1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let A be a 3×3 matrix. If $\det(A) = 0$, then one column of A is a scalar multiple of another column.

(b) Let A and B be square matrices of the same size. If AB is invertible, then both A and B are invertible.

(c) For every square matrices of the same size A and B, $\det(AB - BA) = 0$.

Solution.

(a) False. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\det(A) = 0$, because A has a zero row. Note that any column of A is not a scalar multiple of another one.

(b) True. If AB is invertible, then $\det(AB) = \det(A)\det(B) \neq 0$. Consequently, $\det(A) \neq 0$ and $\det(B) \neq 0$, so both A and B are invertible.

(c) False. Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\det(AB - BA) = \begin{vmatrix} -3 & 0 \\ 0 & 3 \end{vmatrix} \neq 0.$$

2 Use Cramer's rule to solve the system

10 points

$$\begin{aligned} 3x + 3y - 6z &= 1 \\ y + z &= -1 \\ -2y + z &= 0. \end{aligned}$$

Solution.

The system is represented by

$$A\mathbf{x} = \begin{bmatrix} 3 & 3 & -6 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \mathbf{b}.$$

Then, $\det(A) = 9$. Now, by replacing the ith column of A with $\mathbf{b}$, we have

$$A_1 = \begin{bmatrix} 1 & 3 & -6 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 & -6 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A_3 = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Consequently, $\det(A_1) = -6$, $\det(A_2) = -3$, and $\det(A_3) = -6$. Therefore, by Cramer's rule,

$$x = \frac{\det(A_1)}{\det(A)} = -\frac{2}{3}, \quad y = \frac{\det(A_2)}{\det(A)} = -\frac{1}{3}, \quad z = \frac{\det(A_3)}{\det(A)} = -\frac{2}{3}$$