1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let n be a positive integer with $n \geq 2$. Let A be a nonzero $n \times 1$ matrix and B a nonzero $1 \times n$ matrix. Then, 0 is an eigenvalue of AB.

(b) Let A be a 2×2 matrix which is not the identity matrix. If A is orthogonal and 1 is an eigenvalue of A, then $\det(A) = 1$.

(c) Let A be a 2×2 matrix. If

$$A^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

then $\det(A) = 1$.

Solution.

(a) True. Since $n \geq 2$, we can find a nonzero $n \times 1$ matrix C such that $BC = 0$. So, $(AB)C = A \cdot 0 = 0 = 0 \cdot C$. Therefore, 0 is an eigenvalue of AB.

(b) False. Let $H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, i.e., the standard matrix for a reflection about a line through the origin. Then $\det(H_{\theta/2}) = -1$.

(c) True. Denote

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\begin{aligned}
\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} -c & a \\ -d & b \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} 0 & ad - bc \\ bc - ad & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{aligned}$$

Thus, $\det(A) = ad - bc = 1$.

2 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a orthogonal linear operator. Show that $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

10 points

Solution. Let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in $\mathbb{R}^n$. We can derive $\mathbf{x}\cdot\mathbf{y} = \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2)$. Thus,

$$\begin{aligned} T(\mathbf{x}) \cdot T(\mathbf{y}) &= \frac{1}{4}(\|T(\mathbf{x}) + T(\mathbf{y})\|^2 - \|T(\mathbf{x}) - T(\mathbf{y})\|^2) \\ &= \frac{1}{4}(\|T(\mathbf{x} + \mathbf{y})\|^2 - \|T(\mathbf{x} - \mathbf{y})\|^2) \\ &= \frac{1}{4}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2) \qquad [T \text{ IS LENGTH PRESERVING}] \\ &= \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$