det(C)$. Hence $\det(C) = 0$.

i.e., $a_{k1}C_{l1} + a_{k2}C_{l2} + \cdots + a_{kn}C_{ln} = 0$.

Now go back to the proof of (c).

$$\begin{aligned}
\det(B) &= c\left[a_{k1}C_{l1} + a_{k2}C_{l2} + \cdots + a_{kn}C_{ln}\right] \\
&\quad + a_{l1}C_{l1} + a_{l2}C_{l2} + \cdots + a_{ln}C_{ln} \\
&= c \times 0 + \det(A) = \det(A)
\end{aligned}$$

$\square$