# Chapter 8

# Diagonalization

## 8.1 Matrix Representations of Linear Transformations

**Exercise 8.1.** Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear operator given by the formula

$$T(x_1, x_2, x_3,\ x_4, x_5) = (7x_1{+}12x_2{-}5x_3,\ 3x_1{+}10x_2{+}13x_4{+}x_5,\ -9x_1{-}x_3{-}3x_5)$$

and let $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4,\ \mathbf{v}_5\}$ and $B' = \{\mathbf{v}'_1,\ \mathbf{v}'_2,\ \mathbf{v}'_3\}$ be the bases for $\mathbb{R}^5$ and $\mathbb{R}^3$, respectively, in which $\mathbf{v}_1 = (1,\ 1,\ 0,\ 0,\ 0)$, $\mathbf{v}_2 = (0,\ 1,\ 1,\ 0,\ 0)$, $\mathbf{v}_3 = (0,\ 0,\ 1,\ 1,\ 0)$, $\mathbf{v}_4 = (0,\ 0,\ 0,\ 1,\ 1)$, $\mathbf{v}_5 = (1,\ 0,\ 0,\ 0,\ 1)$, $\mathbf{v}'_1 = (1,\ 2,\ -1)$, $\mathbf{v}'_2 = (2,\ 1,\ 3)$, and $\mathbf{v}'_3 = (1,\ 1,\ 1)$.

(a) Find the matrix $[T]_{B',B}$.

(b) For the vector $\mathbf{x} = (3,\ 7,\ -4,\ 5,\ 1)$, find $[\mathbf{x}]_B$ and use the matrix obtained in part (a) to compute $[T(\mathbf{x})]_{B'}$.

(c) Find the factorization of $[T]$ which is the standard matrix for the linear transformation $T$ using Formula (28) in Section 8.1.

***Solution.***

(a)
```
v1 = [1 1 0 0 0]'; v2 = [0 1 1 0 0]'; v3 = [0 0 1 1 0]';
v4 = [0 0 0 1 1]'; v5 = [1 0 0 0 1]';
nv1 = [1 2 -1]'; nv2 = [2 1 3]'; nv3 = [1 1 1]';

T = [7 12 -5 0 0; 3 10 0 13 1; -9 0 -1 0 -3];
B1 = [v1 v2 v3 v4 v5]; B2 = [nv1 nv2 nv3];
format short;
```

```
% Find the matrix representation with respect to the bases B1 and B2.
TB = T*B1;
TB1B2 = B2\TB;

disp('The matrix representation of T with respect to the basis B1 and B2 is');
disp(TB1B2);
```

***MATLAB results.***

```
The matrix representation of T with respect to the basis B1 and B2 is
   34.0000    5.0000  -22.0000  -11.0000   22.0000
   40.0000    2.0000  -40.0000  -25.0000   25.0000
  -95.0000   -2.0000   97.0000   61.0000  -65.0000
```

(b)

```
% Find the coordinate vector of x with respect to the basis B1.
x = [3 7 -4 5 1]';  x_B1 = B1\x;
disp('The coordinate vector of x with respect to the basis B is');
disp(x_B1');

% Find the coordinate vector of T(x) with respect to the basis B2.
Tx_B2 = TB1B2 * x_B1;
disp('The coordinate vector of T(x) with respect to the basis B'' is');
disp(Tx_B2');
```

***MATLAB results.***

```
The coordinate vector of x with respect to the basis B is
     9    -2    -2     7    -6

The coordinate vector of T(x) with respect to the basis B' is
  131.0000  111.0000 -228.0000
```

(c)

```
% Transition matrix from B to the standard basis for R^n.
U=B1;

% Transition matrix from B' to the standard basis for R^m.
V=B2;

T=[7 12 -5 0 0 ; 3 10 0 13 1; -9 0 -1 0 -3];

disp('V'); disp(V);
disp('TB1B2'); disp(TB1B2);
disp('inv(U)'); disp(inv(U));
disp('V*TB1B2*inv(U)');disp(V*TB1B2*inv(U));
disp('T'); disp(T);
```

