1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let V and W be subspaces of $\mathbb{R}^n$. Then $(V \cap W)^{\perp} = V^{\perp} \cup W^{\perp}$.

(b) Let A be an idempotent matrix. The possible eigenvalues of A are 0 and 1.

(c) Suppose P is a symmetric square matrix such that $P^3 = P^2$. Then P^2 is the standard matrix for an orthogonal projection.

Solution.

(a) False.
Suppose $n = 3$, $V = \{(a, 0, b) : a, b \in \mathbb{R}\}$ and $W = \{(c, d, 0) : c, d \in \mathbb{R}\}$. Then clearly V, W are subspaces of $\mathbb{R}^3$. Since $(V \cap W) = \{(x, 0, 0) : x \in \mathbb{R}\}$, $(V \cap W)^{\perp} = \{(0, y, z) : y, z \in \mathbb{R}\}$. Since $V^{\perp} = \{(0, m, 0) : m \in \mathbb{R}\}$ and $W^{\perp} = \{(0, 0, n) : n \in \mathbb{R}\}$, we obtain $(V \cap W)^{\perp} \neq V^{\perp} \cup W^{\perp}$. Note that

$$(0, 1, 1) \in (V \cap W)^{\perp} \text{ but } (0, 1, 1) \notin V^{\perp} \cup W^{\perp}.$$

(b) True.
Since A is idempotent, $A^2 = A$. Let λ be any eigenvalue of A and v be its corresponding eigenvector. Then we have

$$\lambda v = Av = A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = (\lambda)^2 v,$$

thus we have $(\lambda)^2 = \lambda$. Therefore, $\lambda = 0$ or 1.

(c) True.
Since $P^4 = P^3 = P^2$, we have $(P^2)^2 = P^2$. So, P^2 is idempotent. Since $P^T = P$, we obtain $(P^2)^T = (PP)^T = P^T P^T = (P^T)^2 = P^2$. This implies P^2 is symmetric. By Theorem 7.7.6, P^2 is the standard matrix for an orthogonal projection.

2 Find the least squares quadratic fit $y = a_0 + a_1 x + a_2 x^2$ to the given points:

10 points $\{(-1,3),(0,1),(1,3),(2,4)\}$

Solution.

By the given points, let

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \text{ and } y = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

Then by simple calculation,

$$M^T M = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}.$$

Also note that

$$M^T y = \begin{bmatrix} 11 \\ 8 \\ 22 \end{bmatrix}.$$

Hence we get

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

Therefore, the answer is $\frac{7}{4} - \frac{1}{4}x + \frac{3}{4}x^2$.