7.1.2

(a) Easily, $\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1$.

(b) Easily, $\mathbf{v}_3 = 5\mathbf{v}_1 - 5\mathbf{v}_2$.

(c) Easily, $\mathbf{v}_4 = 5\mathbf{v}_1 + 4\mathbf{v}_2 + 8\mathbf{v}_3$.

7.1.6

(a) Following sets of vectors are bases for the given line

$$\{(1, 3)\}, \{(2, 6)\}, \{(-1, -3)\}$$

Those vectors are found by insert $1, 2, -1$ for t.

(b) Following sets of vectors are bases for the given plane

$$\{(1, 1, 3), (1, -1, 2)\}, \{(1, 1, 3), (2, 0, 5)\}, \{(1, -1, 2), (2, 0, 5)\}$$

Those vectors are found by insert $(1, 0), (0, 1), (1, 1)$ for (t_1, t_2).

7.1.10

The reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 2 & -2 & 1 & 3 \\ 0 & 1 & 3 & -1 & -1 \\ 2 & 3 & -7 & 3 & 7 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

is given by

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the canonical basis for the solution space is $\{(-3, 1, 0, 1, 0), (\frac{1}{3}, -1, \frac{2}{3}, 0, 1)\}$ and the dimension of the space is 2.

7.1.D4

The solution space of $A\mathbf{x} = \mathbf{0}$ is nontrivial, in other words, have positive dimension, if and only if $\det(A) = 0$. Now, $\det(A) = 2 \cdot (1-1) - 4 \cdot (-t-3) + t \cdot (t+3) = (t+3)(t+4)$. Hence, the solution space of $A\mathbf{x} = \mathbf{0}$ have positive dimension if and only if $t = -3$ or $t = -4$. For the case $t = -3$, a basis for the solution space is $\{(1, 7, 10)\}$, so it has dimension 1. For the case $t = -4$, a basis for the solution space is $\{(0, 1, 1)\}$, so it has dimension 1.

7.1.P1

(a) If S or S' has the zero vector, then it is trivial. Hence, we may assume S, S' has only nonzero vectors. Thus, $|S| \geq 2$. Then, by Theorem 7.1.2, since S is linearly dependent, there exists $\mathbf{v}_i \in S$ such that $\mathbf{v}_i$ is a linear combination of its predecessors, $\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}$. Now, the predecessors does not changes for S', so it is still true for S', which proves S' is again linearly dependent. Actually, $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + 0\mathbf{w}_1 + \cdots + 0\mathbf{w}_r$ also proves this statement.

(b) Suppose a set of vectors S has a nonempty subset S^* such that linealy dependent. If $\mathbf{0} \in S^*$, then $\mathbf{0} \in S$ which implies S is linearly dependent. Now, if not, then since S^* is linearly dependent, $|S^*| \geq 2$. Then, by Theorem 7.1.2, there is a vector in S^* such that it is a linear combination of its predecessors in S^*. But the set of predecessor vectors in S^* is a subset of predecessor vectors in S, so such vector is also a linear combination of its predecessors in S. Thus, S is linearly dependent.

7.2.6

(a) Consider the matrix $A = \begin{bmatrix} 5 & 3 & 8 \\ 1 & 0 & 1 \\ -8 & 5 & -3 \end{bmatrix}$ such that columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. This matrix has the determinant $5(0-5) - 3(-3+8) + 8(5-0) = 0$, so given vectors do not form a basis.

(b) Consider the matrix $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ such that columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. This matrix has the determinant $(-1)(1-1) - 1(-1-1) + 1(1+1) = 4$, so given vectors form a basis.

7.2.8 (a) For any vector $(x, y, z) \in \mathbb{R}^3$, we have a linear combination

$$(x, y, z) = z\cdot(1,1,1)+(y-z)\cdot(1,1,0)+(x-y)\cdot(1,0,0) = z\mathbf{v}_1+(y-z)\mathbf{v}_2+(x-y)\mathbf{v}_3$$

Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$, so is $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. But from above observation, we have $\mathbf{v}_4 = 2\mathbf{v}_2 + \mathbf{v}_3$ which gives nontrivial linear combination $2\mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = 0$, so $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is linearly dependent, and so is $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

(b) Following expression are the answers.

$$(1,2,3) = 3(1,1,1) + (-1)(1,1,0) + (-1)(1,0,0) = 3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$$
$$(1,2,3) = 3(1,1,1) + 1(1,1,0) + (-1)(3,2,0) = 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_4$$
$$(1,2,3) = 3(1,1,1) + \left(-\frac{1}{2}\right)(1,0,0) + \left(-\frac{1}{2}\right)(3,2,0) = 3\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_3 - \frac{1}{2}\mathbf{v}_4$$

Generally, if we solve by reduced row echelon form, we have

$$(1,2,3) = 3\mathbf{v}_1 - (1+2t)\mathbf{v}_2 - (1+t)\mathbf{v}_3 + t\mathbf{v}_4$$

is answer for any t.

