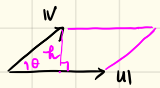
* Geometric interpretation of determinants

Thm 4.3.5 (just as a numerical value ;
we do not consider the unit of area or volume)

(a) For $u, v \in \mathbb{R}^2$, the area of the parallelogram determined by u and v is $|\det(A)|$, where $A = [u \;\; v]$.

(b) For $u, v, w \in \mathbb{R}^3$, the volume of the parallepiped determined by u, v, and w is $|\det(A)|$, where $A = [u \;\; v \;\; w]$.

pf) (a) Assume that u and v are linearly independent.



$h = \|v\| \sin\theta$

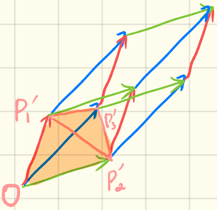
$$(\text{Area})^2 = \|u\|^2 \|v\|^2 \sin^2\theta = \|u\|^2\|v\|^2 - (u \cdot v)^2$$
$$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2$$
$$= (x_1y_2 - x_2y_1)^2 = (\det(A))^2.$$
$$\therefore \text{Area} = |\det(A)|.$$

(b) Exercise. $\left(\text{Hint. } \det[u \;\; v \;\; w] = \det\begin{bmatrix} - u - \\ - v - \\ - w - \end{bmatrix} = u \cdot (v \times w) \right)$

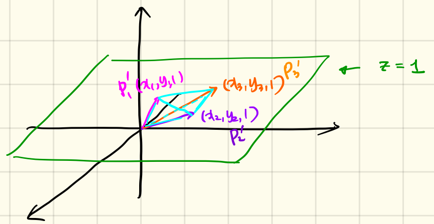
Thm 4.3.6 Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$ be lie on $\mathbb{R}^2$.

The area of the triangle $\triangle P_1P_2P_3$ is

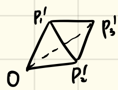
$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



pf)



The volume of the trigonal pyramid



is

$$\frac{1}{3}\triangle P_1P_2P_3 = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$\square$

If n points in the xy-plane have distinct x-coordinates, then there exists a unique polynomial of degree $\leq n-1$ whose graph passes through those points.

pf) Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n) \in \mathbb{R}^2$.

Suppose that

$\exists\ y = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ s.t. $y_i = a_0 + a_1 x_i + \cdots + a_{n-1} x_i^{n-1}$ $(i=1,\cdots,n)$

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad (*)$$

The determinant of this matrix is $\prod_{1 \leq i < j \leq n} (x_i - x_j)$, called the Vandermonde determinant.

$\therefore$ If x_i's are distinct, then we can find the solution of the linear system $(*)$. $\boxed{}$

One can prove it using induction on n.

§4.4. A First look at Eigenvalues and Eigenvectors.

Recall that for an $n\times n$ matrix A,

$\mathbf{x}$ is a fixed point of A if $A\mathbf{x}=\mathbf{x}$.

<u>Thm 4.4.1</u> $A: n\times n$. T.F.A.E.

(a) A has nontrivial fixed points

(b) $I-A$ is singular.

(c) $\det(I-A)=0$.

e.g.) $A=\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \forall t\in\mathbb{R},\ \begin{bmatrix} 2t \\ t \end{bmatrix}$ is a fixed point of A.

<u>Problem</u> If A is an $n\times n$ matrix, for what values of the scalar λ, are there nonzero vectors in $\mathbb{R}^n$ s.t. $A\mathbf{x}=\lambda\mathbf{x}$?

<u>Def.</u> $A: n\times n$

λ: eigenvalue of A if $\exists\ \mathbf{x}\neq\mathbf{0}$ s.t. $A\mathbf{x}=\lambda\mathbf{x}$

$\uparrow$ eigenvector of A corr. to λ.

In the above example, $\lambda=1$ is an eigenvalue of A and

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corr. to $\lambda=1$.

How can we solve the problem above?

