8.3.14 Let $A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$. Find a matrix P that orthogonally diagonalizes A, and determine the diagonal matrix $D = P^T AP$.

Solution. The characteristic polynomial of A is $p(\lambda) = (\lambda - 25)^2(\lambda + 25)^2$. The vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = 25$, and the vectors $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = -25$. These four vectors are mutually orthogonal, so the orthogonal matrix $P = \begin{bmatrix} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} & \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} & \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} & \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} \end{bmatrix}$ has the property that

$$P^T AP = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & -25 \end{bmatrix} = D.$$

□

8.3.18 Find the spectral decomposition of the matrix $A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$.

Solution. The eigenvalues of A are $\lambda_1 = -3$, $\lambda_2 = 25$, and $\lambda_3 = -50$ with corresponding normalized vectors $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}$ respectively. Thus a spectral decomposition of A is

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \lambda_3 \mathbf{u}_3 \mathbf{u}_3^T = -3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 25 \begin{bmatrix} \frac{16}{25} & 0 & -\frac{12}{25} \\ 0 & 0 & 0 \\ -\frac{12}{25} & 0 & \frac{9}{25} \end{bmatrix} - 50 \begin{bmatrix} \frac{9}{25} & 0 & \frac{12}{25} \\ 0 & 0 & 0 \\ \frac{12}{25} & 0 & \frac{16}{25} \end{bmatrix}.$$

$\square$

8.3.23 Consider the matrix $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{bmatrix}$.

(a) Verify that A satisfies its characteristic equation, as guaranteed by the Cayley-Hamilton theorem.

(b) Find an expression for A^4 in terms of A^2, A, and I, and use that expression to evaluate A^4.

(c) Find an expression for A^{-1} in terms of A^2, A, and I.

Solution. (a) The characteristic polynomial of A is $p(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 8$. On the other hand, we have $A^2 = \begin{bmatrix} 8 & -8 & 4 \\ 8 & -12 & 8 \\ 12 & -24 & 16 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 20 & -24 & 12 \\ 24 & -40 & 24 \\ 36 & -72 & 44 \end{bmatrix}$; so

$$A^3 - 6A^2 + 12A - 8I = O.$$

(b) Since $A^3 = 6A^2 - 12A + 8I$, we have $A^4 = 6A^3 - 12A^2 + 8A = 24A^2 - 64A + 48I$.

(c) Since $A^3 - 6A^2 + 12A = 8I$, we have $A(A^2 - 6A + 12I) = 8I$ and $A^{-1} = \frac{1}{8}(A^2 - 6A + 12I)$.

□

8.3.D2 (a) Find a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1,\ \lambda_2 = 3,\ \lambda_3 = 7$ and corresponding eigenvectors $\mathbf{v}_1 = (0,1,-1),\ \mathbf{v}_2 = (1,0,0),\ \mathbf{v}_3 = (0,1,1)$.

(b) Is there a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1,\ \lambda_2 = 3,\ \lambda_3 = 7$ and corresponding eigenvectors $\mathbf{v}_1 = (0,1,-1),\ \mathbf{v}_2 = (1,0,0),\ \mathbf{v}_3 = (1,1,1)$? Explain your reasoning.

Solution. (a) Note that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are mutually orthogonal. Let $P = \begin{bmatrix} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} & \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} & \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} \end{bmatrix}$. We then have

$$A = PDP^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}, \text{ where } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

(b) No, since $\mathbf{v}_2$ and $\mathbf{v}_3$ are not orthogonal.

□

8.3.P2 If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for R^n, and if A can be expressed as

$$A = c_1\mathbf{u}_1\mathbf{u}_1^T + c_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + c_n\mathbf{u}_n\mathbf{u}_n^T$$

then A is symmetric and has eigenvalues $c_1, c_2, \ldots, c_n$.

