1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+4+3 points

(a) Let W be a subspace of $\mathbb{R}^n$. Let M be a $n \times k$ matrix whose column vectors form an orthonormal basis for W. Then $MM^T = I_n$, where I_n is the $n \times n$ identity matrix.

(b) Let W be a subspace of $\mathbb{R}^n$. Let $\mathbf{w} \in W$. Let M be a $n \times k$ matrix whose column vectors form an orthonormal basis for W. Then for any $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \cdot \mathbf{w} = M^T\mathbf{v} \cdot M^T\mathbf{w}$.

(c) Let $\mathbf{v_1} = (4, 1, 1)$, $\mathbf{v_2} = (3, 2, 3)$, $\mathbf{v_3} = (5, 5, 1)$, $\mathbf{w_1} = (1, 1, 0)$, $\mathbf{w_2} = (1, 0, 1)$, and $\mathbf{w_3} = (0, 1, 1)$. Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, $B' = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be ordered bases for $\mathbb{R}^3$. Then the entries of the transition matrix (the change of coordinate matrix) from B to B' ($P_{B \to B'}$) are all integers.

Solution.

(a) FALSE. Choose $M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $MM^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $MM^T \neq I_2$.

(b) TRUE. Note that $MM^T\mathbf{w} = \mathbf{w}$.

$$M^T\mathbf{v} \cdot M^T\mathbf{w} = \left(M^T\mathbf{v}\right)^T M^T\mathbf{w} = \mathbf{v}^T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}.$$

(c) FALSE.

$$[\mathbf{v_3}]_{B'} = \left(\frac{9}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

2 Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v_1} = (1,1,0,0)$, $\mathbf{v_2} = (0,2,1,0)$, and $\mathbf{v_3} = (0,0,3,1)$.

10 points

Solution.

Note that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent. So it is a basis for this subspace.

Now let's use the Gram-Schmidt process. First we construct the orthogonal basis vectors

$$\begin{aligned}
\mathbf{w_1} &= \mathbf{v_1} = (1,1,0,0), \\
\mathbf{w_2} &= \mathbf{v_2} - \frac{\mathbf{v_2}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\mathbf{w_1} = (-1,1,1,0), \\
\mathbf{w_3} &= \mathbf{v_3} - \frac{\mathbf{v_3}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\mathbf{w_1} - \frac{\mathbf{v_3}\cdot\mathbf{w_2}}{\mathbf{w_2}\cdot\mathbf{w_2}}\mathbf{w_2} = (1,-1,2,1).
\end{aligned}$$

Then by normalizing these, we get an orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}\mathbf{w_1}, \frac{1}{\sqrt{3}}\mathbf{w_2}, \frac{1}{\sqrt{7}}\mathbf{w_3}\right\} = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right), \left(\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)\right\}.$$