1 (a) 10 points

Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = (3, -2, 1, 1)$, $\mathbf{v}_2 = (1, -1, 2, 1)$, $\mathbf{v}_3 = (-5, 4, -5, -3)$ and $\mathbf{v}_4 = (4, -3, 3, 2)$.
Find a matrix A with full column rank such that $4\mathbf{v}_2 + 2\mathbf{v}_3$ is a column vector of A and the column vectors of AA^T span W.

Solution. Note that $\text{col}(AA^T) = \text{col}(A)$. Hence, it suffices to find A so that the column vectors of A spans W **(+3 points)**.
In order to find full column rank A, let's find a basis for W which contains $4\mathbf{v}_2 + 2\mathbf{v}_3 = (-6, 4, -2, -2)$.

$$\begin{bmatrix} -6 & 3 & 1 & -5 & 4 \\ 4 & -2 & -1 & 4 & -3 \\ -2 & 1 & 2 & -5 & 3 \\ -2 & 1 & 1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 & -3 & 2 \\ 4 & -2 & -1 & 4 & -3 \\ -2 & 1 & 2 & -5 & 3 \\ -6 & 3 & 1 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 & -3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 & -3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $\{(-6, 4, -2, -2), \mathbf{v}_2 = (1, -1, 2, 1)\}$ is a basis for W **(+3 points)**.

Choose $A = \begin{bmatrix} -6 & 1 \\ 4 & -1 \\ -2 & 2 \\ -2 & 1 \end{bmatrix}$. Then A has a full column rank, $4\mathbf{v}_2 + 2\mathbf{v}_3$ is a column vector of A and the column vectors of AA^T span W **(+4 points)**. □

- If you know that A is a 4×2 matrix, then you will get 3 points.
- If there is a minor mistake, then you lose 2 points.

1 (b) Find a basis for $W^\perp$.

10 points

Solution. $W^\perp$ is a null space of $\begin{bmatrix} -6 & 4 & -2 & -2 \\ 1 & -1 & 2 & 1 \end{bmatrix}$ **(+2 points)**.

$$\begin{bmatrix} -6 & 4 & -2 & -2 \\ 1 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ -6 & 4 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -2 & 10 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & -5 & -2 \end{bmatrix}.$$

(+2 points)

Let $(a_1, a_2, a_3, a_4) \in W^\perp$. Then $\begin{cases} a_1 = 3a_3 + a_4 \\ a_2 = 5a_3 + 2a_4 \end{cases}$ **(+2 points)**. Since dim$W$=2, we see that dim$W^\perp = 2$. So we choose two vectors which are independent.

for example, $\begin{cases} a_3 = 1, a_4 = 0 \to (3, 5, 1, 0) \\ a_3 = 0, a_4 = 1 \to (1, 2, 0, 1) \end{cases}$.

$\{(3, 5, 1, 0), (1, 2, 0, 1)\}$ is linearly independent, so is a basis for $W^\perp$ **(+4 points)**. $\square$

1 (c) 10 points

For the matrix A obtained in (a), let $k = \text{rank}(A)$ and $T : \mathbb{R}^k \to \mathbb{R}^4$ be the linear transformation such that $T(\mathbf{x}) = A\mathbf{x}$. Let $\mathbf{b} = (b_1, b_2, b_3, b_4)$ be a vector in $\mathbb{R}^4$. Find conditions on the numbers b_1, b_2, b_3, b_4 such that $\mathbf{b}$ is in the range of T.

Solution. $k = 2$. In (a), we chose $A = \begin{bmatrix} -6 & 1 \\ 4 & -1 \\ -2 & 2 \\ -2 & 1 \end{bmatrix}$. Let's see that

$$\begin{bmatrix} -6 & 1 & b_1 \\ 4 & -1 & b_2 \\ -2 & 2 & b_3 \\ -2 & 1 & b_4 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & b_4 \\ 4 & -1 & b_2 \\ -2 & 2 & b_3 \\ -6 & 1 & b_1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & b_4 \\ 0 & 1 & b_2 + 2b_4 \\ 0 & 1 & b_3 - b_4 \\ 0 & -2 & b_1 - 3b_4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 0 & b4 - (b_2 + 2b_4) \\ 0 & 1 & b_2 + 2b_4 \\ 0 & 0 & (b_3 - b_4) - (b_2 + 2b_4) \\ 0 & 0 & (b_1 - 3b_4) + 2(b_2 + 2b_4) \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & -b_2 - b_4 \\ 0 & 1 & b_2 + 2b_4 \\ 0 & 0 & -b_2 + b_3 - 3b_4 \\ 0 & 0 & b_1 + 2b_2 + b_4 \end{bmatrix}$$

(+3 points).

