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Review A: $m\times n$ matrix

Dim Thm : $\text{rank}(A) + \text{nullity}(A) = n$

Rank Thm : $\text{rank}(A) = \text{rank}(A^T)$.

Thm 7.5.7 Let A be an $m\times n$ matrix.

(a) $m > n \Rightarrow A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}\in\mathbb{R}^m$.

(b) $m < n \Rightarrow \forall \mathbf{b}\in\mathbb{R}^m$, $A\mathbf{x}=\mathbf{b}$ is either inconsistent or has infinitely many solutions.

pf) (a) $\begin{cases} \dim(\text{col}(A)) \le \#\text{ columns of } A = n < m \\ \text{col}(A) : \text{a subspace of } \mathbb{R}^m \end{cases} \Rightarrow \text{col}(A) \subsetneq \mathbb{R}^m$

(b) $m<n \Rightarrow \dim(\text{col}(A)) \le m < n$

$\Rightarrow$ column vectors of A must be lin. dependent. ∎

Thm 7.5.8+9 Let A be an $m\times n$ matrix.

(a)	$\text{null}(A) = \text{null}(A^TA)$	$\text{null}(A^T) = \text{null}(AA^T)$
(b)	$\text{row}(A) = \text{row}(A^TA)$	$\text{row}(A^T) = \text{row}(AA^T)$
(c)	$\text{col}(A^T) = \text{col}(A^TA)$	$\text{col}(A) = \text{col}(AA^T)$
(d)	$\text{rank}(A) = \text{rank}(A^TA)$	$\text{rank}(A^T) = \text{rank}(AA^T)$

$\therefore \text{rank}(AA^T) = \text{rank}(A^TA)$.

pf) (b) $\Rightarrow$ (d)

(a) $\text{null}(A) \subseteq \text{null}(A^TA)$: clear

$\text{null}(A^TA) \subseteq \text{null}(A)$: $\mathbf{x}_0 \in \text{null}(A^TA)$, i.e., $A^TA\mathbf{x}_0 = 0$

$\Rightarrow (A\mathbf{x}_0)\cdot(A\mathbf{x}_0) = (A\mathbf{x}_0)^TA\mathbf{x}_0 = \mathbf{x}_0^TA^TA\mathbf{x}_0 = 0$

$\Rightarrow \mathbf{x}_0 \in \text{null}(A)$

(b) It comes from (a).

$\text{row}(A) = \text{null}(A)^\perp = \text{null}(A^TA)^\perp = \text{row}(A^TA)$

(c) $\text{col}(A^T) = \text{row}(A) = \text{row}(A^TA) = \text{col}((A^TA)^T) = \text{col}(A^TA)$ ∎

Thm 7.5.10+11 A : $m \times n$ matrix

(a) A has full column rank $\iff$ A^TA is invertible.

(b) A has full row rank $\iff$ AA^T is invertible.

pf) (a) Note that A^TA is an $n \times n$ matrix.

$$\text{rank}(A^TA) = \text{rank}(A) = n$$

(b) Note that AA^T is an $m \times m$ matrix.

$$\text{rank}(AA^T) = \text{rank}(A) = m.$$

$\blacksquare$

Question

A : $n \times n$ matrix with $\text{rank}(A) = \text{rank}(A^2)$

Show ① $\text{null}(A) = \text{null}(A^2)$

② $\text{null}(A) \cap \text{col}(A) = \{\mathbf{0}\}$.

pf) ① clearly $\text{null}(A) \subseteq \text{null}(A^2)$

Since $\text{rank}(A) = \text{rank}(A^2)$, $\text{null}(A) = \text{null}(A^2)$.

② $\mathbf{y} \in \text{null}(A) \cap \text{col}(A) \Rightarrow A\mathbf{y} = \mathbf{0} \Rightarrow \mathbf{x} \in \text{null}(A^2) = \text{null}(A)$

$$A\mathbf{y} = A^2\mathbf{x} \Rightarrow \mathbf{y} = A\mathbf{x} = \mathbf{0}.$$

$\blacksquare$

§7.6. The Pivot theorem.

Problem A: Let $S = \{\mathbf{w}_1, \cdots, \mathbf{w}_s\}$ and $W = \text{span}(S)$. Find a basis $\mathcal{B}$ for W consisting of vectors from S.

i.e., Find a maximal linearly independent subset of S.

Note When U is in a row echelon form of A,

$\text{row}(U) = \text{row}(A)$ but $\text{col}(U) \neq \text{col}(A)$ in general.

Observation

Now assume that A & B are row equivalent matrices:

$A = [\mathbf{c}_1 \cdots \mathbf{c}_n]$ & $B = [\mathbf{c}_1' \cdots \mathbf{c}_n']$. Then $A\mathbf{x} = \mathbf{0}$ & $B\mathbf{x} = \mathbf{0}$

have the same solution space. For $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$x_1 \mathbf{c}_1 + \cdots + x_n \mathbf{c}_n = \mathbf{0} \iff x_1 \mathbf{c}_1' + \cdots + x_n \mathbf{c}_n' = \mathbf{0}.$$

i.e., 1) $\{\mathbf{c}_1, \cdots, \mathbf{c}_n\}$: linearly indep. $\iff$ $\{\mathbf{c}_1', \cdots, \mathbf{c}_n'\}$: linearly indep.

2) $\{\mathbf{c}_1, \cdots, \mathbf{c}_n\}$ & $\{\mathbf{c}_1', \cdots, \mathbf{c}_n'\}$: linearly dependent.

$\Rightarrow$ They have the same dependency relationship.

Thm 7.6.1

Let A & B be row equivalent matrices.

