§8.1. Matrix representations of linear transformations

In this section, we want to solve the following:

1) for a linear operator $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a basis β of $\mathbb{R}^n$,

   find a matrix A s.t. $A[\mathbf{x}]_\beta = [T(\mathbf{x})]_\beta$.

   Such a matrix will be denoted by $[T]_\beta$, and we call $[T]_\beta$ the matrix for T w.r.t. the basis β.

2) for bases β and β' of $\mathbb{R}^n$, find a relationship between $[T]_\beta$ and $[T]_{\beta'}$

3) for a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, a basis β of $\mathbb{R}^n$, and a basis β' of $\mathbb{R}^m$, find a matrix A s.t. $A[\mathbf{x}]_\beta = [T(\mathbf{x})]_{\beta'}$.

   Such a matrix will be denoted by $[T]_{\beta',\beta}$, and we call $[T]_{\beta',\beta}$ the matrix for T w.r.t. the bases β and β'.

<u>Thm 8.1.1</u>

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator and $\beta = \{w_1, \dots, w_n\}$ a basis for $\mathbb{R}^n$. If $A = \big[\,[T(w_1)]_\beta \ \cdots \ [T(w_n)]_\beta\,\big]$, then $\underline{[T(\mathbf{x})]_\beta = A[\mathbf{x}]_\beta \ \forall \mathbf{x} \in \mathbb{R}^n}$ ★. Moreover, A is the only matrix with property ★.

pf) If $[\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, then
$$T(\mathbf{x}) = T(c_1 w_1 + \cdots + c_n w_n) = c_1 T(w_1) + \cdots + c_n T(w_n).$$

$$\therefore [T(\mathbf{x})]_\beta = c_1 [T(w_1)]_\beta + \cdots + c_n [T(w_n)]_\beta$$
$$= \big[\,[T(w_1)]_\beta \ \cdots \ [T(w_n)]_\beta\,\big] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
$$= A[\mathbf{x}]_\beta$$

If $[T(\mathbf{x})]_\beta = C[\mathbf{x}]_\beta$, then $A[\mathbf{x}]_\beta = C[\mathbf{x}]_\beta \ \forall \mathbf{x} \in \mathbb{R}^n$.

$\therefore A = C$. ∎

Thm 8.1.2

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Let $\mathcal{B} = \{w_1, \cdots, w_n\}$ and $\mathcal{B}' = \{w_1', \cdots, w_n'\}$ be bases for $\mathbb{R}^n$. For $P = P_{\mathcal{B}\to\mathcal{B}'}$, we have $[T]_{\mathcal{B}'} = P[T]_{\mathcal{B}}P^{-1}$.

In particular, if $\mathcal{B}$ and $\mathcal{B}'$ are orthonormal bases, then $[T]_{\mathcal{B}'} = P[T]_{\mathcal{B}}P^{T}$.

pf)

$$[T]_{\mathcal{B}'}[x]_{\mathcal{B}'} = [T(x)]_{\mathcal{B}'} = P_{\mathcal{B}\to\mathcal{B}'}[T(x)]_{\mathcal{B}} = P_{\mathcal{B}\to\mathcal{B}'}[T]_{\mathcal{B}}[x]_{\mathcal{B}}$$

$$\parallel$$

$$[T]_{\mathcal{B}'}P_{\mathcal{B}\to\mathcal{B}'}[x]_{\mathcal{B}} \qquad \therefore [T]_{\mathcal{B}'}P_{\mathcal{B}\to\mathcal{B}'}[x]_{\mathcal{B}} = P_{\mathcal{B}\to\mathcal{B}'}[T]_{\mathcal{B}}[x]_{\mathcal{B}} \quad \forall x \in \mathbb{R}^n$$

$$\therefore [T]_{\mathcal{B}'}P_{\mathcal{B}\to\mathcal{B}'} = P_{\mathcal{B}\to\mathcal{B}'}[T]_{\mathcal{B}}$$

$$\text{i.e., } [T]_{\mathcal{B}'} = P_{\mathcal{B}\to\mathcal{B}'}[T]_{\mathcal{B}}P_{\mathcal{B}\to\mathcal{B}'}^{-1}.$$

$\blacksquare$

Thm 8.1.4

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Let $\mathcal{B} = \{w_1, \cdots, w_n\}$ be a basis for $\mathbb{R}^n$ and $\mathcal{B}' = \{w_1', \cdots, w_n'\}$ a basis for $\mathbb{R}^m$.

If $A = \big[\, [T(w_1)]_{\mathcal{B}'} \cdots [T(w_n)]_{\mathcal{B}'} \big]$, then $\underline{[T(x)]_{\mathcal{B}'} = A[x]_{\mathcal{B}} \quad \forall x \in \mathbb{R}^n.}$ $\circledast$

Moreover, A is the only matrix with property $\circledast$.

pf) Exercise. (It is similar to the proof of Thm 8.1.1)

Let $\mathcal{B}_1, \mathcal{B}_2$ be bases for $\mathbb{R}^n$.

