1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+4+3 points

(a) Let W be a subspace of $\mathbb{R}^n$. Let M be a $n \times k$ matrix whose column vectors form an orthonormal basis for W. Then for any $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\| = \|MM^T\mathbf{v}\|$.

(b) Let M be a $n \times n$ matrix whose column vectors form an orthonormal basis for $\mathbb{R}^n$. Then for any $\mathbf{v} \in \mathbb{R}^n$, $\|M\mathbf{v}\| = \|M^T\mathbf{v}\|$.

(c) Let $\mathbf{v_1} = (a, 0, 0)$, $\mathbf{v_2} = (b, b, 0)$, and $\mathbf{v_3} = (c, c, c)$ with $abc \neq 0$. Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis for $\mathbb{R}^3$. Then $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ can be induced by applying the Gram-Schmidt process to $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

Solution.

(a) FALSE. Choose $M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $MM^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. So, $\|\mathbf{v}\| = 1$, but $\|MM^T\mathbf{v}\| = 0$.

(b) TRUE. Note that M is orthogonal, that is, $M^TM = MM^T = I$.

$$\|M\mathbf{v}\|^2 = M\mathbf{v} \cdot M\mathbf{v} = \mathbf{v}^T M^T M\mathbf{v} = \mathbf{v}^T MM^T\mathbf{v} = M^T\mathbf{v} \cdot M^T\mathbf{v} = \|M^T\mathbf{v}\|^2.$$

Hence, the statement is true.

(c) TRUE. Let's apply the Gram-Schmidt process to $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. First, we construct the orthogonal basis vectors

$$\begin{aligned}
\mathbf{w_1} &= \mathbf{v_1} = (a, 0, 0) \\
\mathbf{w_2} &= \mathbf{v_2} - \frac{\mathbf{v_2} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\mathbf{w_1} = (b, b, 0) - \frac{b}{a}(a, 0, 0) = (0, b, 0) \\
\mathbf{w_3} &= \mathbf{v_3} - \frac{\mathbf{v_3} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\mathbf{w_1} - \frac{\mathbf{v_3} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}}\mathbf{w_2} = (c, c, c) - \frac{c}{a}(a, 0, 0) - \frac{c}{b}(0, b, 0) = (0, 0, c).
\end{aligned}$$

and then normalize these to obtain the orthonormal basis vectors. Then we have an orthonormal basis

$$\left\{\frac{1}{a}\mathbf{w_1}, \frac{1}{b}\mathbf{w_1}, \frac{1}{c}\mathbf{w_1}\right\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

which is the standard basis for $\mathbb{R}^3$.

2 (10 points) Let $\mathbf{v_1} = (4,0,0)$, $\mathbf{v_2} = (4,2,0)$, $\mathbf{v_3} = (4,1,1)$, $\mathbf{w_1} = (1,-1,0)$, $\mathbf{w_2} = (0,1,-1)$, and $\mathbf{w_3} = (1,0,1)$. Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, $B' = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be ordered bases for $\mathbb{R}^3$. Find the transition matrix $P_{B\to B'}$ (the change of coordinate matrix) from B to B'.

Solution.

Let $A = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Then $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

We see that $A^{-1}\mathbf{v_1} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, A^{-1}\mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, A^{-1}\mathbf{v_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Hence, $\mathbf{v_1} = A\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v_2} = A\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \mathbf{v_3} = A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. That is,

$$\begin{aligned} \mathbf{v_1} &= 2\mathbf{w_1} + 2\mathbf{w_2} + 2\mathbf{w_3} \\ \mathbf{v_2} &= 1\mathbf{w_1} + 3\mathbf{w_2} + 3\mathbf{w_3} \\ \mathbf{v_3} &= 1\mathbf{w_1} + 2\mathbf{w_2} + 3\mathbf{w_3} \end{aligned}$$

Hence,

$$P_{B\to B'} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \end{bmatrix}.$$