2.1.12 (a) Find a linear equation in the variables x_1 and x_2 whose solution set is given by the equations $x_1 = -3 + t$, $x_2 = 2t$.
(b) Show that the solution set is also given by the equations $x_1 = t$, $x_2 = 2t + 6$.

Solution. (a) Rearranging the equation, we get $t = x_1 + 3$. Since $x_2 = 2t$,

$$x_2 = 2t = 2(x_1 + 3) = 2x_1 + 6.$$

$\square$

(b) Plug in the parametrization of x_1, x_2 to the solution of (a), then

$$x_2 = 2x_1 + 6 \quad \Leftrightarrow \quad 2t + 6 = 2t + 6.$$

$\square$

2.1.26 Describe an elementary row operation that produces B from A, and then describe an elementary row operation that recovers A from B.

(a) $A = \begin{bmatrix} 2 & 0 & -4 \\ -3 & -2 & 6 \\ 2 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 & 1 \\ -3 & -2 & 6 \\ 2 & 0 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 & -4 \\ -3 & -2 & 1 \\ 2 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -4 \\ -3 & -2 & 1 \\ 10 & 0 & 15 \end{bmatrix}$

Solution. (a) B is obtained from A by interchanging the first and third rows. A is obtained from B by interchanging the first and third rows. □

(b) B is obtained from A by multiplying the third row by 5. A is obtained from B by multiplying the third row by $\frac{1}{5}$. □

2.1.28 A manufacturer produces custom steel products from recycled steel in which each ton of the product must contain 4 pounds of chromium, 8 pounds of tungsten, and 7 pounds of carbon. The manufacturer has three sources of recycled steel:

Source 1: Each ton contains 2 pounds of chromium, 8 pounds of tungsten, and 6 pounds of carbon.
Source 2: Each ton contains 3 pounds of chromium, 9 pounds of tungsten, and 6 pounds of carbon.
Source 3: Each ton contains 12 pounds of chromium, 6 pounds of tungsten, and 12 pounds of carbon.

Let x, y, and z denote the percentages of the first, second, and third recycled steel sources that will be melted down for one ton of the product. Find (but do not solve) a linear system in x, y, and z whose solution tells the percentage of each source that must be used to meet the requirements for the finished product.

Solution. Since Source 1, Source 2, Source 3 contain 2, 3, 12 pounds of chromium, respectively, total pound of chromium is $2x + 3y + 12z$ and we get the linear equation $2x + 3y + 12z = 4$. Similary, one can get linear equations of tungsten, carbon as $8x + 9y + 6z = 8$, $6x + 6y + 12z = 7$, respectively. Therefore required linear system is as follow:

$$\begin{cases} 2x + 3y + 12z &= 4 \\ 8x + 9y + 6z &= 8 \\ 6x + 6y + 12z &= 7 \end{cases}. \qquad \square$$

2.1.32 Find (but do not solve) a system of linear equations whose consistency or inconsistency will determine whether the given vector $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$.

$\mathbf{u}_1 = (1, 2, -4, 0, 5)$, $\mathbf{u}_2 = (1, 0, 2, 2, -1)$, $\mathbf{u}_3 = (2, -2, -1, 1, 3)$, $\mathbf{u}_4 = (0, 5, 4, -1, 1)$

(a) $\mathbf{v} = (2, -2, -8, 0, 12)$ (b) $\mathbf{v} = (5, -3, -9, 4, 11)$ (c) $\mathbf{v} = (4, -4, 2, 0, 24)$

Solution. To determine whether the given vector $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$, we need to check every entry separately. Suppose $\mathbf{v}$ is a linear combination of four vectors, then $\exists c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$. In other word,

$$\mathbf{v} = (c_1 + c_2 + 2c_3,\ 2c_1 - 2c_3 + 5c_4,\ -4c_1 + 2c_2 - c_3 + 4c_4,\ 2c_2 + c_3 - c_4,\ 5c_1 - c_2 + 3c_3 + c_4).$$

$$\text{(a) } \mathbf{v} = (2, -2, -8, 0, 12), \text{ so } \begin{cases} c_1 + c_2 + 2c_3 = 2 \\ 2c_1 - 2c_3 + 5c_4 = -2 \\ -4c_1 + 2c_2 - c_3 + 4c_4 = -8 \\ 2c_2 + c_3 - c_4 = 0 \\ 5c_1 - c_2 + 3c_3 + c_4 = 12 \end{cases}. \quad \square$$

