1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let V be the line in $\mathbb{R}^3$ through the origin spanned by $\mathbf{u} = (6, 2, -3)$, and W be the plane in $\mathbb{R}^3$ whose equation is $2x + 3y + 6z = 0$. Then V is a subspace of W.

(b) Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. If $T \circ S \circ T$ is not one-to-one, then there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \notin \operatorname{ran}(T \circ S)$.

(c) Let S be a set of nonzero vectors in $\mathbb{R}^3$ such that $|S| = 4$. ($|A|$ denotes the number of elements of a finite set A.) If $\operatorname{span}(S) = \mathbb{R}^3$, then the number of possible $S_0 \subset S$ such that S_0 is a basis for $\mathbb{R}^3$ is at least 2.

Solution.

(a) True. Since $\mathbf{u} \in W$, $V = \operatorname{span}(\{\mathbf{u}\}) \subset W$.

(b) True. Let A, B be the standard matrices for T, S. Since ABA is the standard matrix for $T \circ S \circ T$, $0 = \det ABA = (\det A)^2(\det B)$. Thus $\det A = 0$ or $\det B = 0$. Since AB is the matrix representation for $T \circ S$ and $\det AB = \det A \det B = 0$, $T \circ S$ is not onto.

(c) True. Note that there exists $S_0 \subset S$ such that S_0 is a basis for $\mathbb{R}^3$. Thus it suffices to show that the number of possible $S_0 \subset S$ such that S_0 is a basis for $\mathbb{R}^3$ cannot be 1. Suppose it is 1. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset S$ be a basis for $\mathbb{R}^3$ and $\mathbf{v}_4$ be the last vector of S. Then $\mathbf{v}_4$ is contained in $\operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2\}), \operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_3\}), \operatorname{span}(\{\mathbf{v}_2, \mathbf{v}_3\})$. Set

$$\mathbf{v}_4 = a\mathbf{v}_1 + b\mathbf{v}_2 = c\mathbf{v}_2 + d\mathbf{v}_3 = e\mathbf{v}_1 + f\mathbf{v}_3.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, $\mathbf{v}_4 = 0$, a contradiction. Thus the number of possible $S_0 \subset S$ such that S_0 is a basis for $\mathbb{R}^3$ is at least 2.

2 (10 points) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (2, 1, -1), \mathbf{v}_2 = (3, 3, 2), \mathbf{v}_3 = (-3, -1, 3)$. Determine whether $\dim(\operatorname{span}(S)) = 3$ or not. Explain your answer.

Solution. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\dim(\operatorname{span}(S)) = 3$. Let $A = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]^T$. Then A is non-singular if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Since $\det A = 1 \neq 0$, A is non-singular. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, which means $\dim(\operatorname{span}(S)) = 3$.