Chapter 3

Matrices and Matrix Algebra

3.1 Operations on Matrices

No MATLAB problems in this section.

3.2 Inverses; Algebraic Properties of Matrices

Exercise 3.1. In this problem, we compute $A^5 - 3A^3 + 7A - 4I$ for the matrix A, where

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 1 & 1 & -2 & 1 \\ 2 & 1 & 3 & 4 \\ -3 & 2 & 2 & -8 \end{bmatrix}.$$

(a) Using the syntax $A\hat{}k$ which produces the k-th power of a square matrix and the command *eye* for the identity matrix, compute the above matrix polynomial.

(b) Using the command *polyvalm*, compute the above matrix polynomial.

(c) Tell what happens if you type the syntax $A.\hat{}k$.

Solution.

```
% Construct the matrix A.
A = [1 2 -3 0; 1 1 -2 1; 2 1 3 4; -3 2 2 -8];

% (a)
result_a = A^5 + (-3)*A^3 + 7*A + (-4)*eye(4);
```

```
% Display the matrix polynomial.
disp('The result of the matrix polynomial is');
disp(result_a)

% (b)
% Coefficient of the matrix polynomial.
coeff_poly = [1 0 -3 0 7 -4];

% Evaluate the matrix polynomial of coefficient
% with coeff_poly vector with the input matrix A.
result_b = polyvalm(coeff_poly, A);

% Display the matrix polynomial.
disp('The result of the matrix polynomial is');
disp(result_b);

% (c)
disp('The result of A.^2 is'); disp(A.^2);
disp('The result of A.^3 is'); disp(A.^3);
disp('The result of A.^4 is'); disp(A.^4);
```

MATLAB results.

```
The result of the matrix polynomial is
         874       -1272         -39        3021
        2580       -2306        -723        7536
        5191       -4121       -2444       14563
      -16852       12539        5649      -46917

The result of the matrix polynomial is
         874       -1272         -39        3021
        2580       -2306        -723        7536
        5191       -4121       -2444       14563
      -16852       12539        5649      -46917

The result of A.^2 is
     1     4     9     0
     1     1     4     1
     4     1     9    16
     9     4     4    64

The result of A.^3 is
     1     8   -27     0
     1     1    -8     1
     8     1    27    64
   -27     8     8  -512
```

```
The result of A.^4 is
           1          16          81           0
           1           1          16           1
          16           1          81         256
          81          16          16        4096
```

From the results, we can see that the syntax $A.\hat{}k$ produces the entrywise k-th powers of the matrix A.

3.3 Elementary Matrices; A Method for Finding A^{-1}

Exercise 3.2. In this problem, we solve the linear system $A\mathbf{x} = \mathbf{b}$ by using matrix inversion, where

$$A = \begin{bmatrix} 3 & 3 & -4 & -3 \\ 0 & 6 & 1 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 3 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 5 \\ 1 \end{bmatrix}.$$

(a) Use the MATLAB command *inv* or the syntax $A\hat{}(-1)$ to find the inverse of A.

(b) Display the output matrix as a rational form, NOT decimally. You may use the command *format*.

(c) Using the result of (a), compute the solution of the linear system $A\mathbf{x} = \mathbf{b}$ by taking $\mathbf{x} = A^{-1}\mathbf{b}$.

Solution.

```
% Construct the matrix A and the right-hand-side vector b.
A = [3 3 -4 -3; 0 6 1 1; 5 4 2 1; 2 3 3 2];
b = [-2 3 5 1]';

% (a)
% Use the command inv.
Inv_A1 = inv(A);

% Use the syntax A^(-1).
Inv_A2 = A^(-1);

% (b)
format rat;
disp('The result of the command inv is'); disp(Inv_A1);
disp('The result of the syntax A^(-1) is'); disp(Inv_A2);
```

```
% (c)
% Since A is invertible, the solution to Ax=b is x=A^(-1)*b.
x = Inv_A1 * b;
disp('The solution to Ax=b is x = A^(-1)*b'); disp(x');
```

MATLAB results.

```
The result of the command inv is
 -7    5   12  -19
  3   -2   -5    8
 41  -30  -69  111
-59   43   99 -159

The result of the syntax A^(-1) is
 -7    5   12  -19
  3   -2   -5    8
 41  -30  -69  111
-59   43   99 -159

The solution to Ax=b is x = A^(-1)*b
    70      -29     -406      583
```

3.4 Subspaces and Linear Independence

Exercise 3.3. (*Sigma notation*)
Compute the linear combination

$$\mathbf{v} = \Sigma_{j=1}^{25} c_j \mathbf{v}_j$$

for $c_j = 1/j$ and $\mathbf{v}_j = (\sin j, \cos j)$.

