1 (3+3+4 points) Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

(a) Let A be a square symmetric matrix and λ_1, λ_2 be different eigenvalues of A. Then $\text{null}(A - \lambda_1 I) \perp \text{null}(A - \lambda_2 I)$.

(b) Let $A = \begin{bmatrix} 0 & -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & 1 & 0 & -2 & -2 \end{bmatrix}$. Then A is neither positive definite nor negative definite.

(c) Let A be a square matrix whose characteristic polynomial is

$$p(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

If $B = 4I - A$, then $p(B) = 0$.

Solution.

(a) True. Since A is symmetric, A is orthogonally diagonalizable and eigenvectors from different eigenspaces are orthogonal.

(b) True. Let $\mathbf{a}_i$ be the ith column vector of A. Then we can check that $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_5 = 0$. Therefore, A is not invertible and 0 is an eigenvalue of A. So A is neither positive definite nor negative definite.

(c) True. By the Caley-Hamilton theorem, we know that $p(A) = 0$. Also we know that $p(4 - \lambda) = -p(\lambda)$. Hence, $p(B) = p(4I - A) = -p(A) = 0$.

2 Let A be an $n \times n$ diagonalizable matrix. Then prove that $\det(e^A) = e^{tr(A)}$.

10 points

Solution. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A. Then

$$A = P^{-1} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} P$$

for some invertible matrix P. So we know that

$$e^A = P^{-1} \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix} P.$$

Therefore,

$$\begin{aligned} \det(e^A) &= \det(P^{-1}) \prod_{i=1}^{n} e^{\lambda_i} \det(P) \\ &= e^{\sum_{i=1}^{n} \lambda_i} \\ &= e^{tr(A)}. \end{aligned}$$