7.2.16

(a) It is enough to check $\mathbf{u}$ and $\mathbf{n}$ are orthogonal because if $\mathbf{u}$ is in W if and only if V is contained in W. Since $(1, 1, 3) \bullet (2, 1, -1) = 2 + 1 - 3 = 0$, V is a subspace of W.

(b) For any $t \neq 0$, we have $3x + 2y + z = 3t + 4t - 5t = 2t \neq 0$, so V is not a subspace of W.

7.2.18

(a) Note that $T(4, 3, 0) = T((1, 0, 0) + 3(1, 1, 0)) = (5, -2, 1, 0) + 3(2, 1, 3, -1) = (11, 1, 10, -3)$

(b) Note that

$$\begin{aligned} T(a, b, c) &= T(c(1, 1, 1) + (b - c)(1, 1, 0) + (a - b)(1, 0, 0)) \\ &= c(3, 2, 0, 1) + (b - c)(2, 1, 3, -1) + (a - b)(5, -2, 1, 0) \\ &= (5a - 3b + c, -2a + 3b + c, a + 2b - 3c, -b + 2c). \end{aligned}$$

(c) From above observation, we get

$$[T] = \begin{bmatrix} 5 & -3 & 1 \\ -2 & 3 & 1 \\ 1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$$

7.2.D1

(a) (T). By Theorem 7.2.6.(d)

(b) (T). By Theorem 7.2.6.(c)

(c) (T). Since every vector in $\mathbb{R}^{\mathrm{n}}$ can be expressed as a linear combination of the vectors in S, so S spans $\mathbb{R}^n$. Also, the way is unique, so S is linearly independent. Hence, $k = n$.

(d) (T). By Theorem 7.2.7 (d) and (p).

(e) (T). By Theorem 7.2.4 (b), if V is a subspace of W, then $V = W$ which contradicts to that V and W are distinct. By same reason, W is not a subspace of V.

7.2.P5

Since B is a basis, for any $\mathbf{v} \in \mathbb{R}^n$ there is unique $c_1, \cdots, c_n$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Then, define

$$T(\mathbf{v}) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$$

This map is linear since

$$\begin{aligned}
T\left(\sum_{i=1}^n c_i\mathbf{v}_i + \sum_{i=1}^n d_i\mathbf{v}_i\right) &= T\left(\sum_{i=1}^n (c_i + d_i)\mathbf{v}_i\right) \\
&= \sum_{i=1}^n (c_i + d_i)\mathbf{w}_i \\
&= \sum_{i=1}^n c_i\mathbf{w}_i + \sum_{i=1}^n d_i\mathbf{w}_i \\
&= T\left(\sum_{i=1}^n c_i\mathbf{v}_i\right) + T\left(\sum_{i=1}^n d_i\mathbf{v}_i\right) \\
T\left(c\sum_{i=1}^n c_i\mathbf{v}_i\right) &= T\left(\sum_{i=1}^n cc_i\mathbf{v}_i\right) \\
&= \sum_{i=1}^n cc_i\mathbf{w}_i \\
&= c\sum_{i=1}^n c_i\mathbf{w}_i \\
&= cT\left(\sum_{i=1}^n c_i\mathbf{v}_i\right)
\end{aligned}$$

Hence T is a linear operator on $\mathbb{R}^n$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for any $1 \le i \le n$. Now, suppose T' is a linear operator such that $T'(\mathbf{v}_i) = \mathbf{w}_i$. Then, for any $\mathbf{v} \in \mathbb{R}^n$, we can write $\mathbf{v} = \sum_{i=1}^n c_i\mathbf{v}_i$, and then, $T(\mathbf{v}) = T(\sum_{i=1}^n c_i\mathbf{v}_i) = \sum_{i=1}^n c_i\mathbf{w}_i = \sum_{i=1}^n c_iT'(\mathbf{v}_i) = T'(\sum_{i=1}^n c_i\mathbf{v}_i) = T'(\mathbf{v})$. Thus, $T = T'$ and so T is the unique linear operator with given property.

Also, let $\mathbf{e}_i$ be a vector in $\mathbb{R}^n$ such that ith component is 1 and others are all 0. Then, define $n \times n$ matrix $W = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix}$ and $V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$. Then, $W\mathbf{e}_i = \mathbf{w}_i$ and $V\mathbf{e}_i = \mathbf{v}_i$. Since $\mathbf{B}$ is a basis, by Theorem 7.2.7, V is an invertible matrix. Moreover, $\mathbf{e}_i = V^{-1}\mathbf{v}_i$. Hence, if we define a linear operator T such that $[T] = WV^{-1}$, then we have $T(\mathbf{v}_i) = WV^{-1}\mathbf{v}_i = W\mathbf{e}_i = \mathbf{w}_i$. Uniqueness is proved as above.

