1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let A be a matrix with rank $(A) = n \neq 0$. Then there exist exactly n pivot columns of A.

(b) Let A, B be two $n \times n$ idempotent matrices and T_A, T_B be linear operators on $\mathbb{R}^n$ corresponding to A, B, respectively. If ran(T_A)=ran(T_B), then $A = B$.

(c) Suppose P is a square matrix such that $P^3 = P^2 \neq 0$. (Here, 0 means $n \times n$ zero matrix.) Then P is the standard matrix for an orthogonal projection.

Solution.

(a) True.
By Theorem 7.5.2, we have rank(A)=rank(A^T). So, by definition of rank, the dimension of column space of A is n. By Theorem 7.6.3, the pivot columns of A form a basis for the column space of A, hence there must exist exactly n pivot columns.

(b) False.
Consider $M_a = \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix}$. Then M_a is 2×2 idempotent matrix for all $a \in \mathbb{R}$. Also, $\text{ran}(T_{M_a}) = \{(0, x) : x \in \mathbb{R}\}$ for any $a \in \mathbb{R}$. This gives many counterexamples, for example, put $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$.

(c) False.
Consider $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then simple calculation gives $P^2 = P^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq 0$, so $P \neq P^2$ thus P is not the standard matrix for an orthogonal projection.

2 Find the least squares solutions of the linear system

10 points

$$\begin{cases} x_1 - x_2 &= 0 \\ x_1 &= 2 \\ x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{cases}$$

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Solution.

The matrix form of the system is $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

Then by simple calculation,

$$A^T A = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}.$$

Also note that

$$A^T b = \begin{bmatrix} 8 \\ 10 \end{bmatrix}.$$

Since $\det(A^T A) \neq 0$, we get

$$\hat{x} = \left(A^T A\right)^{-1} A^T b = \begin{bmatrix} \frac{7}{5} \\ \frac{6}{5} \end{bmatrix}$$

Therefore, the least squares solution is $x_1 = \frac{7}{5}$, $x_2 = \frac{6}{5}$.