1
10 points

Indicate whether the following statements are true(**T**) or false(**F**). You do not need to justify your answer.
Correct answer **2** points, wrong answer **-1** points, no answer **0** point.

(a) There exists a 9×9 symmetric matrix A such that $A \neq 0$ and $A^2 = 0$.

(b) Let λ_1, λ_2 be two distinct eigenvalues of a matrix B and E_{λ_1},E_{λ_2} be the eigenspaces corresponding to λ_1,λ_2 respectively. If $\mathbf{v_1} \in E_{\lambda_1}$ and $\mathbf{v_2} \in E_{\lambda_2}$, then $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent.

(c) There exists a one-to-one linear operator T on $\mathbb{R}^5$ which is not onto.

(d) $\{(x, y, z)|2x + y + z = 1\}$ is a subspace of $\mathbb{R}^3$.

(e) There exists an invertible skew symmetric 5×5 matrix C.

Solution 1. (a) (**F**) Let A be a 9×9 symmetric matrix such that $A^2 = 0$. Let $a_{ij} = [A]_{ij}$. Since A is a symmetric matrix,

$$A^2 = AA^T.$$

By calculation, the trace of AA^T can be expressed as

$$\mathrm{Tr}(AA^T) = \sum_{i=1}^{9} \sum_{j=1}^{9} a_{ij}^2.$$

The trace of zero matrix is zero, thus

$$0 = \sum_{i=1}^{9} \sum_{j=1}^{9} a_{ij}^2.$$

Therefore, $A = 0$.

(b) (**F**) Let $v_1 = v_2 = 0$. Then $\{v_1, v_2\}$ is linearly dependent.

(c) (**F**). By Theorem 6.3.14, if linear operator T is one-to-one then T is onto.

(d) (**F**) Let $S := \{(x, y, z)|2x + y + z = 1\}$. A point $(0, 0, 1) \in S$. However, $(0, 0, 1) + (0, 0, 1) \notin S$. Therefore S is not a subspace.

(e) (**F**) Let C be a skew symmetric 5×5 matrix. By Theorem 4.2.1,

$$\det(C) = \det(C^T).$$

Since C is a skew symmetric, $C = -C^T$. By Theorem 4.2.3,

$$\det(C) = \det(-C^T) = (-1)^5 \det(C^T).$$

Therefore, $\det(C) = 0$. By Theorem 4.2.4, the matrix C is not invertible.

□

2 Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a & 1 & -1 & 0 \\ -1 & 2 & b & -1 \\ 0 & 2 & -1 & a \end{bmatrix}$ with $a, b \in \mathbb{R}$.

20 points

(a) (10 points) Find all pairs (a, b) such that A^T is not invertible.

Solution. A^T is not invertible if and only if A is not invertible by Theorem 3.2.11. Hence, it suffices to check that $\det A = 0$. **(+4 points)**

$$\det A = \begin{vmatrix} 0 & 0 & 1 & 0 \\ a & 1 & -1 & 0 \\ -1 & 2 & b & -1 \\ 0 & 2 & -1 & a \end{vmatrix} = \begin{vmatrix} a & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & a \end{vmatrix} = 2a^2 + 3a = 0. \text{ } \textbf{(+3 points)}$$

Consequently, all pairs (a, b) such that A^T is not invertible are

$$a = 0, \ -\frac{3}{2} \text{ , and } -\infty < b < \infty. \text{ } \textbf{(+3 points)}$$

$\square$

- For each miscalculation, you will get **(−2 points)**.
- You will get full credit if you calculate $\det A^T$ instead of $\det A$.

2 (b) (10 points) Assume that A is invertible and $a = b$. Find all values of a such that the third column vector and the fourth column vector of $(A^T)^{-1}$ are orthogonal.

20 points

Solution 1. Since $(A^T)^{-1} = (A^{-1})^T$ by Theorem 3.2.11, we find the all values of a such that the third row vector and the fourth row vector of A^{-1} are orthogonal.