Note that $A\mathbf{x}=\lambda\mathbf{x} \Leftrightarrow (\lambda I-A)\mathbf{x}=\mathbf{0}$

$\exists\ \mathbf{x}\neq\mathbf{0}$ s.t. $A\mathbf{x}=\lambda\mathbf{x} \Leftrightarrow (\lambda I-A)\mathbf{x}=\mathbf{0}$ has nontrivial solutions.

$\Leftrightarrow \lambda I-A$ is singular.

$\Leftrightarrow \det(\lambda I-A)=0$.

$\therefore$ The eigenvalues of A are the solutions of the equation $\det(\lambda I-A)=0$.

We call $\det(\lambda I-A)=0$ the <u>characteristic equation of A</u>.

We call $\det(\lambda I-A)$ the <u>characteristic polynomial</u> of A.

Thm 4.4.4 A : $n\times n$ matrix, λ : scalar. TFAE

(a) λ is an eigenvalue of A.

(b) λ is a solution of $\det(\lambda I - A) = 0$.

(c) $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

Def. The solution space of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the eigenspace of A corr. to. λ.

e.g.) (Continued) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -2 \\ 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)$

$\therefore$ $\lambda = 0$ & $\lambda = 1$ are eigenvalues of A.

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (t \in \mathbb{R})$$

$\therefore$ $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is the eigenspace of A corr. to $\lambda = 0$

$\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ is the eigenspace of A corr. to $\lambda = 1$.

Thm 4.4.5 If A is triangular, then the eigenvalues of A are the entries on the main diagonal of A.

Thm 4.4.6 λ : an eigenvalue of $A \Rightarrow \lambda^k$: an eigenvalue of A^k

Note $\lambda = 0$ is an eigenvalue of $A \iff A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

Thm 4.4.7 A : invertible $\iff \lambda = 0$ is not an eigenvalue of A.

Rmk A : invertible & $\lambda_1, \cdots, \lambda_k$: eigenvalues of $A \Rightarrow \frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_k}$: eigenvalues of A^{-1}.

Note The characteristic equation of a square matrix with real entries can have imaginary solutions.

e.g.) $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} \Rightarrow \det(\lambda I - A) = \lambda^2 + 1.$

$\Rightarrow$ The eigenvector corr. to $\lambda = i$ is $t\begin{bmatrix} -1 \\ i+2 \end{bmatrix}$ $(t \in \mathbb{C})$

" " " " $\lambda = -i$ " $t\begin{bmatrix} 1 \\ i-2 \end{bmatrix}$ $(t \in \mathbb{C})$

In this case, we consider A as a matrix with complex entries since $\mathbb{R} \subseteq \mathbb{C}$. §8.8 contains complex eigenvalues and complex eigenvectors. However, in this course, we do not cover §8.8. In Linear algebra, we will study vector spaces over an arbitrary field.

If we allow complex eigenvalues, then from the fundamental theorem of algebra, we obtain the following.

Thm 4.4.8 A: an $n \times n$ matrix

$$\Rightarrow \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

where $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues and $m_1 + \cdots + m_k = n$.

Each m_i is called the algebraic multiplicity of λ_i.

e.g.) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(\lambda I - A) = (\lambda - 1)^2.$

$\therefore$ The algebraic multiplicity of $\lambda = 1$ is 2.

The eigenspace of $\lambda = 1$ is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

In general, the dimension of the eigenspace of $\lambda = \lambda_i$ is less than or equal to the algebraic multiplicity of $\lambda = \lambda_i$.

Thm 4.4.12

Let A be an $n\times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. (repeated according to multiplicity). Then

(1) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

(2) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

pf) (1) Since $\det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$,

setting $\lambda = 0$ yeilds $\det(-A) = (-1)^n \lambda_1 \cdots \lambda_n$

$$\det(-A) = (-1)^n \det(A)$$

$\therefore \det(A) = \lambda_1 \cdots \lambda_n$

(2) Let $A = [a_{ij}]_{n\times n}$. Then

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}.$$

If an elementary product from $\det(\lambda I - A)$ contains $-a_{ij}$, $i \neq j$, as a factor, then it is a polynomial of degree at most $(n-2)$.

