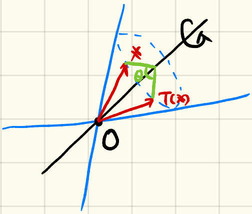
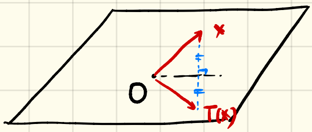
All orthogonal operators on $\mathbb{R}^3$ are one of the following types:

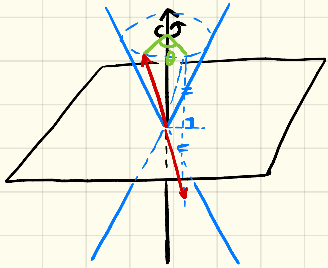
① Rotation about a line through the origin,

② Reflection about a plane through the origin

③ A rotation about a line through the origin followed by a reflection about the plane through the origin that is perpendicular to the line.





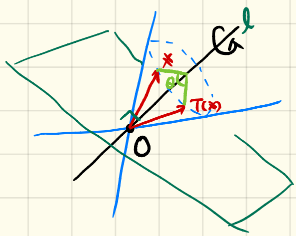


※ $\det([T]) = 1 \Rightarrow$ Type ①

$\det([T]) = -1 \Rightarrow$ Type ② or ③.

Let A be an orthogonal 3×3 matrix with $\det(A) = 1$.

Then T_A is a rotation about a line through the origin.



① How can we find this line?

② How can we find the angle θ?

Let P be the plane orthogonal to l.

Then $\forall w \in P$ with $w \neq 0$, $\theta = \cos^{-1} \dfrac{w \cdot Aw}{\|w\| \, \|Aw\|}$.

Find the eigenspace of A corresponding to the eigenvalue $\lambda = 1$. Denote by l the line.

§6.3. Kernel & Range

Def. For a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

$$\ker(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0} \} \ : \text{ the kernel of } T.$$

$$\operatorname{ran}(T) = \{ T(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n \} \ : \text{ the range of } T.$$

Thm (6.3.2 + 6.3.5 + 6.3.7)

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

(1) $\ker(T)$ is a subspace of $\mathbb{R}^n$.

(2) T maps subspaces of $\mathbb{R}^n$ into subspaces of $\mathbb{R}^m$

(3) $\operatorname{ran}(T)$ is a subspace of $\mathbb{R}^m$.

pf) (1) ① Since $\mathbf{0} \in \ker(T)$, $\ker(T) \neq \phi$

② $\forall \mathbf{u}, \mathbf{v} \in \ker(T)$, $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0}$ $\quad \therefore \mathbf{u}+\mathbf{v} \in \ker(T)$

③ $\forall \mathbf{u} \in \ker(T), \forall c \in \mathbb{R}$ $\quad T(c\mathbf{u}) = c\,T(\mathbf{u}) = \mathbf{0}$ $\quad \therefore c\mathbf{u} \in \ker(T)$

(2) Let S be a subspace of $\mathbb{R}^n$ and let $W = T(S) = \{ T(\mathbf{v}) \mid \mathbf{v} \in S \}$.

① Since $\mathbf{0} \in S$, $T(\mathbf{0}) = \mathbf{0} \in W$ $\quad \therefore W \neq \phi$

② $\forall \mathbf{u}, \mathbf{v} \in W$, $\exists \mathbf{u}_0, \mathbf{v}_0 \in S$ s.t. $\mathbf{u} = T(\mathbf{u}_0)$, $\mathbf{v} = T(\mathbf{v}_0)$

$\therefore \mathbf{u}+\mathbf{v} = T(\mathbf{u}_0+\mathbf{v}_0) \in W$

③ $\forall \mathbf{u} \in W, \forall c \in \mathbb{R}$, $\exists \mathbf{u}_0 \in S$ s.t. $\mathbf{u} = T(\mathbf{u}_0)$

$\therefore c\mathbf{u} = T(c\mathbf{u}_0) \in W$.

