6.3.2

(a) The kernel must be x-axis, which goes to the origin, and the range is y-axis. T is neither one-to-one nor onto.

(b) The kernel must be y-axis, which goes to the origin, and the range is xz-plane. T is neither one-to-one nor onto. Naturally, for the orthogonal projection to a linear subspace, kernel must be the subspace which is orthogonal to given linear subspace and passing through the origin and range must be the given linear subspace.

(c) The kernel is only the origin, and the range is whole plane. T is one-to-one and onto. You may recover original point after applying T by expansion.

(d) The kernel is only the origin, and the range is whole space. T is one-to-one and onto. You may recover original poitnt after applying T by reverse rotation.

6.3.6

To find the kernel, you have to solve the equation

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & -7 & 4 \\ 3 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By row operation, it is equivalent to solve

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $5x_1 = x_3$, $5x_2 = 3x_3$. So $(x_1, x_2, x_3) = t(1, 3, 5)$ for any $t \in \mathbb{R}$. Thus, the kernel is

$$\left\{ t \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

6.3.16

The standard matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

since $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Since this operator has same domain and codomain, it is one-to-one if and only if onto by Theorem 6.3.14. By theorem 6.3.15, you need to check only this matrix is invertible or not, and it is equivalent to check this matrix has nonzero determinant. Since the determinant of this matrix is $1 \cdot (40 - 0) - 2 \cdot (16 - 3) + 3 \cdot (0 - 5) = -1$, it is invertible and so it is one-to-one and onto.

6.3.21

(a) By row operations, solving the linear system $A\mathbf{x} = \mathbf{b}$ is equivalent to solve

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 8 & 8 & -8 \\ 0 & 6 & 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 3b_1 \end{bmatrix}$$

Comparing the second and the third row, to have a solution, it must satisfies $3 \cdot (b_2 - 2b_1) = 4(b_3 - 3b_1)$, so given linear system is consistent only if $6b_1 + 3b_2 - 4b_3 = 0$. Also, if this condition is satisfied, then $(\frac{1}{4}b_2 + \frac{1}{2}b_1, \frac{1}{8}b_2 - \frac{1}{4}b_1, 0, 0)$ is a solution. Hence, given linear system is consistent if and only if $6b_1 + 3b_2 - 4b_3 = 0$.

(b) Note that $\mathbf{b}$ is in the range if and only if given linear system is consistent. Hence, the range is the vectors

$$\begin{bmatrix} 2t - s \\ 2s \\ 3t \end{bmatrix} = s \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

where $s, t \in \mathbb{R}$. Also, from the solution of (a), you may check the first and the second column span every columns for matrix after row operations. Hence, it is true for the original matrix, so you may say the range is the vectors

$$s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$$

You may check two forms are equivalent.

(c) Again, by row operations, especially for making reduced row echelon form, solving $A\mathbf{x} = \mathbf{0}$ is equivalent to solve

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the vector is in the kernel if and only if it satisfies $x_1 + x_3 + x_4 = 0$ and $x_2 + x_3 - x_4 = 0$, which is equivalent to $x_1 = -x_3 - x_4$, $x_2 = -x_3 + x_4$, so vectors

$$s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

for $s, t \in \mathbb{R}$ is the full list of the vectors in the kernel.

6.3.D1

(a) (T). Since $T(\mathbf{0}) = \mathbf{0}$ and T is one-to-one, we get $\mathbf{u} - \mathbf{v} = \mathbf{0}$ so $\mathbf{u} = \mathbf{v}$.

(b) (T). Since $T : \mathbb{R}^n \to \mathbb{R}^n$, is onto, so is one-to-one by Theorem 6.3.14 and hence, by (a), it is true.

(c) (T). It is directly from Theorem 6.3.15.

(d) (T). Since T is not one-to-one, $\ker(T)$ contains a nonzero vector $\mathbf{v}_0$. Then, $\{c\mathbf{v}_0 \mid c \in \mathbb{R}\}$ contains infinitely many vectors, and it is a subset of the $\ker(T)$. Hence, $\ker(T)$ contains infinitely many vectors.

(e) (T). Note that shear operation with minus factor is always the inverse operation of original shear operation. Thus, every shear operation is invertible, so is one-to-one.