8.1.10 Let $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 7x_2 \\ 3x_1 - 4x_2 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 22 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $\mathbf{v}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Find the matrices $[T]_B$ and $[T]_{B'}$ with respect to the bases $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2\}$, respectively, and confirm that these matrices satisfy Formula (14) of Theorem 8.1.2.

Solution. We have $T\mathbf{v}_1 = \begin{bmatrix} 156 \\ -82 \end{bmatrix} = -\frac{86}{45}\begin{bmatrix} 2 \\ 22 \end{bmatrix} + \frac{1798}{45}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = -\frac{86}{45}\mathbf{v}_1 + \frac{1798}{45}\mathbf{v}_2$ and $T\mathbf{v}_2 = \begin{bmatrix} -3 \\ 16 \end{bmatrix} = \frac{61}{90}\mathbf{v}_1 - \frac{49}{45}\mathbf{v}_2$. Similarly, $T\mathbf{v}'_1 = \begin{bmatrix} 22 \\ -9 \end{bmatrix} = -\frac{31}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \frac{75}{2}\begin{bmatrix} -1 \\ -11 \end{bmatrix} = -\frac{31}{2}\mathbf{v}'_1 - \frac{75}{2}\mathbf{v}'_2$ and $T\mathbf{v}'_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix} = \frac{9}{2}\mathbf{v}'_1 + \frac{25}{2}\mathbf{v}'_2$. Therefore,

$$[T]_B = \begin{bmatrix} -\frac{86}{45} & \frac{61}{90} \\ \frac{1798}{45} & -\frac{49}{45} \end{bmatrix} \quad \text{and} \quad [T]_{B'} = \begin{bmatrix} -\frac{31}{2} & \frac{9}{2} \\ -\frac{75}{2} & \frac{25}{2} \end{bmatrix}.$$

Since $\operatorname{rref}[B'|B] = \left[\begin{array}{cc|cc} 1 & 0 & 10 & -\frac{5}{2} \\ 0 & 1 & 8 & -\frac{13}{2} \end{array}\right]$, we have $P = P_{B\to B'} = \begin{bmatrix} 10 & -\frac{5}{2} \\ 8 & -\frac{13}{2} \end{bmatrix}$, and

$$P[T]_B P^{-1} = \begin{bmatrix} 10 & -\frac{5}{2} \\ 8 & -\frac{13}{2} \end{bmatrix} \begin{bmatrix} -\frac{86}{45} & \frac{61}{90} \\ \frac{1798}{45} & -\frac{49}{45} \end{bmatrix} \begin{bmatrix} \frac{13}{90} & -\frac{1}{18} \\ \frac{8}{45} & -\frac{2}{9} \end{bmatrix} = \begin{bmatrix} -\frac{31}{2} & \frac{9}{2} \\ -\frac{75}{2} & \frac{25}{2} \end{bmatrix} = [T]_{B'}$$

□

8.1.16 Let $T : R^3 \to R^3$ be the linear operator that is defined by $T(x, y, z) = (x+y+z, 2y+4z, 4z)$, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis for R^3 in which $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (-1, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$, and let $\mathbf{x} = (2, -3, 4)$.

(a) Find $[T(\mathbf{x})]_B$, $[T]_B$, and $[\mathbf{x}]_B$.

(b) Confirm that $[T(\mathbf{x})]_B = [T]_B[\mathbf{x}]_B$, as guaranteed by Formula (7).

Solution. (a) $T(\mathbf{x}) = (3, 10, 16) = \frac{13}{2}\mathbf{v}_1 + \frac{7}{2}\mathbf{v}_2 + 16\mathbf{v}_3$; thus $[T(\mathbf{x})]_B = (\frac{13}{2}, \frac{7}{2}, 16)$. Also,

$$\mathrm{rref}[B \mid \mathbf{x}] = \begin{bmatrix} 1 & 0 & 0 & \mid & -1/2 \\ 0 & 1 & 0 & \mid & -5/2 \\ 0 & 0 & 1 & \mid & 4 \end{bmatrix}; \text{ thus } [\mathbf{x}]_B = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 4 \end{bmatrix}, \text{ and}$$

$$\mathrm{rref}[B \mid [T] \cdot B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2.5 \\ 0 & 1 & 0 & 0 & 1 & 1.5 \\ 0 & 0 & 1 & 0 & 0 & 4 \end{array}\right]; \text{ thus } [T]_B = \begin{bmatrix} 2 & 1 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 4 \end{bmatrix},$$

where $[T]$ stands for the standard matrix of T. (Note that $[T] \cdot B = [T(\mathbf{v}_1)\ T(\mathbf{v}_2)\ T(\mathbf{v}_3)]$)

