7.4.6

The reduced row echelon form of given A is

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & -\frac{5}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{7}{3} & -\frac{4}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, vectors $(1, 0, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, -\frac{5}{3})$ and $(0, 1, \frac{1}{3}, -\frac{7}{3}, -\frac{4}{3}, \frac{1}{3}, \frac{4}{3})$ form a basis of the row space of A and vectors $(-\frac{1}{3}, -\frac{1}{3}, 1, 0, 0, 0, 0)$, $(-\frac{2}{3}, \frac{7}{3}, 0, 1, 0, 0, 0)$, $(-\frac{2}{3}, \frac{4}{3}, 0, 0, 1, 0, 0)$, $(-\frac{4}{3}, -\frac{1}{3}, 0, 0, 0, 1, 0)$, and $(\frac{5}{3}, -\frac{4}{3}, 0, 0, 0, 0, 1)$ form a basis of the null space of A. Thus, the rank of A is 2 and the nullity of A is 5, and hence, the sum of rank and nullity is 7, which confirm the dimension theorem.

7.4.8

(a) Since the rank of A is 5, the number of pivot variables is 5 and the number of parameters is $9 - 5 = 4$.

(b) Since the nullity of A is 3, the number of parameters is 3 and the number of pivot variables is $6 - 3 = 3$.

(c) Since the number of nonzero rows in a row echelon form is 3, the number of pivot variables, which is equal to the rank, is 3 and the number of parameters is $7 - 3 = 4$.

7.4.12

The reduced row echelon form of the matrix $A = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{19}{8} & \frac{23}{4} \\ 0 & 1 & 0 & \frac{23}{16} & -\frac{15}{8} \\ 0 & 0 & 1 & -\frac{43}{16} & \frac{43}{8} \end{bmatrix}$$

which proves that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Moreover, this gives two vectors, with some rescailing, $(-38, 23, -43, -16, 0)$ and $(46, -15, 43, 0, -8)$ which form a basis of the null space of the matrix A, which extend the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to a basis for $\mathbb{R}^5$.

7.4.19

Since given matrix has rank 1, we have $c_1(x, y, z) + c_2(1, x, y) = 0$ where $(c_1, c_2) \neq 0$. If $c_1 = 0$, then $c_2(1, x, y) = 0$ implies $c_2 = 0$ which is contradiction. Hence, $c_1 \neq 0$, so we have $(x, y, z) = -\frac{c_2}{c_1}(1, x, y) = t(1, x, y)$. Then, $x = t$, $y = tx = t^2$ and $z = ty = t^3$.

7.4.24

(a) Simply,

$$A^2 = \mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T = \mathbf{u}(\mathbf{v} \bullet \mathbf{u})\mathbf{v}^T = (\mathbf{u} \bullet \mathbf{v})\mathbf{u}\mathbf{v}^T = (\mathbf{u} \bullet \mathbf{v})A$$

since $\mathbf{u} \bullet \mathbf{v}$ is a scalar.

(b) First, $A\mathbf{u} = \mathbf{u}\mathbf{v}^T\mathbf{u} = (\mathbf{u} \bullet \mathbf{v})\mathbf{u}$, so $\mathbf{u} \bullet \mathbf{v}$ is an eigenvalue of A. Suppose $A\mathbf{x} = \lambda\mathbf{x}$ for nonzero vector $\mathbf{x}$ and nonzero λ. Then,

$$A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$
$$A^2\mathbf{x} = (\mathbf{u} \bullet \mathbf{v})A\mathbf{x} = \lambda(\mathbf{u} \bullet \mathbf{v})\mathbf{x}$$

Since $\mathbf{x}$ is a nonzero vector, $\lambda^2 = \lambda(\mathbf{u} \bullet \mathbf{v})$. Then, $\lambda \neq 0$, so $\lambda = \mathbf{u} \bullet \mathbf{v}$. Thus, if $\mathbf{u} \bullet \mathbf{v} \neq 0$, then it is the only nonzero eigenvalue of A. Moreover, if $\mathbf{u} \bullet \mathbf{v} = 0$, then there is no nonzero eigenvalue of A. Thus, if A has a nonzero eigenvalue, then it must be $\mathbf{u} \bullet \mathbf{v}$.