By Theorem 4.3.3, $A^{-1} = \frac{1}{\det A}\mathrm{adj}(A)$, so we have

$$\mathbf{r}_3(A^{-1}) = \frac{1}{\det A}\begin{bmatrix} C_{13} & C_{23} & C_{33} & C_{43} \end{bmatrix} = \frac{1}{2a^2+3a}\begin{bmatrix} 2a^2+3a & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

, and

$$\mathbf{r}_4(A^{-1}) = \frac{1}{\det A}\begin{bmatrix} C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} = \frac{1}{2a^2+3a}\begin{bmatrix} 2a^2+2a-1 & -2 & -2a & -2a-1 \end{bmatrix}. \ \textbf{(+6 points)}$$

Consequently, the third column vector and the fourth column vector of $(A^T)^{-1}$ are orthogonal, when

$$\mathbf{r}_3(A^{-1}) \cdot \mathbf{r}_4(A^{-1}) = \frac{2a^2+2a-1}{2a^2+3a} = 0 \ \textbf{(+2 points)}$$

$$\therefore a = \frac{-1 \pm \sqrt{3}}{2} \ \textbf{(+2 points)}$$

□

Solution 2. Since $(A^T)^{-1} = (A^{-1})^T$ by Theorem 3.2.11, we find the all values of a such that the third row vector and the fourth row vector of A^{-1} are orthogonal.

Now, we find the inverse of A.

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ a & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & a & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & a & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} -1 & 2 & a & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ a & 1 & -1 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & -a & 1 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2a+1 & a^2-1 & -a & 0 & 1 & a & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & -a & 1 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2a^2+2a-1 & -2a^2-3a & 0 & 2 & 2a & -2a-1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & -a & 1 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & a & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2a^2+2a-1}{2a^2+3a} & -\frac{2}{2a^2+3a} & -\frac{2a}{2a^2+3a} & \frac{2a+1}{2a^2+3a} \end{array}\right]$$

, since $a \neq 0, -\frac{1}{2}, -\frac{3}{2}$.

Hence, we get

$$\mathbf{r}_3(A^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \mathbf{r}_4(A^{-1}) = \begin{bmatrix} \frac{2a^2+2a-1}{2a^2+3a} & -\frac{2}{2a^2+3a} & -\frac{2a}{2a^2+3a} & \frac{2a+1}{2a^2+3a} \end{bmatrix}. \ \textbf{(+6 points)}$$

Consequently, the third column vector and the fourth column vector of $(A^T)^{-1}$ are orthogonal, when

$$\mathbf{r}_3(A^{-1}) \cdot \mathbf{r}_4(A^{-1}) = \frac{2a^2 + 2a - 1}{2a^2 + 3a} = 0 \ \textbf{(+2 points)}$$

$$\therefore a = \frac{-1 \pm \sqrt{3}}{2} \ \textbf{(+2 points)}$$

$\square$

- For each miscalculation, you will get **($-$2 points)**.

3 Find an LU-decomposition of $B = \begin{bmatrix} 2 & 2 & -4 & 2 \\ -1 & -1 & 5 & -4 \\ 1 & 1 & 7 & -2 \\ 0 & 0 & -6 & 6 \end{bmatrix}$.

10 points

There are several kinds of solutions.

1. $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & 9 & 6 & 0 \\ 0 & -6 & 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for some $t \in \mathbb{R}$.

2. $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & 0 & 9 & 0 \\ 0 & 0 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

3. $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 9 & 0 \\ 0 & -2 & 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for some $t \in \mathbb{R}$.

If you make a minor mistake, you get just **(+5 points)**.

4 Determine whether the vectors

10 points

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

are linearly independent or not.

Solution 1. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and suppose $a, b, c, d \in \mathbb{R}$ satisfy

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{v}_4 = A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.\textbf{(+3 points)}$$

The reduced form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \textbf{(+2 points)}$$

so we get $b = c = d, a = -d$. Take any $d \neq 0$, for example $d = 1$ **(+3 points)**, then we get a nontrivial solution a, b, c, d which satisfies $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{v}_4 = 0$. Hence they are not linearly independent. **(+2 points)** □

5 Let $C = \begin{bmatrix} 1 & 1 & r \\ 1 & r & 1 \\ r & 1 & 1 \end{bmatrix}$ be a real 3×3 matrix.

30 points

(a) (10 points) Find all values of r for which C is invertible.

