< Inverse of a linear transformation >

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a one-to-one linear transformation.

Then $\forall \mathbf{w} \in \operatorname{ran}(T)$, $\exists\, \mathbf{x} \in \mathbb{R}^n$ uniquely such that $T(\mathbf{x}) = \mathbf{w}$.

Then we can define a linear transformation

$$T^{-1} : \operatorname{ran}(T) \longrightarrow \mathbb{R}^n \quad \text{s.t.} \quad T^{-1}(\mathbf{w}) = \mathbf{x} \text{ iff } T(\mathbf{x}) = \mathbf{w}.$$

We call it <u>the inverse of T.</u>

Hence $\mathbf{w} = T(\mathbf{x}) \underset{T:\text{ one-to-one.}}{\Longrightarrow} T^{-1}(T(\mathbf{x})) = \mathbf{x}$ & $T(T^{-1}(\mathbf{w})) = \mathbf{w}$

<u>Thm 6.4.5</u> If T is a one-to-one linear transformation, so is T^{-1}.

pf) $\forall\, \mathbf{w}, \mathbf{w}' \in \operatorname{ran}(T)$ & a scalar c,

$\exists\, \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ s.t. $T(\mathbf{x}) = \mathbf{w}$ & $T(\mathbf{x}') = \mathbf{w}'$.

i.e., $T^{-1}(\mathbf{w}) = \mathbf{x}$ & $T^{-1}(\mathbf{w}') = \mathbf{x}'$.

Then $c\mathbf{w} + \mathbf{w}' = cT(\mathbf{x}) + T(\mathbf{x}') = T(c\mathbf{x} + \mathbf{x}')$.

Since T is one-to-one,
$$\begin{aligned} T^{-1}(c\mathbf{w} + \mathbf{w}') &= T^{-1}(T(c\mathbf{x} + \mathbf{x}')) \\ &= c\mathbf{x} + \mathbf{x}' \\ &= cT^{-1}(\mathbf{w}) + T^{-1}(\mathbf{w}') \end{aligned}$$

$\therefore$ T^{-1} is a linear transformation.

If $\mathbf{w} \neq \mathbf{w}'$ in $\operatorname{ran}(T)$, then $T(\mathbf{x}) \neq T(\mathbf{x}')$ in $\operatorname{ran}(T)$.

i.e., $T(\mathbf{x} - \mathbf{x}') \neq \mathbf{0}$. Since $\ker(T) = \{\mathbf{0}\}$, $\mathbf{x} \neq \mathbf{x}'$ in $\mathbb{R}^n$.

$\therefore$ T^{-1} is one-to-one. ∎

e.g.) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x, y) \longmapsto (x, -y, -x+y)$$

$$\Rightarrow \quad T(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow$ T is one-to-one.

$$\operatorname{ran}(T) = \operatorname{span}\{(1,0,-1), (0,-1,1)\}$$

$\forall\, \mathbb{w} \in \operatorname{ran}(T)$, $\mathbb{w} = w_1(1,0,-1) + w_2(0,-1,1)$.

$$T^{-1} : \operatorname{ran}(T) \longrightarrow \mathbb{R}^2$$

$$T^{-1}(w_1(1,0,-1) + w_2(0,-1,1)) = w_1 \underbrace{T^{-1}(1,0,-1)}_{e_1} + w_2 \underbrace{T^{-1}(0,-1,1)}_{e_2}$$

$$= (w_1, w_2).$$

※ In fact, finding $T^{-1}(\mathbb{w})$ for $\mathbb{w} \in \operatorname{ran}(T)$ is equivalent to expressing $\mathbb{w}$ as a linear combination of the column vectors of $[T]$.

Thm 6.4.6 Let T be a one-to-one linear operator on $\mathbb{R}^n$. Then $[T]$ is invertible & $[T]^{-1} = [T^{-1}]$.

$\forall\, \mathbb{x} \in \mathbb{R}^n$, we have $T^{-1}(T(\mathbb{x})) = \mathbb{x}$ and $T(T^{-1}(\mathbb{x})) = \mathbb{x}$.

Hence $[T][T^{-1}] = [T^{-1}][T] = \mathrm{Id}$.

Thus $[T]^{-1} = [T^{-1}]$. ∎

<u>Thm 6.4.7</u> Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an invertible linear operator.

(a) The image of a line is a line.

(b) The image of a line passes through the origin iff the original line passes through the origin.

(c) The images of two lines are parallel iff the original lines are parallel.

(d) The images of three points lie on a line iff the original points lie on a line.

(e) The images of the line segment joining the images of those points.

<u>Thm 6.4.8</u> Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an invertible linear operator.

Then the image of the unique square is a parallelogram that has a vertex at the origin and whose adjacent sides are $T(e_1)$ and $T(e_2)$. The area of the parallelogram is $|\det([T])|$.

1. (10 points) Let a be a real number and consider the linear system:

$$(*) \quad \underbrace{\begin{bmatrix} 1 & 3 & a \\ 2 & 5 & 1 \\ 3 & 9 & a^2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ a^2 \\ a+3 \end{bmatrix}}_{\mathbb{b}}.$$

A row echelon form of $[A \mid \mathbb{b}]$ is

$$\left[\begin{array}{ccc|c} 1 & 3 & a & 1 \\ 0 & 1 & 2a-1 & 2-a^2 \\ 0 & 0 & a^2-3a & a \end{array}\right].$$

If $a^2-3a=0$ and $a \neq 0$, then $(*)$ is inconsistent.

i.e., $a=3 \Rightarrow (*)$ is inconsistent.

If $a^2-3a = a = 0$, then $(*)$ has infinitely many solution.

If $a^2-3a \neq 0$, then the matrix A is invertible and $(*)$ has exactly one solution.

$\therefore$ $(*)$ is consistent $\Leftrightarrow$ $a \neq 3$.

$(*)$ has infinitely many solution $\Leftrightarrow$ $a=0$.

$(*)$ has exactly one solution $\Leftrightarrow$ $a \neq 0,\ a \neq 3$.