Let $\mathcal{B}_1', \mathcal{B}_2'$ be bases for $\mathbb{R}^m$.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

$$\begin{tikzcd}
(\mathbb{R}^n, \mathcal{B}_1) \arrow[r, "{[T]_{\mathcal{B}_1, \mathcal{B}_1'}}"'] \arrow[d, "P_{\mathcal{B}_1\to\mathcal{B}_2}"'] \arrow[dr, phantom, "\circlearrowleft"] & (\mathbb{R}^m, \mathcal{B}_1') \arrow[d, "P_{\mathcal{B}_1'\to\mathcal{B}_2'}"] \\
(\mathbb{R}^n, \mathcal{B}_2) \arrow[r, "{[T]_{\mathcal{B}_2, \mathcal{B}_2'}}"'] & (\mathbb{R}^m, \mathcal{B}_2')
\end{tikzcd}$$

$$P_{\mathcal{B}_1'\to\mathcal{B}_2'}[T]_{\mathcal{B}_1, \mathcal{B}_1'} = [T]_{\mathcal{B}_2, \mathcal{B}_2'}P_{\mathcal{B}_1\to\mathcal{B}_2}$$

§8.2. Similarity and diagonalizability

Def. For $n \times n$ matrices A and C,

C is similar to A if $\exists$ invertible P s.t. $C = P^{-1}AP$.

Rmk If C is similar to A, then A is similar to C. Hence we say that A and C are similar.

Thm 8.2.2 A and C are similar iff there exist linear operator $T: \mathbb{R}^n \to \mathbb{R}^n$ and bases B and B' of $\mathbb{R}^n$ s.t.

$$A = [T]_B \quad \text{and} \quad C = [T]_{B'}.$$

pf) ($\Rightarrow$) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ with $[T] = A$. Since A and C are similar, $\exists$ inv. P s.t. $C = P^{-1}[T]P$. If $P = [w_1 \ w_2 \cdots w_n]$, then $B = \{w_1, \cdots, w_n\}$ is a basis for $\mathbb{R}^n$. Hence

$$C = P^{-1}[T]P = (P_{B\to S})^{-1} [T]_S P_{B\to S} = [T]_B.$$

($\Leftarrow$) Assume $C = [T]_B$ and $A = [T]_{B'}$ for some linear operator T.

Then
$$C = [T]_B = P_{B'\to B} [T]_{B'} P_{B\to B'} = P_{B\to B'}^{-1} \, A \, P_{B\to B'}$$

$\therefore$ A and C are similar. ■

Note that if A and C are similar, then $\det(A) = \det(C)$.

A property that is shared by similar matrices is called a similarity invariant.

Thm 8.2.3 (Similarity invariants)

(a) determinant (b) rank (c) nullity (d) trace

(e) characteristic polynomial; eigenvalues with algebraic multiplicities pf) (b) & (c) : For two $n \times n$ matrices A and P, if P is invertible,

then $null(A) = null(PA)$. Hence $nullity(A) = nullity(PA)$

and $rank(A) = rank(PA)$.

Since $rank(AP) = rank(P^T A^T) = rank(A^T) = rank(A)$,

$nullity(AP) = nullity(A)$.

Since P and P^{-1} are invertible,

$$rank(P^{-1}AP) = rank(AP) = rank(A)$$

$$nullity(P^{-1}AP) = rank(AP) = nullity(A)$$

(d) : $tr(P^{-1}AP) = tr(PP^{-1}A) = tr(A)$

(e) : $\lambda I - P^{-1}AP = \lambda P^{-1}P - P^{-1}AP = P^{-1}(\lambda I - A)P$

$\therefore \det(\lambda I - P^{-1}AP) = \det(\lambda I - A)$ ▣

<u>Question</u> How about the converse of Thm 8.2.3 ?

$\det(A) = \det(C)$	$\overset{?}{\Longrightarrow}$	A & C are similar.	No
$rank(A) = rank(C)$	$\overset{?}{\Longrightarrow}$	A & C are similar.	No
$nullity(A) = nullity(C)$	$\overset{?}{\Longrightarrow}$	A & C are similar.	No
$tr(A) = tr(C)$	$\overset{?}{\Longrightarrow}$	A & C are similar.	No
$\det(\lambda I - A) = \det(\lambda I - C)$	$\overset{?}{\Longrightarrow}$	A & C are similar.	No.

<u>Def.</u> For an eigenvalue λ_0,

geometric multiplicity of λ_0 = the dim. of the eigenspace corr. to λ_0

Ex) $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \det(\lambda I - A) = (\lambda - 1)^4$ (alg. multip. of $\lambda = 1$ is 4)

$$(I - A)\mathbf{x} = \mathbf{0} \iff \mathbf{x} = t\begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$\therefore$ geom. multip. of $\lambda = 1$ is 1.

In general, $1 \leq$ geom. multip. $\leq$ alg. multip.

Problem Can we find non similar matrices A and B which have the same determinant, rank, trace, and eigenvalues?

Answer Yes!

Consider $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Then $\det(A) = \det(C) = 0$

$\operatorname{rank}(A) = \operatorname{rank}(C) = 2$

$\operatorname{tr}(A) = \operatorname{tr}(C) = 0$

$\det(\lambda I - A) = \det(\lambda I - C) = \lambda^4$

However, A and C are not similar.

$\boxed{\text{Fact}}$ If A and C are similar, then for $n \in \mathbb{N}$, A^n and C^n are similar, too.

Unfortunately, $A^2 = 0$ but $C^2 \neq 0$.

Hence A^2 and C^2 are not similar.