(b) (10 points) Compute $adj(C)$.

(c) (10 points) For those values of r in (a), find the inverse matrix C^{-1} of C.

Solution.

(a)

$$\begin{aligned} det(C) &= 1\begin{vmatrix} r & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ r & 1 \end{vmatrix} + r\begin{vmatrix} 1 & r \\ r & 1 \end{vmatrix} \textbf{ (+3 points)} \\ &= (r-1) - (1-r) + r(1-r^2) \\ &= -r^3 + 3r - 2 \textbf{(+3 points)} \end{aligned}$$

C is invertible, if and only if $det(C) \neq 0$. **(+2 points)**
$0 = -r^3 + 3r - 2 = -(r-1)^2(r+2) \iff r = 1$ or $r = -2$
Therefore, C is invertible when $r \neq 1, 2$. **(+2 points)** □

(b) Compute the minor of each entry of C.

$$\begin{aligned} &M_{11} = r-1, \quad M_{12} = 1-r, \quad M_{13} = 1-r^2, \\ &M_{21} = 1-r, \quad M_{22} = 1-r^2, \quad M_{23} = 1-r, \quad \textbf{(+4 points)} \\ &M_{31} = 1-r^2, \quad M_{32} = 1-r, \quad M_{33} = 1-r. \end{aligned}$$

Then, $C_{ij} = (-1)^{i+j} M_{ij}$. **(+3 points)**

The transpose of this cofactor matrix is $adj(C)$. **(+3 points)**

$$adj(C) = \begin{bmatrix} r-1 & r-1 & 1-r^2 \\ r-1 & 1-r^2 & r-1 \\ 1-r^2 & r-1 & r-1 \end{bmatrix}.$$

□

- If you didn't mention the transpose, **(−3 points)**

(c)

$$C^{-1} = \frac{1}{det(C)} adj(C). \textbf{ (+5 points)}$$

$$\begin{aligned}\therefore C^{-1} &= \frac{1}{-(r-1)^2(r+2)}\begin{bmatrix} r-1 & r-1 & 1-r^2 \\ r-1 & 1-r^2 & r-1 \\ 1-r^2 & r-1 & r-1 \end{bmatrix} \\ &= \frac{1}{(r-1)(r+2)}\begin{bmatrix} -1 & -1 & r+1 \\ -1 & r+1 & -1 \\ r+1 & -1 & -1 \end{bmatrix} \quad \textbf{(+5 points)}\end{aligned}$$

$\square$

6 For given real numbers a, b, c, d, e, f, let $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

20 points

(a) (10 points) Show that A is nilpotent.

(b) (10 points) Show that $I + A^2$ is invertible and find its inverse.

Solution. (a) Since A is an upper triangular matrix, eigenvalues of A are the diagonal entries. Thus $\lambda_i = 0$ for $i = 1, 2, 3, 4$. Thus the characteristic polynomial $p(x)$ of A is $p(x) = (x-0)^4 = x^4$ **(+6 points)**. Thus $p(A) = A^4 = O$**(+4 points)**.

Alternative solution.

A is nilpotent if $A^k = O$ for some positive integer k.**(+4 points)**

$A^2 = \begin{bmatrix} 0 & 0 & ad & ae+bf \\ 0 & 0 & 0 & df \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(+2 points)** $A^3 = \begin{bmatrix} 0 & 0 & 0 & adf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(+2 points)**

$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(+2 points)**

Thus A is nilpotent.

