1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let A be a 2×2 matrix. If $\det(A) = 0$, then one column of A is a scalar multiple of the other one.

(b) The cross product of two unit vectors is an unit vector.

(c) Let A be an $n \times n$ matrix whose column vectors are $\mathbf{a}_1, \cdots, \mathbf{a}_n$. For any $\mathbf{b}$ in $\mathbf{R}^n$, let B be the matrix that results when the nth column of A is replaced by $\mathbf{b}$, *i.e.*, $B = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_{n-1} \ \mathbf{b}]$. If $A\mathbf{x} = \mathbf{b}$ has a nontrivial solution, then $\det(B) = 0$.

Solution.

(a) True. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ are two column vectors of A. If $\det A = 0$, then $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent by Theorem 4.2.7. Therefore, one column of A should be a scalar multiple of the other one.

(b) False. Let $\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Then $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, but $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

(c) False. Let $A = I_n$ and $\mathbf{b} = \mathbf{e}_n$. Then, $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = \mathbf{e}_n$. The matrix B that results when the last column of $A = I_n$ is replaced by $\mathbf{b} = \mathbf{e}_n$ is equal to I_n, since $\mathbf{e}_n$ is the last column of I_n. Then $\det(B)$ is not zero.

2 Find the value of k for which the matrix M is singular.

10 points

$$M = \begin{bmatrix} k+1 & -1 & 2 & 1 & 3 \\ -2 & k & 1 & 0 & -1 \\ 0 & 0 & k & 0 & 1 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 2 & 0 & k-2 \end{bmatrix}$$

(Hint: If a square matrix M is partitioned into block traiangular form as $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ in which A and B are square, then $\det(M) = \det(A)\det(B)$.)

Solution.

By Theorem 4.2.4, we should find the value of k such that $\det(M) = 0$. Since the matrix M is partitioned into the form of $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ where $A = \begin{bmatrix} k+1 & -1 \\ -2 & k \end{bmatrix}$ and $B = \begin{bmatrix} k & 0 & 1 \\ 0 & k & 0 \\ 2 & 0 & k-2 \end{bmatrix}$, we get

$$\begin{aligned} \det M &= \det A \det B \\ &= \begin{vmatrix} k+1 & -1 \\ -2 & k \end{vmatrix} \begin{vmatrix} k & 0 & 1 \\ 0 & k & 0 \\ 2 & 0 & k-2 \end{vmatrix} \\ &= [k(k+1)-2][k^2(k-2)-2k] \\ &= k(k-1)(k+2)(k^2-2k-2) \end{aligned}$$

by using the hint. Therefore, $k = 0, 1, -2$, or $k = 1 \pm \sqrt{3}$.