4.3.34 (a) Find the area of the triangle having vertices $A(1,0,1)$, $B(0,2,3)$, and $C(2,1,0)$.
(b) use the result of part (a) to find the length of the altitude from vertex C to side AB.

Solution. We first calculate the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}.$$

Then the area of triangle can be calculated as follows:

$$\text{area } \Delta ABC = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{\sqrt{26}}{2}.$$

□

(b) Let h be a length of required altitude from vertex C to side AB. Then, by (a),

$$\text{area } \Delta ABC = \frac{1}{2}\|\overrightarrow{AB}\|h = \frac{3}{2}h \;\Leftrightarrow\; h = \frac{\sqrt{26}}{3}.$$

□

4.3.50 Consider the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{R}^3$. The expression $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

(a) Show that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

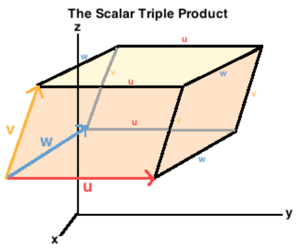
(b) Give a geometric interpretation of $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ (vertical bars denote absolute value).

Solution. (a) We have $\mathbf{v} \times \mathbf{w} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$, thus

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

□

(b) We claim that $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is equal to the volume of the parallelepiped having the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ as adjacent edges.



By the definition of scalar product, $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}||\mathbf{v} \times \mathbf{w}| \cos \phi$ where ϕ is the angle between vector $\mathbf{u}$ and plane $\mathbf{v} \times \mathbf{w}$ (which is both normal to $\mathbf{v}$ and $\mathbf{w}$). Therefore $|\mathbf{v} \times \mathbf{w}|$ is the area of parallelogram and $|\mathbf{u}| \cos \phi$ is the height of parallelepiped with respect to $\mathrm{span}(\mathbf{v}, \mathbf{w})$, i.e., $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped. □

4.3.57 Use Cramer's rule to find a polynomial of degree 3 that passes through the points $(0,1)$, $(1,-1)$, $(2,-1)$, and $(3,7)$.

Solution. The polynomial $p(x) = ax^3 + bx^2 + cx + d$ passes through the points $(0,1)$, $(1,-1)$, $(2,-1)$, and $(3,7)$ if and only if

$$\begin{cases} d = 1 \\ a + b + c + d = -1 \\ 8a + 4b + 2c + d = -1 \\ 27a + 9b + 3c + d = 7 \end{cases}$$

which corresponds to the augmented matrix $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 8 & 4 & 2 & 1 & -1 \\ 27 & 9 & 3 & 1 & 7 \end{array}\right]$. Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix}$.

By Cramer's rule, the solution of this system is given by

$$a = \frac{\begin{vmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 4 & 2 & 1 \\ 7 & 9 & 3 & 1 \end{vmatrix}}{\det(A)} = \frac{12}{12} = 1 \qquad b = \frac{\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 8 & -1 & 2 & 1 \\ 27 & 7 & 3 & 1 \end{vmatrix}}{\det(A)} = \frac{-24}{12} = -2$$

$$c = \frac{\begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 8 & 4 & -1 & 1 \\ 27 & 9 & 7 & 1 \end{vmatrix}}{\det(A)} = \frac{-12}{12} = -1 \qquad d = \frac{\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 8 & 4 & 2 & -1 \\ 27 & 9 & 3 & 7 \end{vmatrix}}{\det(A)} = \frac{12}{12} = 1.$$

Thus the interpolating polynomial is $p(x) = x^3 - 2x^2 - x + 1$. □

4.3.P1 Prove that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero, nonorthogonal vectors in $\mathbb{R}^3$, and θ is the angle between them, then

$$\tan\theta = \frac{\|\mathbf{u}\times\mathbf{v}\|}{(\mathbf{u}\cdot\mathbf{v})}.$$

Solution. Recall Theorem 1.2.8 and Theorem 4.3.10(a). We know that

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$
$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta.$$

Then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\|\mathbf{u}\times\mathbf{v}\|}{(\mathbf{u}\cdot\mathbf{v})}$ follows directly. □

4.3.P2 Prove that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^2$ and α and β are the angles in the accompanying figure, then

$$\cos(\alpha - \beta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

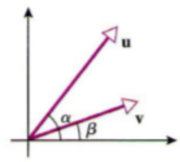


Figure Ex-P2

Solution. The angle between the vectors is $\theta = \alpha - \beta$; thus

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\alpha - \beta) \quad \Leftrightarrow \quad \cos(\alpha - \beta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

$\square$

4.4.26 For what value(s) of x, if any, will the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & x & 2 \\ 0 & 2 & x \end{bmatrix}$$

have at least one repeated eigenvalue?