(b) Factorize $A^4 - I = A^4 - I^4 = (A^2+I)(A^2-I)$**(+6 points)**. Then we get $-I = (A^2+I)(A^2-I)$ **(+2 points)**. Hence $I + A^2$ is invertible,**(+1 points)** and $I - A^2$ is the inverse.**(+1 points)**

Alternative solution.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & ad & ae+bf & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & df & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \to r_1 - ad r_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & ae+bf & 1 & 0 & -ad & 0 \\ 0 & 1 & 0 & df & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_1 \to r_1 - (ae+bf) r_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -ad & -ae-bf \\ 0 & 1 & 0 & df & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \to r_2 - df r_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -ad & -ae-bf \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -df \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

(+8 points)

Hence $A+2I$ is invertible **(+1 points)** and $(A+2I)^{-1} = \begin{bmatrix} 1 & 0 & -ad & -ae-bf \\ 0 & 1 & 0 & -df \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(+1 points)**

- There are 2 points for each matrices in Gauss Jordan elimination .
- If you directly calculate the inverse you will get 9 points for calculating the inverse matrix, and 1 point for showing invertible. In this case there is no partial point.

7 Let B be a 3×3 matrix whose characteristic polynomial is $p(\lambda) = \lambda^3 - 5\lambda^2 - 11\lambda + 15$

20 points

(a) Compute $\det(B)^3$.

(b) Compute $\mathrm{tr}(B^3)$.

Solution. (a) Recall that the constant term of the characteristic polynomial of a matrix B is $(-1)^n \det(B)$. Therefore, the determinant of B is -15 **(+5 points)**. The determinant of B can also be computed by finding the eigenvalues of B and multiplying them. Since the determinant is multiplicative, then we have

$$\det(B^3) = \det(B)^3 = (-15)^3 = -3375 \quad \textbf{(+5 points)}.$$

Alternatively, by solving the equation $p(\lambda) = 0$, one can find that the eigenvalues of B are 1, $2 \pm \sqrt{19}$ **(+4 points)**. Also, if λ is an eigenvalue of B with eigenvector $\mathbf{x}$, then $B^3\mathbf{x} = \lambda^3\mathbf{x}$, so λ^3 is an eigenvalue of B, and this means that the eigenvalues of B^3 are

$$1^3 = 1, \quad (2+\sqrt{19})^3 = 122 + 31\sqrt{19}, \quad (2-\sqrt{19})^3 = 122 - 31\sqrt{19} \quad \textbf{(+3 points)}.$$

Multiplying these eigenvalues together then yields the determinant of B, which is -3375 **(+3 points)**.

(b) The trace is equal to the sum of its eigenvalues. As observed in the previous solution, the eigenvalues of B^3 are the cubes of the eigenvalues of B **(+3 points)**. The eigenvalues can then be computed as above by solving $p(\lambda) = 0$ **(+4 points)**, and lastly adding their cubes yields

$$\mathrm{tr}(B^3) = 1^3 + (2+\sqrt{19})^3 + (2-\sqrt{19})^3 = 245 \quad \textbf{(+3 points)}.$$

Alternatively, from the given characteristic polynomial, the coefficient of the term with the second highest term is $-\mathrm{tr}(B)$ **(+2 points)**. By partially solving $p(\lambda) = 0$ (or by guess and check), we can see that $\lambda_1 = 1$ is an eigenvalue **(+2 points)**. If λ_2, λ_3 represent the other two eigenvalues, then we have that

$$\mathrm{tr}(B) = \lambda_1 + \lambda_2 + \lambda_3 \implies \lambda_2 + \lambda_3 = \mathrm{tr}(B) - \lambda_1 = 4. \quad \textbf{(+2 points)}$$

Lastly, the following computation determines $\mathrm{tr}(B^3)$:

$$\begin{aligned}
\mathrm{tr}(B) &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \\
&= \lambda_1^3 + (\lambda_2 + \lambda_3)^3 - 3\lambda_2\lambda_3(\lambda_2 + \lambda_3) \\
&= 1 + (\lambda_2 + \lambda_3) + 3\lambda_1\lambda_2\lambda_3(\lambda_2 + \lambda_3) \\
&= 1 + (\lambda_2 + \lambda_3) + 3\det(B)(\lambda_2 + \lambda_3) \\
&= 1 + (4)^3 + 3 \cdot (-15) \cdot 4 \\
&= 245 \quad \textbf{(+4 points)}
\end{aligned}$$

