MAS109: Introduction to Linear Algebra **Fall 2019**

Lecture 1: September 3

Lecturer: Seonjeong Park (psjeong@kaist.ac.kr)

Text Book: *Contemporary Linear Algebra, H. Anton and R.C. Busby, John Wiley & Sons, Inc.*

Office Hours: *TTh. 10:30∼11:20 (Please send an email if you would like to visit outside of office hours.)*

Introduction

Linear Algebra is central to almost all areas of mathematics. Of course, abstract algebra is based on linear algebra. Linear algebra is also fundamental in geometry. Moreover, functional analysis can be viewed as an application of linear algebra to spaces of functions. Linear algebra is also used in most sciences and engineering area.

Let us see the book cover of our text book. We will understand the meaning of this cover in the final week when we study singular value decomposition in Section 8.6. Briefly speaking, each photo can be thought as a matrix whose components are real numbers between 0 and 1. By using a singular value decomposition of a matrix, we can compress the photo.

Notation. We will use **boldface** letters in order to distinguish the *vector* $\mathbf{a}$ from the *scalar* a, the single real number a.

- scalar $a, b, c, \ldots$ zero 0
- vector $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ zero vector $\mathbf{0} = (0, \ldots, 0)$

Definition 1.1

1. *$\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \parallel \mathbf{w}$ if $\mathbf{v} = k\mathbf{w}$ for some $k \in \mathbb{R}$. (Hence, $\forall \mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \parallel \mathbf{0}$.)*
2. *$\mathbf{w} \in \mathbb{R}^n$ is a* linear combination *of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if*
$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k, \qquad c_i \in \mathbb{R},$$
where $c_1, \ldots, c_k$ are called the coefficients in the linear combination.
3. *length/norm/magnitude of $\mathbf{v} = (v_1, \ldots, v_n)$*
$$||\mathbf{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}$$

4. *dot product/Euclidean inner product*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

5. *$\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} \perp \mathbf{v}$ if $\mathbf{u} \cdot \mathbf{v} = 0$ (Hence, $\forall \mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \perp \mathbf{0}$.)*

6. *Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthonormal if $\mathbf{u} \perp \mathbf{v}$ and $||\mathbf{u}|| = ||\mathbf{v}|| = 1$.*

Alternative notation for vectors: for a vector $\mathbf{v} = (v_1, \ldots, v_n)$

- row-vector form $[v_1, \ldots, v_n]$
- column-vector form $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

1.1 Systems of linear equations

1.1.1 Introduction of systems of linear equations

We define a linear equation in the n variables $x_1, \ldots, x_n$ to be an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \tag{1.1}$$

for $a_1, \ldots, a_n, b \in \mathbb{R}$. If $b = 0$, then (1.1) is called *homogeneous*. (If $(v_1, \ldots, v_n)$ is a solution of (1.1), then $k(v_1, \ldots, v_n)$ is also a solution of (1.1) for every $k \in \mathbb{R}$. Furthermore, $\mathbf{0}$ is a solution of (1.1).)

Note that if $n = 2$, then the solution set of (1.1) forms a straight line in $\mathbb{R}^2$; if $n = 3$, then the solution set of (1.1) forms a plane in $\mathbb{R}^3$. In general, the solution set of (1.1) in $\mathbb{R}^n$ is called a *hyperplane*. We will study in Section 3.5.

A finite set of linear equations is called a *system of linear equations* (or *linear system*), and the variables in a linear system are called the *unknowns*.

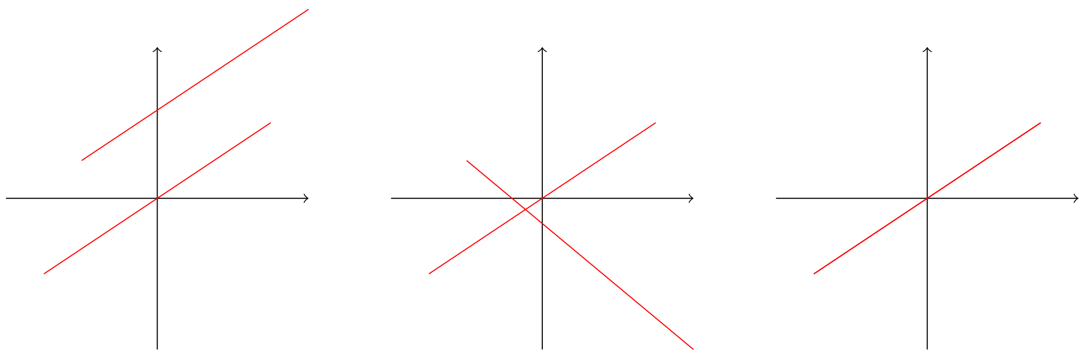
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1.2}$$

If $b_1 = \cdots = b_m = 0$, then (1.2) is called a *homogeneous linear system*. In this course, we will study various ways to find a solution set of a system of linear equations

Consider a linear system of two unknowns

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \tag{1.3}$$

There there are three possible cases: (a) No solution, (b) One solution, (c) Infinitely many solutions



(a) parallel (b) Intersecting at a point (c) coincident

Definition 1.2 *A linear system is* consistent *(respectively, inconsistent) if it has at least one solution (respectively, if it has no solution).*

We will prove later that the above phenomenon holds for all linear systems.

Theorem 1.3 *Every system of linear equations has zero, one, or infinitely many solutions; there are no other possibilities.*

We will see the proof of the above theorem in Section 3.5.

For the system (1.2), we can obtain a matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}, \tag{1.4}$$

which is called the *augmented matrix* for the system.

Note that the basic method for solving a linear system is to produce a succession of simpler systems by eliminating unknowns systemically:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another.

