4.4 A First Look at Eigenvalues and Eigenvectors

Exercise 4.8. (*Eigenvalues and Eigenvectors*)
Use the MATLAB command *eig* to find the eigenvalues and the associated eigenvectors of the matrix A, where

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 3 & 0 & -1 & 4 \\ 1 & 6 & 5 & 6 \end{bmatrix}.$$

Display the results with long digits.

Solution.

```
% Construct the matrix A.
A=[2 -3 1 0; 1 1 2 2; 3 0 -1 4; 1 6 5 6];

% Find the eigenvalues and eigenvectors of A by using eig.
% This command gives AQ = QD.
[Q D] = eig(A);
lambda1 = D(1,1); lambda2 = D(2,2);
lambda3 = D(3,3); lambda4 = D(4,4);

% Extract each column vector as an eigenvector of A.
x1 = Q(:,1); x2 = Q(:,2); x3 = Q(:,3); x4 = Q(:,4);

% Display the result with long digits.
format long;
disp('lambda1 is'); disp(lambda1);
disp('The eigenvector corresponding to lambda1 is'); disp(x1');
disp('lambda2 is'); disp(lambda2);
disp('The eigenvector corresponding to lambda2 is'); disp(x2');
disp('lambda3 is'); disp(lambda3);
disp('The eigenvector corresponding to lambda3 is'); disp(x3');
disp('lambda4 is'); disp(lambda4);
disp('The eigenvector corresponding to lambda4 is'); disp(x4');
```

MATLAB results.

```
lambda1 is
  9.561855032395805
The eigenvector corresponding to lambda1 is
 -0.067716707308095  0.278176502030497  0.322465582156500  0.902246213399589
lambda2 is
 -3.364648937746373
The eigenvector corresponding to lambda2 is
  0.275562522991092  0.197508356444458 -0.885771126913498  0.316962546342283
lambda3 is
```

```
  1.802793905350564
The eigenvector corresponding to lambda3 is
 -0.833621905475750 -0.103812731179200 -0.147042873144503  0.522183711938150
lambda4 is
 -3.860931435448914e-16
The eigenvector corresponding to lambda4 is
 -0.705886578756789 -0.456750139195570  0.041522739926871  0.539795619049310
```

Remark. In fact, if we compute λ_4 by hand, we can obtain that $\lambda_4 = 0$. However, from the result, we see that the resulting value of λ_4 seems to be nonzero even though it is small enough. This is due to roundoff errors in arithmetic operations. Please refer to the help command of *eps*, then you can see that $eps = 2.220446049250313e{-}016$ is floating-point relative accuracy, which means that *eps* value is the allowable tolerance when we do numerical computations with rounding floating-point number off. (*i.e.*, *eps* is an upper bound on the relative error due to rounding in floating point arithmetic.) Therefore, we can regard the resulting value of λ_4 as zero.

Exercise 4.9. (*Eigenvalues and Eigenvectors*)

Define an nth-order checkboard matrix C_n to be a matrix that has a 1 in the upper left corner and alternates between 1 and 0 along rows and columns (see the figure below). Find the eigenvalues of $C_1, C_2, \cdots$ to make a conjecture about the eigenvalues of C_n. What can you say about the eigenvalues of C_n?

1	0	1	0
0	1	0	1
1	0	1	0
0	1	0	1

Solution.

```
format short;
n=10;   % Set the size of the large check board

% Construct your checkboard
CheckBoard=zeros(n);
CheckBoard(1:2:n, 1:2:n)=1;
CheckBoard(2:2:n, 2:2:n)=1;
for i=1:n
    Cn=CheckBoard(1:i, 1:i);
    [Qn Dn]=eig(Cn);     % Eigenvectors and eigenvalues
    fprintf('The size of the checkboard is %d \n',i);
```

```
    disp(diag(Dn)');
end
```

MATLAB results.

```
The size of the checkboard is 1
     1
The size of the checkboard is 2
     1 1
The size of the checkboard is 3
     0 1 2
The size of the checkboard is 4
     0 0 2 2
The size of the checkboard is 5
   -0.0000 -0.0000  0.0000  2.0000  3.0000
The size of the checkboard is 6
   -0.0000  -0.0000  -0.0000  -0.0000  3.0000  3.0000
The size of the checkboard is 7
   -0.0000  -0.0000   0.0000   0.0000  0.0000  3.0000  4.0000
The size of the checkboard is 8
   -0.0000  -0.0000  -0.0000  0.0000  0.0000  0.0000  4.0000  4.0000
The size of the checkboard is 9
   -0.0000  -0.0000  -0.0000 -0.0000   0   0.0000  0.0000  4.0000  5.0000
The size of the checkboard is 10
   -0.0000  -0.0000  -0.0000  0  0.0000  0.0000  0.0000  0.0000  5.0000  5.0000
```

We may conclude that the eigenvalues of C_n are given as follows:

$$\begin{cases} 1 & \text{if } n = 1, \\ k,\, k,\, \underbrace{0,\, 0,\, \cdots,\, 0}_{(n-2)} & \text{if } n = 2k, \\ k,\, k+1,\, \underbrace{0,\, 0,\, \cdots,\, 0}_{(n-2)} & \text{if } n = 2k+1, \end{cases}$$

where k is a positive integer.