$$\text{(b) } \mathbf{v} = (5, -3, -9, 4, 11), \text{ so } \begin{cases} c_1 + c_2 + 2c_3 = 5 \\ 2c_1 - 2c_3 + 5c_4 = -3 \\ -4c_1 + 2c_2 - c_3 + 4c_4 = -9 \\ 2c_2 + c_3 - c_4 = 4 \\ 5c_1 - c_2 + 3c_3 + c_4 = 11 \end{cases}. \quad \square$$

$$\text{(c) } \mathbf{v} = (4, -4, 2, 0, 24), \text{ so } \begin{cases} c_1 + c_2 + 2c_3 = 4 \\ 2c_1 - 2c_3 + 5c_4 = -4 \\ -4c_1 + 2c_2 - c_3 + 4c_4 = 2 \\ 2c_2 + c_3 - c_4 = 0 \\ 5c_1 - c_2 + 3c_3 + c_4 = 24 \end{cases}. \quad \square$$

2.1.D8 Indicate whether the statement is true (T) of false (F). Justify your answer.

(a) Every matrix with two or more columns is an augmented matrix for some system of linear equations.

(b) If two linear systems have exactly the same solutions, then they have the same augmented matrix.

(c) A row of an augmented matrix can be multiplied by zero without affecting the solution set of the corresponding linear system.

(d) A linear system of two equations in three unknowns cannot have exactly one solution.

Solution. (a) True. If there are $n \geq 2$ columns, then the first $n-1$ columns correspond to the coefficients of the variables that appear in the equations and the last column corresponds to the constants that appear on the right-hand side of the equal sign. $\square$

(b) False. Consider the Example 6 of textbook (p.44). We have a linear system

$$\begin{cases} x + \ \ y + 2z = & 9 \\ 2x + 4y - 3z = & 1 \\ 3x + 6y - 5z = & 0 \end{cases}.$$

Cleary the augmented matrix of above linear system is $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$. This linear system has a unique solution $\begin{cases} x & = & 1 \\ \ \ y & = & 2 \\ \ \ \ \ z = & & 3 \end{cases}$ and its corresponding augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. Both linear systems have exactly the same solutions, but they have different augmented matrix. $\square$

(c) False. Multiplying a row of the augmented matrix by zero corresponds to multiplying both sides of the corresponding equation by zero. But this is equivalent to discarding one of the equations! $\square$

(d) True. If the system is consistent, one can solve for two of the variables in terms of the third or (if further redundancy is present) for one of the variables in terms of the other two. In any case, there is at least one "free" variable that can be made into a parameter in describing the solution set of the sytstem. Thus if the system is consistent, it will have infinitely many solutions. $\square$

2.2.12 Suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row ecchelon form. Solve the system. Assume that the variables are named $x_1, x_2, \cdots$ from left to right.

$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. The given matrix corresponds to the system

$$\begin{cases} x_1 - 3x_2 \quad = 0 \\ \qquad\qquad x_3 = 0 \\ \qquad\qquad\ 0 = 1 \end{cases}.$$

Last equation $0 = 1$ is always false whatever x_1, x_2, and x_3 are. Hence this linear system is inconsistent. $\square$

2.2.26 Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} \quad\;\; -2b+3c = 2 \\ 3a+6b-3c = -2 \\ 6a+6b+3c = 5 \end{cases}$$

Solution. The augmented matrix of the system is

$$\begin{bmatrix} 0 & -2 & 3 & 2 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}.$$

Since row 2 not row 1 has the first nonzero entry, we interchange rows 1 and 2. To make the first entry "1", multiply the new row 1 by $\frac{1}{3}$. To eliminate other rows' first entry, add -6 times the new row 1 to row 3, then we get

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 2 \\ 0 & -6 & 9 & 9 \end{bmatrix}.$$

Now only row 1 has the first entry "1", and we do the similar process for second entry. Multiply row 2 by $-\frac{1}{2}$ and add 6 times the new row 2 to row 3, then we get

