1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator whose standard matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 3 & 9 \\ 2 & 0 & 1 \end{bmatrix}$$

Then, there is a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that

$$[T]_B = \begin{bmatrix} 8 & 8 & 0 \\ 6 & 6 & 2 \\ 5 & 0 & 1 \end{bmatrix}.$$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}.$$

Then,

$$[T]_{B',B} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ -5 & -3 \end{bmatrix}$$

for $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, and B' is the standard basis of $\mathbb{R}^3$.

(c) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then, A is similar to B.

Solution.

(a) False. If such basis B exists, then $[T]$ and $[T]_B$ are similar. Since $\det([T]) = -9 \neq 80 = \det([T]_B)$, $[T]$ can not be similar to $[T]_B$ by theorem 8.2.3.

(b) True.

$$T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

(c) False. Note that $\text{tr}(A) = 3 \neq -3 = \text{tr}(B)$. Thus, A is not similar to B by theorem 8.2.3.

2 Let

10 points

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}.$$

Find all the eigenvalues of A and their algebraic and geometric multiplicity.

Solution.

The characteristic polynomial of A is $(\lambda - 2)(\lambda - 3)^2$. So, $\lambda = 2$ has algebraic multiplicity 1 and $\lambda = 3$ has algebraic multiplicity 2. Note that, for $\lambda = 2$,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} \iff \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 8x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}.$$

This implies that $\lambda = 2$ has geometric multiplicity 1. Note that, for $\lambda = 3$,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix} \iff \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This implies that $\lambda = 2$ has geometric multiplicity 1. Hence,

$$\lambda = 2 : \text{algebraic multiplicity} = 1 \quad \text{and} \quad \text{geometric multiplicity} = 1$$

$$\lambda = 3 : \text{algebraic multiplicity} = 2 \quad \text{and} \quad \text{geometric multiplicity} = 1$$