Hence for any $(b_1, b_2, b_3, b_4) \in \mathbb{R}^4$,

$$\begin{cases} -b_2 + b_3 - 3b_4 = 0 \\ b_1 + 2b_2 + b_4 = 0. \end{cases} \iff \text{there exists } (x_1, x_2) \in \mathbb{R}^2 \text{ such that } \begin{bmatrix} -6 & 1 \\ 4 & -1 \\ -2 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

$$\iff (b_1, b_2, b_3, b_4) \text{ is in the range of } T.$$

(+7 points). □

2 Let

10 points

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -a & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \text{ with } a, b \in \mathbb{R}.$$

Find all pairs of (a, b) such that A and B are similar.

Solution. Note that a symmetric matrix is (orthogonally) diagonalizable. **(+2 points)** If two matrices are similar, then their characteristic polynomials are same. **(+2 points)** Hence we deduce that two symmetric matrices are similar if and only if their characteristic polynomials are same.

Now we will compute the characteristic polynomials f_A and f_B of A and B, respectively.

$$f_A = \det(xI - A) = (x-1)(x^2 - 2x - 9), \textbf{(+3 points)}$$

and

$$f_B = \det(xI - B) = (x-1)\big(x^2 + (a-1)x - (a^2 + a + b^2)\big). \textbf{(+3 points)}$$

Therefore, A and B are similar if and only if $a - 1 = -2$ and $a^2 + a + b^2 = 9$. All possible pairs of (a, b) are $(-1, 3)$ and $(-1, -3)$. □

- **(−1 points)** if you only compute the determinants and traces of A and B rather than the characteristic functions.
- **(−1 points)** for each mistake to compute characteristic functions, determinants or traces.
- **(−1 points)** if the answer is wrong.

3 Let

10 points

$$f(x, y, z) = cx^2 + cy^2 + cz^2 + 4xy + 4yz + 4xz$$

where $c \in \mathbb{R}$. Find all values of c such that f is negative definite.

Solution. Let A be a matrix corresponding to the quadratic form $f(x, y, z)$. Then we know that

$$A = \begin{bmatrix} c & 2 & 2 \\ 2 & c & 2 \\ 2 & 2 & c \end{bmatrix}$$

(+3 points) and A should be negative definite. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} \lambda - c & -2 & -2 \\ -2 & \lambda - c & -2 \\ -2 & -2 & \lambda - c \end{vmatrix} = (\lambda - (c+4))(\lambda - (c-2))^2$$

So all eigenvalues of A are $c+4, c-2$ **(+3 points)**. Also we know that all eigenvalues of A should be negative **(+3 points)**. So we know that $c < -4$. **(+1 points)**

□

4 Let $P = \begin{bmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{bmatrix}$.

20 points

(a) Show that P is the standard matrix for the orthogonal projection of $\mathbb{R}^3$ onto $\text{col}(P)$.

10 points

Solution (a). **Theorem 7.7.5** If W is a nonzero subspace of R^n, and if M is any matrix whose column vectors form a basis for W, then $\text{proj}_W \mathbf{x} = M(M^T M)^{-1} M^T \mathbf{x}$ for every column vector $\mathbf{x}$ in R^n.

By Gauss-Jordan elimination, one can show that the first two column vectors of P form a basis for $\text{col}(P)$. **(+5 points)** Let

$$M = \begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \\ -1/6 & -1/3 \end{bmatrix}.$$

By computation, one can show that $P = M(M^T M)^{-1} M^T$. By Theorem 7.7.5, P is the standard matrix for the orthogonal projection of R^3 onto $\text{col}(P)$. **(+5 points)**

- You get only 5 points if you use a wrong basis for $\text{col}(P)$ with the correct direction. **(−5 points)**
- Although you show that P is an orthogonal projection of $\mathbb{R}^3$ onto $\text{col}(P)$, if you show that P is standard matrix, then you lose 5 points. **(−5 points)**
- Computational mistakes with the correct direction lose 2 points. **(−2 points)**
- If you compute the rank of P incorrectly, then you lose 1 point. **(−1 points)**
- An answer without justification would get no points.