□

- Any answer without justification gets no points. **(−2 points)**
- Any calculation error loses points **(−2 points)**

8 Find the solution set of the following linear system using *Cramer's rule.*

10 points

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -3 \end{bmatrix}$$

Solution. By Cramer's rule,

$$x = \frac{\det A_1}{\det A}, y = \frac{\det A_2}{\det A}, z = \frac{\det A_3}{\det A}$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 2 & 1 \\ -3 & 3 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 1 \\ -1 & -3 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & 3 & -3 \end{bmatrix}.$$

(+5 points)

$$\det A = 1 \times (2 \times 1 - 3 \times 1) + 1 \times (0 \times 3 - 2 \times (-1)) = 1 \textbf{(+1 point)}$$

$$\det A_1 = -1 \times (2 \times 1 - 3 \times 1) + 1 \times ((-3) \times 3 - (-3) \times 2) = -2 \textbf{(+1 point)}$$

$$\det A_2 = 1 \times ((-3) \times 1 - (-3) \times 1) - 1 \times ((-1) \times 1 - (-3) \times 1) = -2 \textbf{(+1 point)}$$

$$\det A_3 = 1 \times (2 \times (-3) - 3 \times (-3)) - 1 \times ((-3) \times 0 - (-1) \times 2) = 1 \textbf{(+1 point)}$$

Therefore, $x = -2, y = -2, z = 1$. **(+1 point)**

□

- If your idea is correct, you get 5 points.
- When you mistake some calculation, to get the score for idea : It is ok that you don't describe Cramer's rule explicitly. But your process should be written clearly.
- If you do not use Cramer's rule, you cannot get more than 2 points for total score.
- Each correct answer for $\det A, \det A_i$ and (x, y, z) give you 1 point. (No partial score for each)
- If you claim $x = \det A_1, y = \det A_2, z = \det A_3$ (not divided by $\det A$) without explaining $\det A = 1$, you get only 5 points for total score. (idea : 1 point, calculation : 4 points)
- If you calculate x, y, z correctly using Cramer's rule but you explain something more which is not necessary, for example your answer is $\{t(-2, -2, 1) | t \in \mathbb{R}\}$ or you claim $\{(-2, -2, 1) + t\mathbf{v} | t \in \mathbb{R}\}$ for some vector $\mathbf{v}$, you lose 2 points for idea because it means you do not understand Cramer's rule exactly. You can lose more point for calculation part.

9 Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation whose standard matrix is

20 points

$$C = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{bmatrix}$$

.

(a) (**10 points**) Is $\mathbf{b} = \begin{bmatrix} 45 \\ -36 \\ 72 \end{bmatrix}$ in the range of T?

(b) (**10 points**) Find the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^5$ such that $\ker(T) = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

Solution. By theorem 6.3.8 (p.299), the range of T is the column space of C. So, by theorem 3.5.3, it is enough to show that the linear system $C\mathbf{x} = \mathbf{b}$ is consistent.

Now, consider the augmented matrix of $C\mathbf{x} = \mathbf{b}$ and its reduced form. Note that the reduced form is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 5 & 45 \\ -2 & -4 & 0 & 4 & -2 & -36 \\ 1 & 2 & 2 & 4 & 9 & 72 \end{array}\right] \Longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & 1 & 18 \\ 0 & 0 & 1 & 3 & 4 & 27 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \textbf{(+6 points)}$$

by the Gauss-Jordan elimination. By the last row, we know that the linear system $C\mathbf{x} = \mathbf{b}$ is consistent **(+4 points)**. Thus, $\mathbf{b}$ is in the range of T.

□

Solution. Note that the RREF of C is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \textbf{(+2 points)}.$$

Thus,

$$C\mathbf{x} = 0 \Leftrightarrow \begin{cases} x_1 + 2x_2 - 2x_4 + x_5 = 0 \\ x_3 + 3x_4 + 4x_5 = 0 \end{cases} \quad \textbf{(+2 points)}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$. Let $x_2 = r, x_4 = s, x_5 = t$ be free variables for $r, s, t \in \mathbb{R}$. Then, $x_1 = -2r + 2s - t$ and $x_3 = -3s - 4t$.