Solution.

```
v=zeros(1,2);
for i=1:25
    v=v+(1/i)*[sin(i), cos(i)];
end
disp(v);
```

MATLAB results.

```
1.0322    0.0553
```

Exercise 3.4. Let $\mathbf{v_1} = (4,3,2,1)$, $\mathbf{v_2} = (5,1,2,4)$, $\mathbf{v_3} = (7,1,5,3)$, $\mathbf{x} = (16,5,9,8)$, and $\mathbf{y} = (3,1,2,7)$. Determine whether $\mathbf{x}$ and $\mathbf{y}$ lie in span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

Solution.

```
% Construct v1, v2, v3, x, y
v1=[4 3 2 1]'; v2=[5 1 2 4]'; v3=[7 1 5 3]';
x=[16 5 9 8]'; y=[3 1 2 7]';
```

```
% Augmented matrices [v1|v2|v3|x] and [v1|v2|v3|y]
X=[v1 v2 v3 x];
Y=[v1 v2 v3 y];

disp('Reduced row echelon form of [v1 v2 v3 x] is');
disp(rref(X));
disp('Reduced row echelon form of [v1 v2 v3 y] is');
disp(rref(Y));
```

MATLAB results.

```
Reduced row echelon form of [v1 v2 v3 x] is
       1              0              0              1
       0              1              0              1
       0              0              1              1
       0              0              0              0

Reduced row echelon form of [v1 v2 v3 y] is
       1              0              0              0
       0              1              0              0
       0              0              1              0
       0              0              0              1
```

Therefore, $\mathbf{x}$ lies in span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $\mathbf{y}$ does not lie in span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

3.5 The Geometry of Linear Systems

No MATLAB problems in this section.

3.6 Matrices with Special Forms

Exercise 3.5. (*Inverting* $(I - A)$)

(a) (*Inverting* $(I - A)$ *when* A *is nilpotent*) Using MATLAB, show that the matrix

$$A = \begin{bmatrix} 2 & 11 & 3 \\ -2 & -11 & -3 \\ 8 & 35 & 9 \end{bmatrix}$$

is nilpotent, and then use Theorem 3.6.6 in the text book to compute $(I - A)^{-1}$. Check your answer by computing the inverse directly in MATLAB.

(b) (*Approximating* $(I - A)^{-1}$ *by a power series*) Using MATLAB, confirm that the matrix

$$A = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{10} \\ \frac{1}{8} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}$$

satisfies the condition in Theorem 3.6.7 of the text book. You may use the command *sum*. Since A satisfies that condition, $(I - A)$ is invertible and can be expressed by the series in Formula (18) in Section 3.6 of the text book. Compute the approximation

$$(I - A)^{-1} \approx I + A + A^2 + A^3 + \cdots + A^{10},$$

and compare it with the inverse of $I - A$ produced directly by MATLAB. To how many decimal places do the results agree? You may use the command *format* to display the output with long digits.

Solution.

(a)
```
% (a)-i
A = [ 2 11 3 ; -2 -11 -3; 8 35 9];  % Construct the matrix A.
% Compute the A^2, A^3, ... , and display.
disp('A^2 is'); disp(A^2);
disp('A^3 is'); disp(A^3);

% (a)-ii Comparing two result

% By Theorem 3.6.6, (I-A)^(-1)=I+A+A^2.
result1=eye(3)+A+A^2;

% Compute the inverse of (I-A) directly.
result2=inv(eye(3)-A);
disp('I+A+A^2 is'); disp(result1);
```

```
disp('(I-A)^(-1) is'); disp(result2);

% Display as a rational form.
format rat;
disp('Rational form of (I-A)^(-1) is');disp(result2);
```

MATLAB results.

```
A^2 is
     6     6     0
    -6    -6     0
    18    18     0

A^3 is
     0     0     0
     0     0     0
     0     0     0

I+A+A^2 is
     9    17     3
    -8   -16    -3
    26    53    10

(I-A)^(-1) is
    9.0000   17.0000    3.0000
   -8.0000  -16.0000   -3.0000
   26.0000   53.0000   10.0000

Rational form of (I-A)^(-1) is
       9             17              3
      -8            -16             -3
      26             53             10
```

Since $A^3 = \mathbf{0}$, A is nilpotent. By the Theorem 3.6.6, since $A^3 = \mathbf{0}$, $I - A$ is invertible and $(I-A)^{-1} = I + A + A^2$. To check answer by computing the inverse directly in MATLAB, we implement as in the next page.