Solution (a). **Theorem 7.7.6** An $n \times n$ matrix P is the standard matrix for an orthogonal projection of R^n onto a k-dimensional subspace of R^n if and only if P is symmetric, idempotent, and had rank k.

It is obvious that P has rank as the dimension of $\mathrm{col}(P)$. P is symmetric. **(+5 points)** By computation, one can show that $P^2 = P$, that is, P is idempotent. By Theorem 7.7.6, P is the standard matrix for the orthogonal projection of $\mathbb{R}^3$ onto $\mathrm{col}(P)$. **(+5 points)**

- Although you show that P is an orthogonal projection of $\mathbb{R}^3$ onto $\mathrm{col}(P)$, if you show that P is standard matrix, then you lose 5 points. **(−5 points)**
- Computational mistakes with the correct direction lose 2 points. **(−2 points)**
- If you compute the rank of P incorrectly, then you lose 1 point. **(−1 points)**
- An answer without justification would get no points.

(b) Find a matrix M such that $P = MM^T$ and the columns of M are orthonormal.

10 points

Solution (b). **Theorem 7.9.3** If P is the standard matrix for an orthogonal projection of $\mathbb{R}^n$ onto a subspace of $\mathbb{R}^n$, then $\operatorname{tr}(P) = \operatorname{rank}(P)$. By Theorem 7.7.5, it suffices to find an orthonormal basis for $\operatorname{col}(P)$. By Theorem 7.9.3, $\operatorname{tr}(P) = \operatorname{rank}(P) = 2$. By Gauss-Jordan elimination, one can show that the first two column vectors of P form a basis for $\operatorname{col}(P)$. **(+5 points)** By Gram-Schmidt process, one can find a desired matrix M as follows.

$$M = \begin{bmatrix} \frac{5}{\sqrt{30}} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{30}} & \frac{-2}{\sqrt{5}} \end{bmatrix}. \textbf{ (+5 points)}$$

- You lose 5 points if you use a wrong basis for $\operatorname{col}(P)$ with the correct direction. **(-5 points)**
- Computational mistakes with the correct direction lose 2 points. **(-2 points)**
- An answer with another desired matrix M with the correct direction will get full points.
- An answer without justification would get no points.

5 10+10 points

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$.

(a) (10 points) Find a QR-decomposition of A.

Solution. Let $v_1 = (1,1,1,-1)$ and $v_2 = (2,1,1,0)$. (Note that v_1 and v_2 are column vectors of the matrix A.) Since v_1 and v_2 are linear independent, A has full column rank. So A has a QR-decomposition. To find Q, we can use the Gram-Schmidt process. **(+3 points)**

Put $u_1 = v_1 = (1,1,1,-1)$. By the Gram-Schmidt process, we can find

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = (2,1,1,0) - \frac{4}{4}(1,1,1,-1) = (1,0,0,1).$$

By normalizing we obtain

$$e_1 = \frac{1}{2}(1,1,1,-1) \text{ and } e_2 = \frac{\sqrt{2}}{2}(1,0,0,1).$$

Recall that $\{e_1, e_2\}$ is an orthonormal basis of the vector space spanned by $\{v_1, v_2\}$. Now let

$$Q = \begin{bmatrix} e_1{}^T & \vdots & e_2{}^T \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} \\ 1 & 0 \\ 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix}. \textbf{(+2 points)}$$

Then $Q^T Q = I_2$. So, since $A = QR$, we have

$$R = I_2 R = Q^T Q R = Q^T A. \textbf{(+3 points)}$$

So,

$$R = Q^T A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ \sqrt{2} & 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & \sqrt{2} \end{bmatrix} \textbf{(+2 points)}.$$

Therefore,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} \\ 1 & 0 \\ 1 & 0 \\ -1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & \sqrt{2} \end{bmatrix} = QR.$$

□

- The answer without justifications gets no points.
- You can calculate R by using formula

$$R = \begin{bmatrix} v_1 \cdot e_1 & v_2 \cdot e_1 \\ 0 & v_2 \cdot e_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & \sqrt{2} \end{bmatrix}.$$

- A QR-decomposition is not unique.
(For example,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{2} \\ 1 & 0 \\ 1 & 0 \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & -\sqrt{2} \end{bmatrix} = QR.)$$

In your solution, if there is no logical defect, you get full points.

