8.6.4 Find the distinct singular values of $A = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix}$

Solution. $A^T A = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ and the eigenvalues of $A^T A$ are $\lambda = 4,\ 1$. Thus the singular values of A are $\sigma_1 = 2$, and $\sigma_2 = 1$. $\square$

8.6.12 Find a singular value decomposition of $A = \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}$.

Solution. The eigenvalues of $A^TA = \begin{bmatrix} 52 & 24 \\ 24 & 16 \end{bmatrix}$ are $\lambda_1 = 64$ and $\lambda_2 = 4$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = 8$ and $\sigma_2 = 2$, so we have $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. We choose $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3. This results in the following singular value decomposition:

$$A = \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} = U\Sigma V^T.$$

□

8.6.14 Use the singular value decomposition of $A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ and the method of Example 3 to find a polar decomposition of A.

Solution. The eigenvalues of $A^TA = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix}$ are $\lambda_1 = 36$ and $\lambda_2 = 0$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ respectively. The only singular value of A is $\sigma_1 = 6$, and we have $\mathbf{u}_1 = \dfrac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. The vector $\mathbf{u}_2$ must be chosen so that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for R^2, e.g., $\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. This results in the following singular value decomposition:

$$A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = U\Sigma V^T.$$

Hence, we have the following polar decomposition of A:

$$A = (U\Sigma U^T)(UV^T) = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = PQ.$$

□

8.6.16 Use the singular value decomposition of $A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$ and the method of Example 5 to find a reduced singular value decomposition of A and a reduced singular value expansion of A.

Solution. The eigenvalues of $A^TA = \begin{bmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{bmatrix}$ are $\lambda_1 = 18$ and $\lambda_2 = \lambda_3 = 0$. The unit vector $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 18$. The vectors $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} \frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$ form an orthonormal basis for the eigenspace corresponding to $\lambda = 0$. The only singular value of$A$ is $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, and we have $\mathbf{u}_1 = \dfrac{1}{\sigma_1}A\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. We must choose the vector $\mathbf{u}_2$ so that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for R^2, e.g., $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. This results in the following singular value decomposition:

$$A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} = U\Sigma V^T.$$

Since A has rank 1, the corresponding reduced singular value decomposition is

$$A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = U_1\Sigma_1V_1^T$$

and the reduced singular value expansion is

$$A = 3\sqrt{2} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T.$$

□

8.6.18 Find an eigenvalue decomposition of the symmetric matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$, and then use the method of Example 2 to find a singular value decomposition of A.

Solution. The characteristic polynomial of A is $(\lambda+1)(\lambda+2)(\lambda-4)$. Thus the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = 4$, and the corresponding unit eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$ respectively. The matrix $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ orthogonally diagonalizes A and the eigenvalue decomposition of A is

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{bmatrix} = PDP^T.$$

The corresponding singular value decomposition of A is obtained by shifting the negative signs from the diagonal factor to the second orthogonal factor:

$$A = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{bmatrix} = U\Sigma V^T.$$

$\square$

8.6.19 Suppose that A has the singular value decomposition

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

(a) Find orthonormal bases for the four fundamental spaces of A.

(b) Find the reduced singular value decomposition of A.

Solution. (a) We have $\operatorname{rank}(A) = 2$; thus $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ form a basis for $\operatorname{col}(A)$, $\mathbf{u}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ and $\mathbf{u}_4 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ form a basis for $\operatorname{col}(A)^\perp = \operatorname{null}(A^T)$, $\mathbf{v}_1^T = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ and $\mathbf{v}_2^T = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ form a basis for $\operatorname{row}(A)$, and $\mathbf{v}_3 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$ forms a basis for $\operatorname{row}(A)^\perp = \operatorname{null}(A)$.

(b) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 & 0 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = U_1 \Sigma_1 V_1^T.$

□