(3) It follows from (2). ▨

Thm 6.3.3 For an $m \times n$ matrix A,

$\ker(T_A)$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

Def. For an $m \times n$ matrix A,

the <u>null space</u> of A = $\operatorname{null}(A)$ = the solution space of "$A\mathbf{x}=\mathbf{0}$" = $\ker(T_A)$.

Thm 6.3.8 For a matrix A, $\operatorname{ran}(T_A) = \operatorname{col}(A)$.

☺ Let A be an $m\times n$ matrix, and $A = [\mathbb{a}_1\ \mathbb{a}_2 \cdots \mathbb{a}_n]$

① $\operatorname{ran}(T_A) \subseteq \operatorname{col}(A)$: $\forall \mathbb{b} \in \operatorname{ran}(T_A)$, $\exists\, \mathbb{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ s.t. $A\mathbb{x} = \mathbb{b}$.

Since $A\mathbb{x} = x_1\mathbb{a}_1 + x_2\mathbb{a}_2 + \cdots + x_n\mathbb{a}_n$, it implies that $\mathbb{b} \in \operatorname{col}(A)$.

② $\operatorname{col}(A) \subseteq \operatorname{ran}(T_A)$: $\mathbb{b} \in \operatorname{col}(A) \Rightarrow \exists$ a solution of $A\mathbb{x} = \mathbb{b}$

Since $T_A(\mathbb{x}) = A\mathbb{x}$, it implies that $\mathbb{b} \in \operatorname{ran}(T_A)$. ∎

Def. A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **onto** if $\operatorname{ran}(T) = \mathbb{R}^m$.

" " " " **one-to-one** if

$\mathbb{x}_1 \neq \mathbb{x}_2$ in $\mathbb{R}^n$ implies $T(\mathbb{x}_1) \neq T(\mathbb{x}_2)$ in $\mathbb{R}^m$.

Ex: $(x, y) \xrightarrow{T} (x, 0)$ is neither one-to-one nor onto.

Thm 6.3.11 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. TFAE

(a) T is one-to-one

(b) $\ker(T) = \{\mathbb{0}\}$

pf) (a) $\Rightarrow$ (b): Since T is linear $T(\mathbb{0}) = \mathbb{0}$.

$\forall \mathbb{x} \neq \mathbb{0}$ in $\mathbb{R}^n$, $T(\mathbb{x}) \neq \mathbb{0}$. $\therefore \ker(T) = \{\mathbb{0}\}$.

(b) $\Rightarrow$ (a): If $\mathbb{x}_1 \neq \mathbb{x}_2$ in $\mathbb{R}^n$, then $\mathbb{x}_1 - \mathbb{x}_2 \neq \mathbb{0}$.

Hence $T(\mathbb{x}_1 - \mathbb{x}_2) \neq \mathbb{0}$

$$\begin{gathered} T(\mathbb{x}_1 - \mathbb{x}_2) \\ \| \\ T(\mathbb{x}_1) - T(\mathbb{x}_2) \end{gathered}$$

$\therefore T(\mathbb{x}_1) \neq T(\mathbb{x}_2)$. ∎

Thm 6.3.12 ~ 13 A: $m\times n$ matrix

(a) $T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one-to-one $\Leftrightarrow$ $A\mathbb{x} = \mathbb{0}$ has only the trivial solution.

(b) " " onto $\Leftrightarrow$ $A\mathbb{x} = \mathbb{b}$ is consistent $\forall \mathbb{b} \in \mathbb{R}^m$.

Thm 6.3.14 T: a linear operator on $\mathbb{R}^n$

T: one-to-one $\Leftrightarrow$ T: onto.

pf) T: one-to-one $\Leftrightarrow$ $[T]\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$\Leftrightarrow$ $[T]\mathbf{x} = \mathbf{b}$ is consistent $\forall \mathbf{b} \in \mathbb{R}^n$.

