Chapter 4

Determinants

4.1 Determinants; cofactor Expansion

Exercise 4.1. Compute the determinants of the matrix A:

$$A = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

How can you construct A brilliantly?

Solution.

```
A=ones(5)-5*eye(5);
disp('A is'); disp(A);
disp('Determinant of A is'); disp(det(A));
```

MATLAB results.

```
A is
    -4     1     1     1     1
     1    -4     1     1     1
     1     1    -4     1     1
     1     1     1    -4     1
     1     1     1     1    -4

Determinant of A is
  -5.5511e-14
```

Exercise 4.2. Show that

$$\det\left(\begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}\right) = (a^2+b^2+c^2+d^2)^2.$$

Solution.

```
syms a b c d;

A=[a b c d; -b a d -c; -c -d a b; -d c -b a];

disp('Given matrix is'); disp(A);
disp('Determinant of the given matrix is');
disp(simplify(det(A)));
```

MATLAB results.

```
Given matrix is
[  a,  b,  c,  d]
[ -b,  a,  d, -c]
[ -c, -d,  a,  b]
[ -d,  c, -b,  a]

Determinant of the given matrix is
(a^2 + b^2 + c^2 + d^2)^2
```

Exercise 4.3. The nth-order **Fibonacci matrix** [named for the Italian mathematician (circa 1170 - 1250)] is the $n \times n$ matrix F_n that has 1's on the main diagonal, 1's along the diagonal immediately above the main diagonal, -1's along the diagonal immediately below the main diagonal, and zeros everywhere else. Construct the sequence

$$\det(F_1), \det(F_2), \det(F_3), \cdots, \det(F_7).$$

Make a conjecture about the relationship between a term in the sequence and its two immediate predecessors, and then use your conjecture to make a guess at $\det(F_8)$. Check your guess by calculating this number.

Solution.

```
% Construct the 10x10 Fibonacci matrix F.
N=10; nOnes=ones(N, 1);
F=diag(nOnes)+diag(nOnes(1:N-1),1)-diag(nOnes(1:N-1),-1);

for n=1:7 % n is from 1 to 7
    Fn=F(1:n,1:n); % nxn Fibonacci matrix is selected from F.
    disp(det(Fn));
end
```

MATLAB results.

```
     1
     2
     3
     5
     8
    13
    21
```

The constructed sequence satisfies the relationship $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$, for $\det(F_1) = 1$ and $\det(F_2) = 2$. From that, we may guess that $\det(F_8) = 34$. MATLAB gives us the same output value 34 as our guess.

Exercise 4.4. Let A_n be the $n \times n$ matrix that has 2's along the main diagonal, 1's along the diagonals immediately above and below the main diagonal, and zeros everywhere else. Make a conjecture about the relationship between n and $\det(A_n)$.

Solution.

```
format rat;
% Construct the 10x10 matrix A satisfying given conditions.
n=10; nOnes=ones(n, 1);
A=2*diag(nOnes)+diag(nOnes(1:n-1),1)+diag(nOnes(1:n-1),-1);

for i=1:10 % i is from 1 to 10
    Ai=A(1:i,1:i); % A_i matrix is selected from A.
    disp(det(Ai));
end
```

MATLAB results.

```
       2
       3
       4
       5
       6
       7
       8
       9
      10
      11
```

From the outputs, we make a conjecture about the relationship between n and $\det(A_n)$ as follows:

$$\det(A_n) = n + 1.$$

4.2 Properties of Determinants

Exercise 4.5. (*Determinants with LU-decomposition*) In this problem, we find the determinant of the matrix A by using the LU-decomposition of A, where

$$A = \begin{bmatrix} -2 & 2 & -4 & -6 \\ -3 & 6 & 3 & -15 \\ 5 & -8 & -1 & 17 \\ 1 & 1 & 11 & 7 \end{bmatrix}.$$

(a) Compute the determinant of A directly by using the MATLAB command *det* for A.