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Since the last row corresponds to the equation $0 = 1$, this linear system is inconsistent. □

2.2.41 Solve the given homogeneous system of linear equations by any method.

$$\begin{cases} 2I_1 - \ \ I_2 + 3I_3 + 4I_4 = 0 \\ \ \ I_1 \qquad\quad - 2I_3 + 7I_4 = 0 \\ 3I_1 - 3I_2 + \ \ I_3 + 5I_4 = 0 \\ 2I_1 + \ \ I_2 + 4I_3 + 4I_4 = 0 \end{cases}$$

Solution. We will solve the system by Gaussian elimination, i.e., by reducing the augmented matrix of the system to a row-echelon form: The augmented matrix of the original system is

$$\begin{bmatrix} 2 & -1 & 3 & 4 & 0 \\ 1 & 0 & -2 & 7 & 0 \\ 3 & -3 & 1 & 5 & 0 \\ 2 & 1 & 4 & 4 & 0 \end{bmatrix}.$$

To find an echelon form, interchange row 1 and 2, add -2 times the new row 1 to the new row 2. Add -3 times the new row 1 to row 3, add -2 times the new row 1 to row 4.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 0 \\ 0 & -1 & 7 & -10 & 0 \\ 0 & -3 & 7 & -16 & 0 \\ 0 & 1 & 8 & -10 & 0 \end{bmatrix}.$$

Multiply row 2 by -1, add 3 times the new row 2 to row 3, add -1 times the new row 2 to row 4.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 0 \\ 0 & 1 & -7 & 10 & 0 \\ 0 & 0 & -14 & 14 & 0 \\ 0 & 0 & 15 & -20 & 0 \end{bmatrix}.$$

Multiply row 3 by $-\frac{1}{14}$, add -15 times the new row 3 to row 4, multiply the new row 4 by $-\frac{1}{5}$.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 0 \\ 0 & 1 & -7 & 10 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

From the row 4, we get $I_4 = 0$. Then the row 3 gives $I_3 = 0$, continuously the row 2 gives $I_2 = 0$ and the row 1 gives $I_1 = 0$. Therefore, this system has only the trivial solution. $\square$

2.2.46 Determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

$$\begin{cases} x + y + 7z = -7 \\ 2x + 3y + 17z = -16 \\ x + 2y + (a^2+1)z = 3a \end{cases}$$

Solution. The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & -16 \\ 1 & 2 & a^2+1 & 3a \end{bmatrix}.$$

Add row 1 to row 3, add -1 times row 2 to row 3, add -2 times row 1 to row 2. Then we get the reduced matrix

$$\begin{bmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & a^2-9 & 3a+9 \end{bmatrix}.$$

The last row corresponds to the equation $(a^2-9)z = 3a+9 \Leftrightarrow (a-3)(a+3)z = 3(a+3)$. If $a = -3$ this becomes $0 = 0$, and the system will have infinitely many solutions. If $a = 3$, then the last row corresponds to $0 = 18$; the system is inconsistent. If $a \neq \pm 3$, then $z = \frac{3}{a-3}$ and, from the back substitution of linear system, x and y are uniquely determined as well; the system has exactly one solution. □

2.2.D8 Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) A matrix may be reduced to more than one row echelon form.

(b) A matrix may be reduced to more than one reduced row echelon form.

(c) If the reduced row echelon form of the augmented matrix of a system of equations has a row consisting entirely of zeros, then the system of equations has infinitely many solutions.

(d) A nonhomogeneous sytstem of equations with more equations than unknowns must be inconsistent.

Solution. (a) True. For example $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ can be reduced to either $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. □

(b) False. The reduced row echelon form of a matrix is unique. □

(c) False. The appearance of a row of zeros means that there was some redundancy in the system. But the remaining equations may be inconsistent, have exactly one solution, or have infinitely many solutions. All of these are possible. For example, consider the augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Though this reduced row echelon form has a row consisting entirely of zeros, first two equations imply $x = 2$ and $y = -1$, which is the unique solution. □

(d) False. There may be redundancy in the system. For example, the system consisting of the equations $x + y = 1$, $2x + 2y = 2$, and $3x + 3y = 3$ has infinitely many solutions with following augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}.$$

□