Solution. Since $(\mathbf{u}_j\mathbf{u}_j^T)^T = \mathbf{u}_j^{TT}\mathbf{u}_j^T = \mathbf{u}_j\mathbf{u}_j^T$, it follows that $A^T = A$; thus A is symmetric. Furthermore, since $\mathbf{u}_i^T\mathbf{u}_j = \mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$, we have

$$A\mathbf{u}_j = (c_1\mathbf{u}_1\mathbf{u}_1^T + c_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + c_n\mathbf{u}_n\mathbf{u}_n^T)\mathbf{u}_j = c_j\mathbf{u}_j$$

for each $j = 1, 2, \ldots, n$. Thus $c_1, c_2, \ldots, c_n$ are eigenvalues of A. □

8.3.P3 Prove that if A is a symmetric matrix whose spectral decomposition is

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

then

$$f(A) = f(\lambda_1) \mathbf{u}_1 \mathbf{u}_1^T + f(\lambda_2) \mathbf{u}_2 \mathbf{u}_2^T + \cdots + f(\lambda_n) \mathbf{u}_n \mathbf{u}_n^T.$$

Solution. Let $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$, so that $A = PDP^T$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Therefore,

$$\begin{aligned} f(A) &= Pf(D)P^T = P \,\mathrm{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)) P^T \\ &= f(\lambda_1) \mathbf{u}_1 {\mathbf{u}_1}^T + f(\lambda_2) \mathbf{u}_2 {\mathbf{u}_2}^T + \cdots + f(\lambda_n) \mathbf{u}_n {\mathbf{u}_\mathrm{n}}^T. \end{aligned}$$

$\square$

8.3.P4 (a) Assume that A is a symmetric $n \times n$ matrix. One way to prove that A is diagonalizable is to show that for each eigenvalue λ_0 the geometric multiplicity is equal to the algebraic multiplicity. For this purpose, assume that the geometric multiplicity of λ_0 is k, let $B_0 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for the eigenspace corresponding to λ_0, extend this to an orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for R^n, and let P be the matrix having the vectors of B as columns. As shown in Exercise P6(b) of Section 8.2, the product AP can be written as

$$AP = P \begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

Use the fact that B is orthonormal basis to prove that $X = 0$ [a zero matrix of size $n \times (n-k)$].

(b) It follows from part (a) and Exercise P6(c) of Section 8.2 that A has the same characteristic polynomial as

$$C = \begin{bmatrix} \lambda_0 I_k & 0 \\ 0 & Y \end{bmatrix}.$$

Use this fact and Exercise P6(d) of Section 8.2 to prove that the algebraic multiplicity of λ_0 is the same as the geometric multiplicity of λ_0. This establishes that A is diagonalizable.

(c) Use part (b) of Theorem 8.3.4 and the fact that A is diagonalizable to prove that A is orthogonally diagonalizable.

Solution. (a) Since $P^TAP = \begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$ and P^TAP is symmetric, we have $X = 0$.

(b) The characteristic polynomial of C is $(\lambda - \lambda_0)^k p_Y(\lambda)$ where $p_Y(\lambda)$ is the characteristic polynomial of Y. We will now prove that $p_Y(\lambda_0) \neq 0$ and so that the algebraic multiplicity of λ_0 is exactly k. The proof is by contradiction:

Suppose $p_Y(\lambda_0) = 0$, then there is a nonzero vector $\mathbf{y}$ in R^{n-k} such that $Y\mathbf{y} = \lambda_0 \mathbf{y}$. Let $\mathbf{x} = \begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix}$ be the vector in R^n whose first k components are 0 and whose last $n-k$ components are those of $\mathbf{y}$. Then

$$C\mathbf{x} = \begin{bmatrix} \lambda_0 I_k & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_0 \mathbf{y} \end{bmatrix} = \lambda_0 \mathbf{x}$$

and so $\mathbf{x}$ is an eigenvector of C corresponding to λ_0. Since $AP = PC$, it follows that $P\mathbf{x}$ is an eigenvector of A corresponding to λ_0. But note that $\mathbf{e}_1, \ldots, \mathbf{e}_k$ are also eigenvectors of C corresponding to λ_0, and that the set $\{\mathbf{e}_1, \ldots \mathbf{e}_k, \mathbf{x}\}$ is linearly independent. It follows that the set $\{P\mathbf{e}_1, \ldots, P\mathbf{e}_k, P\mathbf{x}\}$ consists of linearly independent eigenvectors of A corresponding to λ_0. However, this implies that the geometric multiplicity of λ_0 is greater than k, a contradiction.

