**4.1.24** Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}.$$

Find

(a) $M_{32}$ and $C_{32}$ (b) $M_{44}$ and $C_{44}$

(c) $M_{41}$ and $C_{41}$ (d) $M_{24}$ and $C_{24}$

*Solution.* Recall that $M_{ij}$, the minor of entry $a_{ij}$, is defined to be the determinant of the submatrix that remains when the $i$th row and $j$th column of $A$ are deleted. The cofactor of entry $a_{ij}$ is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

(a) $M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = -30, \;\; C_{32} = (-1)^{3+2} M_{32} = 30.$ □

(b) $M_{44} = \begin{vmatrix} 2 & 3 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 13, \;\; C_{44} = (-1)^{4+4} M_{44} = 13.$ □

(c) $M_{41} = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 3 \\ -2 & 1 & 0 \end{vmatrix} = -1, \;\; C_{41} = (-1)^{4+1} M_{41} = 1.$ □

(d) $M_{24} = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 0, \;\; C_{24} = (-1)^{2+4} M_{24} = 0.$ □

**4.1.33** Show that the value of the determinant

$$\begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta)-\cos(\theta) & \sin(\theta)+\cos(\theta) & 1 \end{vmatrix}$$

is independent of $\theta$.

*Solution.* By expanding along third column, we have

$$\begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta)-\cos(\theta) & \sin(\theta)+\cos(\theta) & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{vmatrix} = \sin^2(\theta)+\cos^2(\theta) = 1.$$

$\square$

**4.1.36** Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix}$$

for any $2 \times 2$ matrix $A$.

*Solution.* If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathrm{tr}(A) = a+d$, $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$, and $\mathrm{tr}(A^2) = a^2+2bc+d^2$. Thus,

$$\begin{aligned} \frac{1}{2}\begin{vmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{vmatrix} &= \frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2+2bc+d^2 & a+d \end{vmatrix} \\ &= \frac{1}{2}((a+d)^2 - (a^2+2bc+d^2)) \\ &= ad - bc \\ &= \det(A). \end{aligned}$$

□

**4.1.P1** Prove that $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ are collinear points if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

*Solution.* If the three points lie on a vertical line then $x_1 = x_2 = x_3 = a$ and we have $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a & y_1 & 1 \\ a & y_2 & 1 \\ a & y_3 & 1 \end{vmatrix} = 0$. Thus, without loss of generality, we may assume that the points do not lie on a vertical line. In this case the points are collinear if and only if $\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$. The latter condition is equivalent to $(y_3 - y_1)(x_2 - x_1) = (y_2 - y_1)(x_3 - x_1)$ which can be written as:

$$(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) = 0.$$

On the other hand, expanding along the third column, we have:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1).$$

Thus the three points are collinear if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$ □

**4.2.16** Evaluate the determinant of the matrix by reducing it to row echelon form.

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

*Solution.* We first calculate the row echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

The above calculation consists of two times of interchange of two rows and several times of addition of rows. Then

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)^2 \cdot \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 6 \end{vmatrix} = 6.$$

□

**4.2.22** Find the determinant of the matrix

$$\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

*Solution.* By the direct calculation,

$$\begin{aligned}
\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} &= \begin{vmatrix} a & b & b & b \\ b & a & b & b \\ 0 & a-b & -(a-b) & 0 \\ 0 & a-b & 0 & -(a-b) \end{vmatrix} \\
&= \begin{vmatrix} a & b & b & b \\ a-b & -(a-b) & 0 & 0 \\ 0 & a-b & -(a-b) & 0 \\ 0 & 0 & -(a-b) & a-b \end{vmatrix} \\
&= (a-b)^3 \begin{vmatrix} a & b & b & b \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix} \\
&= (a-b)^3 \left( 1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} b & b & b \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} + (-1) \cdot (-1)^{2+2} \cdot \begin{vmatrix} a & b & b \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} \right) \\
&= (a-b)^3 \left( -b \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} - a \cdot \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} \right) \\
&= (a-b)^3(a+3b).
\end{aligned}$$

□

**4.2.35** (a) Show that if $A$ is a square matrix, then $\det(A^T A) = \det(AA^T)$.
(b) Show that $A$ is invertible if and only if $A^T A$ is invertible.

