1 (3+3+4 points) Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

(a) If A and B are square matrices of the same size, then AB^TBA^T is orthogonally diagonalizable.

(b) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Then A is negative definite.

(c) The quadratic form $Q = 9x^2 + 4xy + y^2$ represents a hyperbola.

Solution.

(a) True. Since AB^TBA^T is symmetric, it is orthogonally diagonalizable.

(b) False. A has a positive eigenvalue.

(c) False. $Q = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and the characteristic polynomial of $A = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}$ is $\lambda^2 - 10\lambda + 5$. If λ_1 and λ_2 are eigenvalues of A, then $\lambda_1\lambda_2 = 5 > 0$. So Q dose not represent a hyperbola.

2 10 points

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Then, find $I + A + A^2 + \cdots + A^{100}$.

Solution.

Since A is symmetric, A is orthogonally diagonalizable.

So we can get

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} & I + A + A^2 + \cdots + A^{100} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{3^{101}-1}{2} & 0 \\ 0 & 101 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3^{101}+201}{4} & \frac{3^{101}-203}{4} \\ \frac{3^{101}-203}{4} & \frac{3^{101}+201}{4} \end{bmatrix}. \end{aligned}$$