- In this solution, Q is a 4×2 matrix and R is a 2×2 matrix. Also we can express $A = QR$ where Q is a 4×4 matrix and R is a 4×2 matrix. To do this, find the basis $\{b_1, b_2, b_3, b_4\}$ for $\mathbb{R}^4$ where $b_1 = v_1$ and $b_2 = v_2$. For the set $\{b_1, b_2, b_3, b_4\}$, by applying Gram-Schmidt process, we can obtain Q which is 4×4 matrix. Remaining part is same as above.

5 Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$.

10+10 points

(b) (10 points) Find the least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 3 \end{bmatrix}$.

Solution. By part (a), recall that A has full column rank. So, the least squares solution $\hat{\mathbf{x}}$ is a solution of the normal equation $A^T Ax = A^T b$ where $b = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 3 \end{bmatrix}$. By Theorem 7.10.2 in textbook, the normal equation is given by

$$Rx = Q^T b.\textbf{(+5 points)}$$

So, we have

$$\begin{bmatrix} 2 & 2 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ \sqrt{2} & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{2}\sqrt{2} \end{bmatrix}.$$

Simple calculation gives that the solution is $\begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \end{bmatrix}$. So, the answer is

$$\hat{\mathbf{x}} = \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \end{bmatrix}.\textbf{(+5 points)}$$

- The answer without justifications gets no points.
- You can solve this problem just by solving the normal equation $A^T Ax = A^T b$ directly. The normal equation is given by
$$\begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

6 For $a, b, c, d \in \mathbb{R}$, let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear operator defined by

10 points

$$T(x, y, z, w) = ((a-b)x + dw, ax + (b-c)y, by + (c-d)z, cz + (d-a)w).$$

Find all values of a, b, c, and d such that $\dim(\ker(T)) \geq 3$.

Solution. From the definition, the standard matrix for T is

$$[T] = \begin{pmatrix} a-b & 0 & 0 & d \\ a & b-c & 0 & 0 \\ 0 & b & c-d & 0 \\ 0 & 0 & c & d-a \end{pmatrix}. \quad \textbf{(+2 points)}$$

By the dimension theorem, $\dim(\ker(T)) = \dim(\text{null}([T])) = \text{nullity}([T]) = 4 - \text{rank}([T]) \geq 3$ and we get $\text{rank}([T]) \leq 1$. Therefore, $\text{rank}([T]) = 0$ or 1. **(+3 points)**

(i) $\text{rank}([T]) = 1$ case.
Since the rank of $[T]$ is one, there is one nonzero row such that the other rows are just scalar multiples of this row. **(+1 points)** By the symmetry of $[T]$, suppose the first row $[a-b,\ 0,\ 0,\ d]$ is nonzero. Then,

2nd row is a scalar multiple of the first row. $\Rightarrow\ b - c = 0 \quad \Rightarrow b = c.$
3rd row is a scalar multiple of the first row. $\Rightarrow\ b = c - d = 0 \quad \Rightarrow b = 0$ and $c = d$.

$\therefore b = c = d = 0.$ and $[T] = \begin{pmatrix} a & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}.$

Finally, the 4th row is also a scalar multiple of the first row. $\Rightarrow\ -a = 0$
$\therefore\ a = b = c = d = 0.$

Thus $[T]$ becomes a zero matrix, but this contradicts to the assumption that the first row of $[T]$ is nonzero. Therefore, there is no such pair of (a, b, c, d) satisfying $\text{rank}([T]) = 1$. **(+2 points)**

(ii) $\text{rank}([T]) = 0$ case.
Easily, $[T]$ is a zero matrix and thus $a = b = c = d = 0$. **(+2 points)**

By (i) and (ii), $a = b = c = d = 0$ is the only possible value. □

- The answer without justifications gets no points.
- If you did not show the contradiction for the rank one case, you get **(−2 points)**.
- Each minor mistake gets **(−1 points)**.
- Alternative solutions are also allowed.

7 Find a 3×3 matrix A such that $A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = 2\mathbf{v}_2, A\mathbf{v}_3 = 3\mathbf{v}_3$, where

10 points

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Solution. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then $\mathcal{B}$ is a basis of $\mathcal{R}^3$. So, it is enough to find $A = [D]_{\mathcal{B}}$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then we know that

$$A = [D]_{\mathcal{B}} = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

So we get

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

and it satisfies the condition.

