1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. If $T \circ T \circ T \circ T = 0$, then $S \circ T$ is not one-to-one.

(b) There exists $t \in \mathbb{R}$ such that the solution space of $A\mathbf{x} = 0$ has zero dimension where

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & t-1 & t \end{bmatrix}.$$

(c) Let S be a set of linearly independent nonzero vectors in $\mathbb{R}^3$ such that $|S| = 2$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. ($|A|$ denotes the number of elements of a finite set A.) If $S \cup \{\mathbf{a}\}$ is a basis for $\mathbb{R}^3$ and $\mathbf{a} - \mathbf{b} \in \operatorname{span}(S)$, then $S \cup \{\mathbf{b}\}$ is also a basis for $\mathbb{R}^3$.

Solution.

(a) True. Let A, B be the standard matrices for T, S. Then $A^4 = 0$ and so $\det A = 0$. Since BA is the standard matrix for $S \circ T$ and $\det BA = \det B \det A = 0$, $S \circ T$ is not one-to-one.

(b) False. $\det A = 0$ for any $t \in R$. However $A\mathbf{x} = 0$ has the solution space of dimension zero if and only if A is non-singular.

(c) True. Set $\mathbf{S} = \{\mathbf{u}, \mathbf{v}\}$. It suffices to show that $\{\mathbf{u}, \mathbf{v}, \mathbf{b}\}$ is linearly independent. Set $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{b} = 0$. Since $\mathbf{a} - \mathbf{b} \in \operatorname{span}(\mathbf{S})$, $\mathbf{b} = \mathbf{a} + k_4\mathbf{u} + k_5\mathbf{v}$. Thus $k_1\mathbf{u} + k_2\mathbf{v} + k_3(\mathbf{a} + k_4\mathbf{u} + k_5\mathbf{v}) = 0$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{a}\}$ is linearly independent, $k_3 = 0$. Then $k_1\mathbf{u} + k_2\mathbf{v} = 0$, so $k_1 = k_2 = 0$. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{b}\}$ is linearly independent.

2
10 points

Let $\mathbf{v}_1 = (-2, 1, 3)$, $\mathbf{v}_2 = (1, 0, 4)$, and $\mathbf{v}_3 = (3, -4, 2)$. Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ or not. Explain your answer.

Solution. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then it is a basis for $\mathbb{R}^3$. Let $A = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]^T$. Then A is non-singular if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Since $\det A = -34 \neq 0$, A is non-singular. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.