1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) A nonhomogeneous linear system with more unknowns than equations always has infinitely many solutions.

(b) Every square matrix can be expressed as sum of symmetric matrix and skew-symmetric matrix.

(c) For a matrix A, AA^T is invertible if and only if A^TA is invertible.

Solution.

(a) False.

$$\begin{cases} x + y + z &= 1 \\ 2x + 2y + 2z &= 1 \end{cases}$$

is nonhomogeneous linear system with more unknowns than equations but there is no solution.

(b) True. For any square matrix A, $(A + A^T)/2$ is symmetric and $(A - A^T)/2$ is skew-symmetric. $A = (A + A^T)/2 + (A - A^T)/2$

(c) False. For example let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Then $AA^T = \begin{bmatrix} 1 \end{bmatrix}$ which is invertible, but $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which is not invertible.

2 (10 points) For given real numbers $a, b,$ and c, let $A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$. Find $(I+A)^{-1}$.

Solution.

By simple calculation,

$$A^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ , and } \quad A^3 = 0.$$

Thus, we get $(I+A)^{-1} = (I-(-A))^{-1} = I + (-A) + (-A)^2 = I - A + A^2$ by Theorem 3.6.6. Therefore,

$$(I+A)^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$