Hence, we can write that

$$\mathbf{x} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \quad \textbf{(+6 points)}.$$

Therefore, $\ker(T) = \text{null}(C) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$\square$

- The answer without justifications gets no points.
- Computational mistakes in solving the RREF. **(−4 points)** in (a), and **(−3 points)** in (b).
- Other minor mistakes. **(−2 points)**

10

10 points

Let $u = (2, 6, -2)$, $v = (1, 2, 2)$ and $w = (1, 1, 1)$ be vectors in $\mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that $T(u) = (2, 4, 0, -2)$, $T(v) = (-3, 2, 1, 2)$ and $T(w) = (1, -2, 2, 1)$. Find the standard matrix for T.

Solution.

Let standard matrix for T be A. Then $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$. **(+5 points)**

$e_1 = (1, 0, 0) = 2w - v$. Since T is a linear transformation, $T(e_1) = 2T(w) - T(v)$. **(+1 point)** So we get $T(e_1) = (5, -6, 3, 4)$.

$e_2 = \frac{1}{8}u + \frac{1}{2}v - \frac{3}{4}w$, so $T(e_2) = \frac{1}{8}T(u) + \frac{1}{2}T(v) - \frac{3}{4}T(w)$ **(+1 point)**.

$e_3 = -\frac{1}{8}u + \frac{1}{2}v - \frac{1}{4}w$, thus $T(e_3) = -\frac{1}{8}T(u) + \frac{1}{2}T(v) - \frac{1}{4}T(w)$ **(+1 point)**.

Therefore, $T(e_2) = (-2, 3, -1, -2)$, $T(e_3) = (-2, 1, 0, 1)$.

Standard matrix for T is $A = \begin{bmatrix} 5 & -2 & -2 \\ -6 & 3 & 1 \\ 3 & -1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$ **(+2 points)**

□

- The standard matrix is $4\times$ 3 matrix. If your answer is $3\times$ 4 transposed matrix, you get no points. **(−2 points)**
- If your answer's few entry is wrong, you get 1 point deducted **(−1 point)**. If your answer has more than five wrong entries, you get no point. **(−2 points)**

11 20 points

Let $A = \begin{bmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{bmatrix}$

(a) (10 points) Find all eigenvalues λ of A.

(b) (10 points) For each of all eigenvalues λ of A in (a), find the eigenspace E_λ corresponding to λ

Solution (a). Any eigenvalue $\lambda \in \mathbb{R}$ of A has its corresponding nonzero eigenvectors $\mathbf{x} \in \mathbb{R}^3$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$. Equivalently, its characteristic polynomial $\det(x \cdot I_3 - A)$ in indeterminant x has a root $x = \lambda$ where I_3 is the 3×3 identity matrix **(+4 points)**. By the definition of the determinant function,

$$\begin{aligned}
& \det(x \cdot I_3 - A) \\
= & \begin{vmatrix} x-1 & 4 & 4 \\ -8 & x+11 & 8 \\ 8 & -8 & x-5 \end{vmatrix} \\
= & (x-1) \cdot \begin{vmatrix} x+11 & 8 \\ -8 & x-5 \end{vmatrix} - 4 \cdot \begin{vmatrix} -8 & 8 \\ 8 & x-5 \end{vmatrix} + 4 \cdot \begin{vmatrix} -8 & x+11 \\ 8 & -8 \end{vmatrix} \quad \textbf{(+4 points)} \\
= & (x-1)(x^2+6x+9) - 4(-8x-24) + 4(-8x-24) \\
= & (x-1)(x+3)^2.
\end{aligned}$$

Therefore all eigenvalues of A are 1 **(+1 points)** and -3 **(+1 points)**.

