Review October 31, 2019.

Let V be a subspace of $\mathbb{R}^n$.

- A subset S of V is a <u>basis</u> if it is linearly independent and spans V.
- $\{0\}$ has no basis.
- All the bases of $V(\neq\{0\})$ have the same number of vectors.
- We define the dimension of V, denoted by $\dim(V)$, by the number of a basis for V.
- If A is an $n\times n$ matrix, then the following are equivalent.
  1) A is invertible.
  2) The column vectors of A form a basis for $\mathbb{R}^n$.
  3) The row vectors of A form a basis for $\mathbb{R}^n$.

§ 7.3. The fundamental spaces of a matrix

For an $m\times n$ matrix A,

row(A) = the subspace of $\mathbb{R}^n$ that is spanned by row vectors of A.

col(A) = the subspace of $\mathbb{R}^n$ that is spanned by column vectors of A.

null(A) = the solution space of $A\mathbf{x}=\mathbf{0}$.

Consider A and A^T together.

row(A) col(A) null(A)

row(A^T) col(A^T) null(A^T)

row(A), col(A), null(A), null(A^T) are called the fundamental spaces of A.

Def. • $\text{rank}(A) = \dim(\text{row}(A))$

• $\text{nullity}(A) = \dim(\text{null}(A))$

We will show that $\underbrace{\text{rank}(A)}_{\substack{\| \\ \#\text{ leading variable}}} + \underbrace{\text{nullity}(A)}_{\substack{\| \\ \#\text{ free variables.}}} = n$ ← This is called the dimension theorem for matrices.

Def. Let S be a nonempty set in $\mathbb{R}^n$.

Set $S^\perp = \{ x \in \mathbb{R}^n \mid a \cdot x = 0 \quad \forall a \in S \}$

We call $S^\perp$ the <u>orthogonal complement</u> of S.

Ex. For an $m \times n$ matrix, if S is the set of row vectors of A,

then $S^\perp = \text{null}(A)$.

Ex. $S = \{ v_1 = (1, -2, 1), v_2 = (3, -7, 5) \}$,

$S^\perp =$ the solution space of $\begin{bmatrix} 1 & -2 & 1 \\ 3 & -7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thm 7.3.3 If S is a nonempty set in $\mathbb{R}^n$, then $S^\perp$ is a subspace of $\mathbb{R}^n$.

proof) (i) $S^\perp \neq \phi$ ($\because 0 \in S^\perp$)

(ii) $\forall u, v \in S^\perp$, $\quad (u+v) \cdot a = 0 \quad \forall a \in S$.

(iii) $\forall u \in S^\perp$, $\forall c \in \mathbb{R}$, $\quad (cu) \cdot a = 0 \quad \forall a \in S$. ∎

Thm 7.3.4

(a) If W is a subspace of $\mathbb{R}^n$, then $W^\perp \cap W = \{0\}$

(b) If S is a nonempty subset of $\mathbb{R}^n$, then $S^\perp = \text{span}(S)^\perp$

(c) If W is a subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp = W$.

pf) (a) If $v \in W^\perp \cap W$, then $v \cdot v = 0$. Hence $v = \{0\}$.

(b) Since $S^\perp \supseteq \text{span}(S)^\perp$, it is enough to show $S^\perp \subseteq \text{span}(S)^\perp$.

Set $S = \{ w_1, \cdots, w_k \}$. If $v \in S^\perp$, i.e., $v \cdot w_i = 0$ for $1 \le i \le k$,

then $v \cdot (c_1 w_1 + \cdots + c_k w_k) = 0 \quad \forall c_1, \cdots, c_k \in \mathbb{R}$. Hence $v \in \text{span}(S)^\perp$.

(c) We will prove it later. ∎

We say that W and $W^\perp$ are orthogonal complements, since $(W^\perp)^\perp = W$.

Question For a nonempty subset S in $\mathbb{R}^n$, what is $(S^\perp)^\perp$?

$\left(\dfrac{\text{Thm 7.3.5}}{\text{Thm 7.3.6}}\right)$ If A is an $m \times n$ matrix, then

row(A) and null(A) are orthogonal complements, and

col(A) and null(A^T) " " " .

($\because$ (b) implies that $\text{row}(A)^{\perp} = \text{null}(A)$ and $\text{col}(A)^{\perp} = \text{row}(A^T)^{\perp} = \text{null}(A^T)$.)

Thm 7.3.7

(a) Elementary row operations do not change the row space of a matrix.

(b) Elementary row operations do not change the null space of a matrix.

(c) The nonzero row vectors in any row echelon form of a matrix form a basis for the row space of a matrix.

Proof Set $B = EA$, where E is an elementary row matrix.

Then $\mathbf{r}_i(B) = k\,\mathbf{r}_i(A)$, $\mathbf{r}_j(A)$, or $\mathbf{r}_i(A) + k\,\mathbf{r}_j(A)$.