(b) Compute the determinant of A by using the MATLAB command *lu* for A. Confirm that you get the same results.

Solution.

```
%(a)
A = [-2 2 -4 -6; -3 6 3 -15; 5 -8 -1 17; 1 1 11 7];

det_A = det(A); % Find the determinant of A by using the command det.

disp('The determinant of A by direct use of the command det is');
disp(det_A);

%(b)
[L U P] = lu(A); % We have a PLU-decomposition of A. (i.e., PA=LU ).

% Since the determinant of a triangular matrix is
% just a product of diagonal entries,

det_L = prod(diag(L)); % The product of diagonal entries of L.
% Or, you may use the command det for L, directly. (i.e., det_L = det(L)).

det_U = prod(diag(U)); % The product of diagonal entries of U.
% Or, you may use the command det for U, directly. (i.e., det_U = det(U)).

% If you observe the permutation matrix P, you can see that
% P is an odd permutation. Thus, we have det(P) = -1.
det_P = -1;
% Or, you may use the command det for P, directly. (i.e., det_P = det(P)).

% Since PA = LU, det(P)*det(A) = det(L)*det(U).
det_A = det_P * det_L * det_U;

disp('The determinant of A by using the LU-decomposition is'); disp(det_A);
```

MATLAB results.

```
The determinant of A by direct use of the command det is
   24.0000

The determinant of A by using the PLU-decomposition is
   24.0000
```

Exercise 4.6. (*Effects of Elementary Row Operations on the Determinant*)
Using the MATLAB command *det*, confirm the formulas (a)-(c) in Theorem 4.2.2 of Section 4.2 for the matrix A given in the problem 31 of Exercise set 4.1.

Solution.

```
A = [3 3 0 5; 2 2 0 -2; 4 1 -3 0; 2 10 3 2];

% (a). Multiply the second row of A by 2 and call it A2.
% Initialize the matrix A2 as A.
A2 = A;
% Multiply the second row of A by 2.
A2(2,:) = 2*A(2,:);
disp('The determinant of A2 is'); disp(det(A2));
disp('2*det(A) = '); disp(2*det(A));

% (b). Interchange the rows 2 and 4 of A and call it A24.
% Initialize the matrix A24 as A.
A24 = A;
% Interchange the rows 2 and 4 of A.
A24(2, :) = A(4, :) ; A24(4, :) = A(2, :);
disp('The determinant of A24 is'); disp(det(A24));
disp('-det(A) = '); disp(-det(A));

% (c). Add 2 times row 3 to row 4 of A and call it A234.
% Initialize the matrix A234 as A.
A234 = A;
% Add 2 times row 3 of A to row 4.
A234(4, :) = 2 * A(3, :) + A(4, :);
disp('The determinant of A234 is'); disp(det(A234));
disp('det(A) = '); disp(det(A));
```

MATLAB results.

```
The determinant of A2 is
  -480
2*det(A) =
 -480.0000

The determinant of A24 is
  240.0000
-det(A) =
  240.0000

The determinant of A234 is
 -240.0000
det(A) =
 -240.0000
```

Exercise 4.7. Use a determinant to show that if a, b, c, and d are not all zeros, then the vectors

$$\begin{aligned}\mathbf{v}_1 &= (a,\, b,\, c,\, d)\\ \mathbf{v}_2 &= (-b, a, d, -c)\\ \mathbf{v}_3 &= (-c, -d, a, b)\\ \mathbf{v}_4 &= (-d, c, -b, a)\end{aligned}$$

are linearly independent.

Solution.

```
syms a b c d;
v1=[a b c d];
v2=[-b a d -c];
v3=[-c -d a b];
v4=[-d c -b a];

V=[v1; v2; v3; v4];
disp('det(V) is'); disp(simplify(det(V)));
```

MATLAB results.

```
det(V) is
(a^2 + b^2 + c^2 + d^2)^2
```

4.3 Cramer's Rule; Formula for A^{-1}; Applications

No MATLAB problems in this section.