□

- The answer without justifications gets no points.
- If you changed A in a way that does not preserve eigenvalues, then you lose 5 points. For example, taking some conjugate of A is possible, but applying row/column operations is not allowed. **(−5 points)**
- If you wrote one of the definition of eigenvalues, the characteristic polynomial of A, or the determinant of $A - \lambda \cdot I_3$, you will get 3 points. **(+4 points)**
- If you wrote the inductive definition of the determinant function or the formula for the determinant of 3×3 matrix correctly, then you can get 3 points. **(+4 points)**
- You can get scores if you computed eigenvalues logically even without using the determinant.
- If you made some mistakes in computation, you will lose 1 point. **(−1 points)**

Solution (b). Recall that the eigenvalues of A are 1 and -3. Also remind that E_λ consists of all vectors $\mathbf{x} = [x\ y\ z]^t \in \mathbb{R}^3$ such that $A\mathbf{x} = \lambda \cdot \mathbf{x}$, i.e. $(A - \lambda \cdot I_3)\mathbf{x} = [0\ 0\ 0]^t$ **(+2 points)**.

1. Consider $\lambda = 1$ first. Then above equation becomes

$$\begin{bmatrix} 0 & -4 & -4 \\ 8 & -12 & -8 \\ -8 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After applying some row operations, one can see that it is equivalent to solve the equation

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $z = -y$ and $x = \frac{1}{2}y$. Therefore $E_1 = \{y \cdot (\frac{1}{2}, 1, -1) \in \mathbb{R}^3 : y \in \mathbb{R}\}$ **(+4 points)**.

2. Next consider $\lambda = -3$. Then we have

$$\begin{bmatrix} 4 & -4 & -4 \\ 8 & -8 & -8 \\ -8 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By applying suitable row operations, it is equivalent to say that

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $x = y + z$, and $E_{-3} = \{y \cdot (1, 1, 0) + z \cdot (1, 0, 1) \in \mathbb{R}^3 : y, z \in \mathbb{R}\}$ **(+4 points)**.

$\square$

- The answer without justifications gets no points.
- If you write the definition or concept of eigenspaces correctly, then you can get 2 points. **(+2 points)**
- No point will be given to the eigenspace corresponding to a wrong eigenvalue.
- If you tried to compute the eigenspace for a correct eigenvalue but failed because of minor mistakes in computation, you will lose 1 point (for each eigenvalue). **(−1 points)**

12 (MATLAB programming problem)

2 point for each blank except (a)-4. 4 points for (a)-4.

(a) Fill in the blanks (1) - (4).

(b) Fill in the blanks (1) - (5).

Solution.

(a) (1) `size(A)`

(2) Any list whose length is equal to $m - 2$.

(3) Any list whose length is equal to $n - 2$.

(4) One of the followings:

```
- A(i-1:i+1, j-1:j+1) .* W
- diag(A(i-1:i+1, j-1:j+1) * W')
- diag(W * A(i-1:i+1, j-1:j+1)')
```

All variants of the indices according to blanks 2 and 3 are allowed.

(b) (1) One of the followings:

```
- syms x y z
- syms x
  syms y
  syms z
```

(2) One of the followings:

```
- diff(f, x, x)
- diff(f, 'x', 'x')
- diff(diff(f, x), x)
- diff(diff(f, 'x'), 'x')
- diff(f, x, 2)
- diff(f, 2)
- diff(diff(f))
```

(3) One of the followings:

```
- subs(fxx, [x, y], [3, -5])
- subs(subs(fxx, x, 3), y, -5)
- subs(fxx, {x, y}, {3, -5})
```

- `fxx` (only if the (2) already contains `subs`)

(4) One of the followings:

```
- g, z, 0, 'order', 10
- g, z, 'order', 10
- g, ''order'', 10
- g, z, 0, ''order'', 10
- g, 'order', 10
- g, 0, 'order', 10
```

(5) One of the followings:

```
- int(abs(g - T)^2, [-1, 1])
- int(abs(g - T)^2, -1, 1)
- int((g - T)^2, -1, 1)
- int(abs(g - T).^2, [-1, 1])
- int(abs(g - T)^2, z, -1, 1)
- int((subs(g - T, z)^2), -1, 1)
```