(b)
```
% Construct the matrix A.
A=[0 1/4 1/8; 1/4 1/8 1/10; 1/8 1/10 1/10];

% Check that the condition in Theorem 3.6.7
% of the text book is satisfied for matrix A.
column_sum=sum(abs(A),1);   % column-wise sum
row_sum=sum(abs(A),2);  % row-wise sum
disp('The sum of the absolute values of the entries in each column is');
disp(column_sum);
disp('The sum of the absolute values of the entries in each row is');
disp(row_sum);
```

```
result3=eye(size(A))+A+A^2+A^3+A^4+A^5+A^6+A^7+A^8+A^9+A^10;
result4=inv(eye(3)-A);

format long; % Display the result with long digits
disp('With format long');
disp('Approximated inv(I-A) is'); disp(result3);
disp('Exact inv(I-A) is'); disp(result4);
```

MATLAB results.

```
The sum of the absolute values of the entries in each column is
       3/8           19/40          13/40

The sum of the absolute values of the entries in each row is
       3/8
      19/40
      13/40

With format long
Approximated inv(I-A) is
   1.108587459181130   0.338615080927493   0.191581699462210
   0.338615080927493   1.260966638806045   0.187122081247432
   0.191581699462210   0.187122081247432   1.158500720998029

Exact inv(I-A) is
   1.108610894508188   0.338643199287067   0.191600757491367
   0.338643199287067   1.261000334187368   0.187144925921800
   0.191600757491367   0.187144925921800   1.158516208087334
```

The approximation result agrees with the exact result to 2 decimal places.

3.7 Matrix Factorizations; LU-Decomposition

Exercise 3.6. (*LU-decompositions*) In this problem, we find an LU-decomposition of A, where A is given in the Example 2 of the Section 3.7.

(a) Find an LU-decomposition of A by following the procedure given in the Example 2.

(b) Solve the linear system $A\mathbf{x} = \mathbf{b}$ by using the LU-decomposition of A obtained in (a), where $\mathbf{b} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

(c) Tell what happens if you use the MATLAB command *lu* of A. Explain why this result differs from the result in (a).

Solution.

```
%(a)
A = [6 -2 0; 9 -1 1; 3 7 5]; % Set the matrix A.

format rat; % Display results as a rational form.

% Initialization of U and L.
U = A; L = eye(3);

% Multiply the first row by 1/6.
U(1,:)=(1/6)*U(1,:);
% L(1,1) is the inverse of 1/6.
L(1,1)=(1/6)^(-1);

% Add (-9) times the first to the second.
U(2,:)=((-9)*U(1,:))+U(2,:);
% L(2,1) is the negative of (-9).
L(2,1)=-(-9);

% Add (-3) times the first to the third.
U(3,:)=((-3)*U(1,:))+U(3,:);
% L(3,1) is the negative of (-3).
L(3,1)=-(-3);

% Multiply the second row by 1/2.
U(2,:)=(1/2)*U(2,:);
% L(2,2) is the inverse of 1/2.
L(2,2)=(1/2)^(-1);

% Add (-8) times the second to the third.
U(3,:)=((-8)*U(2,:))+U(3,:);
% L(3,2) is the negative of (-8).
L(3,2)=-(-8);

disp('A is'); disp(A);
disp('The Lower Triangular part L is'); disp(L);
disp('The Upper Triangular part U is'); disp(U);
disp('The product L*U is'); disp(L*U);

%(b)
% Solve the linear system Ax=b
% by using the LU-decomposition obtained in (a).

% First, let us solve L*y = b by forward substitution.
 % Set the right-hand-side vector b.
b = [0 -2 1]';
```

```
 % Initialization of the solution vector y.
y = zeros(3, 1);
y(1) = b(1) / L(1, 1);
y(2) = (b(2) - (L(2, 1)*y(1))) / L(2, 2);
y(3) = (b(3) - (L(3, 1)*y(1)) - (L(3, 2)*y(2))) / L(3, 3);

% Next, let us solve U*x = y by backward substitution.
x = zeros(3, 1); % Initialization of the solution vector x.
x(3) = y(3) / U(3, 3);
x(2) = (y(2) - (U(2, 3)*x(3))) / U(2, 2);
x(1) = (y(1) - (U(1, 3)*x(3)) - (U(1, 2)*x(2))) / U(1, 1);

disp('The solution to Ax=b by the LU-decomposition is'); disp(x');

% (c)
fprintf('Using MATLAB command lu\n');
% LU decomposition of A with a permutation matrix.
[L U P] = lu(A);

disp('Lower triangular part L is'); disp(L);
disp('Upper triangular part U is'); disp(U);
disp('The permutation matrix P is'); disp(P);
disp('PA='); disp(P*A); disp('LU='); disp(L*U);
```