(c) By Theorem 7.4.7, A is a rank 1 matrix if and only if there exists nonzero vector $\mathbf{u}, \mathbf{v}$ such that $A = \mathbf{u}\mathbf{v}^T$. If $A^2 = A$, then by (a), $\mathbf{u} \bullet \mathbf{v} = 1$. Hence, 1 is an eigenvalue by (b), so there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$. Now, $(I - A)\mathbf{x} = \mathbf{x} - A\mathbf{x} = \mathbf{0}$, so the homogeneous equation $(I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and hence, $I - A$ is not invertible by Theorem 7.2.7. Conversely, if $I - A$ is not invertible, then there is a nonzero vector $\mathbf{x}$ such that $(I - A)\mathbf{x} = \mathbf{0}$. Hence, $A\mathbf{x} = I\mathbf{x} = \mathbf{x}$, so 1 is an eigenvalue of A. Then, by additional conclusion of (b) $\mathbf{u} \bullet \mathbf{v} = 1$. Hence, by (a), $A^2 = (\mathbf{u} \bullet \mathbf{v})A = A$. Thus, $I - A$ is invertible if and only if $A^2 \neq A$.

7.4.D4

The number of leading 1's in the reduced row echelon form of A is rank. Since the rank of A is at most 3, so the first blank is 3. The number of parameter in a general solution of $A\mathbf{x} = 0$ is 3 minus the rank. Thus, the second blank is 3.

7.4.P4

By assumption, V is a basis of $\text{row}(A)$ and W is a basis of $\text{null}(A)$. Since $\dim(\mathbb{R}^n) = n$ and V, W has n elements in total, it is enough to show $\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{w}_{k+1}, \mathbf{w}_n$ is linearly independent. Suppose $\sum_{i=1}^{k} c_i \mathbf{v}_i + \sum_{i=k+1}^{n} c_i \mathbf{w}_i = 0$. Then

$$\text{row}(A) \ni \sum_{i=1}^{k} c_i \mathbf{v}_i = \sum_{i=k+1}^{n} (-c_i)\mathbf{w}_i \in \text{null}(A).$$

Since $\text{row}(A) \cap \text{null}(A) = \text{row}(A) \cap \text{row}(A)^{\perp} = \{\mathbf{0}\}$, we get

$$\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0} = \sum_{i=k+1}^{n} (-c_i)\mathbf{w}_i$$

Then, since V is linearly independent, $c_1 = \cdots = c_k = 0$ and since W is linearly independent, $c_{k+1} = \cdots = c_n = 0$. Thus, $\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{w}_{k+1}, \mathbf{w}_n$ is linearly independent, so done. This is genarlly true, that for a fixed given global vector space, if V is a subspace of dimension m and W is a subspace of dimension n such that $V \cap W = \{\mathbf{0}\}$, then for any basis S of V and T of W, $S \cup T$ is a basis of $\text{span}(V \cup W)$ and $\dim(\text{span}(V \cup W)) = m + n$.

7.4.P5 Exercise T4 says if A has n columns and B has n rows, then

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \leq \operatorname{rank}(AB) \leq \operatorname{rank}(A)$$
$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \leq \operatorname{rank}(AB) \leq \operatorname{rank}(B)$$

Hence, if A, B are $n \times n$ square matrices, then

$$\begin{aligned}
\operatorname{nullity}(A) &= n - \operatorname{rank}(A) \\
&\leq n - \operatorname{rank}(AB) \\
&= \operatorname{nullity}(AB) \\
&\leq n - (\operatorname{rank}(A) + \operatorname{rank}(B) - n) \\
&= (n - \operatorname{rank}(A)) + (n - \operatorname{rank}(B)) \\
&= \operatorname{nullity}(A) + \operatorname{nullity}(B)
\end{aligned}$$

Thus,

$$\operatorname{nullity}(A) \leq \operatorname{nullity}(AB) \leq \operatorname{nullity}(A) + \operatorname{nullity}(B)$$

and similarily,

$$\operatorname{nullity}(B) \leq \operatorname{nullity}(AB) \leq \operatorname{nullity}(A) + \operatorname{nullity}(B).$$

7.5.10

(a) Since the number of rows is larger than the number of columns, A cannot have full row rank. Since the first and the second column are linearly independent, it has full column rank.

