§7.1. Basis & Dimension

Def. Let V be a subspace of $\mathbb{R}^n$.

A subset S in V is called a basis for V if it is linearly independent and spans V.

e.g.) ① Let V be a line through the origin of $\mathbb{R}^n$

$\Rightarrow$ For every $w \in V \setminus \{0\}$, $\{w\}$ is a basis for V.

② Let V be a plane through the origin of $\mathbb{R}^n$

If $\begin{pmatrix} w_1, w_2 \in V \setminus \{0\} \\ w_1 \ \& \ w_2 \text{ are not parallel} \end{pmatrix}$, then $\{w_1, w_2\}$ is a basis for V.

③ The subspace $\{0\}$ has no basis.

④ $\{e_1, e_2, \cdots, e_n\}$ is a basis for $\mathbb{R}^n$, which is called the standard basis for $\mathbb{R}^n$.

For this reason, e_i's are called the standard basis vectors.

⑤ For every $a \in \mathbb{R}$, $\{e_1, ae_1 + e_2\}$ is a basis for $\mathbb{R}^2$.

⑥ For an $n \times n$ matrix A, both $\{c_1(A), \cdots, c_n(A)\}$ and $\{r_1(A), \cdots, r_n(A)\}$ are bases for $\mathbb{R}^n$. The converse is also true.

Thm 7.1.2 Let $S = \{w_1, w_2, \cdots, w_k\} \subseteq \mathbb{R}^n$ with $k \geq 2$.

Then S is linearly dependent iff some vector in S is a linear combination of its predcessors.

pf) ($\Leftarrow$) Clear. (Thm 3.4.6)

($\Rightarrow$) $\exists\, t_1, \cdots, t_k \in \mathbb{R}$ s.t. $t_1 w_1 + \cdots + t_k w_k = 0$.

Let t_j be the nonzero scalar having the largest index.

Then w_j can be written as a linear combination of $w_1, \cdots, w_{j-1}$.

Thm 7.1.3 If V is a nonzero subspace of $\mathbb{R}^n$, then there exists a basis for V that has at most n vectors.

Proof) **Step 1** Choose $\mathbb{v}_1 \in V \setminus \{0\}$. If $V = \text{span}\{\mathbb{v}_1\}$, then we are done.

If not, we are going to Step 2.

Step 2 choose $\mathbb{v}_2 \in V \setminus \text{span}\{\mathbb{v}_1\}$. If $V = \text{span}\{\mathbb{v}_1, \mathbb{v}_2\}$, then we are done.

If not, we are going to Step 3.

Step 3 choose $\mathbb{v}_3 \in V \setminus \text{span}\{\mathbb{v}_1, \mathbb{v}_2\}$. ...

Repeat the same process.

Since a linearly independent set in $\mathbb{R}^n$ has at most n vectors, the process will be end at k stage, $k \le n$. $\boxed{/\!/}$

Thm 7.1.4 All bases for a nonzero subspace of $\mathbb{R}^n$ have the same number of vectors.

pf) Let V be a nonzero subspace of $\mathbb{R}^n$.

Let $\mathcal{B}_1 = \{\mathbb{v}_1, \cdots, \mathbb{v}_k\}$ and $\mathcal{B}_2 = \{\mathbb{w}_1, \cdots, \mathbb{w}_m\}$ be bases for V.

Suppose that $k \ne m$. WMA $k < m$.

Since span $\mathcal{B}_1 = V$ and $\mathcal{B}_2 \subset V$, every $\mathbb{w}_i \in \mathcal{B}_2$ can be written as a linear combination of $\mathbb{v}_1, \cdots, \mathbb{v}_k$. i.e.,

$$\mathbb{w}_1 = a_{11}\mathbb{v}_1 + \cdots + a_{k1}\mathbb{v}_k$$
$$\vdots$$
$$\mathbb{w}_m = a_{1m}\mathbb{v}_1 + \cdots + a_{km}\mathbb{v}_k$$

$$\begin{bmatrix} | & & | \\ \mathbb{w}_1 & \cdots & \mathbb{w}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbb{v}_1 & \cdots & \mathbb{v}_k \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{km} \end{bmatrix}}_{A}$$

Since $k < m$, $A\mathbb{x} = \mathbb{0}$ has infinitely many solutions. i.e., $\exists\, \mathbb{c} \ne 0$ s.t. $A\mathbb{c} = \mathbb{0}$.

