Chapter 6

Linear Transformations

6.1 Matrices as Transformations

Exercise 6.1. (*Linear Transformation: Rotation*)
Make a function file with a function name *reflect_pt* to find the reflection of the point (a, b) about the line through the origin of the xy-plane that makes an angle of $\theta\,^\circ$ with the positive x-axis. Make a, b, and θ the inputs to the function and the reflection point (x, y) the output. Using this function, find the result of this function for $(a, b) = (1, 3)$ and $\theta = 12$ by the following commands:

```
>> [x, y]=reflect_pt(1, 3, 12)
```

Solution.

```
%--- The following is the function file 'reflect_pt.m'. ---%
function [x, y]=reflect_pt(a, b, ang)
  theta=ang*(pi/180);
  T = [cos(2*theta) sin(2*theta); sin(2*theta) -cos(2*theta)];

  result=T*[a b]';
  x=result(1); y=result(2);
end
```

MATLAB results.

```
x =
    2.1338

y =
   -2.3339
```

Exercise 6.2. (*Linear Transformation: Projection*)
Consider successive rotations of $\mathbb{R}^3$ by an angle θ_1 degree about the x-axis, then by an angle θ_2 degree about the y-axis, and then by an angle θ_3 degree about

the z-axis. Make a function file named *comp_Rot* to find an appropriate axis and angle of rotation that achieves the same result in one rotation. Make θ_1, θ_2, and θ_3 degrees the inputs to the function and the axis and angle of rotation the outputs. Give the output axis as a unit vector. You may make the standard matrix for the composition of the rotations by multiplication of the standard matrices for the rotations about the position x-, y-, and z-axes, respectively. Using the function *comp_Rot*, check the result of the following commands:

```
>> [L ang]=comp_Rot(45, 45, 45)
```

Solution.

```
%--- The following is the function file 'comp_Rot.m'. ---%
function [L, ang] = comp_Rot(ang1, ang2, ang3)

  % Angle as a degree.
  angle=[ang1 ang2 ang3];

  % Convert angles from degrees to radians.
  theta=angle*(pi/180);

  % Rotation about x-axis, y-axis, and z-axis.
  Rx = [1 0 0;
        0 cos(theta(1)) -sin(theta(1));
        0 sin(theta(1)) cos(theta(1))];
  Ry = [cos(theta(2)) 0 sin(theta(2));
        0 1 0;
        -sin(theta(2)) 0 cos(theta(2))];
  Rz = [cos(theta(3)) -sin(theta(3)) 0;
        sin(theta(3)) cos(theta(3)) 0;
        0 0 1];

  % Composition of the three rotation matrices R = Ry*Rx*Rz.
  R = Rz*Ry*Rx;

  % Find the axis of rotation of R,
  % by finding the eigenvector of R
  % corresponding to the eigenvalue lambda=1.

  % Find a nonzero vector satisfying Rx = x.
  L = null(eye(3) - R);

  % Make the axis of rotation unit vector.
  L = L/norm(L);

  % Find the angle of rotation of R.
  % Note that w=(-x2,x1,0) is
```

```
  % orthogonal to the axis of rotation x=(x1,x2,x3).
  w = [-L(2) L(1) 0]';
  rot_theta = acos((dot(w, R*w))/((norm(w)*norm(R*w))));
  ang = ((rot_theta)*(180/pi));
end
```

MATLAB results.

```
L =
   -0.3574
   -0.8629
   -0.3574

ang =
   64.7368
```

6.2 Geometry of Linear Operators

Exercise 6.3. (*Rotation as an Orthogonal Transformation*)
Let

$$A = \begin{bmatrix} -\frac{3}{7} & -\frac{2}{7} & -\frac{6}{7} \\ \frac{6}{7} & -\frac{3}{7} & -\frac{2}{7} \\ -\frac{2}{7} & -\frac{6}{7} & \frac{3}{7} \end{bmatrix}.$$

Show that A represents a rotation, and use Formulas (16) and (17) in Section 6.2 to find the axis and angle of rotation.