$\square$

- The minor computation error **$(-2$ points)**
- If there is an idea which try to use basis transition, diagonalization but it is wrong, **$(+2$ points)**

8.(a) 10 points

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation represented by

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

with respect to the bases

$\mathcal{B}_1 = \{(1,0,0),(1,-1,0),(1,1,1)\}$ and $\mathcal{B}_2 = \{(1,0,0,0),(1,0,1,1),(1,1,0,0),(1,1,-1,0)\}$

of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.
Find $T(x,y,z)$.

Solution. $A = [T]_{\mathcal{B}_2,\mathcal{B}_1}$
To calcuate the standard form of linear transform as a function form, we need the representing matrix form of a give linear transform with respect to the standard bases for each domain, range space
Let S_3,S_4 be standard bases for $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.
Let $P_{\mathcal{B}_1 \to S_3}$, $P_{\mathcal{B}_2 \to S_4}$ be transition matrix with respect to the indexed bases.
Then,

$$P_{\mathcal{B}_1 \to S_3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{\mathcal{B}_2 \to S_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

So the transition matrix from S_3 to $\mathcal{B}_1$ is $(P_{\mathcal{B}_1 \to S_3})^{-1}$

$$P_{S_3 \to \mathcal{B}_1} = (P_{\mathcal{B}_1 \to S_3})^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(+4 points)}$$

Therefore, the representing matrix for T with respect to the standard bases is

$$\begin{aligned} [T]_{S_4,S_3} &= P_{\mathcal{B}_2 \to S_4}[T]_{\mathcal{B}_2,\mathcal{B}_1}P_{S_3 \to \mathcal{B}_1} \\ &= P_{\mathcal{B}_2 \to S_4} A P_{S_3 \to \mathcal{B}_1} \quad \textbf{(+4 points)} \\ &= \begin{bmatrix} 7 & 3 & -4 \\ 4 & 4 & -6 \\ -1 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix} \end{aligned}$$

$\therefore T(x,y,z) = (7x+3y-4z, 4x+4y-6z, -x-y+3z, 2x+y-z)$ **(+2 points)**

- If you end in the result of matrix form, it will be graded as the same as the functino form, 2 points
- For any calculation error, you get -2 points at the end.

□

Solution. $A = [T]_{\mathcal{B}_1,\mathcal{B}_2}$. Let $\mathcal{B}_1 = \{v_1, v_2, v_3\}$ and $\mathcal{B}_2 = \{w_1, w_2, w_3, w_4\}$.

Then, $A = [[T(v1)]_{\mathcal{B}_2}\ [T(v2)]_{\mathcal{B}_2}\ [T(v3)]_{\mathcal{B}_2}]$. For the first column of A, $[T(v_1)]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$

$$\begin{aligned} [T(v_1)]_{S_4} &= w_1 + 2w_2 + w_3 + 3w_4 \\ &= \begin{bmatrix} 7 \\ 4 \\ -1 \\ 2 \end{bmatrix} \\ &= [T(e_1)]_{S_4}\ \textbf{(+4 points)} \end{aligned}$$

for e_1, the first member of S_3.

For the second column of A, $[T(v_2)]_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

$$\begin{aligned} [T(v_2)]_{S_4} &= [T(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix})]_{S_4} \\ &= [T(e_1)]_{S_4} + [T(-e_2)]_{S_4} \\ &= 3w_1 + w_2 - w_3 + w_4 \\ &= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

for e_2, the second member of S_3

$$[T(e_2)]_{S_4} = -(\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} - [T(e_1)]_{S_4}) = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 1 \end{bmatrix}\ \textbf{(+4 points)}$$

