7.9.6 Find the orthogonal projection of $\mathbf{x} = (1, 2, 0, -1)$ on the subspace of R^4 spanned by the given orthonormal vectors:

(a) $\mathbf{v}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\mathbf{v}_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

(b) $\mathbf{v}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\mathbf{v}_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $\mathbf{v}_3 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

Solution. (a) $\text{proj}_W \mathbf{x} = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{x} \cdot \mathbf{v}_2)\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 = (\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$

(b) $\text{proj}_W \mathbf{x} = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{x} \cdot \mathbf{v}_2)\mathbf{v}_2 + (\mathbf{x} \cdot \mathbf{v}_3)\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2 = (\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$

□

7.9.13 Find the standard matrix for the orthogonal projection onto the subspace of R^3 spanned by $\mathbf{v}_1 = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$, $\mathbf{v}_2 = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$.

Solution. Using Formula (6), we have $P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{5}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{8}{9} \end{bmatrix}$ $\square$

7.9.16 Use the matrix obtained in Exercise 7.9.13 to find the orthogonal projection of $\mathbf{w} = (4, -5, 1)$ onto $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, and check the resulting Formula (7).

Solution. Using the matrix P found in Exercies 13, the orthogonal projection of $\mathbf{w}$ onto $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$P\mathbf{w} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{5}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{8}{9} \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ -\frac{11}{9} \\ \frac{26}{9} \end{bmatrix}.$$

On the other hand, using Formula (7), we have

$$\text{proj}_W \mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w} \cdot \mathbf{v}_2)\mathbf{v}_2 = (\frac{5}{3})(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}) + (-\frac{8}{3})(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}) = (\frac{2}{9}, -\frac{11}{9}, \frac{26}{9}),$$

so it coincides. $\square$

7.9.26 Let $P = \begin{bmatrix} \frac{8}{9} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix}$.

Use Theorem 7.7.6 to confirm that P is the standard matrix for an orthogonal projection, and then use Theorem 7.9.3 to find the dimension of the range of that projection.

Solution. P is symmetric, and it's easy to check that $P^2 = P$. Then, by Theorem 7.9.3,

$$rank(P) = tr(P) = \frac{8}{9} + \frac{5}{9} + \frac{5}{9} = 2.$$

□

7.9.32 Use the Gram-Schmidt process to transform the basis $\mathbf{w}_1 = (1,2,1,0)$, $\mathbf{w}_2 = (1,1,2,0)$, $\mathbf{w}_3 = (0,1,1,-2)$, $\mathbf{w}_4 = (1,0,3,1)$ into an orthonormal basis.

Solution. Let $\mathbf{v}_1 = \mathbf{w}_1 = (1,2,1,0)$, $\mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,1,2,0) - (\frac{5}{6})(1,2,1,0) = (\frac{1}{6}, -\frac{2}{3}, \frac{7}{6}, 0)$,

$$\begin{aligned}\mathbf{v}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_3\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 &= (0,1,1,-2) - (\frac{3}{6})(1,2,1,0) - \left(\frac{\frac{1}{2}}{\frac{11}{6}}\right)\left(\frac{1}{6}, -\frac{2}{3}, \frac{7}{6}, 0\right)\\ &= \left(-\frac{6}{11}, \frac{2}{11}, \frac{2}{11}, -2\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_4 &= \mathbf{w}_4 - \frac{\mathbf{w}_4\cdot\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_4\cdot\mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\mathbf{w}_4\cdot\mathbf{v}_3}{\|\mathbf{v}_3\|^2}\mathbf{v}_3\\ &= (1,0,3,1) - (\frac{4}{6})(1,2,1,0) - \left(\frac{\frac{11}{3}}{\frac{11}{6}}\right)\left(\frac{1}{6}, -\frac{2}{3}, \frac{7}{6}, 0\right) - \left(-\frac{2}{\frac{48}{11}}\right)\left(-\frac{6}{11}, \frac{2}{11}, \frac{2}{11}, -2\right)\\ &= \left(-\frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an orthogonal basis for R^4, and the vectors

