1 (3+3+4 points) Indicate whether the following statements are true(T) or false(F). You do **not** need to justify your answer.

(a) There is a system of equations that has exactly one solution such that the reduced row echelon form of the augmented matrix of the system has a row consisting of entirely zeros.

(b) For any matrix A, two matrices A^TA and AA^T are always invertible.

(c) If A, B and $A+B$ are invertible matrices, then there are matrices P and Q such that

$$P(A^{-1}+B^{-1})Q = I.$$

Solution.

(a) True. For example, consider the following system of equations:

$$\begin{cases} x_1 \qquad\quad = 0, \\ \qquad\ x_2 = 0, \\ x_1 + x_2 = 0. \end{cases}$$

The reduced row echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the system has a unique solution $x_1 = x_2 = 0$.

(b) False. For example, if $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$, then $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

(c) True. Let $P = A$ and $Q = B(A+B)^{-1}$. It is easy to see that

$$P(A^{-1}+B^{-1})Q = A(A^{-1}+B^{-1})B(A+B)^{-1} = I.$$

2 (10 points) Find the conditions on the b_i's that will ensure that the following system is consistent.

$$\begin{cases} x_1 & - x_2 & + 3x_3 & + 2x_4 & = b_1, \\ -2x_1 & + x_2 & + 5x_3 & + x_4 & = b_2, \\ -3x_1 & + 2x_2 & + 2x_3 & - x_4 & = b_3, \\ 4x_1 & - 3x_2 & + x_3 & + 3x_4 & = b_4 \end{cases}$$

Solution. The augmented matrix of this system is

$$\begin{bmatrix} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{bmatrix}.$$

Using the Gauss-Jordan elimination, we can reduce the matrix as follows:

$$\begin{bmatrix} 1 & 0 & -8 & -3 & -b_1 - b_2 \\ 0 & 1 & -11 & -5 & -2b_1 - b_2 \\ 0 & 0 & 0 & 0 & b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & -2b_1 + b_2 + b_4 \end{bmatrix}.$$

Thus the conditions are

$$b_1 - b_2 + b_3 = 0 \quad \text{and} \quad -2b_1 + b_2 + b_4 = 0.$$