1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 3 & 9 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 8 & 0 \\ 6 & 6 & 2 \\ 5 & 0 & 1 \end{bmatrix}.$$

Then, A is similar to B.

(b) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 3 & 9 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then, there is an eigenvalue λ of A whose geometric multiplicity is 2.

(c) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & -5 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then, A is diagonalizable .

Solution.

(a) False. Since $\det(A) = -9 \neq 80 = \det(B)$, A can not be similar to B by theorem 8.2.3.

(b) False. Note that $\det(\lambda I_3 - A) = (\lambda - 3)^2(\lambda + 1)$. But

$$\begin{bmatrix} 1 & 0 & 2 \\ 6 & 3 & 9 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix} \implies \begin{cases} x_1 + 2x_3 = 3x_1 \\ 6x_1 + 3x_2 + 9x_3 = 3x_2 \\ 2x_1 + x_3 = 3x_3 \end{cases} \implies x_1 = x_3 = 0$$

Thus,

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} \quad \text{is an eigenvector of } \lambda = 3 \text{ for all } t \in \mathbb{R}.$$

That is, for eigenvalue $\lambda = 3$, the geometric multiplicity of λ is 1.

(c) True. Note that $\det(\lambda I_3 - A) = (\lambda - 2)(\lambda - 3)(\lambda + 1)$, thus there are 3 distinct real eigenvalues. Hence, A is diagoanlizable by theorem 8.2.8.

2 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator whose standard matrix is

10 points

$$[T] = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Find the matrix for T with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Solution. The image of the basis vector under the operator T are

$$T(\mathbf{v}_1) = [T]\mathbf{v}_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \mathbf{v}_1 = \mathbf{v}_1 + 0\mathbf{v}_2,$$

$$T(\mathbf{v}_2) = [T]\mathbf{v}_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix} = 5\mathbf{v}_2 = 0\mathbf{v}_1 + 5\mathbf{v}_2.$$

So the coordinate matrices of these vectors with repect to B are

$$[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Thus,

$$[T]_B = \Big[[T(\mathbf{v}_1)]_B | [T(\mathbf{v}_2)]_B\Big] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$