December 05, 2019

$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Question How can we decide whether a given $n \times n$ matrix A is diagonalizable?

$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)^2$

1) Find the eigenvalues of A, say $\lambda_1, \cdots, \lambda_k$.

$\lambda_1 = 1, \quad \lambda_2 = 2$.

2) Compute the eigenspace E_i of A corresponding to λ_i, $1 \le i \le k$. Say $m_i := \dim E_i$.

$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \ \forall t$

3) A is diagonalizable iff $\sum_{i=1}^{k} m_i = n$.

$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

If A is diagonalizable, by finding bases of the eigenspaces, we can diagonalize A.

4) If $\{\mathbb{p}_1^i, \cdots, \mathbb{p}_{m_i}^i\}$ is a basis for E_k, then form the matrix

$$P = \left[\mathbb{p}_1^1 \cdots \mathbb{p}_{m_1}^1 \ \cdots \ \mathbb{p}_1^k \cdots \mathbb{p}_{m_k}^k \right],$$

then
$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & \\ & \ddots \lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_k & \\ & & & & \ddots \lambda_k \end{bmatrix}$$

$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

< Method for orthogonally diagonalizing a symmetric matrix >

Step 1 Find a basis for each eigenspace of A.

Step 2 Apply G-S process to each of bases to produce orthonormal bases for the eigenspaces.

Step 3 Form the matrix $P = [\mathbb{p}_1 \ \mathbb{p}_2 \cdots \mathbb{p}_n]$ whose columns are the vectors constructed in Step 2.

Step 4 The matrix P^TAP will be diagonal;

$$P^TAP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{with} \quad A\mathbb{p}_i = \lambda_i \mathbb{p}_i \quad (i = 1, \cdots, n).$$

Let A be a symmetric matrix and $P = [u_1 \; u_2 \cdots u_n]$ s.t.

$$P^T A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = D : \text{ diagonal.}$$

Then $A = PDP^T = [u_1 \cdots u_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - u_1^T \\ \vdots \\ - u_n^T \end{bmatrix}$

$$= \underbrace{\lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T}$$

a spectral decomposition of A

(or an eigenvalue decomposition of A)

<u>Remark</u>

- If $P^{-1}AP = D$, then $(P^{-1}AP)^k = P^{-1}A^kP = D^k$, and hence $A^k = PD^kP^{-1}$.
- If $A = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$, then

$$A^k = PD^kP^T = \lambda_1^k u_1 u_1^T + \cdots + \lambda_n^k u_n u_n^T.$$

<u>Thm 8.3.5</u> (Cayley–Hamilton theorem)

If A is an $n \times n$ matrix whose characteristic eqn is

$$\lambda_1^n + c_1 \lambda_2^{n-1} + \cdots + c_n = 0,$$

then $A^n + c_1 A^{n-1} + \cdots + c_n I = 0.$

※ Note that $\mathbb{R} \subset \mathbb{C}$ and every polynomial of degree n with complex coefficents can be factorized into n factors. i.e., $\exists$ an invertible matrix P whose coefficients are complex numbers s.t. $P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, $\lambda_i \in \mathbb{C}$. Since $D^n + c_1 D^{n-1} + \cdots + c_n I = O$, $A^n + c_1 A^{n-1} + \cdots + c_n I = O$.

e.g.) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc$

and $A^2 - (a+d)A + (ad-bc) I = O.$

Remark If A is invertible, then $c_n \neq 0$.

Note $c_n I = A(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I)$.

Hence $A^{-1} = \frac{1}{c_n}(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I)$ for $c_n \neq 0$.

Now assume that a function f is represented by its Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{m}(0)}{m!}x^m + \cdots$$

on some interval.

Then $f(A) = f(0)I + f'(0)A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^{(m)}(0)}{m!}A^m + \cdots$

If A is diagonalizable, $P^{-1}AP = D$, then

$$f(D) = f(P^{-1}AP) = P^{-1}f(A)P$$

$$\therefore f(A) = Pf(D)P^{-1} = P\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}P^{-1}.$$

Ex. Compute e^{tA}, $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Recall $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$

$$\therefore e^{tA} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} e^{tA} & & \\ & e^{2tA} & \\ & & e^{2tA} \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^t - e^{2t} & 0 & 2e^t - 2e^{2t} \\ e^{2t} - e^t & e^{2t} & e^{2t} - e^t \\ e^{2t} - e^t & 0 & 2e^{2t} - e^t \end{bmatrix}$$

< Matrix decomposition & linear system >

$A\mathbf{x} = \mathbf{b}$ & $P^{-1}AP = D$

$\Rightarrow$ Letting $y = P^{-1}\mathbf{x}$, we obtain $\underbrace{Dy = P^{-1}\mathbf{b}}_{\text{easy!}}$.

Many algorithms for solving linear systems are based on this idea.

However, computing P^{-1} magnifies roundoff error.

If A is symmetric, $Dy = P^T\mathbf{b}$ & P^T does not magnify roundoff error.