Solution.

```
A = [-3/7 -2/7 -6/7; 6/7 -3/7 -2/7; -2/7 -6/7 3/7]; format short;

% Check that A*A' is the identity matrix.
A*A'

% Although the off diagonal entries are not exactly all zeros,
% the scaling factor suggests that roundoff error prevents
% the computed matrix from being the identity matrix.
% You can see that the product is exactly the identity matrix,
% when the symbolic computation is used.

% Check that det(A)=1 to conclude that A is orthogonal.
det(A)

% Since A*A'=I and det(A) = 1, A represents a rotation.

% By (16) of Section 6.2,
% find the angle of rotation
% and convert the angle from radians to degrees.
theta = acos((trace(A)-1)/2); ang = ((theta)*(180/pi));
disp('The angle of rotation in degrees is'); disp(ang);

% By (17) of Section 6.2, find the axis of rotation.
% The initial point at the origin.
e1 = [1 0 0]';

% v is along the axis of rotation.
v = (A+A'+((1-trace(A))*eye(3))) * e1;
disp('The axis of rotation passes through the point'); disp(v');
```

MATLAB results.

```
The angle of rotation in degrees is
```

```
  135.5847
The axis of rotation passes through the point
    0.5714    0.5714   -1.1429
```

6.3 Kernel and Range

Exercise 6.4. (*Invertible Matrix as a Bijective Linear Transformation*)
Consider the matrix

$$A = \begin{bmatrix} 3 & -5 & -2 & 2 \\ -4 & 7 & 4 & 4 \\ 4 & -9 & -3 & 7 \\ 2 & -6 & -3 & 2 \end{bmatrix}.$$

Referring to the Theorem 6.3.15 in Section 6.3, show that $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ is onto in at least four different ways. You may use several related MATLAB commands.

Solution.

```
A = [3 -5 -2 2; -4 7 4 4; 4 -9 -3 7; 2 -6 -3 2];
format short;

% By (a) in Theorem 6.3.15, use the command rref of A.
rref(A)

% Since the reduced row echelon form of A is the identity matrix,
% the linear transformation T is onto.

% By (d) in Theorem 6.3.15, use the command null of A.

% Find a basis for the null space of A.
null(A, 'r')

% Since the null space contains only the zero vector,
% the linear transformation T is onto.

% By (g) in Theorem 6.3.15, use the command rank of A.

% Find the number of linearly independent columns of A.
rank(A)

% Since the column vectors of A are linearly independent,
% the linear transformation T is onto.

% By (i) in Theorem 6.3.15, use the command det of A.
```

```
% Find the determinant of A.
det(A)

% Since det(A) is nonzero,
% the linear transformation T is onto.

% By (j) in Theorem 6.3.15, use the command eig of A.

% Find the eigenvalues A.
eig(A)

% Since 0 is not an eigenvalue of A,
% the linear transformation T is onto.
```

6.4 Composition and Invertibility of Linear Transformations

Exercise 6.5. (*Compositions of Linear Transformations*)
Consider successive rotations of $\mathbb{R}^3$ by 30° about the z-axis, then by 60° about the x-axis, and then by 45° about the y-axis. If it is desired to execute the three rotations by a single rotation about an appropriate axis, what axis and angle should be used?

Solution.

```
% Angle as a degree.
ang1=30; ang2=60; ang3=45;

% Convert angles from degrees to radians.
theta1=((ang1)*(pi/180));
theta2=((ang2)*(pi/180));
theta3=((ang3)*(pi/180));

format short;

% Rotation about z-axis with the angle 30.
Rz = [cos(theta1) -sin(theta1) 0; sin(theta1) cos(theta1) 0; 0 0 1];
% Rotation about x-axis with the angle 60.
Rx = [1 0 0; 0 cos(theta2) -sin(theta2); 0 sin(theta2) cos(theta2)];
% Rotation about y-axis with the angle 45.
Ry = [cos(theta3) 0 sin(theta3); 0 1 0; -sin(theta3) 0 cos(theta3)];

% Composition of the three rotation matrices R = Ry*Rx*Rz.
R = Ry*Rx*Rz;

% Find the axis of rotation of R,
```

```
% by finding the eigenvector of R corresponding to the eigenvalue lambda=1.

% Find a nonzero vector satisfying Rx = x.
x = null(eye(3) - R);

% Find the angle of rotation of R.
% Note that w=(-x2,x1,0) is orthogonal to the axis of rotation x=(x1,x2,x3).
w = [-x(2) x(1) 0]';
theta = acos((dot(w, R*w))/((norm(w)*norm(R*w))));

% Convert the angle from radians to degrees.
ang = ((theta)*(180/pi));

disp('The angle of rotation in degrees is'); disp(ang);
disp('The axis of rotation is'); disp(x');
```

MATLAB results.

```
The angle of rotation in degrees is
   69.3559

The axis of rotation is
   -0.9350   -0.3525   -0.0391
```