Solution. The characteristic polynomial of A is

$$p(\lambda) = \begin{vmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda - x & -2 \\ 0 & -2 & \lambda - x \end{vmatrix} = (\lambda - 3)(\lambda^2 - 2\lambda x + x^2 - 4).$$

Note that the second factor in this polynomial cannot have a double root (for any value of x) since the discriminant $(-2x)^2 - 4(x^2 - 4) = 16 \neq 0$. Thus the only possible repeated eigenvalue of A is $\lambda = 3$, and this occurs if and only if $\lambda = 3$ is a root of the second factor of $p(\lambda)$, i.e., if and only if $9 - 6x + x^2 - 4 = 0$. The roots of this quadratic equation are $x = 1$ or $x = 5$. For these values of x, $\lambda = 3$ is an eigenvalue of multiplicity 2. $\square$

4.4.27 Show that if A is a 2×2 matrix such that $A^2 = I$ and if $\mathbf{x}$ is any vector in $\mathbb{R}^2$, then $\mathbf{y} = \mathbf{x} + A\mathbf{x}$ and $\mathbf{z} = \mathbf{x} - A\mathbf{x}$ are eigenvectors of A. Find the corresponding eigenvalues.

Solution. If $A^2 = I$, then $A(\mathbf{x} + A\mathbf{x}) = A\mathbf{x} + A^2\mathbf{x} = A\mathbf{x} + \mathbf{x} = \mathbf{x} + A\mathbf{x}$; thus $\mathbf{y} = \mathbf{x} + A\mathbf{x}$ is an eigenvector of A corresponding to $\lambda = 1$. Similarly, $A(\mathbf{x} - A\mathbf{x}) = A\mathbf{x} - A^2\mathbf{x} = A\mathbf{x} - \mathbf{x} = -(\mathbf{x} - A\mathbf{x})$ implies $\mathbf{z} = \mathbf{x} - A\mathbf{x}$ is an eigenvector of A corresponding to $\lambda = -1$. $\square$

4.4.32 Show that if λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, then

$$\lambda = \frac{(A\mathbf{x}) \cdot \mathbf{x}}{\|\mathbf{x}\|^2}.$$

Solution. If $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq 0$, then

$$(A\mathbf{x}) \cdot \mathbf{x} = (\lambda\mathbf{x}) \cdot \mathbf{x} = \lambda(\mathbf{x} \cdot \mathbf{x}) = \lambda\|\mathbf{x}\|^2$$

so $\lambda = \frac{(A\mathbf{x}) \cdot \mathbf{x}}{\|\mathbf{x}\|^2}$. □

4.4.33 (a) Show that the characteristic polynomial of the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

is $p(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n$. [*Hint*: Evaluate all required determinants by adding a suitable multiple of the second row to the first to introduce a zero at the top of the first column, and then expand by cofactors along the first column. Then repeat the process].

(b) The matrix in part (a) is called the ***companion matrix*** of the polynomial

$$p(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n.$$

Thus we see that if $p(\lambda)$ is any polynomial whose highest power has a coefficient of 1, then there is some matrix whose characteristic polynomial is $p(\lambda)$, namely its companion matrix. Use this observation to find a matrix whose characteristic polynomial is $p(\lambda) = 2 - 3\lambda + \lambda^2 - 5\lambda^3 + \lambda^4$.

Solution. (a) The characteristic polynomial of the matrix C is

$$p(\lambda) = \det(\lambda I - C) = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & c_0 \\ -1 & \lambda & 0 & \cdots & 0 & c_1 \\ 0 & -1 & \lambda & \cdots & 0 & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \lambda + c_{n-1} \end{vmatrix}.$$

Add λ times the second row to the first row, then expand by cofactors along the first column:

$$p(\lambda) = \begin{vmatrix} 0 & \lambda^2 & 0 & \cdots & 0 & c_0 + c_1\lambda \\ -1 & \lambda & 0 & \cdots & 0 & c_1 \\ 0 & -1 & \lambda & \cdots & 0 & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \lambda + c_{n-1} \end{vmatrix} = \begin{vmatrix} \lambda^2 & 0 & \cdots & 0 & c_0 + c_1\lambda \\ -1 & \lambda & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda + c_{n-1} \end{vmatrix}.$$

