6.4.4

(a) Naturally,

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so

$$[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}, [T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$$

(b) By computation,

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$$

and

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$

(c) By (b),

$$T_1(T_2(x_1, x_2, x_3)) = [T_1 \circ T_2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (4x_1 + 8x_2, -2x_1 - 4x_2 - x_3, -x_1 - 2x_2 + 3x_3)$$

$$T_2(T_1(x_1, x_2, x_3)) = [T_2 \circ T_1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (2x_2, x_1 + 3x_2, 17x_1 + 3x_2)$$

Of course, this result matches with direct computation. For example,

$$\begin{aligned} T_2(T_1(x_1, x_2, x_3)) &= T_2(4x_1, -2x_1 + x_2, -x_1 - 3x_2) \\ &= (4x_1 + 2(-2x_1 + x_2), -(-x_1 - 3x_2), 4(4x_1) - (-x_1 - 3x_2)) \\ &= (2x_2, x_1 + 3x_2, 17x_1 + 3x_2) \end{aligned}$$

6.4.10

(a) The standard matrix for given operation is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

This result matches when you try to get this matrix by observing the movements of the standard basis. In other words,

$$\begin{aligned} &(1,0,0) \to (1,0,0) \to (0,0,-1) \to (0,0,-1) \\ &(0,1,0) \to (0,0,-1) \to (-1,0,0) \to (1,0,0) \\ &(0,0,1) \to (0,1,0) \to (0,1,0) \to (0,-1,0) \end{aligned}$$

(b) The standard matrix for given operation is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Again, you may observe this result by

$$\begin{aligned} &(1,0,0) \to (\sqrt{3}/2, 1/2, 0) \to (\sqrt{3}/2, 1/2, 0) \to (0, 1/2, 0) \\ &(0,1,0) \to (-1/2, \sqrt{3}/2, 0) \to (-1/2, \sqrt{3}/2, 0) \to (0, \sqrt{3}/2, 0) \\ &(0,0,1) \to (0,0,1) \to (0,0,-1) \to (0,0,-1) \end{aligned}$$

6.4.12

At first, following descriptions are not the unique description for each problem.

(a) Note that

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, multiplication by A is a compression by a factor $\frac{1}{2}$ in the x-direction followed by the compression by a factor $\frac{1}{3}$ in the y-direction.

(b) Note that

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Hence, multiplication by A is a shear in the x-direction with factor 2, followed by reflection about the line $y = x$.

(c) Note that

$$\begin{bmatrix} 0 & 2 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, multiplication by A is the expansion in the x-direction by the factor 5 and reflection about the y-axis, followed by the expansion in the y-direction by the factor of 2, followed by the reflection about the line $y = x$.

(d) Note that

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Hence, multiplication by A is the shear in the x-direction with factor 4, followed by the shear in the y-direction with factor 2.

6.4.22

It is enough to check whether

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 7 & 4 & 5 \end{bmatrix}$$

is invertible or not, and to find the inverse of it, if exists. For the invertibility, there are many ways. One of them is to check determinant, which is $1 \cdot (5-16) - 2 \cdot (10-28) + 1 \cdot (8-7) = 26 \neq 0$. Hence, given linear operator is one-to-one. Now

$$\begin{aligned} -11w_1 - 6w_2 + 7w_3 &= -11(x_1 + 2x_2 + x_3) - 6(2x_1 + x_2 + 4x_3) + 7(7x_1 + 4x_2 + 5x_3) = 26x_1 \\ 9w_1 - w_2 - w_3 &= 9(x_1 + 2x_2 + x_3) - (2x_1 + x_2 + 4x_3) - (7x_1 + 4x_2 + 5x_3) = 13x_2 \\ w_1 + 10w_2 - 3w_3 &= (x_1 + 2x_2 + x_3) + 10(2x_1 + x_2 + 4x_3) - 3(7x_1 + 4x_2 + 5x_3) = 26x_3 \end{aligned}$$

Hence, the standard matrix for the inverse is

$$\begin{bmatrix} -\frac{11}{26} & -\frac{3}{13} & \frac{7}{26} \\ \frac{9}{13} & -\frac{1}{13} & -\frac{1}{13} \\ \frac{1}{26} & \frac{5}{13} & -\frac{3}{26} \end{bmatrix}$$

and $T^{-1}(w_1, w_2, w_3)$ is

$$T^{-1}(w_1, w_2, w_3) = \frac{1}{26}(-11w_1 - 6w_2 + 7w_3, 18w_1 - 2w_2 - 2w_3, w_1 + 10w_2 - 3w_3)$$

6.4.24

(a) Note that the reflection about the x-axis followed by the reflection about the y-axis is exactly the reflection about the origin, which is same as the reflection about the y-axis followed by the reflection about the x-axis. In computation,

$$\begin{aligned}[T_1 \circ T_2] = [T_1][T_2] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [T_2][T_1] = [T_2 \circ T_1]\end{aligned}$$

Thus, $T_1 \circ T_2 = T_2 \circ T_1$.

(b) For the vector $(1,0)$, $T_2 \circ T_1(1,0) = T_2(1,0)$, which is not on the x-axis if $\theta \neq n\pi$. But $T_1 \circ T_2(1,0) = T_1(T_2(1,0))$ is always on the x-axis. Thus, $T_2 \circ T_1 \neq T_1 \circ T_2$ if $\theta \neq n\pi$. If $\theta = n\pi$, then you may check $T_1 \circ T_2 = T_2 \circ T_1$ easily. In computation,

$$\begin{aligned}[T_1 \circ T_2] = [T_1][T_2] &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix} \\ [T_2 \circ T_1] = [T_2][T_1] &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}\end{aligned}$$

(c) Note that $[T_1] = kI$, hence $[T_1 \circ T_2] = [T_1][T_2] = kI[T_2] = k[T_2] = [T_2](kI) = [T_2][T_1] = [T_2 \circ T_1]$. Thus, $T_1 \circ T_2 = T_2 \circ T_1$.

6.4.26

Since $H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$, we have $H_{\pi/4} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $H_{\pi/8} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$. Then

$$H_{\pi/8}H_{\pi/4} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} = R_{-\pi/4}$$

is the standard matrix of the clockwise rotation, by the angle $\pi/4$.

6.4.D1

(a) (T). If $T_2 \circ T_1$ is one-to-one, then $T_1(\mathbf{x}) = T_1(\mathbf{y})$ implies $T_2 \circ T_1(\mathbf{x}) = T_2(T_1(\mathbf{x})) = T_2(T_1(\mathbf{y})) = T_2 \circ T_1(\mathbf{y})$, so $\mathbf{x} = \mathbf{y}$ since $T_2 \circ T_1$ is one-to-one. Hence, T_1 is one-to-one if $T_2 \circ T_1$ is one-to-one, so given statement is true.

(b) (F). Let $T_2 : \mathbb{R}^2 \to \mathbb{R}$ as $T_2(x, y) = x$ and $T_1 : \mathbb{R} \to \mathbb{R}^2$ as $T_1(x) = (x, 0)$. Then, T_1 is not onto, but $T_2 \circ T_1(x) = x$ is onto.

(c) (F). For the example of (b), T_2 is not one-to-one but $T_2 \circ T_1$ is one-to-one.

(d) (T). If $T_2 \circ T_1$ is onto, then $\mathbb{R}^k = \text{ran } (T_2 \circ T_1) = T_2(T_1(\mathbb{R}^n)) \subseteq T_2(\mathbb{R}^m) = \text{ran } T_2 \subseteq \mathbb{R}^k$. Hence, ran $T_2 = \mathbb{R}^k$, so T_2 is also onto.