For the third column of A, $[T(v_3)]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} [T(v_3)]_{S_4} &= [T(e_1) + T(e_2) + T(e_3)]_{S_4} \\ &= 2w_1 + 2w_2 + w_3 + w_4 \\ &= \begin{bmatrix} 6 \\ 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [T(e_3)]_{S_4} &= \begin{bmatrix} 6 \\ 2 \\ 1 \\ 2 \end{bmatrix} - T(e_1)_{S_4} - T(e_2)_{S_4} \\ &= \begin{bmatrix} -4 \\ -6 \\ 3 \\ -1 \end{bmatrix} \end{aligned}$$

for e_3, the third member of S_3

$$\begin{aligned} \therefore [T]_{S_4,S_3} &= [[T(e_1)]_{S_4}, [T(e_2)]_{S_4}, [T(e_3)]_{S_4}] \\ &= \begin{bmatrix} 7 & 3 & -4 \\ 4 & 4 & -6 \\ -1 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix} \textbf{(+2 points)} \end{aligned}$$

- In the second solution, only when you express calculated vectors are $[T(v1)]_{S_4} = T(e_1)$ and $[T(v2)]_{S_4} = T(e_1) - T(e_2)$, you can get full 4 points.
- For any calculation error, you get -2 points at the end.

□

8.(b) 10 points Let $\mathcal{B}_3 = \{(2,1,1,2),(1,0,1,0),(0,-1,1,1),(0,1,0,-1)\}$ be a basis of $\mathbb{R}^4$. Find the transition matrix from $\mathcal{B}_2$ to $\mathcal{B}_3$.

Solution. Let $P_{\mathcal{B}_2\to\mathcal{B}_3}$ be a transition matrix from $\mathcal{B}_2$ to $\mathcal{B}_3$ and $P_{\mathcal{B}_3\to S_4}$ be a transition matrix from a $\mathcal{B}_3$ to a standard matrix S_4

$$P_{\mathcal{B}_3\to S_4} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

So, the transition matrix $P_{S_4\to\mathcal{B}_3}$ from S_4 to $\mathcal{B}_3$ is $(P_{\mathcal{B}_3\to S_4})^{-1}$

$$P_{S_4\to\mathcal{B}_3} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ -1 & \frac{1}{3} & 1 & \frac{1}{3} \\ -1 & 1 & 1 & 0 \end{bmatrix} \textbf{(+4 points)}$$

$$\begin{aligned} \therefore\ P_{\mathcal{B}_2\to\mathcal{B}_3} &= P_{S_4\to\mathcal{B}_3}P_{\mathcal{B}_2\to S_4}\textbf{(+4 points)} \\ &= \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ -1 & \frac{1}{3} & 1 & \frac{1}{3} \\ -1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} \\ -1 & 0 & 0 & -1 \end{bmatrix} \textbf{(+2 points)} \end{aligned}$$

- For any calculation error in calculating inverse matrix or matrix multiplication, you get -2 points at the end.
- If you define a transition matrix as a transpose form(rows of matrix consist of each member of basis), then you cannot get points.(incorrect definition of a transition matrix)

□

Solution. $P_{\mathcal{B}_3\to S_4}P_{\mathcal{B}_2\to\mathcal{B}_3} = P_{\mathcal{B}_2\to S_4}$
Thus, using *Gauss–Jordan method* for $[P_{\mathcal{B}_3\to S_4}|P_{\mathcal{B}_2\to S_4}]$**(+4 points)** to get $[I_4|P_{\mathcal{B}_2\to\mathcal{B}_3}]$**(+4 points)**

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 2 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$\rightarrow$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \end{array}\right] \textbf{(+2 points)}$$

- If you calculate as a transition form as I mentioned at the above, you cannot have points.
- For any calculation error in calculating inverse matrix or matrix multiplication, you get -2 points at the end.

$\square$

8.(c) 10 points Assume that the transition matrix from another basis $\mathcal{B}_4$ to $\mathcal{B}_1$ is given by the matrix $-3I_3$. Find the matrix representation for T with respect to the bases $\mathcal{B}_4$ and $\mathcal{B}_3$.

Solution. For a Given transition matrix $P_{\mathcal{B}_4->\mathcal{B}_1}$ from $\mathcal{B}_4$ to $\mathcal{B}_1$ is $-3I_3$ and let $[T]_{\mathcal{B}_3,\mathcal{B}_4}$ be a matrix representing for a give linear transform T with respect to the bases $\mathcal{B}_4$ and $\mathcal{B}_3$

$$\begin{aligned}
\therefore\ [T]_{\mathcal{B}_3,\mathcal{B}_4} &= P_{\mathcal{B}_2->\mathcal{B}_3}[T]_{\mathcal{B}_2,\mathcal{B}_1}P_{\mathcal{B}_4->\mathcal{B}_1} \\
&= P_{\mathcal{B}_2->\mathcal{B}_3}AP_{\mathcal{B}_4->\mathcal{B}_1}\ \textbf{(+5 points)} \\
&= \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} (-3I_3) \\
&= \begin{bmatrix} -6 & -1 & -4 \\ -9 & -10 & -10 \\ 18 & 11 & 11 \\ 12 & 12 & 9 \end{bmatrix}\ \textbf{(+5 points)}
\end{aligned}$$

- You can get 5 points for the only correct calculation at the last matrix multiplication.
- If you use the commutative diagrams to depict changing bases and result, you can get 5 full points.

