1 Indicate whether the following statements are true(**T**) or false(**F**). You do **not** need to justify your answer.

3+3+4 points

(a) A set with less than n vectors in $\mathbb{R}^n$ is linearly independent.

(b) The product of two symmetric matrices is symmetric.

(c) If a square matrix A satisfies $(A + kI)^n = 0$ for a positive integer n and a nonzero scalar k, then A is invertible.

Solution.

(a) False. For example $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ is not linearly independent.

(b) False. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}$ are symmetric but $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$ which is not symmetric.

(c) True. By Theorem 3.6.6., $I - (\frac{1}{k}A + I)$ is invertible sicne $(\frac{1}{k}A + I)^n = 0$. Thus A is invertible.

2 Find all 2×2 upper triangular matrices A such that $A^2 + 7A + 10I = 0$.

10 points

Solution.

Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a, b,$ and c are real. Then

$$A^2 + 7A + 10I = \begin{bmatrix} (a+2)(a+5) & b(a+c+7) \\ 0 & (c+2)(c+5) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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So $a = -2$ or $a = -5$. And $c = -2$ or $c = -5$.

If $a = c = -2$ or $a = c = -5$, then $b = 0$. Otherwise, b can be any value. Therefore,

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \text{ or } \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \text{ or } \begin{bmatrix} -2 & i \\ 0 & -5 \end{bmatrix} \text{ or } \begin{bmatrix} -5 & j \\ 0 & -2 \end{bmatrix}$$

where i and j are any real numbers.