*Solution.* (a) Since $\det(A) = \det(A^T)$, we have

$$\det(A^T A) = \det(A^T)\det(A) = (\det(A))^2 = \det(A)\det(A^T) = \det(AA^T).$$

$\square$

(b) Since $\det(A^T A) = (\det(A))^2$, it follows that $\det(A^T A) = 0$ if and only if $\det(A) = 0$. Thus, from Theorem 4.2.4, $A^T A$ is invertible if and only if $A$ is invertible. $\square$

**4.2.38** It can be proved that if a square matrix $M$ is partitioned into ***block triangular form*** as

$$M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

in which $A$ and $B$ are square, then $\det(M) = \det(A)\det(B)$. Use this result to compute the determinants of the matrix.

(a) $M = \left[\begin{array}{cc:cc} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ \hdashline 5 & 12 & 3 & 3 \\ 11 & -8 & 2 & 1 \end{array}\right]$

(b) $M = \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 8 & 6 & -9 \\ 2 & 5 & 0 & 4 & 7 & 5 \\ -1 & 3 & 2 & 6 & 9 & -2 \\ \hdashline 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 8 & -4 \end{array}\right]$

*Solution.* (a) We have $\det \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 1$ and $\det \begin{bmatrix} 3 & 3 \\ 2 & 1 \end{bmatrix} = -3$, thus $\det(M) = 1 \cdot (-3) = -3$. □

(b) We have $\det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ -1 & 3 & 2 \end{bmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 2$ and $\det \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 8 & -4 \end{bmatrix} = 3 \cdot \begin{vmatrix} 1 & 0 \\ 8 & -4 \end{vmatrix} = -12$, thus $\det(M) = 2 \cdot (-12) = -24$. □

**4.2.D4** What can you say about the determinant of an $n \times n$ matrix with the following form?

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

*Solution.* For convenience call the given matrix $A_n$. If $n = 2$ or 3, then $A_n$ can be reduced to the identity matrix by interchanging the first and last rows. Thus $\det(A_n) = -1$ if $n = 2$ or 3. If $n = 4$ or 5, then two row interchanges are required to reduce $A_n$ to the identity (interchange the first and last rows, then interchange the second and next to last rows). Thus $\det(A_n) = +1$ if $n =$ 4 or 5. This pattern continues and can be summarized as follows:

$$\det(A_{2k}) = \det(A_{2k+1}) = -1 \quad \text{for } k = 1, 3, 5, \cdots$$
$$\det(A_{2k}) = \det(A_{2k+1}) = +1 \quad \text{for } k = 2, 4, 6, \cdots$$

□

**4.2.D9** If $A = A^2$, what can you say about the determinant of $A$? What can you say if $A = A^3$?

*Solution.* If $A = A^2$, then $\det(A) = \det(A^2) = (\det(A))^2$ and so $\det(A) = 0$ or $\det(A) = 1$. If $A = A^3$, then $\det(A) = \det(A^3) = (\det(A))^3$ and so $\det(A) = 0$ or $\det(A) = \pm 1$. □

**4.2.D10** Let $A$ be a matrix of the form

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}.$$

How many different values can you obtain for $\det(A)$ by substituting numerical values (not necessarily all the smae) for the $*$'s? Explain your reasoning.

*Solution.* Each elementary product of this matrix must include a factor that comes from the $3 \times 3$ block of zeros on the upper right. Thus all of the elementary products are zero. It follows that $\det(A) = 0$, no matter what values are assigned to the starred quantities. $\square$

**4.2.P1** Let $A$, $B$ and $C$ be $n \times n$ matrices of the form

$$A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{c_n} \end{bmatrix}, B = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{y} & \cdots & \mathbf{c_n} \end{bmatrix},$$
$$C = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{x}+\mathbf{y} & \cdots & \mathbf{c_n} \end{bmatrix}$$

where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$ are the $j$th column vectors. Use cofactor expansions to prove that $\det(C) = \det(A) + \det(B)$.

*Solution.* If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ then, using cofactor expansions along the $j$th column, we have

$$\begin{aligned} \det(C) &= (x_1 + y_1)C_{1j} + (x_2 + y_2)C_{2j} + \cdots + (x_n + y_n)C_{nj} \\ &= (x_1 C_{1j} + x_2 C_{2j} + \cdots + x_n C_{nj}) + (y_1 C_{1j} + y_2 C_{2j} + \cdots + y_n C_{nj}) \\ &= \det(A) + \det(B). \end{aligned}$$

□