(c) It follows from part (b) that A is diagonalizable. Since A is symmetric, by Theorem 8.3.4.(b), eigenvectors of A from different eigenspaces are orthogonal. Thus we can form an orthonormal basis for R^n by choosing an orthonormal basis for each of the eigenspaces and joining them together. Therefore, A is orthogonally diagonalizable.

$\square$

8.4.8 Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form

$$Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3,$$

and express Q in terms of the new variables.

Solution. The given quadratic form can be expressed as $Q = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$.

The characteristic polynomial of A is $p(\lambda) = (\lambda - 1)^2(\lambda - 10)$. The vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = 1$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 10$. Application of the Gram-Schmidt process to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ produces orthonormal eigenvectors $\mathbf{p}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} -\frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$. Thus the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}$ orthogonally diagonalizes A and the change of variable $\mathbf{x} = P\mathbf{y}$ eliminates the cross product terms in Q:

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T (P^T AP)\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + y_2^2 + 10y_3^2.$$

□

8.4.10 Write the quadratic equation in the matrix form $\mathbf{x}^T A\mathbf{x} + K\mathbf{x} + f = 0$, where $\mathbf{x}^T A\mathbf{x}$ is the associated quadratic form and K is an appropriate matrix:

(a) $x^2 - xy + 5x + 8y - 3 = 0$

(b) $5xy = 8$

Solution. (a) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [5 \quad 8] \begin{bmatrix} x \\ y \end{bmatrix} - 3 = 0$

(b) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & \frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 8 = 0$

□

8.4.28 Express the symmetric positive definite matrices in the form $A = B^2$, where B is symmetric and positive definite, and where

(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Solution. (a) The matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ orthogonally diagonalizes A, and the matrix

$$B = P \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix} P^T = \begin{bmatrix} \frac{1+\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} \\ \frac{-1+\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} \end{bmatrix}$$

has the property that $B^2 = A$.

(b) The matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Thus $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A, and

$$B = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} P^T = \begin{bmatrix} \frac{5}{6} + \frac{\sqrt{3}}{2} & -\frac{1}{3} & \frac{5}{6} - \frac{\sqrt{3}}{2} \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\ \frac{5}{6} - \frac{\sqrt{3}}{2} & -\frac{1}{3} & \frac{5}{6} + \frac{\sqrt{3}}{2} \end{bmatrix}$$

has the property that $B^2 = A$.

□

8.4.31 Consider the matrix $A = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}$.

(a) Show that the matrix A is positive definite and find a symmetric positive definite matrix B such that $A = B^2$.

(b) Find an invertible upper triangular matrix C such that $A = C^T C$.

Solution. (a) The matrix A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 15$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus A is positive definite, the matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ orthogonally diagonalizes A, and the matrix

$$B = P \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{15} \end{bmatrix} P^T = \begin{bmatrix} \frac{\sqrt{3}}{2} + \frac{\sqrt{15}}{2} & -\frac{\sqrt{3}}{2} + \frac{\sqrt{15}}{2} \\ -\frac{\sqrt{3}}{2} + \frac{\sqrt{15}}{2} & \frac{\sqrt{3}}{2} + \frac{\sqrt{15}}{2} \end{bmatrix}$$

has the property that $B^2 = A$.

(b) The LDU-decomposition (p159-160) of the matrix A is

$$A = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix} = LDU.$$

We may write $\sqrt{D} = \begin{bmatrix} 3 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$ and, since $L = U^T$, we have

$$A = (L\sqrt{D})(\sqrt{D}U) = (\sqrt{D}U)^T(\sqrt{D}U) = C^T C$$

where $C = \sqrt{D}U$ is invertible and upper-triangular.