(b) $$[T]_B[\mathbf{x}]_B = \begin{bmatrix} 2 & 1 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{13}{2} \\ \frac{7}{2} \\ 16 \end{bmatrix} = [T(\mathbf{x})]_B.$$

□

Note. Given an (ordered) basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of R^n, by abusing the notation, one can regard B as the matrix with $\mathbf{v}_1, \dots, \mathbf{v}_n$ as columns in that order.

8.1.22 Consider the bases

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \quad \text{and} \quad B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$$

for R^4 and R^3, respectively, in which $\mathbf{v}_1 = (0,1,1,1)$, $\mathbf{v}_2 = (2,1,-1,-1)$, $\mathbf{v}_3 = (1,4,-1,2)$, $\mathbf{v}_4 = (6,9,4,2)$, $\mathbf{v}'_1 = (0,8,8)$, $\mathbf{v}'_2 = (-7,8,1)$, $\mathbf{v}'_3 = (-6,9,1)$, and let $T : R^4 \to R^3$ be the linear transformation whose matrix with respect to B and B' is

$$[T]_{B',B} = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ -3 & 0 & 7 & 1 \end{bmatrix}$$

(a) Find $[T(\mathbf{v}_1)]_{B'}$, $[T(\mathbf{v}_2)]_{B'}$, $[T(\mathbf{v}_3)]_{B'}$, $[T(\mathbf{v}_4)]_{B'}$.

(b) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, $T(\mathbf{v}_3)$, and $T(\mathbf{v}_4)$.

(c) Find a formula for $T(x_1, x_2, x_3, x_4)$.

(d) Use the formula obtained in part (c) to compute $T(2,2,0,0)$.

Solution. (a) Take columns:

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \ [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}, \ [T(\mathbf{v}_3)]_{B'} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \ [T(\mathbf{v}_4)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(b) By using (a),

$$T\mathbf{v}_1 = 3\mathbf{v}'_1 + \mathbf{v}'_2 - 3\mathbf{v}'_3 = (11,5,22),\ T\mathbf{v}_2 = -2\mathbf{v}'_1 + 6\mathbf{v}'_2 = (-42,32,-10),$$
$$T\mathbf{v}_3 = \mathbf{v}'_1 + 2\mathbf{v}'_2 + 7\mathbf{v}'_3 = (-56,87,17),\ T\mathbf{v}_4 = \mathbf{v}'_2 + \mathbf{v}'_3 = (-13,17,2)$$

(c) The standard matrix of T is

$$[T] = B' \cdot [T]_{B',B} \cdot B^{-1} = \begin{bmatrix} -\frac{253}{10} & \frac{49}{5} & \frac{241}{10} & -\frac{229}{10} \\ \frac{115}{2} & -39 & -\frac{65}{2} & \frac{153}{2} \\ 66 & -60 & -9 & 91 \end{bmatrix}.$$

The formula comes from this matrix in the obvious way.

(d) From (c), we have $T(2,2,0,0) = (-31,37,12)$.

$\square$

Note. Given an (ordered) basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of R^n, by abusing the notation, one can regard B as the matrix with $\mathbf{v}_1, \dots, \mathbf{v}_n$ as columns in that order.

8.1.26 Show that if $T : R^n \to R^n$ is a contraction or a dilation of R^n (see Section 6.2), then the matrix for T with respect to any basis for R^n is a positive scalar multiple of the identity matrix.

Solution. There is a scalar $k > 0$ such that $T(\mathbf{x}) = k\mathbf{x}$ for all $\mathbf{x} \in R^n$. Thus for any basis $B = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ of R^n, $T\mathbf{v}_i = k\mathbf{v}_i$ for each i, so that $[T]_B = kI$. □

8.1.30 Let $T : R^3 \to R^3$ be the linear operator whose standard matrix is

$$[T] = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 2 & 0 & 0 \end{bmatrix}$$

and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the orthonormal basis for R^3 in which

$$\mathbf{v}_1 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \quad \mathbf{v}_2 = (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \quad \mathbf{v}_3 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}).$$

Find $[T]_B$, and use that matrix to describe the geometric effect of the operator T.

