6.1.22

(a) $w_1 = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Hence, $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so the standard matrix for the linear transformation T is

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix}$$

(b) By substitution, we get $T(-1, 2, 4) = (2 \cdot (-1) - 3 \cdot 2 + 4, 3 \cdot (-1) + 5 \cdot 2 - 4) = (-4, 3)$. By standard matrix, we get

$$T\left(\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

6.1.31

At first, when θ is given as **Figure Ex-31**, note that $\cos\theta = \frac{1}{\sqrt{1+m^2}}$, $\sin\theta = \frac{m}{\sqrt{1+m^2}}$, which gives $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{1-m^2}{1+m^2}$, $\sin 2\theta = 2\cos\theta\sin\theta = \frac{2m}{1+m^2}$

(a) By (18),

$$H_L = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

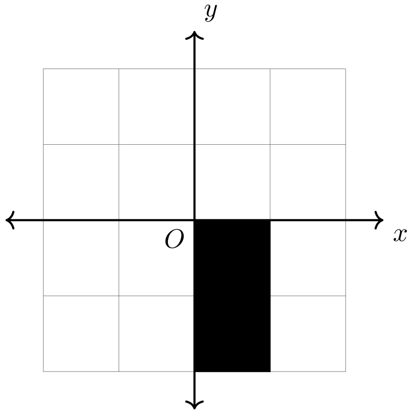
(b) By (21),

$$P_L = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}.$$

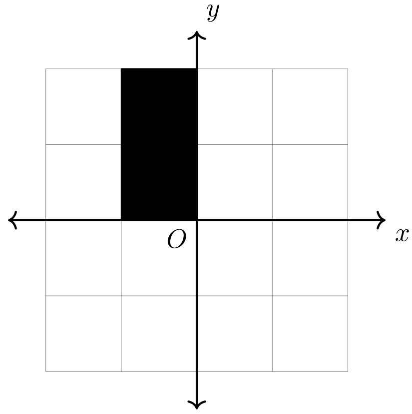
6.1.36 Since $x = \frac{1}{2}((6x+2y)-2\cdot(2x+y)) = \frac{1}{2}t-s$ and $y = 3\cdot(2x+y)-(6x+2y) = 3s-t$, any point (x, y) satisfying $x+y = 1$ maps to a point (s, t) which satisfying $\frac{1}{2}t-s+3s-t = 1$. Hence, the image of the line $x + y = 1$ is the line $4s - t = 2$. Also, if you figure out that $T(x, 1-x) = (x+1, 4x+2)$ and $4 \cdot (x+1) - (4x+2) = 2$, it will also prove that the image is the line $4s - t = 2$.

6.1.38

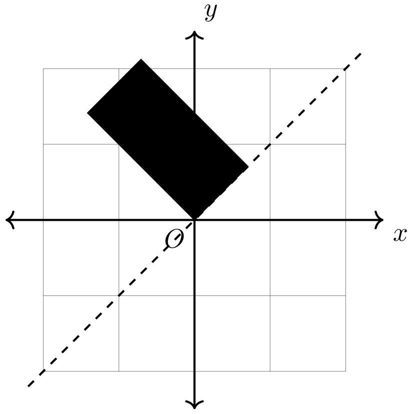
(a)



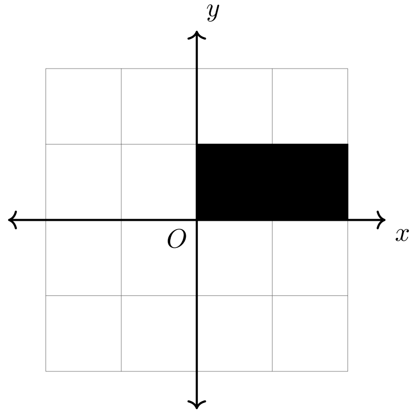
(b)



(c)



(d)



6.1.D1

(a) (F). If $T : \mathbb{R} \to \mathbb{R}$ is defined as $T(x) = x^2$, then $T(0) = 0$ but $T(1) + T(1) = 2 \neq T(1+1)$, so is not linear.

(b) (T). For any scalar c and vector $\mathbf{u} \in \mathbb{R}^n$, $T(c\mathbf{u}) = T(c\mathbf{u} + 0\mathbf{u}) = cT(\mathbf{u}) + 0T(\mathbf{u}) = cT(\mathbf{u})$ and for any vector $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $T(\mathbf{u} + \mathbf{v}) = T(1 \cdot \mathbf{u} + 1 \cdot \mathbf{v}) = 1 \cdot T(\mathbf{u}) + 1 \cdot T(\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Thus, T is linear.

(c) (F). If $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ is defined as $T_i(x) = i \cdot x$, which are linear, then $T_i(-\mathbf{v}) = i \cdot (-\mathbf{v}) = -i \cdot \mathbf{v} = -T_i(\mathbf{v})$, so both T_1, T_2 satisfying given condition. Actually, for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(-\mathbf{v}) = T((-1) \cdot \mathbf{v}) = (-1) \cdot T(\mathbf{v}) = -T(\mathbf{v})$.