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, 0\right), \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{\sqrt{66}}{66}, -\frac{2\sqrt{66}}{33}, \frac{7\sqrt{66}}{66}, 0\right)$$
$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(-\frac{\sqrt{33}}{22}, \frac{\sqrt{33}}{66}, \frac{\sqrt{33}}{66}, -\frac{\sqrt{33}}{6}\right), \quad \mathbf{q}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}\right)$$

form an orthonormal basis for R^4. □

7.9.38 Express $\mathbf{w} = (-1, 2, 6, 0)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in the subspace W of R^4 spanned by the vectors $\mathbf{u}_1 = (-1, 0, 1, 2)$ and $\mathbf{u}_2 = (0, 1, 0, 1)$, and $\mathbf{w}_2$ is orthogonal to W.

Solution. By applying the Gram-Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2\}$, we can find an orthonormal basis

$$\mathbf{q}_1 = (-\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}), \qquad \mathbf{q}_2 = (\frac{1}{\sqrt{12}}, \frac{3}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}})$$

for W. So the orthogonal projection of $\mathbf{w}$ onto W is given by

$$\mathbf{w}_1 = \text{proj}_W \mathbf{w} = (\mathbf{w} \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{w} \cdot \mathbf{q}_2)\mathbf{q}_2 = (-\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4})$$

and the component of $\mathbf{w}$ orthogonal to W is

$$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = (-1, 2, 6, 0) - (-\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4}) = (\frac{1}{4}, \frac{9}{4}, \frac{19}{4}, -\frac{9}{4}).$$

□

7.9.D5

(a) We know from Formula (6) that the standard matrix for the orthogonal projection of R^n onto a subspace W can be expressed in the form $P = MM^T$. What is the relationship between the column spaces of M and P?

(b) If you know P, how might you find a matrix M for which $P = MM^T$?

(c) Is the matrix M unique?

Solution. (a) $\text{col}(M) = \text{col}(P) = W$.

(b) Find an orthonormal basis for $\text{col}(P)$ and use these vectors as the columns of the matrix M.

(c) No. Any orthonormal basis for $\text{col}(P)$ can be used to form the columns of M.

□

7.9.P2 If A is symmetric and idempotent, then A can be factored as $A = UU^T$, where U has orthonormal columns.

This problem is wrong.
[counterexample] A=zero matrix

□

If A is symmetric and idempotent nonzero matrix, then A is the standard matrix of an orthogonal projection operator; namely the orthogonal projection of R^n onto W = col(A). Thus A = UU^T where U is any $n \times k$ matrix whose column vectors form an orthonormal basis for W .

7.10.6 Find a QR-decomposition of $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

Solution. Application of the Gram-Schmidt process to the column vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of A yields

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad \mathbf{q}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \mathbf{q}_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$

where $\mathbf{w}_1 = 2\mathbf{q}_1$, $\mathbf{w}_2 = -\mathbf{q}_1 + \mathbf{q}_2$, and $\mathbf{w}_3 = \frac{1}{2}\mathbf{q}_1 + \frac{3}{2}\mathbf{q}_2 + \frac{1}{\sqrt{2}}\mathbf{q}_3$. This yields the following QR-decomposition of A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = QR$$

□

7.10.10 Use the QR-decomposition of the matrix A in Exercise 7.10.6 and the method of Example 2 to find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$, where $b = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$.