***MATLAB results.***

```
V
     1     2     1
     2     1     1
    -1     3     1
```

```
TB1B2
   34.0000    5.0000  -22.0000  -11.0000   22.0000
   40.0000    2.0000  -40.0000  -25.0000   25.0000
  -95.0000   -2.0000   97.0000   61.0000  -65.0000

inv(U)
    0.5000    0.5000   -0.5000    0.5000   -0.5000
   -0.5000    0.5000    0.5000   -0.5000    0.5000
    0.5000   -0.5000    0.5000    0.5000   -0.5000
   -0.5000    0.5000   -0.5000    0.5000    0.5000
    0.5000   -0.5000    0.5000   -0.5000    0.5000

V*TB1B2*inv(U)
    7.0000   12.0000   -5.0000         0         0
    3.0000   10.0000         0   13.0000    1.0000
   -9.0000   -0.0000   -1.0000   -0.0000   -3.0000

T
     7    12    -5     0     0
     3    10     0    13     1
    -9     0    -1     0    -3
```

## 8.2 Similarity and Diagonalizability

**Exercise 8.2.** (a) Show that the matrix

$$A = \begin{bmatrix} -13 & -60 & -60 \\ 10 & 42 & 40 \\ -5 & -20 & -18 \end{bmatrix}$$

is diagonalizable by finding the nullity of $\lambda I - A$ for each eigenvalue $\lambda$ with the use of Theorem 8.2.11 in the Section 8.2.

(b) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$.

***Solution.***

(a)
```
% For the exact computation of the eigenvalues,
% we use symbolic computation.

% Set A as a symbolic matrix.
A = sym([-13 -60 -60; 10 42 40; -5 -20 -18]);

n = length(A);

% Find the eigenvalues of A by using the command eig.
```

```
eigenvalues = eig(A);

for j = 1 : n
    fprintf('The eigenvalue lambda is '); disp(eigenvalues(j));

    % nullity(lambda*I - A) = n - rank(lambda*I - A);
    nullity = n - rank((eigenvalues(j) * eye(n)) - A);

    fprintf('The nullity of (lambda*I - A) is '); disp(nullity);
end

% Since the geometric multiplicity of each eigenvalue of A
% is the same as the algebraic multiplicity,
% by the Theorem 8.2.11, A is diagonalizable.
```

***MATLAB results.***

```
The eigenvalue lambda is 2
The nullity of (lambda*I - A) is      2
The eigenvalue lambda is 2
The nullity of (lambda*I - A) is      2
The eigenvalue lambda is 7
The nullity of (lambda*I - A) is      1
```

(b)

```
% Since the eigenvalue = 2 of A has the multiplicity = 2,
% find two linearly independent eigenvectors of A corresponding to lambda = 2.

%Find a basis for the null space of (2*I-A).
eigvec12=null((2 * eye(n)) - A);

% Since the eigenvalue = 7 of A has the multiplicity = 1,
% find an eigenvector of A corresponding to lambda = 7.

%Find a basis for the null space of (7*I-A).
eigvec3=null((7 * eye(n)) - A);

p1 = eigvec12(:, 1); p2 = eigvec12(:, 2); p3 = eigvec3(:, 1);

% By the Theorem 8.2.7, since the eigenvectors corresponding to
% distinct eigenvalues are linearly independent,
% the three obtained eigenvectors {p1, p2, p3} form a basis for R^{3}.

disp('A basis {p1, p2, p3} for R^{3} consisting of the eigenvectors of A is');
fprintf('p1 ='); disp(p1');
fprintf('p2 ='); disp(p2');
fprintf('p3 ='); disp(p3');
```

***MATLAB results.***

```
A basis {p1, p2, p3} for R^{3} consisting of the eigenvectors of A is
```

```
p1 =[ -4, 1, 0]
p2 =[ -4, 0, 1]
p3 =[ 3, -2, 1]
```