1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) A linear system of four equations in four unknowns cannot have infinitely many solutions.

(b) A square matrix A is said to be idempotent if $A^2 = A$. If A is idempotent, then A is invertible and $A^{-1} = I - A$.

(c) There is only one matrix A for which

$$A \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution.

(a) False. For example, consider the following system of equations:

$$\begin{aligned} x + w &= 0, \\ y + w &= 0, \\ z + w &= 0, \\ 2z + 2w &= 0. \end{aligned}$$

This system has infinitely many solutions.

(b) False. $A(I - A) = A - A^2 = A - A = \mathit{0}$.

(c) False. Let $A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 & -3 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Then

$$A_1 \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & -1 \\ -1 & 1 \end{bmatrix} = A_2 \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 Solve the given homogeneous system of linear equations.

10 points

$$\begin{cases} x_1 & & +2x_3 & & = 0, \\ 3x_1 & & +6x_3 & +x_4 & = 0, \\ -x_1 & +x_2 & -2x_3 & -3x_4 & = 0, \\ & 2x_2 & & -6x_4 & = 0 \end{cases}$$

Solution. The augmented matrix of this system is **(+2 points)**

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 3 & 0 & 6 & 1 & 0 \\ -1 & 1 & -2 & -3 & 0 \\ 0 & 2 & 0 & -6 & 0 \end{array}\right].$$

Using the Gauss-Jordan elimination, we can reduce the matrix as follows:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence we get

$$\begin{cases} J_1 = -2t, \\ J_2 = 0, \\ J_3 = t, \\ J_4 = 0, \end{cases}$$

where $-\infty < t < \infty$.