Review Quiz (True or False ?)

① $A^2 = A \Leftrightarrow A(A-I) = O \Leftrightarrow A = O$ or $A = I$

② $(AB)^T = A^T B^T$ ③ $tr(AB) = tr(BA)$

④ $AB = I \Leftrightarrow A = B^{-1}$ & $B = A^{-1}$.

⑤ A : invertible $\Leftrightarrow A^T$: invertible

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A can be expressed as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.*

$\rightarrow$ We can find the inverse matrix of an invertible matrix A.

If A is invertible, then the reduced row echelon form of $[A|I]$ is $[I|A^{-1}]$.

$$[A|I] \xrightarrow{E_1} \xrightarrow{E_2} \cdots \xrightarrow{E_k} [I|A^{-1}] \qquad \underbrace{E_k \cdots E_2 E_1}_{= A^{-1}} A = I$$

For $S \subseteq \mathbb{R}^n$ with $S \neq \emptyset$,

S is a <u>subspace</u> of $\mathbb{R}^n$ if it satisfies

(i) $v, w \in S \Rightarrow v + w \in S$ (closed under $+$)

(ii) $v \in S,\ k \in \mathbb{R} \Rightarrow kv \in S$. (closed under scalar multiplication)

e.g.) $\{0\}$ & $\mathbb{R}^n$: trivial subspaces of $\mathbb{R}^n$

Note that every subspace of $\mathbb{R}^n$ contains 0.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be vectors in $\mathbb{R}^n$. Then the set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ is a subspace of $\mathbb{R}^n$.

pf) Let $W = \{ t_1\mathbf{v}_1 + \cdots + t_s\mathbf{v}_s \mid t_1, \cdots, t_s \in \mathbb{R} \}$.

① Since $\mathbf{0} \in W$, $W \neq \phi$.

② $\mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_s\mathbf{v}_s \in W$

$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_s\mathbf{v}_s \in W$

$k \in \mathbb{R}$

$\Rightarrow \quad \mathbf{x} + \mathbf{y} = (t_1 + c_1)\mathbf{v}_1 + \cdots + (t_s + c_s)\mathbf{v}_s \in W$

$k\mathbf{x} = (kt_1)\mathbf{v}_1 + \cdots + (kt_s)\mathbf{v}_s \in W$ $\boxed{//}$

Let $W = \{\mathbf{x} \mid \mathbf{x} = t_1\mathbf{v}_1 + \cdots + t_s\mathbf{v}_s, \forall t_i \in \mathbb{R}\}$. The subspace W is called the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_s$ and is denoted by

$$W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}.$$

We also say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ *span* W.

e.g.) $\mathbb{R}^n = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, $\mathbf{e}_i := (0, \cdots, 0, \underset{i\text{th}}{1}, 0, \cdots, 0)$

$(\because \forall \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n, \quad \mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.)$

Theorem

If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system with n unknowns, then its solution set is a subspace of $\mathbb{R}^n$.

pf) Let $S = \{ \mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0} \}$. ← The solution space of the system $A\mathbf{x}=\mathbf{0}$.

Since $\mathbf{0} \in S$, $S \neq \phi$.

If $\mathbf{x}_1, \mathbf{x}_2 \in S$, then $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0}$

$\therefore\ \mathbf{x}_1 + \mathbf{x}_2 \in S$

If $\mathbf{x} \in S$ & $k \in \mathbb{R}$, then $A(k\mathbf{x}) = kA\mathbf{x} = k \cdot \mathbf{0} = \mathbf{0}$.

$\therefore\ k\mathbf{x} \in S$. $\blacksquare$

Ex.

$$(*)\quad \underbrace{\begin{bmatrix} 0 & 0 & -2 & 0 & 7 \\ 2 & 4 & -10 & 6 & 12 \\ 2 & 4 & -5 & 6 & -5 \end{bmatrix}}_{\parallel \atop A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

reduced row echelon form of $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

leading variables : x_1, x_3, x_5
free variables: $x_2 = s$, $x_4 = t$ $(s, t \in \mathbb{R})$

general sol.: $x_1 = -2s - 3t$, $x_2 = s$, $x_3 = 0$, $x_4 = t$, $x_5 = 0$

$(x_1, x_2, x_3, x_4, x_5) = s(-2, 1, 0, 0, 0) + t(-3, 0, 0, 1, 0)$

$\therefore$ The solution set of the linear system $(*)$ is spanned by

$(-2, 1, 0, 0, 0)$ and $(-3, 0, 0, 1, 0)$.

