1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let A be an $m \times n$ matrix. Then, A and AA^T have the same null space.

(b) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then,

$$\operatorname{rank}(AB) = \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

(c) Let n and k be positive integers. Let A be an $(n-k) \times n$ matrix. Let $\{v_1, \ldots, v_{n-k}\}$ be the set of row vectors of A. If A has full row rank, then there exist k vectors $\{w_1, \ldots, w_k\}$ such that $Aw_i = 0$ and $\{v_1, \ldots, v_{n-k}, w_1, \ldots, w_k\}$ is linearly independent set.

Solution.

(a) False. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Let $v = (1, -1)$. Then, $v \in \operatorname{null}(A)$ and $v \notin \operatorname{null}(AA^T)$. Thus, the null space of A is not equal to the null space of AA^T.

(b) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then, rank of A and B are 1. However, $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, thus the rank of AB is 0.

(c) True. Since A has full row rank, the set of row vectors of A is linearly independent set. By the dimension theorem, the rank of null space of A is k, thus we can choose k linearly independent vectors $\{w_1, \ldots, w_k\}$. Now, we assume that $\sum_{i=1}^{n-k} c_i v_i + \sum_{j=1}^{k} d_j w_j = 0$. By assumption, $\sum_{i=1}^{n-k} c_i v_i = -\sum_{j=1}^{k} d_j w_j$ and then $\sum_{i=1}^{n-k} c_i v_i \in \operatorname{row}(A)$, $-\sum_{j=1}^{k} d_j w_j \in \operatorname{null}(A)$. By Theorem 7.3.4, $\operatorname{row}(A) \cap \operatorname{null}(A) = \operatorname{row}(A) \cap \operatorname{row}(A)^T = \{0\}$, thus

$$\sum_{i=1}^{n-k} c_i v_i = 0 = \sum_{j=1}^{k} d_j w_j.$$

Since $\{v_1, \ldots, v_{n-k}\}$ and $\{w_1, \ldots, w_k\}$ are linearly independent sets, $c_1 = \cdots = c_{n-k} = 0 = d_1 = \cdots = d_k$.

2 Let A be a 3×3 matrix. Assume that $\operatorname{rank}(A^3) = 2$. Prove that

10 points

$$\operatorname{null}(A) \cap \operatorname{col}(A) = \{0\}.$$

Solution.

For a matrix A and B,

$$\operatorname{null}(B) \subset \operatorname{null}(AB)$$

since for $v \in \operatorname{null}(B)$, $ABv = A(Bv) = 0$. Thus,

$$\operatorname{null}(A) \subset \operatorname{null}(A^2) \subset \operatorname{null}(A^3). \tag{1}$$

If $\operatorname{rank}(A) = 3$, then A is an invertible matrix. It implies that A^3 is an invertible matrix. It contradicts to $\operatorname{rank}(A^3) = 2$. Thus, the rank of A is 2. By the dimension theorem and (1), we get

$$2 = \operatorname{rank}(A) \geq \operatorname{rank}(A^2) \geq \operatorname{rank}(A^3) = 2.$$

Thus, $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ and by (1), $\operatorname{null}(A) = \operatorname{null}(A^2)$.

Let $v \in \operatorname{null}(A) \cap \operatorname{col}(A)$. Then, $Av = 0$ and there exists v_1 such that $v = Av_1$. It implies that $A(Av_1) = A^2 v_1 = 0$. Thus, $v_1 \in \operatorname{null}(A^2) = \operatorname{null}(A)$. Therefore, $v = Av_1 = 0$.