Add λ^2 times the second row to the first row, then expand by cofactors along the first column:

$$p(\lambda) = \begin{vmatrix} 0 & \lambda^3 & \cdots & 0 & c_0 + c_1\lambda + c_2\lambda^2 \\ -1 & \lambda & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda + c_{n-1} \end{vmatrix} = \begin{vmatrix} \lambda^3 & \cdots & 0 & c_0 + c_1\lambda + c_2\lambda^2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & \lambda + c_{n-1} \end{vmatrix}.$$

Continuing in this fashion for $n-2$ steps, we obtain

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} \lambda^{n-1} & c_0 + c_1\lambda + \cdots + c_{n-2}\lambda^{n-2} \\ -1 & \lambda + c_{n-1} \end{vmatrix} \\ &= c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n. \end{aligned}$$

□

(b) The matrix $C = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ has $p(\lambda) = 2 - 3\lambda + \lambda^2 - 5\lambda^3 + \lambda^4$ as its characteristic polynomial. □

4.4.D4 If $p(\lambda) = (\lambda-3)^2(\lambda+2)^3$ is the characteristic polynomial of a matrix A, then $\det(A) =$ ________ and $\operatorname{tr}(A) =$ ________.

Solution. The eigenvalues of A (with multiplicity) are 3, 3 and -2, -2, -2. Thus, from Theorem 4.4.12, we have

$$\begin{aligned} \det(A) &= (3)(3)(-2)(-2)(-2) = -72 \\ \operatorname{tr}(A) &= 3 + 3 + (-2) + (-2) + (-2) = 0. \end{aligned}$$

□

4.4.D8 Indicate whether the statement is true (T) or false (F). Justify your answer. In each part, A is assumed to be square.

(a) The eigenvalues of A are the same as the eigenvalues of the reduced row echelon form of A.

(b) If eigenvectors corresponding to distinct eigenvalues are added, then the resulting vector is not an eigenvector of A.

(c) A 3×3 matrix has at least one real eigenvalue.

(d) If the characteristic polynomial of A is $p(\lambda) = \lambda^n + 1$, then A is invertible.

Solution. (a) False. For example, the reduced row echelon form of $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which has the only eigenvalue 1 while A has 1 and 2. $\square$

(b) True. We have $A(\mathbf{x}_1 + \mathbf{x}_2) = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2$, and if $\lambda_1 \neq \lambda_2$ it can be shown (since $\mathbf{x}_1$ and $\mathbf{x}_2$ must be linearly independent) that $\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 \neq \beta(\mathbf{x}_1 + \mathbf{x}_2)$. $\square$

(c) True. The characteristic polynomial of A is a cubic polynomial, and every cubic polynomial has at least one real root. $\square$

(d) True. If $p(\lambda) = \lambda^n + 1$, then $\det(A) = (-1)^n p(0) = \pm 1 \neq 0$; thus A is invertible. $\square$

4.4.P2 (a) Prove that if A is a square matrix, then A and A^T have the same characteristic polynomial. [*Hint*. Consider the characteristic equation $\det(\lambda I - A) = 0$ and use properties of the determinant.]

(b) Show that A and A^T need not have the same eigenspaces by considering the matrix.

$$\begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$$

Solution. (a) Using previously established properties, we have

$$\det(\lambda I - A^T) = \det(\lambda I^T - A^T) = \det((\lambda I - A)^T) = \det(\lambda I - A).$$

Thus A and A^T have the same characteristic polynomial. $\square$

(b) The eigenvalues are 2 and 3 in each case. The eigenspace of A corresponding to $\lambda = 2$ is obtained by solving the system $\begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; whereas the eigenspace of A^T corresponding to $\lambda = 2$ is obtained by solving $\begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus the eigenspace of A corresponding to $\lambda = 2$ is the line $y = -2x$, whereas the eigenspace of A^T corresponding to $\lambda = 2$ is $y = 0$. Similarly, for $\lambda = 3$, the eigenspace of A is $x = 0$, whereas the eigenspace of A^T is $y = \frac{1}{2}x$. $\square$

4.4.P3 Prove that if λ is an eigenvalue of an invertible matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1}, and $\mathbf{x}$ is a corresponding eigenvector. [*Hint*: Begin with the equation $A\mathbf{x} = \lambda\mathbf{x}$.]

Solution. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq 0$ and A is invertible. Then $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x} = \lambda A^{-1}\mathbf{x}$ and since $\lambda \neq 0$ (because A is invertible), it follows that $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. Thus $1/\lambda$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector. $\square$