(d) (T). If we let $\mathbf{u} = \mathbf{0}$, then $T(\mathbf{v}) = T(-\mathbf{v}) = -T(\mathbf{v})$. Hence, $2T(\mathbf{v}) = \mathbf{0}$, so $T(\mathbf{v}) = \mathbf{0}$ for any vector $\mathbf{v} \in \mathbb{R}^n$. Thus, the zero transformation is the only linear transformation with given property.

(e) (F). Since $T(\mathbf{0}) = \mathbf{v}_0 \neq \mathbf{0}$, T is not a linear transformation.

6.1.D6

Note that $f(x) + f(y) = m(x + y) + 2b = f(x + y) + b$, $f(cx) = cmx + b = cf(x) + (1 - c)b$. Hence, if $b = 0$, then f is additive and homogeneous. But if $b \neq 0$, then when you consider the case that $c = 0$, you may prove that f is not additive nor homogeneous.

6.2.14

It is enough to observe how the standard basis changes.

$$
\begin{aligned}
(1,0,0) &\to (1,0,0) \to (1,0,0) \to (-1,0,0) \\
(0,1,0) &\to (0,1,0) \to (0,-1,0) \to (0,-1,0) \\
(0,0,1) &\to (0,0,-1) \to (0,0,-1) \to (0,0,-1)
\end{aligned}
$$

Hence, $[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. With results from chapter 6.4, you may also compute as

$$
[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

with matrices provided by Table 6.2.5.

6.2.24

(a) Rotation matrix with standard basis is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Hence, the image of the vector $(-2, 1, 2)$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{1-2\sqrt{3}}{2} \\ \frac{2+\sqrt{3}}{2} \end{bmatrix}$$

(b) Rotation matrix with standard basis is $\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$. Hence, the image of the vector $(-2, 1, 2)$ is

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ 1 \\ 1+\sqrt{3} \end{bmatrix}$$

(c) Rotation matrix with standard basis is $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence, the fimage of the vector $(-2, 1, 2)$ is

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \\ 2 \end{bmatrix}$$

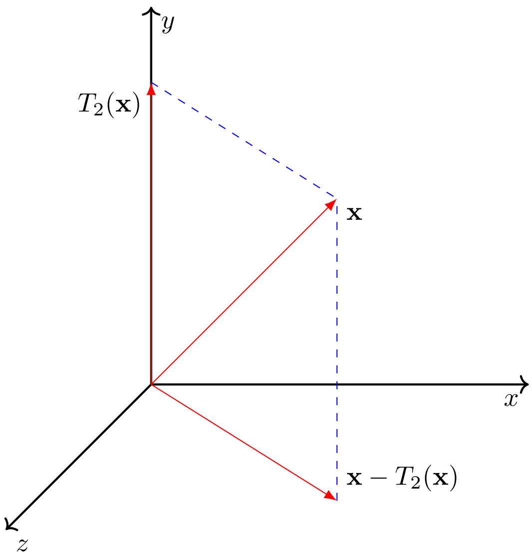
6.2.29

(a) Since $T_1((x_1, y_1, z_1) + (x_2, y_2, z_2)) = (x_1 + x_2, 0, 0) = (x_1, 0, 0) + (x_2, 0, 0) = T_1(x_1, y_1, z_1) + T_1(x_2, y_2, z_2)$ and $T_1(c(x, y, z)) = c(x, 0, 0) = cT_1(x, y, z)$ which proves T_1 is a linear operator. It is similar to prove T_2, T_3 are linear opeartors. Also, you may check

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which proves T_1 is linear, and $M_1 = [T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Similarily, $M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) For T_1, $T_1(x, y, z) \bullet ((x, y, z) - T_1(x, y, z)) = (x, 0, 0) \bullet (0, y, z) = 0$ so $T_1(\mathbf{x})$ and $\mathbf{x} - T_1(\mathbf{x})$ are orthogonal. It is similar to prove for T_2, T_3. Visually, T_2 and $I - T_2$ can be shown as following.



6.2.30

(a) You may check

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+kz \\ y+kz \\ z \end{bmatrix}$$

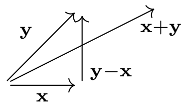
so the standard matrix for the shear in the xy-direction with factor k is

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

(b) It is natural to define shear in the xz-direction with factor k as $(x, y, z) \mapsto (x + ky, y, z + ky)$ which gives the standard matrix $\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$ and shear in the yz-direction with factor k as $(x, y, z) \mapsto (x, y + kx, z + kx)$ which gives the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$

6.2.D8

Following picture is for $\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}, \mathbf{y}-\mathbf{x}$.



You may observe that $\mathbf{x}+\mathbf{y}$ and $\mathbf{y}-\mathbf{x}$ are diagonals of the parellelogram which is constructed by $\mathbf{x}$ and $\mathbf{y}$. Hence $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$ means $\mathbf{x}$ and $\mathbf{y}$ make a parallelogram which has diagonals of equal length, which is equivalent to say that given parallelogram be a rectangle, which means $\mathbf{x} \bullet \mathbf{y} = 0$.