(b) Since the number of columns is larger than the number of rows, A cannot have full column rank. Since the first and the second row are linearly independent, it has full row rank.

(c) It cannot have full column rank. Since the second row is the twice of the first row, two rows are linearly dependent, so it does not have full row rank, also.

(d) Since the row rank and the column rank are same, A has full row rank if and only if it has full column rank. Since two rows are linearly independent, A has both full column rank and full row rank.

7.5.14

Consider the augmented matrix $\left[A \mid \mathbf{b}\right]$ and make reduced row echelon A, which gives

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{5}{3}b_1 + \frac{4}{3}b_2 \\ 0 & 1 & \frac{2}{3}b_1 - \frac{1}{3}b_2 \\ 0 & 0 & b_1 - 2b_2 + b_3 \end{array}\right]$$

Hence, if $b_1 - 2b_2 + b_3 \neq 0$, then the system $A\mathbf{x} = \mathbf{b}$ has no solution. If $b_1 - 2b_2 + b_3 = 0$, then the system has unique solution. Thus, the system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$ in $\mathbb{R}^3$. Moreover, $A\mathbf{x} = \mathbf{0}$ has already a solution $\mathbf{x} = \mathbf{0}$ which is trivial, so it has only the trivial solution.

7.5.16

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -6 \end{bmatrix}$$

so $\text{row}(A) = \text{span}(\{(1,0,7),(0,1,-6)\})$. Now, $A^TA = \begin{bmatrix} 5 & 7 & -7 \\ 7 & 10 & -11 \\ -7 & -11 & 17 \end{bmatrix}$ where reduced row echelon form of it is

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\text{row}(A) = \text{row}(A^TA)$ and $\text{null}(A) = \text{row}(A)^{\perp} = \text{row}(A^TA)^{\perp} = \text{null}(A^TA) = \text{span}(\{(-7,6,1)\})$

7.5.18

Note that $(1, 2, 3, -1)$ and $(3, -1, 1, 2)$ are linearly independent, but $(4, 1, 4, 1) = (1, 2, 3, -1) + (3, -1, 1, 2)$. Hence, if we use row echelon form, we have $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $b_3 - b_1 - b_2 = 0$. And for such case, $\frac{1}{7}(b_1 + 2b_2, 3b_1 - b_2, 0, 0) + s(-5, -8, 7, 0) + t(-3, 5, 0, 7)$ is a solution for any s, t, so given underdetermined linear system has infinitely many solutions.

7.5.D2

Note that $\mathrm{rank}(A) = \dim(\mathrm{row}(A))$ by definition and $\mathrm{rank}(A) = \mathrm{rank}(A^T A) = \mathrm{rank}(AA^T)$ by Theorem 7.5.8 and Theorem 7.5.9. Hence, both blanks are filled by k.

7.5.D4

(a) (F). The row rank is always equal to the column rank.

(b) (F). It is true for every matrix.

(c) (T). Since A is invertible, the column space of A already contains $\mathbf{b}$. Hence, the column rank of A is same as the augmented matrix.

(d) (T). Since the column rank is same as the row rank, the number of rows must be equal to the number of columns if it has both full row rank and full column rank.

(e) (T). For $m \times n$ matrix A, AA^T is a $m \times m$ matrix and $A^T A$ is a $n \times n$ matrix. Now, if AA^T and $A^T A$ are invertible, then $m = \text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^T A) = n$, so A is square.

(f) (T). Since the sum of the rank and the nulity is 3 and both are integer, they cannot be same.

7.5.P5

If $A\mathbf{x} = \mathbf{0}$, then $A^2\mathbf{x} = \mathbf{0}$. Hence, $\text{null}(A) \subseteq \text{null}(A^2)$. Now, since A and A^2 have same rank, they have same nullity. Thus, $\text{null}(A) = \text{null}(A^2)$. Now, note that, $\mathbf{0} \in \text{null}(A) \cap \text{col}(A)$ naturally. Then, suppose $\mathbf{b} \in \text{null}(A) \cap \text{col}(A)$. From that, there exists $\mathbf{x}$ such that $\mathbf{b} = A\mathbf{x}$ since $\mathbf{b} \in \text{col}(A)$. Now, $A^2\mathbf{x} = A\mathbf{b} = \mathbf{0}$ since $\mathbf{b} \in \text{null}(A)$. Then, $\text{null}(A) = \text{null}(A^2)$, so $A^2\mathbf{x} = \mathbf{0}$ implies $A\mathbf{x} = \mathbf{0}$. Hence, $\mathbf{b} = A\mathbf{x} = \mathbf{0}$. Thus, $\{\mathbf{0}\} = \text{null}(A) \cap \text{col}(A)$.