Solution. Note first that $[T]/2$ is an orthogonal matrix with determinant 1, so it is a rotation matrix. By easy calculation, we have

$$T\mathbf{v}_1 = 2\mathbf{v}_1,\ T\mathbf{v}_2 = -\mathbf{v}_2 + \sqrt{3}\mathbf{v}_3,\ T\mathbf{v}_3 = -\sqrt{3}\mathbf{v}_2 - \mathbf{v}_3;$$

thus $[T]_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & -\sqrt{3} \\ 0 & \sqrt{3} & -1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 120° & -\sin 120° \\ 0 & \sin 120° & \cos 120° \end{bmatrix}$. From this we see that the effect of the operator T is to rotate vectors counterclockwise by an angle of 120° about the $\mathbf{v}_1$ axis, then stretch by a factor of 2.

□

8.1.D4 Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are linear operators, and if $[T_1]_{B',B} = [T_2]_{B',B}$ with respect to two bases B and B' for R^n, then $T_1(\mathbf{x}) = T_2(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n

(b) If $T_1 : R^n \to R^n$ is a linear operator, and if $[T_1]_B = [T_1]_{B'}$ with respect to two bases B and B' for R^n, then $B = B'$.

(c) If $T : R^n \to R^n$ is a linear operator, and if $[T]_B = I_n$ with respect to some basis B for R^n, then T is the identity operator on R^n.

(d) If $T : R^n \to R^n$ is a linear operator, and if $[T]_{B',B} = I_n$ with respect to two bases B and B' for R^n, then T is the identity operator on R^n.

Solution. (a) True. We have $[T_1(\mathbf{x})]_{B'} = [T_1]_{B',B}[\mathbf{x}]_B = [T_2]_{B',B}[\mathbf{x}]_B = [T_2(\mathbf{x})]_{B'}$; thus $T_1(\mathbf{x}) = T_2(\mathbf{x})$.

(b) False. For example, the zero operator has the same matrix (zero matrix) with respect to any basis for R^2.

(c) True. Write $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then we have $T(\mathbf{v}_k) = \mathbf{v}_k$ for each k.

(d) False. For example, let $B = \{\mathbf{e}_1, \mathbf{e}_2\}$, $B' = \{\mathbf{e}_2, \mathbf{e}_1\}$, and $T(x, y) = (y, x)$. Then $[T]_{B',B} = I_2$ but T is not the identity operator.

□

8.1.P2 Suppose that $T_1 : R^n \to R^k$ and $T_2 : R^k \to R^m$ are linear transformations, and suppose that $B,\ B'$, and B'' are bases for $R^n,\ R^k$, and R^m, respectively. Prove that

$$[T_2 \circ T_1]_{B'',B} = [T_2]_{B'',B'}[T_1]_{B',B}$$

Solution. $[T_2 \circ T_1(\mathbf{x})]_{B''} = [T_2]_{B'',B'}[T_1(\mathbf{x})]_{B'} = [T_2]_{B'',B'}[T_1]_{B',B}[\mathbf{x}]_B$ for every $\mathbf{x} \in R^n$. □

8.2.6 The characteristic polynomial of a matrix A is given. Find the size of the matrix, list its eigenvalues with their algebraic multiplicities, and discuss the possible dimensions of the eigenspaces.

(a) $\lambda(\lambda-1)(\lambda+2)(\lambda-3)^2$

(b) $\lambda^2(\lambda-6)(\lambda-2)^3$

Solution. (a) The matrix is 5×5 with eigenvalues $\lambda=0$ (multiplicity 1), $\lambda=1$ (multiplicity 1), $\lambda=-2$ (multiplicity 1), and $\lambda=3$ (multiplicity 2). The eigenspaces corresponding to $\lambda=0,1,-2$ each have dimension 1. The eigenspace corresponding to $\lambda=3$ has dimension 1 or 2.

(b) The matrix is 6×6 matrix with eigenvalues $\lambda=0$ (multiplicity 2), $\lambda=6$ (multiplicity 1), and $\lambda=2$ (multiplicity 3). The eigenspace corresponding to $\lambda=6$ has dimension 1, the eigenspace corresponding to $\lambda=0$ has dimension 1 or 2, and the eigenspace corresponding to $\lambda=2$ has dimension 1, 2, or 3.

□

8.2.12 Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & -6 & 11 \\ 1 & -4 & 7 \end{bmatrix}$. Find the geometric multiplicities of the eigenvalues of A by computing the rank of $\lambda I - A$ for each eigenvalue by row reduction and then using the relationship between rank and nullity.