(a) If some subset of column vectors of A is linearly independent then the corresponding column vectors of B are linearly indep., and conversely.

(b) If some subset of column vectors of A is linearly dependent then the corr. column vectors from B are linearly dependent., and conversely. Moreover, the column vectors in the two matrices have the same dependency relationship.

Ex Let $\mathbf{w}_1 = (1, 1, 3, 0)$, $\mathbf{w}_2 = (-2, 1, -9, 6)$, $\mathbf{w}_3 = (1, -4, 8, -10)$,
$\mathbf{w}_4$ $(3, -2, 14, -10)$,

Find a subset of these vectors that forms a basis for $W = \text{span}\{\mathbf{w}_1, \cdots, \mathbf{w}_4\}$.

Sol.) Let $A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 1 & 1 & -4 & -2 \\ 3 & -9 & 8 & 14 \\ 0 & 6 & -10 & -10 \end{bmatrix}$.

Do the Gauss-Jordan elimination.

$$\rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 3 & -5 & -5 \\ 0 & -3 & 5 & 5 \\ 0 & 6 & -10 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -5/3 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{7}{3} & -\frac{1}{3} \\ 0 & 1 & -5/3 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\{\mathbf{w}_1, \mathbf{w}_2\}$: a basis for W.

$$\mathbf{w}_3 = -\frac{7}{3}\mathbf{w}_1 - \frac{5}{3}\mathbf{w}_2$$

$$\mathbf{w}_4 = -\frac{1}{3}\mathbf{w}_1 - \frac{5}{3}\mathbf{w}_2$$

THEREFORE, for a given matrix A, we can find bases for row(A), null(A), and col(A) via the Gauss-Jordan elimination.

Def. The column vectors of a matrix A that lie in the column positions where the leading 1's occur in the row echelon forms of A are called the <u>pivot columns</u> of A.

Thm 7.6.3 (The Pivot theorem)

The pivot columns of a nonzero matrix A form a basis for col(A).

Algorithm 1 on p. 372 explains a way to find a desired basis in Problem A via the Gauss-Jordan elimination.

Now we are going to study how to find a basis for $\mathrm{null}(A^T)$.

Let A be an $m \times n$ matrix.

i) $\mathrm{rank}(A) = m \Rightarrow \mathrm{nullity}(A^T) = m - \mathrm{rank}(A^T) = 0$

$\therefore \mathrm{null}(A^T) = \{0\}$ & $\nexists$ basis for $\mathrm{null}(A^T)$.

ii) $\mathrm{rank}(A) = k < m \Rightarrow$ We obtain the following procedure.

Algorithm 2

I. Adjoin I_m to the right side of A to create $[A \mid I_m]$.

II. Apply the Gaussian elimination to $[A \mid I_m]$:

$$E[A \mid I_m] = [U \mid E]$$

III. Partition $[U \mid E]$ by adding a horizontal rule to split off the zero rows of U. This yields a matrix of the form

$$[U \mid E] = \begin{array}{c} \underbrace{\left[\begin{array}{c|c} V & E_1 \\ \hline 0 & E_2 \end{array}\right]}_{n \quad\quad m} \end{array} \begin{array}{l} \}\, k \\ \}\, m-k \end{array}$$

IV. The row vectors of E_2 form a basis for $\mathrm{null}(A^T)$.

$\because \begin{bmatrix} V \\ \hline 0 \end{bmatrix} = U = EA = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} A = \begin{bmatrix} E_1 A \\ E_2 A \end{bmatrix}$

$\therefore E_2 A = 0 \Rightarrow \mathrm{row}(E_2) \subseteq \mathrm{col}(A)^\perp = \mathrm{null}(A^T)$.

Since E is invertible, the rows of E are linearly independent.

$\therefore E_2$ has full row rank.

$\mathrm{rank}(E_2) = m-k = \mathrm{rank}(\mathrm{null}(A^T))$

$\therefore \mathrm{row}(E_2) = \mathrm{null}(A^T)$.

$$A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 1 & 1 & -4 & -2 \\ 3 & -9 & 8 & 14 \\ 0 & 6 & -10 & -10 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 3 & -5 & -5 \\ 0 & -3 & 5 & 5 \\ 0 & 6 & -10 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & -5/3 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{7}{3} & -\frac{1}{3} \\ 0 & 1 & -5/3 & -5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -4 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ -4 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix}$$

$$[A \mid I_4] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -7/3 & -1/3 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 1 & -5/3 & -5/3 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 1 \end{array}\right]$$

The bases for the fundamental spaces of A are given as follows:

row(A) : $\{(1, -2, 1, 3),\ (0, 1, -5/3, -5/3)\}$

col(A) : $\{(1, 1, 3, 0),\ (-2, 1, -9, 6)\}$

null(A) : $\left\{\left(\frac{7}{3}, \frac{5}{3}, 1, 0\right),\ \left(\frac{1}{3}, \frac{5}{3}, 0, 1\right)\right\}$

null(A^T) : $\{(-4, 1, 1, 0),\ (2, -2, 0, 1)\}$

Thm 7.6.4 (Colum-Row factorization)

A: a nonzero $m \times n$ matrix of rank A.

$\Rightarrow$ $A = CR$, C: $m \times k$ matrix whose columns are the pivot columns of A

R: $k \times n$ matrix whose rows are the nonzero rows in the reduced row echelon form of A.

Ex.

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 1 & 1 & -4 & -2 \\ 3 & -9 & 8 & 14 \\ 0 & 6 & -10 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 3 & -9 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/3 & -1/3 \\ 0 & 1 & -5/3 & -5/3 \end{bmatrix}$$