Note that if A : $m \times n$ matrix, then $A^T A$: $n \times n$ matrix, AA^T : $m \times m$ matrix, and both $A^T A$ and AA^T are symmetric.

A : $n \times n$ matrix

If $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$, then $A^T = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}$

$$A^T A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_1^T \mathbf{a}_1 & \mathbf{a}_1^T \mathbf{a}_2 & \cdots & \mathbf{a}_1^T \mathbf{a}_n \\ \mathbf{a}_2^T \mathbf{a}_1 & \mathbf{a}_2^T \mathbf{a}_2 & \cdots & \mathbf{a}_2^T \mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^T \mathbf{a}_1 & \mathbf{a}_n^T \mathbf{a}_2 & \cdots & \mathbf{a}_n^T \mathbf{a}_n \end{bmatrix}$$

$$\therefore \ A^T A = I_n \iff \mathbf{a}_i^T \mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

<u>Thm 3.6.5</u> Let A be a square matrix.

A, $A^T A$, and AA^T are either all invertible or all singular.

pf) ① A is invertible $\iff$ A^T : invertible. ①

② A, B : square matrices & AB : invertible $\Rightarrow$ A, B : invertible.

By ①, A : invertible $\Rightarrow$ $A^T A$ & AA^T are invertible.

By ②, AA^T : invertible

$\Rightarrow$ A and A^T are invertible.

$\Rightarrow$ A and $A^T A$ are invertible.

Similarly, $A^T A$: invertible $\Rightarrow$ A and AA^T are invertible.

$\blacksquare$

For an $n\times n$ matrix A, $\mathbf{x}\in\mathbb{R}^n$ is a <u>fixed point</u> of A if $A\mathbf{x}=\mathbf{x}$.

$\therefore$ The zero vector $\mathbf{0}$ is a fixed point of every A.

Note that $A\mathbf{x}=\mathbf{x} \iff (I-A)\mathbf{x}=\mathbf{0}$.

$\therefore$ The set of fixed points of A

= the solution set of $(I-A)\mathbf{x}=\mathbf{0}$.

* Invertibility of $I-A$. (Note that $1-x^k=(1-x)(1+x+\cdots+x^{k-1})$)

<u>Case 1</u> If A is nilpotent and the nilpotency of A is k,

then $$I=I-\mathbf{0}=I-A^k=(I-A)(I+A+A^2+\cdots+A^{k-1}).$$

Hence $I-A$ is invertible and $(I-A)^{-1}=I+A+\cdots+A^{k-1}$.

e.g.) $A=\begin{bmatrix}0&2&1\\0&0&3\\0&0&0\end{bmatrix} \Rightarrow A^3=\mathbf{0}$ & $I-A=\begin{bmatrix}1&-2&-1\\0&1&-3\\0&0&1\end{bmatrix}$

$$\therefore (I-A)^{-1}=I+A+A^2=\begin{bmatrix}1&2&7\\0&1&3\\0&0&1\end{bmatrix}$$

※ Every strictly lower or upper triangular matrix is nilpotent.

<u>Case 2</u> A is not nilpotent. (Note that $\frac{1}{1-x}=1+x+x^2+\cdots$)

If the sum of the absolute values of the entries in each column (or each row) of A is less than 1, then $I-A$ is invertible. ($\because$ $I+A+A^2+\cdots$ converges to some matrix.)

§3.7. Matrix factorization ; LU-decomposition.

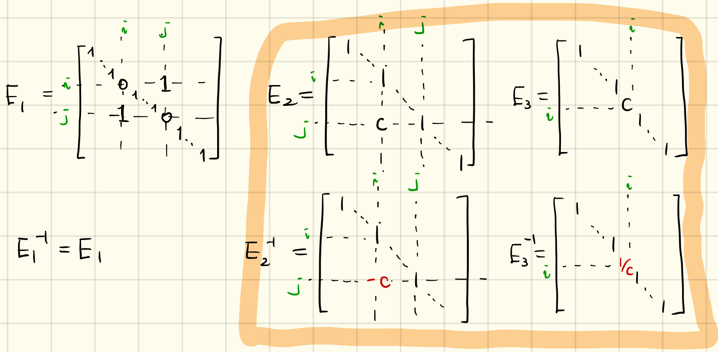
Recall that there are three kinds of elementary operations

① Interchange the ith row and the jth row. $(1 \le i, j \le n)$

② Add a multiple of the ith row to the jth row. $(1 \le i < j \le n)$

③ Multiply a constant to the ith row. $(1 \le i \le n)$

Elementary matrices are the matrix obtained from I_n by doing an elementary operation.



↑ These matrices are lower triangular.

Recall that a product of lower triangular matrices is also lower triangular.

A row echelon form of a square matrix is upper triangular.