Solution. The normal system for $A\mathbf{x} = \mathbf{b}$ can be expressed as $R\mathbf{x} = Q^T\mathbf{b}$, which is:

$$\begin{bmatrix} 2 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \\ -\sqrt{2} \end{bmatrix}.$$

Solving this system by back substitution yields $x_3 = -2,\ x_2 = \frac{11}{2},\ x_1 = \frac{9}{2}$. □

7.10.16 Find the standard matrix H for the reflection of R^4 about the hyperplane $\mathbf{a}^\perp = (-1,2,3,1)^\perp$.

Solution. By Formula (16),

$$H = I - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I - \frac{2}{15}\begin{bmatrix} 1 & -2 & -3 & -1 \\ -2 & 4 & 6 & 2 \\ -3 & 6 & 9 & 3 \\ -1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{15} & \frac{4}{15} & \frac{2}{5} & \frac{2}{15} \\ \frac{4}{15} & \frac{7}{15} & -\frac{4}{5} & -\frac{4}{15} \\ \frac{2}{5} & -\frac{4}{5} & -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{15} & -\frac{4}{15} & -\frac{2}{5} & \frac{13}{15} \end{bmatrix}$$

□

7.11.10 Find the coordinate vector of $\mathbf{w} = (4, 3, 0, -2)$ with respect to the orthonormal basis formed by $\mathbf{v}_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, 0)$, $\mathbf{v}_2 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}})$, $\mathbf{v}_3 = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, -\frac{2}{\sqrt{6}})$, $\mathbf{v}_4 = (\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, 0)$.

Solution. $(\mathbf{w} \cdot \mathbf{v}_1, \mathbf{w} \cdot \mathbf{v}_2, \mathbf{w} \cdot \mathbf{v}_3, \mathbf{w} \cdot \mathbf{v}_4) = (\frac{14}{3}, -\frac{\sqrt{3}}{3}, \frac{5\sqrt{6}}{6}, \frac{7\sqrt{2}}{6})$. □

7.11.11 Let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the orthonormal basis for R^2 for which $\mathbf{v}_1 = (\frac{3}{5}, -\frac{4}{5})$, $\mathbf{v}_2 = (\frac{4}{5}, \frac{3}{5})$.

(a) Find the vectors $\mathbf{u}$ and $\mathbf{v}$ that have coordinate vectors $(\mathbf{u})_B$=(1,1) and $(\mathbf{v})_B = (-1, 4)$.

(b) Compute $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\mathbf{u} \cdot \mathbf{v}$ by applying Theorem 7.11.2 to the coordinate vectors $(\mathbf{u})_B$ and $(\mathbf{v})_B$, and then check the results by performing the computations directly with $\mathbf{u}$ and $\mathbf{v}$.

Solution. (a) $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2 = (\frac{7}{5}, -\frac{1}{5})$, $\mathbf{v} = -\mathbf{v}_1 + 4\mathbf{v}_2 = (\frac{13}{5}, \frac{16}{5})$.

(b)

- Using Theorem 7.11.2: $\|\mathbf{u}\| = \|(\mathbf{u})_B\| = \sqrt{2}$, $\|\mathbf{v}\| = \|(\mathbf{v})_B\| = \sqrt{17}$, and $\mathbf{u} \cdot \mathbf{v} = (\mathbf{u})_B \cdot (\mathbf{v})_B = 3$.
- Computing directly: use (a). The result will be the same.

□

7.11.18 Let S be the standard basis for R^3, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis in which $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 5, 0)$, and $\mathbf{v}_3 = (3, 3, 8)$.

(a) Find the transition matrix $P_{B\to S}$ by inspection.

(b) Find the transition matrix $P_{S\to B}$ using row reduction (see Example 7).

(c) Confirm that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Find the coordinate matrix for $\mathbf{w} = (5, -3, 1)$ with respect to B, and use the matrix $P_{B\to S}$ to compute $[\mathbf{w}]_S$ from $[\mathbf{w}]_B$.

(e) Find the coordinate matrix for $\mathbf{w} = (3, -5, 0)$ with respect to S, and use the matrix $P_{S\to B}$ to compute $[\mathbf{w}]_B$ from $[\mathbf{w}]_S$.

