1 (3+3+4 points) Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

(a) Let n be a positive integer with $n \geq 2$. Let A be a nonzero $n \times 1$ matrix and B a nonzero $1 \times n$ matrix. Then $\det(AB) = 0$.

(b) Let A be a square matrix. If A is orthogonal, then 1 is an eigenvalue of A.

(c) Let A be a $n \times n$ matrix with a positive integer $n \geq 2$. If A is nilpotent, then 1 is an eigenvalue of $I - A$.

Solution.

(a) True. Since $n \geq 2$, we can find a $n \times 1$ matrix C so that $BC = 0$. Then $(AB)C = A(BC) = A \cdot 0 = 0 = 0 \cdot C$. Therefore, C is an eigenvector of AB and the corresponding eigenvalue is 0. Thus, $\det(AB) = 0$.

(b) False. Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then -1 is only one eigenvalue of A.

(c) True. Since A is nilpotent, $\det(A) = 0$. By *Thm 4.4.12*, 0 is an eigenvalue and there is a corresponding nonzero eigenvector $\mathbf{v}$. Then $(I - A)\mathbf{v} = I\mathbf{v} - A\mathbf{v} = I\mathbf{v} = \mathbf{v}$. Therefore, 1 is an eigenvalue of $I - A$.

2 (10 points) For positive integers m, n, let A be an $m \times n$ matrix. Suppose that $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Show that the column vectors of A are orthonormal.

Solution. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the set of standard unit vectors in $\mathbb{R}^n$. Then, $\{A\mathbf{e}_1, A\mathbf{e}_2, \cdots, A\mathbf{e}_n\}$ is the set of column vectors of A. By the assumption, for all i, $1 = \mathbf{e}_i \cdot \mathbf{e}_i = A\mathbf{e}_i \cdot A\mathbf{e}_i$ and whenever $i \neq j$, $0 = \mathbf{e}_i \cdot \mathbf{e}_j = A\mathbf{e}_i \cdot A\mathbf{e}_j$.