$\Leftrightarrow$ T is onto.

$\blacksquare$

Thm 6.3.15

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*
9. $\det(A) \neq 0$.
10. $\lambda = 0$ is not an eigenvalue of A.
11. T_A is one-to-one.
12. T_A is onto.

§6.4 Composition & Invertibility of linear transformations.

$T_A \circ T_B = T_{AB}$ $\quad T_A^{-1} = T_{A^{-1}}$ $\quad T_A$

$\ast$ The composition of T_2 with T_1 is

$$(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x})).$$

Thm 6.4.1 $T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, $T_2 : \mathbb{R}^k \longrightarrow \mathbb{R}^m$: linear transf.

$\Rightarrow T_2 \circ T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a linear transf.

pf) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ & c : scalar.

$$\begin{aligned} T_2 \circ T_1 (c\mathbf{u} + \mathbf{v}) &= T_2(c\, T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= c\, T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) \\ &= c\, T_2 \circ T_1(\mathbf{u}) + T_2 \circ T_1(\mathbf{v}). \end{aligned}$$

$\therefore T_2 \circ T_1$ is also a linear transformation. $\boxtimes$

Q: What is $[T_2 \circ T_1]$?

A: $[T_2 \circ T_1] = [T_2][T_1]$

$(\because \; T_2 \circ T_1(\mathbf{e}_i) = T_2(T_1(\mathbf{e}_i)) = [T_2][T_1]\mathbf{e}_i \;)$

Thm 6.4.2 $A : k \times n$ & $B : m \times k \Rightarrow T_{BA} = T_B \circ T_A.$

Rmk $T_B \circ T_A = T_A \circ T_B \Longleftrightarrow BA = AB.$

e.g.) $\begin{cases} R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1 + \theta_2} = R_{\theta_2} \circ R_{\theta_1} \\ H_{\theta_1} \circ H_{\theta_2} = R_{2(\theta_1 - \theta_2)} \end{cases}$

Thm 6.4.3 $T_1, \cdots, T_k$: rotations about axes through the origin of $\mathbb{R}^3$

$\rightarrow T_k \circ T_{k-1} \circ \cdots \circ T_1$ is a single rotation

($\because$ $[T_k \circ T_{k-1} \circ \cdots \circ T_1] = [T_k][T_{k-1}] \cdots [T_1]$ is orthogonal and has determinant 1.)

Note that every 2x2 elementary matrix is one of the following forms :

$$\underbrace{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}}_{\text{shear of } \mathbb{R}^2}, \underset{\substack{\uparrow \\ \text{reflection} \\ \text{abut } y=x}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underbrace{\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}}_{\substack{0 \le k < 1 \quad \text{compression} \\ k > 1 \quad \text{expansion}}}$$

If $k < 0$, then $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

& $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -k \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Note that $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ represents the reflection about y-axis and

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ " " " " x-axis.

Thm 6.4.4 For an invertible 2x2 matrix A,

T_A is a composition of shears, compressions, and expansions in the direction of the coordinate axes and the reflection about the coordinate axes and about the line $y=x$.

< Inverse of a linear transformation >

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a one-to-one linear transformation.

Then $\forall \mathbb{w} \in ran(T)$, $\exists\, \mathbb{x} \in \mathbb{R}^n$ uniquely such that $T(\mathbb{x}) = \mathbb{w}$.

Then we can define a linear transformation

$$T^{-1} : ran(T) \longrightarrow \mathbb{R}^n \quad \text{s.t.} \quad T^{-1}(\mathbb{w}) = \mathbb{x} \text{ iff } T(\mathbb{x}) = \mathbb{w}.$$

We call it the <u>inverse of T</u>.

Hence $\mathbb{w} = T(\mathbb{x}) \underset{T:\text{ one-to-one.}}{\Longrightarrow} T^{-1}(T(\mathbb{x})) = \mathbb{x}$ & $T(T^{-1}(\mathbb{w})) = \mathbb{w}$