Note that there are only three kinds of subspaces in $\mathbb{R}^2$.
(four) ($\mathbb{R}^3$)

1. $\{0\}$
2. Lines passing thru the origin
3. $\mathbb{R}^2$

(3. Planes passing thru. the origin
4. $\mathbb{R}^3$)

The solution space of a homogeneous linear system in two unknowns is one of the following types
(three)

1. $\{0\}$
2. Lines passing thru the origin
3. $\mathbb{R}^2$

(3. Planes passing thru. the origin
4. $\mathbb{R}^3$)

Theorem

1. *If A is a matrix with n columns, then the solution space of $A\mathbf{x} = \mathbf{0}$ is all of $\mathbb{R}^n$ if and only if $A = \mathbf{0}$.*
2. *If A and B are matrices with n columns, then $A = B$ if and only if $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.*

pf) (a) ($\Leftarrow$) Trivial

($\Rightarrow$) Since $A\mathbf{e}_i = \mathbf{0}$ $\forall i = 1, \cdots, n$,

$$A = AI = A[\mathbf{e}_1\ \mathbf{e}_2 \cdots \mathbf{e}_n] = [A\mathbf{e}_1\ A\mathbf{e}_2 \cdots A\mathbf{e}_n]$$
$$= [\mathbf{0}\ \mathbf{0} \cdots \mathbf{0}] = \mathbf{0}$$

(b) $A\mathbb{x} = B\mathbb{x} \quad \forall\, \mathbb{x} \in \mathbb{R}^n$

$$\iff (A-B)\mathbb{x} = \mathbb{0} \quad \forall\, \mathbb{x} \in \mathbb{R}^n$$

$$\underset{(a)}{\iff} A - B = \mathbb{0}$$

$$\iff A = B$$

$\blacksquare$

<u>Def.</u> A nonempty subset $S = \{\mathbb{v}_1, \cdots, \mathbb{v}_s\} \subseteq \mathbb{R}^n$ is <u>linearly independent</u> if we have

$$(*) \quad c_1\mathbb{v}_1 + \cdots + c_s\mathbb{v}_s = \mathbb{0} \iff c_1 = \cdots = c_s = \mathbb{0}.$$

If one of c_i's is nonzero, S is <u>linearly dependent</u>.

Ex)
- $\{\mathbb{0}\}$ is linearly dependent.
- If $\mathbb{0} \in S$, then S is linearly dependent.
- $\{\mathbb{e}_1, \mathbb{e}_2, \cdots, \mathbb{e}_n\}$ is linearly independent.
- $\{(1,0), (1,1)\}$ is linearly independent.
- $\{(-2,1,0,0,0), (-3,0,0,1,0)\}$ is linearly independent.

Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_s\} \subseteq \mathbb{R}^n$, $s \geq 2$. Then S is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

pf) ($\Rightarrow$) $\exists\ c_1, \cdots, c_s$, not all zero, s.t.

$$c_1\mathbf{v}_1 + \cdots + c_s\mathbf{v}_s = \mathbf{0}.$$

WLOG, WMA $c_1 \neq 0$.

Then $\mathbf{v}_1 = \left(-\frac{c_2}{c_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{c_s}{c_1}\right)\mathbf{v}_s$.

($\Leftarrow$) WLOG, WMA $\mathbf{v}_1 := c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s$.

Then $-\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0}$.

$\therefore$ S is linearly dependent. $\blacksquare$

Suppose that we have three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ such that at least one of them is a linear combination of the other two. Then they lie in a plane through the origin. The converse is also true.

Suppose that $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_s\}$ is linearly independent in $\mathbb{R}^n$.

Set $A := \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_s \\ | & | & & | \end{bmatrix}$. Then

$$A \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s.$$

Theorem

A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the column vectors of A are linearly independent.

Theorem

A set with more than n vectors in $\mathbb{R}^n$ is linearly dependent.

$S = \{\mathbf{v}_1, \cdots, \mathbf{v}_s\}$, $s > n$. $\longleftrightarrow$ $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_s \\ | & | & & | \end{bmatrix}$

$\Rightarrow$ # leading variables $\leq n$

$\Rightarrow$ # free variables $\geq s - n > 0$.

$\therefore$ $A\mathbf{x} = \mathbf{0}$ has nonzero solutions.

Theorem

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3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*

※ Abbreviation

iff $\longleftrightarrow$ if and only if

WLOG $\longleftrightarrow$ without loss of generality

WMA $\longleftrightarrow$ we may assume

ETS $\longleftrightarrow$ Enough to show