Unfortunately, A is not always symmetric, so we decompose A into other simple forms.

Schur's thm A : $n\times n$ matrix with real entries and real eigenvalues

$\Rightarrow$ $\exists$ orthogonal P s.t. P^TAP is upper triangular.

Hessenberg's thm A : $n\times n$ matrix

$\Rightarrow$ $\exists$ orthogonal P s.t. $P^TAP = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 0 \; * & * \end{bmatrix}$

§8.4. Quadratic forms.

quadratic form on $\mathbb{R}^n$ $\longleftrightarrow$ $n\times n$ symmetric matrix

$$\left(\because \; a_1x_1^2 + \cdots + a_nx_n^2 + \sum_{1\le i<j\le n} a_{ij}x_ix_j = [x_1 \cdots x_n] \begin{bmatrix} a_1 & a_{12}/2 & \cdots & a_{1n}/2 \\ a_{12}/2 & a_2 & \cdots & a_{2n}/2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}/2 & a_{2n}/2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right)$$

For a symmetric matrix A, we denote $Q_A(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$.

Recall that every symmetric matrix is orthogonally diagonalizable.

$\therefore Q_A(\mathbf{x}) = \mathbf{x}^TP^TDP\mathbf{x} = (P\mathbf{x})^TD(P\mathbf{x})$.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ P\downarrow & \circlearrowleft & \downarrow P \\ \mathbb{R}^n & \xrightarrow{D} & \mathbb{R}^n \end{array}$$

Example Consider the quadratic form $Q = 5x^2 - 4xy + 8y^2$.

Then $Q = [x \ \ y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $\qquad A := \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$

$$\Rightarrow \quad \begin{vmatrix} \lambda-5 & 2 \\ 2 & \lambda-8 \end{vmatrix} = \lambda^2 - 13\lambda + 40 - 4 = (\lambda-4)(\lambda-9)$$

$$(4I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} \frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} \end{bmatrix} \qquad s \in \mathbb{R}$$

$$(9I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -\frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \end{bmatrix} \qquad t \in \mathbb{R}.$$

$$\therefore \quad \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\underbrace{\begin{bmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix}}_{P} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

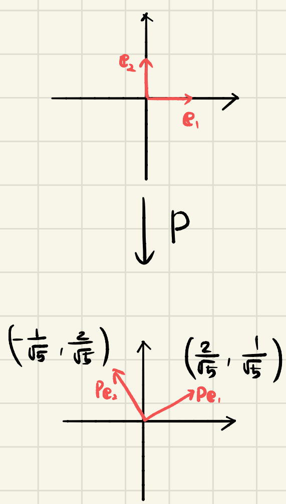
$$Q_D(x', y') = 4x'^2 + 9y'^2.$$

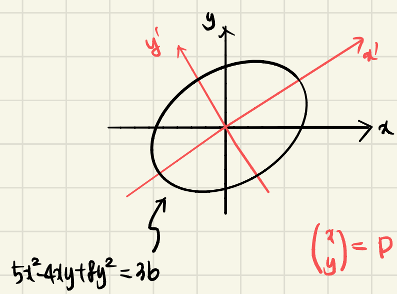
I'll explain this example in the next class.

Draw the curve satisfying $5x^2 - 4xy + 8y^2 = 36$.

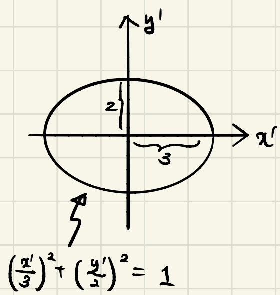
Since $\det(P) = 1$, P represents a rotation and

$4x'^2 + 9y'^2 = 36$ represents the ellipse $\left(\frac{x'}{3}\right)^2 + \left(\frac{y'}{2}\right)^2 = 1$





$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x' \\ y' \end{pmatrix}$$



If A is a symmetric $n \times n$ matrix, then A is orthogonally diagonalizable. i.e., $\exists$ orthogonal P s.t. $P^TAP = D$

$$\left(P = [\mathbb{p}_1 \ \mathbb{p}_2 \cdots \mathbb{p}_n] \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad A\mathbb{p}_i = \lambda_i \mathbb{p}_i \right)$$

Letting $\mathbb{x} = P\mathbb{y}$, $\mathbb{x}^TA\mathbb{x}$ yields

$$\mathbb{y}^TD\mathbb{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Thm 8.4.1 (The principal axes theorem)

If A is symmetric, then there is an orthogonal change of variables that transform $\mathbb{x}^TA\mathbb{x}$ into $\mathbb{y}^TD\mathbb{y}$ with no cross product terms.

Question

If $n=2$ or 3, what kind of a curve or a surface is represented by $\mathbb{x}^TA\mathbb{x} = k$?

Answer $(n=2)$ circle, ellipse, or hyperbola

$(n=3)$ sphere, ellipsoid, or hyperboloid.