□

Solution.

$$\begin{aligned}
P_{\mathcal{B}_1\to S_3}P_{\mathcal{B}_4\to\mathcal{B}_1} &= P_{\mathcal{B}_4\to S_3} \\
&= -3[v_1\ v_2\ v_3]
\end{aligned}$$

Then, by using the result of (a) and (b),

$$\begin{aligned}
[T]_{\mathcal{B}_3,\mathcal{B}_4} &= P_{S_4\to\mathcal{B}_3}[T]_{S_4,S_3}P_{\mathcal{B}_4\to S_3}\ \textbf{(+5 points)} \\
&= -3P_{S_4\to\mathcal{B}_3}[T]_{S_4,S_3}[v_1\ v_2\ v_3] \\
&= \begin{bmatrix} -6 & -1 & -4 \\ -9 & -10 & -10 \\ 18 & 11 & 11 \\ 12 & 12 & 9 \end{bmatrix}\ \textbf{(+5 points)}
\end{aligned}$$

- You can get 5 points for the only correct calculation at the last matrix multiplication.
- If you use the commutative diagrams to depict changing bases and result, you can get 5 full points.

□

Solution. Let $\mathcal{B}_4 = \{a_1\ a_2\ a_3\} = \{-3v_1\ \ -3v_2\ \ -3v_3\}$. So, it could be calculated directly by a vector notation

$$\begin{aligned}[T]_{\mathcal{B}_3,\mathcal{B}_4} &= [[T(a_1)]_{\mathcal{B}_3}\ [T(a_2)]_{\mathcal{B}_3}\ [T(a_3)]_{\mathcal{B}_3}] \\ &= [[T(-3v_1)]_{\mathcal{B}_3}\ [T(-3v_2)]_{\mathcal{B}_3}\ [T(-3v_3)]_{\mathcal{B}_3}] \\ &= -3[[T(v_1)]_{\mathcal{B}_3}\ [T(v_2)]_{\mathcal{B}_3}\ [T(v_3)]_{\mathcal{B}_3}]\end{aligned}$$

$$\begin{aligned}[T(v_1)]_{\mathcal{B}_3} &= P_{S_4\to\mathcal{B}_3}T(v_1) \\ &= \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} & 0 & -\frac{2}{3} \\ -1 & \frac{1}{3} & 1 & \frac{1}{3} \\ -1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 3 & -4 \\ 4 & 4 & -6 \\ -1 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \\ -6 \\ -4 \end{bmatrix} \textbf{(+5 points)}\end{aligned}$$

Similarly,

$$[T(v_2)]_{\mathcal{B}_3} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{3} \\ -\frac{11}{3} \\ -4 \end{bmatrix} \quad [T(v_3)]_{\mathcal{B}_3} = \begin{bmatrix} \frac{4}{3} \\ \frac{10}{3} \\ -\frac{11}{3} \\ -3 \end{bmatrix}$$

$$\therefore [T]_{\mathcal{B}_3,\mathcal{B}_4} = \begin{bmatrix} -6 & -1 & -4 \\ -9 & -10 & -10 \\ 18 & 11 & 11 \\ 12 & 12 & 9 \end{bmatrix} \textbf{(+5 points)}$$

- If you write $[T]_{\mathcal{B}_3,\mathcal{B}_4} = [[T(a_1)]_{\mathcal{B}_3}\ [T(a_2)]_{\mathcal{B}_3}\ [T(a_3)]_{\mathcal{B}_3}]$ $= [[T(-3v_1)]_{\mathcal{B}_3}\ [T(-3v_2)]_{\mathcal{B}_3}\ [T(-3v_3)]_{\mathcal{B}_3}]$ $= -3[[T(v_1)]_{\mathcal{B}_3}\ [T(v_2)]_{\mathcal{B}_3}\ [T(v_3)]_{\mathcal{B}_3}]$ only without further expansion or calculation, you cannot get points since this idea is so global then I do not consider that this is a key idea.
- If you expands any further calculation probably with error, you can get 5 points.