$\left(\because \quad \text{Row echelon form of } A = \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \quad \right.$ A leading 1 is in the jth column. Then it is in the row i with $i \le j$. $\left. \right)$

Suppose that a square matrix A can be reduced to row echelon form by Gaussian elimination with no row interchanges.

i.e., $\underbrace{E_k \cdots E_2 E_1}_{\text{No row interchange}} A = U$: row echelon form

$\Rightarrow$ U : upper triangular & $E_k \cdots E_2 E_1$: lower triangular.

$\Rightarrow$ Since $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is lower triangular,

A can be decomposed into LU.

lower triangular upper triangular.

This proves Thm 3.7.2.

Thm 3.7.2 If a square matrix A can be reduced to row echelon form by Gaussian elimination with no row interchanges, then A has LU-decomposition.

Def. A square matrix A has LU-decomposition (or LU-factorization) if $A = LU$

lower triangular upper triangular.

※ Not every square matrix A has a LU-decomposition.

※ LU-decomposition is not unique if it exists

Ex

$$A=\begin{bmatrix}6&-2&0\\9&-1&1\\3&7&5\end{bmatrix}\xrightarrow{①\times\frac{1}{6}}\begin{bmatrix}1&-\frac{1}{3}&0\\9&-1&1\\3&7&5\end{bmatrix}\xrightarrow[①\times(-3)+③]{①\times(-9)+②}\begin{bmatrix}1&-\frac{1}{3}&0\\0&2&1\\0&8&5\end{bmatrix}$$

$$\xrightarrow[②'\times(-8)+③]{②\times\frac{1}{2}}\begin{bmatrix}1&-\frac{1}{3}&0\\0&1&\frac{1}{2}\\0&0&1\end{bmatrix}$$

$$L=\begin{bmatrix}6&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\9&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&8&1\end{bmatrix}$$

$$=\begin{bmatrix}6&0&0\\9&2&0\\3&8&1\end{bmatrix}$$

We can solve $A\mathbf{x}=\mathbf{b}$ using LU-decomposition.

i) Rewrite $A\mathbf{x}=\mathbf{b}$ as $LU\mathbf{x}=\mathbf{b}$.

ii) Define a new unknown $\mathbf{y}$ by letting $U\mathbf{x}=\mathbf{y}$.
Then we obtain $L\mathbf{y}=\mathbf{b}$.

iii) Solve $L\mathbf{y}=\mathbf{b}$.

iv) Substitute the new-unknown vector $\mathbf{y}$ into $U\mathbf{x}=\mathbf{y}$ and solve for $\mathbf{x}$.

Ex Solve the following linear system using LU-decomposition.

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -18 \\ 30 \end{bmatrix}$$

Remark on elementary row operations.

For $i > j$, if we add a multiple of the ith row to the jth row of I_n, then the result is

$$\begin{array}{c} \\ j \\ \\ i \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \ & j & \ & i & \ \end{array} \\ \begin{bmatrix} 1 & & & & \\ & 1 & \cdots & c & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \end{array},$$

and it equals the product

$$\begin{array}{c} \\ j \\ \\ i \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \ & j & \ & i & \ \end{array} \\ \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & c & & 1 & \\ & & & & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \ & i & \ & j & \ \end{array} \\ \begin{bmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & 1 \end{bmatrix} \end{array}.$$

Hence, when we define the second type of elementary row operation, we assume that $i < j$, as follows.

① Interchange the ith row and the jth row. $(1 \le i, j \le n)$

② Add a multiple of the ith row to the jth row. $(1 \le i < j \le n)$

③ Multiply a constant to the ith row. $(1 \le i \le n)$

THEREFORE, every elementary matrix corresponding to the second or the third kind of elementary row operation is lower triangular.

A permutation matrix is obtained from I_n by permuting its rows, and it can be written as a product of elementary matrices $E_1 E_2 \cdots E_k$, where each E_i is the the elementary matrix obtained from I_n by interchanging two rows.

For a square matrix A and a permutation matrix P, PA is the matrix obtained by permuting the rows of A, and AP is the matrix obtained by permuting the columns of A.

For every square matrix A, by permuting the rows of A, we can get a matrix PA that have LU-decomposition.

i.e., $PA = LU$.

Then $A = P^{-1}LU$ and P^{-1} is also a permutation matrix.

Def. A square matrix A has PLU-decomposition if

$$A = \underbrace{P}_{\text{permutation matrix}}\ \underbrace{L}_{\text{lower triangular}}\ \underbrace{U}_{\text{upper triangular.}}$$

Consider a lower triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

If $a_{11} \neq 0$, $a_{22} \neq 0$, $\cdots$, $a_{nn} \neq 0$, then

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21}/a_{11} & 1 & 0 & \cdots & 0 \\ a_{31}/a_{11} & a_{32}/a_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1}/a_{11} & a_{n2}/a_{22} & \cdots & & 1 \end{bmatrix}}_{\text{lower triangular.}} \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}}_{\text{diagonal}}$$

In the proof of Thm 3.7.2, the lower triangular matrix in its LU-decomposition satisfies that every diagonal entry is nonzero. Hence if a matrix has an LU-decomposition, then it also has an LDU-decomposition.