7.5.P6

Let $\mathbf{e}_{i,n}$ be the column vector in $\mathbb{R}^n$ such that only ith component is 1 and other are 0. Then, for any matrix with proper size to product be defined, $A\mathbf{e}_{i,n}$ is ith column of A and $\mathbf{e}_{i,n}^T A$ is ith row of A. Now, if A is a $m\times n$ matrix, define $R_{i_1,i_2,\cdots,i_k} = \begin{bmatrix} \mathbf{e}_{i_1,m}^T \\ \vdots \\ \mathbf{e}_{i_k,m}^T \end{bmatrix}$ and $C_{j_1,\cdots,j_k} = \begin{bmatrix} \mathbf{e}_{j_1,n} & \cdots & \mathbf{e}_{j_k,n} \end{bmatrix}$. Then, by above observation, for $1 \le a, b \le k$, the a, b-entry of $R_{i_1,\cdots,i_k}AC_{j_1,\cdots,j_k}$ is $\mathbf{e}_{i_a,m}^T A\mathbf{e}_{j_b,n}$, which is same as i_a, j_b-entry of A. Hence, $R_{i_1,\cdots,i_k}AC_{j_1,\cdots,j_k}$ is the $k \times k$ submatrix of A where it is consists of i_1th, $\cdots$, i_kth rows of A and j_1, $\cdots$, j_k columns of A. Also, for any $k \times k$ submatrix B of A, there exists $I = \{i_1, \cdots, i_k\}$, $J = \{j_1, \cdots, j_k\}$ such that $B = R_I AC_J$.

First, suppose there exists $k\times k$ invertible submatrix B of A. Then, $B = R_I AC_J$ for some I, J. Note that for any matrix C, D we have $\text{rank}(CD) \le \text{rank}(C), \text{rank}(D)$. Hence, $k = \text{rank}(B) = \text{rank}(R_I AC_J) \le \text{rank}(A)$. Thus, if A has a $k \times k$ invertible submatrix, then $\text{rank}(A) \ge k$.

Second, suppose $\text{rank}(A) = k$. From Theorem 7.2.2.(a), A has k columns that spans $\text{col}(A)$. Choose such columns as J. Then, AC_J is an $m \times k$ matrix which consists of the chosen k columns of A. Hence, the column rank of AC_J is k, which is same as the row rank of AC_J. Then, again, by Theorem 7.2.2.(a), AC_J has k rows that spans $\text{row}(AC_J)$. Choose such rows as I. Then, $R_I AC_J$ is a $k \times k$ matrix which consists of the chosen k rows of AC_J. Hence, the row rank of $R_I AC_J$ is k. Then, $R_I AC_J$ is a $k \times k$ matrix with $\text{rank}(R_I AC_J) = k$, so $R_I AC_J$ is a $k \times k$ invertible submatrix of A. Thus, if $\text{rank}(A) = k$, then there exists a $k \times k$ invertible submatrix.

Those two facts proves our problem. First, if A is a nonzero matrix with rank k, by the second fact, there exists at least one invertible $k \times k$ submatrix. Moreover, if there is square submatrix with size $\ge k + 1$ which is nonsingular, then by the first fact, $\text{rank}(A) \ge k + 1$ which is contradiction. Thus, all square submatrices with larger size are singular. Conversely, if the size of the largest invertible submatrix of a nonzero matrix A is $k \times k$, then by the first fact, $\text{rank}(A) \ge k$. But if $\text{rank}(A) \ge k + 1$, then by the second fact, there exists a $(k + 1)\times(k + 1)$ invertible submatrix which is contradiction that $k \times k$ is the largest. Thus, $\text{rank}(A) = k$.