$\therefore$ The coeff. of λ^{n-1} in $\det(\lambda I - A)$ is the coeff. of λ^{n-1} in $(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$.

$\therefore \quad -(\lambda_1 + \cdots + \lambda_n) = -(a_{11} + a_{22} + \cdots + a_{nn})$.

$\therefore \quad tr(A) = \lambda_1 + \cdots + \lambda_n$ $\blacksquare$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\lambda I - A) = \lambda^2 - tr(A)\lambda + \det(A)$.

$\therefore$ A has real eigenvalues iff $D = tr(A)^2 - 4\det(A) \geq 0$.

Thm 4.4.10 A symmetric 2×2 matrix with real entries has real eigenvalues. The matrix A has repeated eigenvalues iff $A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

pf) Let $A=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $a,b,d\in\mathbb{R}$.

$$D=(a+d)^2-4(ad-b^2)=(a-d)^2+4b^2\geq 0.$$

$$D=0 \iff b=0 \ \& \ a=d. \quad \blacksquare$$

Thm 4.4.11

(a) $A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \Rightarrow$ The eigenspace corr. to $\lambda=a$ is $\mathbb{R}^2$.

(b) $A=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ $(b\neq 0 \text{ or } a\neq d)$

$\Rightarrow$ The eigenspaces corr. to eigenvalues are perpendicular lines through the origin of $\mathbb{R}^2$.

pf) (a) $aI-A=O$.

(b) If $b=0$, then the eigenspace corr. to $\lambda=a$ is x-axis, and

〃 〃 〃 〃 $\lambda=b$ 〃 y-axis.

If $b\neq 0$, then 〃 〃 〃 〃 $\lambda=\lambda_i$ 〃 $(\lambda_i-a)x+by=0$.

$$\left(\left(1, \frac{\lambda_1-a}{b}\right)\cdot\left(1, \frac{\lambda_2-a}{b}\right) = 1+\frac{\overset{ad-b^2}{\lambda_1\lambda_2}-a\overset{a+d}{(\lambda_1+\lambda_2)}+a^2}{b^2} = 1+\frac{(ad-b^2)-a(a+d)+a^2}{b^2} = 1-1=0. \right)$$

§ 6.1. Matrices as transformations.

Def. A <u>transformation</u> is a function whose domain and codomain are the set of vectors, and it is denoted by capital letters such as F, T, or L.

If T is a transformation that maps $\mathbf{x}$ into $\mathbf{w}$, then $\mathbf{w} = T(\mathbf{x})$ can be written as $\mathbf{x} \xrightarrow{T} \mathbf{w}$.

If T is a transformation whose domain is $\mathbb{R}^n$ and codomain is $\mathbb{R}^m$, then we will write $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

If $T: \mathbb{R}^n \to \mathbb{R}^n$, then T is called an <u>operator</u> on $\mathbb{R}^n$.

A matrix transformation is a transformation induced by a matrix.

For an $m \times n$ matrix A, the matrix transformation T_A is

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \& \quad \mathbf{x} \xrightarrow{T_A} A\mathbf{x}.$$

If A is $n \times n$, then T_A is called a matrix operator on $\mathbb{R}^n$.

E.g.) ① 0 : $m \times n$ zero matrix

$\Rightarrow$ T_0 is the zero transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

② I : $n \times n$ identity matrix

$\Rightarrow$ T_I is the identity operator on $\mathbb{R}^n$.

$\boxed{\text{Note}}$ For an $m \times n$ matrix,

$\boxed{\text{Solve a linear system } A\mathbf{x} = \mathbf{b}}$

$= \boxed{\text{Find } \mathbf{x} \in \mathbb{R}^n \text{ whose image under } T_A \text{ is } \mathbf{b} \in \mathbb{R}^n}$