Hence $\mathbf{v} \in \text{row}(B) \underset{(a)}{\iff} \mathbf{v} \in \text{row}(A)$, and

$$\text{null}(A) = \text{row}(A)^{\perp} = \text{row}(B)^{\perp} = \text{null}(B).$$

This proves (b).

Now let $R = E_k \cdots E_1 A$, where $E_1, \cdots, E_k$ are elementary row matrices, and assume R is in row echelon form.

Then the nonzero row vectors in R are linearly independent.

Since $\text{row}(R) = \text{row}(A)$ by (a), the nonzero row vectors of R form a basis for row(A). This proves (c). $\boxtimes$

THEREFORE, $\text{rank}(A) = \#$ nonzero rows of R

$= \#$ leading variables.

Note : $\underset{\text{nullity}(A)}{\overset{\|}{\#\text{ free variables}}} = n - \text{rank}(A)$, $n = \#$ columns of A.

$\uparrow$ from the dimension theorem for homogeneous linear systems.

$\therefore$ $\text{rank}(A) + \text{nullity}(A) = n$ ($\longleftarrow$ Theorem 7.4.1)

Thm 7.3.8

Assume that A and B have the same number of columns. TFAE.

(a) $\text{row}(A) = \text{row}(B)$

(b) $\text{null}(A) = \text{null}(B)$

(c) The row vectors of A are linear combinations of the row vectors of B, and conversely.

Problem A

Let $S = \{ \mathbb{w}_1, \cdots, \mathbb{w}_m \}$ be a set in $\mathbb{R}^n$ and let $W = \text{span}(S)$.

(a) Find a basis for W.

(b) Find a basis for $W^{\perp}$.

How to solve? Construct a matrix $A = \begin{bmatrix} - \mathbb{w}_1 - \\ - \mathbb{w}_2 - \\ \vdots \\ - \mathbb{w}_m - \end{bmatrix}$.

Find R, a row echelon form of A.

Then $\begin{cases} W = \text{row}(A) = \text{row}(R). \\ W^{\perp} = \text{null}(A) = \text{null}(R) \end{cases}$

Problem B

(a) Given a set of vectors $S = \{ \mathbb{w}_1, \cdots, \mathbb{w}_n \}$ in $\mathbb{R}^m$, find conditions on the numbers $b_1, \cdots, b_m$ under which $\mathbb{b} = (b_1, \cdots, b_m)$ will lie in $\text{span}(S)$.

$$A = \begin{bmatrix} | & | & & | \\ \mathbb{w}_1 & \mathbb{w}_2 & \cdots & \mathbb{w}_n \\ | & | & & | \end{bmatrix}_{m \times n} \qquad \underset{\underset{\mathbb{b}}{\cup}}{\text{span}(S)} = \text{col}(A)$$

(b) Given an $m \times n$ matrix A, find conditions on the numbers $b_1, \cdots, b_m$ under which $\mathbb{b} = (b_1, \cdots, b_m)$ will lie in $\text{col}(A)$.

(c) Given a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, find conditions on the numbers $b_1, \cdots, b_m$ under which $\mathbb{b} = (b_1, \cdots, b_m)$ will lie in $\text{ran}(T)$.

$$\text{ran}(T) = \text{col}([T])$$

All the problems are equivalent to determine whether $A\mathbb{x} = \mathbb{b}$ is consistent.

Q: Let R be a row echelon form of A. Is $\mathrm{col}(A) = \mathrm{col}(R)$?

A: No! $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\mathrm{col}(A) = \mathrm{span}\{e_2\}$ $\quad$ $\mathrm{col}(R) = \mathrm{span}\{e_1\}$.

§7.4. The dimension theorem & its implications

Recall that $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$ for an $m \times n$ matrix.

Thm 7.4.2 $\quad A : m \times n, \quad \mathrm{rank}(A) = k$.

(a) $\mathrm{nullity}(A) = n - k$

(b) Every row echelon form of A has k nonzero rows.

(c) $A\mathbf{x} = \mathbf{0}$ has k leading variables and $n-k$ free variables.

Thm 7.4.3 (Dimension theorem for a subspace)

If W is a subspace of $\mathbb{R}^n$, then $\dim(W) + \dim(W^\perp) = n$.

pf) If $W = \{0\}$, then $W^\perp = \mathbb{R}^n$ and hence $\dim(W) + \dim(W^\perp) = n$.

If $W \neq \{0\}$, then choose a basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ for W.

Then W is the row space of the matrix $A = \begin{bmatrix} - \mathbf{w}_1 - \\ \vdots \\ - \mathbf{w}_k - \end{bmatrix}$

and $W^\perp = \mathrm{null}(A)$. Hence $\dim(W) + \dim(W^\perp) = \mathrm{rank}(A) + \mathrm{nullity}(A) = n$.

$\blacksquare$

Thm 7.4.4 $\quad A : n \times n$

A: invertible $\iff \mathrm{rank}(A) = n \iff \mathrm{nullity}(A) = 0$.