Since the rows of an augmented matrix correspond to the equations in the corresponding system, the above three operations correspond to the operations on the rows of the augmented matrix as follows:

1. Multiply a **row** through by a nonzero constant.
2. Interchange two **rows**.
3. Add a multiple of one **row** to another.

These are called *elementary row operations* on a matrix.

Example 1.4 *Consider the system of linear equations and the associated augmented matrix*

$$\begin{cases} 2x + 4y - 3z = 1 \\ x + y + 2z = 9 \\ 3x + 6y - 5z = 0 \end{cases} \qquad \textit{and} \quad \begin{bmatrix} 2 & 4 & -3 & 1 \\ 1 & 1 & 2 & 9 \\ 3 & 6 & -5 & 0 \end{bmatrix}.$$

The system has the unique solution $x = 1, y = 2, z = 3$.

Example 1.5 *Determine whether the vector* $\mathbf{w} = (9, 1, 0)$ *can be expressed as a linear combination of the vectors*

$$\mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (1, 4, 6), \mathbf{v}_3 = (2, -3, -5)$$

and, if so, find such a linear combination.

Solution *By the definition of a linear combination, if the vector* $\mathbf{w}$ *is a linear combination of* $\mathbf{v}_1$, $\mathbf{v}_2$, *and* $\mathbf{v}_3$, *then there are real numbers* c_1, c_2, c_3 *such that* $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. *That is,*

$$\begin{aligned}(9, 1, 0) &= c_1(1, 2, 3) + c_2(1, 4, 6) + c_3(2, -3, -5) \\ &= (c_1 + c_2 + 2c_3, 2c_1 + 4c_2 - 3c_3, 3c_1 + 6c_2 - 5c_3).\end{aligned}$$

Then we have a linear system

$$\begin{cases} c_1 + c_2 + 2c_3 = 9 \\ 2c_1 + 4c_2 - 3c_3 = 1 \\ 3c_1 + 6c_2 - 5c_3 = 0. \end{cases}$$

From the previous example, we already know that the above system of linear equations has the unique solution $c_1 = 1, c_2 = 2, c_3 = 3$. *Therefore,* $\mathbf{w} = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$.

Note that from the above example, one can see that $x_1 = c_1, \ldots, x_n = c_n$ is a solution of a linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

is consistent if and only if

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Suggested problems: (§2.1) 12, 26, 28, 32, D8

1.2 Solving linear systems by row reduction

For a system of linear equations, if we have a small numbers of unknowns, then we can solve it by hands. However, in order to solve large systems, we need to use a computer. For large systems, we need special techniques. In this section, we study a method by row reduction. Almost all methods for solving a large system are based on this method.

Definition 1.6 *A matrix is in* reduced row echelon form *if it satisfies the following properties:*

1. *If a row does not consist entirely of zeros, then the first nonzero number in the row is a* 1*. We call this a* leading 1*.*
2. *If there are any rows that consist entirely of zeros, then they are groups together at the bottom of the matrix.*
3. *In any two successive rows that do not consist entirely of zeros, the leading* 1 *in the lower row occurs farther to the right than the leading* 1 *in the higher row.*
4. *Each column that contains a leading* 1 *has zeros everywhere else.*

If a matrix has the first three properties, then is said to be in row echelon form*.*

For example, the matrix
$$\begin{bmatrix} 1 & 1 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
is in reduced row echelon form. Consider the linear system whose augmented matrix is in the above:
$$\begin{cases} x_1 + 2x_2 + 3x_4 = 7 \\ \qquad x_3 = 1 \\ \qquad\qquad x_5 = 2. \end{cases}$$
Since x_1, x_3, x_5 correspond to the leading 1's in the augmented matrix, we call these the *leading variables*. The remaining variables x_2 and x_4 are called *free variables*. For the free variables x_2 and x_4 we can assign arbitrary values, say $x_2 = s$ and $x_4 = t$. Then
$$x_1 = -2s - 3t + 7, x_2 = s, x_3 = 1, x_4 = t, x_5 = 2 \qquad \text{for } s, t \in \mathbb{R}.$$
That is,
$$(x_1, \ldots, x_5) = s(-2, 1, 0, 0, 0) + t(-3, 0, 0, 1, 0) + (7, 0, 1, 0, 2) \qquad \text{for } s, t \in \mathbb{R}.$$

<u>How can we solve a linear system?</u>

1. Find the augmented matrix A of the linear system.

2. By using the elementary row operations, reduce A to the reduced row echelon form (respectively, row echelon form).

3. Then we can obtain a general solution of the system.

We call this algorithm *Gauss-Jordan elimination* (respectively, *Gaussian elimination*).

Now we consider a homogeneous linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases} \tag{1.5}$$

Note that every homogeneous linear system is consistent since

$$x_1 = 0, \ldots, x_n = 0$$

is a solution of (1.5). This is called the *trivial solution*. All other solutions, if any, are called *nontrivial solution*. If (1.5) has a nontrivial solution

$$x_1 = s_1, \ldots, x_n = s_n,$$

then

$$x_1 = ks_1, \ldots, x_n = ks_n (\forall k \neq 0)$$

is also a solution of (1.5). Therefore, we get the following.

Theorem 1.7 *A homogeneous linear system has only the trivial solution or infinitely many solutions; there are no other possibilities.*

The following two theorems do not hold in general, but they are always true only for homogeneous linear systems.

Theorem 1.8 *If the reduced row echelon form has r nonzero rows, then the system has $n-r$ free variables.*

Theorem 1.9 *If $m < m$, then the system has infinitely many solutions.*

Suggested problems: (§2.2) 12, 26, 41, 46, D8.