□

9 (10+10 points) Let A be a 3×3 matrix.

1. (10 points) Show that $\text{null}(A^2) \subseteq \text{null}(A^3)$.

2. (10 points) Show that if $\text{null}(A) = \text{null}(A^2)$, then $\text{null}(A^2) = \text{null}(A^3)$.

Solution.

1. For any $v \in \text{null}(A^2)$, $A^3v = A\left(A^2v\right) = A0 = 0$ and $v \in \text{null}(A^3)$.

2. For any $v \in \text{null}(A^3)$, we have that $A^2(Av) = A^3v = 0$ and $Av \in \text{null}(A^2) = \text{null}(A)$. Therefore $A^2v = A(Av) = 0$ and $v \in \text{null}(A^2)$.

$\square$

- There is **(−5 points)** for unclear argument.

10 Find a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

10 points

Solution. We first compute $A^TA = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ **(+2 points)**. Note that this matrix is symmetric. The characteristic polynomial of A^TA is $(\lambda-2)(\lambda^2-4\lambda+2)$, so we compute the eigenvalues of A^TA as

$$\lambda_1 = 2+\sqrt{2},\ \lambda_2 = 2,\ \lambda_3 = 2-\sqrt{2} \quad \textbf{(+1 points)}.$$

Corresponding eigenvectors v_i' of λ_i $(i = 1,2,3)$ can be computed as

$$v_1' = [1\ \sqrt{2}\ -1]^T,\ v_2' = [1\ 0\ 1]^T,\ v_3' = [-1\ \sqrt{2}\ 1]^T \quad \textbf{(+1 points)}.$$

After **normalization**, we get

$$v_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{bmatrix},\ v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, v_3 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}, \text{ and } V = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \quad \textbf{(+1 points)}.$$

The singular values of A are $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $\sigma_3 = \sqrt{\lambda_3}$. Since $u_i = \frac{1}{\sigma_i}Av_i$ **(+2 points)**, we may compute that

$$U = \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & 0 & -\frac{\sqrt{2-\sqrt{2}}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2-\sqrt{2}}}{2} & 0 & -\frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix} \quad \textbf{(+2 points)}.$$

Therefore, we get a singular value decomposition of A as $A = U\Sigma V^T$, where

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma 3 \end{bmatrix},$$

and U and V as written above **(+1 points)**.

□

Solution. One can compute the vectors u_i as eigenvectors of the matrix AA^T also. Then, using the formula $A^Tu_i = \sigma_i v_i$, the vectors v_i may be computed from this.

□

- The answer without justifications gets no points.
- (very) Simple computational mistakes will get **(−1 points)** for each mistake.

11
20 points

Let S be an $n \times k$ non-zero matrix and let $\mathcal{B}$ be the set of all column vectors of S. If a MATLAB function file `MAS109_Final.m` takes S as an input, then it produces an $n \times n$ matrix V whose column vectors form a basis $\mathcal{C}$ for $\mathbb{R}^n$. Here, the basis $\mathcal{C}$ should contain a maximal linearly independent subset of $\mathcal{B}$. Fill in the blanks (1) - (6).
[Do not use MATLAB built-in functions `null` and `rank`.]

Solution.

(1) `size(S)`

(2) `S(:, pivotCol_S)`

(3) `length(pivotCol_S)` *or* `size(pivotCol_S, 2)` *or* `length(col_S)` *or* `size(col_S, 1)`

(4) `rank_S == n` *or* `size(col_S) == n` *or* `size(col_S) == [n, n]`

(5) `col_S`
`V = col_S` was also allowed.

(6) Who those seem to know the right direction : **(+1 points)**
Who those got an intermediate result : **(+3 points)**
Who those got a result : **(+6 points)**

Example of solutions :

```
aug_S = [S, eye(n)];
rref_aug_S = rref(aug_S);
new_S = rreg_aug_S(rank_S + 1:n, k + 1:k + n);
```

or

```
[~, ~, V] = svd(S');
new_S = V(:, rank_S+1:n);
```

□