Solution. (a) Since S is a standard basis, $P_{B\to S} = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

(b) The matrix $[B|S] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$ has $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ as its reduced row echelon form; thus $P_{S\to B} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

(c) Just multiply them.

(d) Note that v_1, v_2, and v_3 are not orthogonal.

The matrix $[B|\mathbf{w}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 5 & 3 & -3 \\ 1 & 0 & 8 & 1 \end{array}\right]$ has $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -239 \\ 0 & 1 & 0 & 77 \\ 0 & 0 & 1 & 30 \end{array}\right]$ as its reduced row echelon form; thus $[\mathbf{w}]_B = \begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix}$ and $[\mathbf{w}]_S = P_{B\to S}[\mathbf{w}]_B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

(e) $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$ and $[\mathbf{w}]_B = P_{S\to B}[\mathbf{w}]_S = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -200 \\ 64 \\ 25 \end{bmatrix}$

$\square$

7.11.20 Let $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the bases for R^2 in which $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (2, 3)$, $\mathbf{v}_1 = (1, 3)$, and $\mathbf{v}_2 = (1, 4)$.

(a) Find the transition matrix $P_{B_2 \to B_1}$ by row reduction.

(b) Find the transition matrix $P_{B_1 \to B_2}$ by row reduction.

(c) Confirm that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Find the coordinate matrix for $\mathbf{w} = (0, 1)$ with respect to B_1, and use the matrix $P_{B_1 \to B_2}$ to compute $[\mathbf{w}]_{B_2}$ from $[\mathbf{w}]_{B_1}$.

(e) Find the coordinate matrix for $\mathbf{w} = (2, 5)$ with respect to B_2, and use the matrix $P_{B_2 \to B_1}$ to compute $[\mathbf{w}]_{B_1}$ from $[\mathbf{w}]_{B_2}$.

Solution. (a) $\text{rref}[B_1|B_2] = \text{rref}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right]$; thus $P_{B_2 \to B_1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$

(b) Similarly, $\text{rref}[B_2|B_1] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right]$; thus $P_{B_1 \to B_2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

(c) Just multiply them.

(d) $(0, 1) = 2\mathbf{u}_1 - \mathbf{u}_2$; thus $[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $[\mathbf{w}]_{B_2} = P_{B_1 \to B_2}[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(e) $(2, 5) = 3\mathbf{v}_1 - \mathbf{v}_2$; thus $[\mathbf{w}]_{B_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $[\mathbf{w}]_{B_1} = P_{B_2 \to B_1}[\mathbf{w}]_{B_2} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

□

7.11.26 Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the vectors in S are reflected about the line that makes an angle θ with the positive x-axis.

(a) Find the transition matrix $P_{B\to S}$.

(b) Let $P = P_{B\to S}$ and show that $P^T = P_{S\to B}$. Give a geometric explanation of this.

Solution. (a) We have $\mathbf{v}_1 = (\cos 2\theta, \sin 2\theta)$ and $\mathbf{v}_2 = (\sin 2\theta, -\cos 2\theta)$; thus $P_{B\to S} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

(b) Since P is an orthogonal matrix (by Theorem 7.11.7), $P^T = P^{-1} = P_{S\to B}$. Geometrically, this corresponds to the fact that reflection about the line preserves length and thus is an orthogonal transformation. □

7.11.P2 Let B be a basis for R^n. Prove that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span R^n if and only if the vectors $(\mathbf{v}_1)_B, (\mathbf{v}_2)_B, \ldots, (\mathbf{v}_k)_B$ span R^n.

Solution. Let $\mathbf{v} \in R^{\text{n}}$ be a vector, and write $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ for some scalars $c_i \in R$. Since $(\mathbf{v})_B = c_1(\mathbf{v}_1)_B + \cdots + c_k(\mathbf{v}_k)_B$ and the coordinate mapping $\mathbf{v} \to (\mathbf{v})_B$ is bijective, it finishes the proof.

$\square$