MATLAB results.

```
A is
       6             -2              0
       9             -1              1
       3              7              5

The Lower Triangular part L is
       6              0              0
       9              2              0
       3              8              1

The Upper Triangular part U is
       1             -1/3            0
       0              1              1/2
       0              0              1

The product L*U is
       6             -2              0
       9             -1              1
       3              7              5
```

```
The solution to Ax=b by the LU-decomposition is
     -11/6          -11/2          9

Using MATLAB command lu

Lower triangular part L is
       1              0              0
       1/3            1              0
       2/3           -2/11           1
Upper triangular part U is
       9             -1              1
       0             22/3           14/3
       0              0              2/11
The permutation matrix P is
       0              1              0
       0              0              1
       1              0              0
PA=
       9             -1              1
       3              7              5
       6             -2              0
LU=
       9             -1              1
       3              7              5
       6             -2              0
```

Since the permutation matrix P is not the identity matrix, the MATLAB command *lu* gave us an LU-decomposition after multiplying A by the permutation matrix P, hence, this decomposition is a PLU-decomposition of A because $PA = LU$. Since at least one row interchange of A occurred in the process of LU-decomposition, this result is different from the previous decomposition obtained in (a).

Exercise 3.7. (*LU-decomposition*)

(a) The MATLAB command *lu* is used to find the LU-decomposition of a matrix A. Tell what happens if you use the command *lu* for A, where A is given in the Example 2 of the Section 3.7. Explain why this result differs from the result in the textbook.

(b) Using MATLAB, observe what happens when you try to find an LU-decomposition of a singular matrix.

Solution.

```
% (a)
```

```
% Construct the matrix A.
A=[6 -2 0; 9 -1 1; 3 7 5];

% LU decomposition of A.
[L U P]=lu(A);
disp('[L U P]=lu(A)');
disp('L'); disp(L); disp('U'); disp(U); disp('P'); disp(P);

% (b)
% Construct the some singular matrices.
A1=[1 0 0; -2 0 0; 4 6 1];
A2=[1 -2 7; -4 8 5; 2 -4 3];
A3=[1 0 0; -2 0 0; 4 6 1];

% LU decompositions of them.
[L1 U1 P1]=lu(A1); [L2 U2 P2]=lu(A2); [L3 U3 P3]=lu(A3);
disp('[L1 U1 P1]=lu(A1)'); disp('L1');disp(L1);disp('U1');disp(U1);
disp('[L2 U2 P2]=lu(A2)'); disp('L2');disp(L2); disp('U2');disp(U2);
disp('[L3 U3 P3]=lu(A3)'); disp('L3');disp(L3); disp('U3');disp(U3);
```

MATLAB results.

```
[L U P]=lu(A)
L
    1.0000         0         0
    0.3333    1.0000         0
    0.6667   -0.1818    1.0000

U
    9.0000   -1.0000    1.0000
         0    7.3333    4.6667
         0         0    0.1818

P
     0     1     0
     0     0     1
     1     0     0


[L1 U1 P1]=lu(A1)
L1
    1.0000         0         0
   -0.5000    1.0000         0
    0.2500   -0.5000    1.0000
U1
    4.0000    6.0000    1.0000
         0    3.0000    0.5000
```

```
        0         0         0

[L2 U2 P2]=lu(A2)
L2
    1.0000         0         0
   -0.2500    1.0000         0
   -0.5000         0    1.0000
U2
   -4.0000    8.0000    5.0000
         0         0    8.2500
         0         0    5.5000

[L3 U3 P3]=lu(A3)
L3
    1.0000         0         0
   -0.5000    1.0000         0
    0.2500   -0.5000    1.0000
U3
    4.0000    6.0000    1.0000
         0    3.0000    0.5000
         0         0         0
```

Remark on (a). Since the permutation matrix P is not the identity matrix, the MATLAB command *lu* gave us an LU-decomposition after multiplying A by the permutation matrix P, hence, this decomposition is a PLU-decomposition of A because $PA = LU$. Since at least one row interchange of A occurred in the process of LU-decomposition, this result is different from the decomposition result in the textbook.

Remark on (b). When we try LU-decomposition of the sigular matrices using the MATLAB command *lu*, the resulting upper triangular matrices are singular.