Then
$$\begin{bmatrix} | & & | \\ \mathbb{w}_1 & \cdots & \mathbb{w}_m \\ | & & | \end{bmatrix} \mathbb{c} = \begin{bmatrix} | & & | \\ \mathbb{v}_1 & \cdots & \mathbb{v}_k \\ | & & | \end{bmatrix} A\mathbb{c} = \mathbb{0}.$$

This is a contradiction that $\{\mathbb{w}_1, \cdots, \mathbb{w}_m\}$ are linearly independent. $\boxed{/\!/}$

Def. For a nonzero subspace V of $\mathbb{R}^n$,

$$\dim(V) := \text{the number of vectors in a basis for } V.$$

If $V = \{0\}$, $\dim(V) = 0$.

Ex A line passing through the origin has dimension 1.

A plane " " " " " " 2.

We can find a basis for the solution space of $A\mathbf{x} = \mathbf{0}$ by using the Gauss-Jordan elimination :

i.e., $A\mathbf{x} = \mathbf{0} \rightsquigarrow \mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k$,

where $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.

$\Rightarrow$ $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is the canonical basis for the solution space of $A\mathbf{x} = \mathbf{0}$. ※ k = the number of free variables.

Def. For $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we define

$$\mathbf{a}^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{x} = 0\}$$

($\mathbf{a}$ perp)

the solution space of the homogeneous linear system $[-\ \mathbf{a}\ ]\begin{bmatrix} \mathbf{x} \\ | \end{bmatrix} = \mathbf{0}$

Thm 7.1.6 For $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\dim(\mathbf{a}^\perp) = n-1$.

§7.2. Properties of Bases.

Thm 7.2.1 If S is a basis for a subspace V of $\mathbb{R}^n$, then every vector $\mathbf{w} \in V$ can be expressed in exactly one way as a linear combination of the vectors in S.

$(\because \ \mathbf{w} = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = t_1'\mathbf{v}_1 + \cdots + t_k'\mathbf{v}_k \Rightarrow \mathbf{0} = \mathbf{w} - \mathbf{w} = (t_1 - t_1')\mathbf{v}_1 + \cdots + (t_k - t_k')\mathbf{v}_k)$

Thm 7.2.2 Let S be a finite set of vectors in a nonzero subspace V of $\mathbb{R}^n$.

(a) If $\mathrm{span}(S) = V$ but S is not a basis for V, then there is a basis T for V s.t. $T \subsetneq S$.

(b) If S is linearly indep. but S is not a basis for V, then there is a basis T for V s.t. $S \subsetneq T$.

__Thm 7.2.3__ For a nonzero subspace V of $\mathbb{R}^n$, $\dim(V)$ is the maximal number of linearly independent vectors in V.

__Def.__ Let V and W be subspaces of $\mathbb{R}^n$. If $V \subseteq W$, then V is a subspace of W.

__Thm 7.2.4__ Let V, W be subspaces of $\mathbb{R}^n$ & $V \subseteq W$.

(a) $0 \leq \dim(V) \leq \dim(W) \leq n$

(b) $V = W \iff \dim(V) = \dim(W)$.

__Thm 7.2.5__

S : a nonempty set of vectors in $\mathbb{R}^n$ & $S' \supseteq S$

(a) If every $w \in S' \setminus S$ belongs to $\mathrm{span}(S)$, then $\mathrm{span}(S) = \mathrm{span}(S')$.

(b) If $\mathrm{span}(S) = \mathrm{span}(S')$, then every $w \in S' \setminus S$ belongs to $\mathrm{span}(S)$.

(c) If $\dim(\mathrm{span}(S')) = \dim(\mathrm{span}(S))$, then every $w \in S' \setminus S$ belongs to $\mathrm{span}(S)$ and $\mathrm{span}(S) = \mathrm{span}(S')$.

__Thm 7.2.6__

Let V be a nonzero k-dim'l subspace of $\mathbb{R}^n$ and $S = \{w_1, \cdots, w_s\} \subseteq V$.

(a) If $s = k$ & S : linearly indep., then S is a basis.

(b) If $s < k$, then $\mathrm{span}(S) \subsetneq V$.

(c) If $s > k$, then S is linearly dep. $\not\Rightarrow$ $\mathrm{span}(S) = V$.