Solution. The characteristic polynomial of A is $(\lambda - 1)(\lambda^2 - 2\lambda + 2)$; thus $\lambda = 1$ is the only real eigenvalue of A. the reduced row echelon form of the matrix $I - A$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$; thus $\text{rank}(I - A) = 2$ and the geometric multiplicity of $\lambda = 1$ is $\text{nullity}(I - A) = 3 - 2 = 1$. $\square$

8.2.23 Let $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. Determine whether A is diagonalizable. If so, find a matrix P that diagonalizes the matrix A, and determine $P^{-1}AP$.

Solution. The characteristic polynomial of A is $p(\lambda) = (\lambda - 3)^2(\lambda + 2)^2$. Since

$$\operatorname{rref}(3I - A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$\operatorname{nullity}(3I - A) = 4 - \operatorname{rank}(3I - A) = 1$, so the geometric multiplicity of $\lambda = 3$ is strictly less than its algebraic multiplicity 2. Therefore, A is not diagonalizable by Theorem 8.2.10. $\square$

8.2.26 Show that if a 3×3 matrix has a three-dimensional eigenspace, then it must be diagonal. State a generalization of this result.

Solution. If A is an $n \times n$ matrix with an n-dimensional eigenspace then A has only one eigenvalue, say λ. It follows that $A\mathbf{x} = \lambda\mathbf{x}$ for all $\mathbf{x} \in R^n$, and so $A = \lambda I$ is a diagonal matrix. $\square$

8.2.29 Consider the linear operator $T : R^3 \to R^3$ defined by the formula

$$T(x_1, x_2, x_3) = (-2x_1 + x_2 - x_3, x_1 - 2x_2 - x_3, -x_1 - x_2 - 2x_3).$$

Find the eigenvalues of T and show that T is diagonalizable.

Solution. The standard matrix of T is $A = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$, and the characteristic polynomial of A is $p(\lambda) = \lambda^3 + 6\lambda^2 + 9\lambda = \lambda(\lambda + 3)^2$. Hence the eigenvalues of T are $\lambda = 0$ and $\lambda = -3$, with algebraic multiplicities 1 and 2 respectively. It is easy to see that $\mathrm{rank}(-3I - A) = 1$ by computing $\mathrm{rref}(-3I - A)$, so the geometric multiplicity of $\lambda = -3$ is 2. Therefore, T is diagonalizable by Theorem 8.2.10. $\square$

8.2.D3 Indicate whether the statement is true (T) or false (F). Justify your answer.

(a) Singular matrices are not diagonalizable.

(b) If A is diagonalizable, then there is a unique matrix P such that $P^{-1}AP$ is a diagonal matrix.

(c) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero vectors that come from different eigenspaces of A, then it is impossible to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(d) If an invertible matrix A is diagonalizable, then A^{-1} is also diagonalizable.

(e) If R^n has a basis of eigenvectors for the matrix A, then A is diagonalizable.

Solution. (a) False. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) False. $P^{-1}AP = (2P)^{-1}A(2P)$.

(c) True, since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

(d) True. If an invertible matrix A has a diagonalization $A = PDP^{-1}$, then D is also invertible since the diagonal entries are nonzero. Hence, $A^{-1} = PD^{-1}P^{-1}$ is a diagonalization of A^{-1}.

(e) True, since A has n linearly independent eigenvectors.

□

8.2.D4 Suppose that the characteristic polynomial of a matrix A is

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

(a) What size is A?

(b) What can you say about the dimensions of the eigenspaces of A?

(c) What can you say about the dimensions of the eigenspaces if you know that A is diagonalizable?

(d) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of eigenvectors of A all of which correspond to the same eigenvalue A, what can you say about that eigenvalue?

Solution. (a) $1 + 2 + 3 = 6$.

(b) The eigenspace corresponding to $\lambda = 1$ has dimension 1. The eigenspace corresponding to $\lambda = 3$ has dimension 1 or 2. The eigenspace corresponding to $\lambda = 4$ has dimension 1, 2, or 3.

(c) If A is diagonalizable, then the eigenspace corresponding to $\lambda = 1$, $\lambda = 3$, and $\lambda = 4$ have dimensions 1, 2, and 3 respectively.

(d) These vectors must correspond to the eigenvalue $\lambda = 4$.

□