□

8.4.34 In statistics the quantities

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n), \text{ and } s_x^2 = \frac{1}{n-1}[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2]$$

are called, respectively, the sample mean and sample variance of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

(a) Express the quadratic form s_x^2 in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where A is symmetric.

(b) Is s_x^2 a positive definite quadratic form? Explain.

Solution. (a) In the formula of s_x^2, the coefficient of x_i^2 is $\dfrac{1}{n-1}(1 - \dfrac{2}{n} + \dfrac{1}{n^2}n) = \dfrac{1}{n}$, and the coefficient of $x_i x_j$ for $i \neq j$ is $\dfrac{1}{n-1}(-\dfrac{2}{n} - \dfrac{2}{n} + \dfrac{2}{n^2}n) = -\dfrac{2}{n(n-1)}$. If follows that $s_x^2 = \mathbf{x}^T A\mathbf{x}$ where

$$A = \begin{bmatrix} \frac{1}{n} & -\frac{1}{n(n-1)} & \cdots & -\frac{1}{n(n-1)} \\ -\frac{1}{n(n-1)} & \frac{1}{n} & \cdots & -\frac{1}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n(n-1)} & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \end{bmatrix}.$$

(b) We have $s_x^2 \geq 0$ from the formula, and $s_x^2 = 0$ if and only if $x_i = \bar{x}$ for all i, i.e., if and only if $x_1 = x_2 = \cdots = x_n$. Therefore, s_x^2 is not positive definite, but positive semidefinite.

□

8.4.D1 Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) A symmetric matrix with positive entries is positive definite.

(b) $x_1^2 - x_2^2 + x_3^2 + 4x_1x_2x_3$ is a quadratic form.

(c) $(x_1 - 3x_2)^2$ is a quadratic form.

(d) A positive definite matrix is invertible

(e) A symmetric matrix is either positive definite, negative definite, or indefinite.

(f) If A is positive definite, then $-A$ is negative definite.

Solution. (a) False. The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has eigenvalues -1 and 3.

(b) False. The term $4x_1x_2x_3$ is not quadratic in the variables x_1, x_2, x_3.

(c) True. Easy.

(d) True. The determinant of a positive definite matrix is also positive.

(e) False. For example the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is positive semidefinite.

(f) True. λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of $-A$.

□

8.4.D2 Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) If $\mathbf{x}$ is a vector in R^n, then $\mathbf{x} \cdot \mathbf{x}$ is a quadratic form.

(b) If $\mathbf{x}^T A\mathbf{x}$ is a positive definite quadratic form, then so is $\mathbf{x}^T A^{-1}\mathbf{x}$

(c) If A is a matrix with positive eigenvalues, then $\mathbf{x}^T A\mathbf{x}$ is a positive definite quadratic form.

(d) If A is a symmetric 2×2 matrix with positive entries and a positive determinant, then A is positive definite.

(e) If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form with no cross product terms, then A is a diagonal matrix.

(f) If $\mathbf{x}^T A\mathbf{x}$ is a positive definite quadratic form in x and y, and if $c \neq 0$, then the graph of the equation $\mathbf{x}^T A\mathbf{x} = c$ is an ellipse.

Solution. (a) True. $\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T I\mathbf{x}$.

(b) True. A^{-1} has also positive eigenvalues.

(c) True. See Theorem 8.4.3(a).

(d) True. The eigenvalues of A is $\dfrac{\mathrm{tr}(A) \pm\sqrt{\mathrm{tr}(A)^2 - 4\,\det(A)}}{2}$ and it is easy to see that these values are greater than 0. Or we can just apply Theorem 8.4.5.

(e) False. If $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $\mathbf{x}^T A\mathbf{x} = x^2 + y^2$ where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. On the other hand, the statement is true if A is assumed to be symmetric.

(f) False. If $c < 0$ the graph is empty.

□