7.2.P8

Let $\dim(V) = k = \dim(W)$ and $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be vectors which form a basis of V. Since it is a basis, given vectors are linearly independent. Also, $V \subseteq W$, so these vectors are linearly independent k vectors in W which is a k-dimensional subspace. Thus, by Theorem 7.2.6.(a), the set $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a basis for W, which proves $W = \text{span}\,(S) = V$.

7.3.8

Note that W is a solution space of $x + y + z = 0$, $x - y + z = 0$ which is a line satisfying $x = -t$, $y = 0$ and $z = t$. Hence, $W^{\perp}$ is a plane orthogonal to this line, which is equivalent to that orthogonal to the direction $(-1, 0, 1)$. Hence $W^{\perp}$ is a set of vectors satisfying $-x + z = 0$, which can be parametrized as (s, r, s) with parameter r, s.

7.3.14

The reduced row echelon form of the matrix

$$\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \\ \mathbf{v}_4^T \\ \mathbf{v}_5^T \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 & 3 & 5 \\ 0 & 1 & 6 & -7 & 1 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 5 & 5 & -1 & 5 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -243 \\ 0 & 1 & 0 & 0 & 57 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which proves W has a basis $\{(1,0,0,0,-243),(0,1,0,0,57),(0,0,1,0,-7),(0,0,0,1,2)\}$ and $W^\perp$ has a basis $(243,-57,7,-2,1)$. This is possible since row operations do not change the row space and the null space.

7.3.24

Consider the augmented matrix

$$\left[\begin{array}{cccc|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{array}\right] = \left[\begin{array}{cccc|c|c|c} 0 & 1 & -1 & 3 & 3 & -2 & 3 \\ 1 & 1 & 0 & -2 & -1 & 0 & 2 \\ 0 & 3 & 2 & 1 & 7 & -1 & 6 \\ 2 & 1 & 1 & 0 & 2 & 2 & 4 \\ 0 & -1 & 1 & 1 & 1 & 2 & 1 \end{array}\right]$$

If we make reduced row echelon form for $\mathbf{v}_1, \cdots, \mathbf{v}_4$ then it becomes

$$\left[\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 0 & -\frac{1}{5} & 1 & 0 \\ 0 & 1 & 0 & 0 & \frac{6}{5} & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{6}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, $\mathbf{b}_1, \mathbf{b}_2$ lie in the space spanned by the $\mathbf{v}$'s but $\mathbf{b}_3$ is not.

7.3.26

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{7}{5} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{6}{5} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives a basis of $\mathrm{row}(A)$ as $\{(20,0,0,5,-28),(0,10,0,5,12),(0,0,20,5,12)\}$ and a basis of $\mathrm{null}(A)$ as $\{(-1,-2,-1,4,0),(7,-6,-3,0,5)\}$. To check two space are orthogonal complement, it is enough to check bases for two spaces are orthogonal. Then

$$\begin{bmatrix} 20 & 0 & 0 & 5 & -28 \\ 0 & 10 & 0 & 5 & 12 \\ 0 & 0 & 20 & 5 & 12 \end{bmatrix} \begin{bmatrix} -1 & 7 \\ -2 & -6 \\ -1 & -3 \\ 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

proves that $\mathrm{null}(A)$ and $\mathrm{row}(A)$ are subset of orthogonal complement to each other respectively. But from the fact that those 5 vectors, from the basis of $\mathrm{row}(A)$ and $\mathrm{null}(A)$, form a basis of $\mathbb{R}^5$, you may prove $\mathrm{row}(A)$ and $\mathrm{null}(A)$ are really orthogonal complement to each other easily.

7.3.32

(a) The column space of A is spanned by $(1, 0, 0)$, and $(0, 1, 0)$ which is xy-plane and the null space of A is spanned by $(0, 0, 1)$ which is z-axis.

(b) Any matrix of type

$$\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right]$$

with invertible B has x-axis as the null space and yz-plane as the column space.

7.3.D2

(a) (T). By Theorem 7.2.7, row vectors and column vectors form bases respectively, so both row space and column space are $\mathbb{R}^n$.

(b) (F). If W is a subspace of V, then for any vector in $V^\perp$ is orthogonal to every vectors in V, so is orthogonal to every vectors in W. Hence, $V^\perp$ is a subspace of $W^\perp$. But $W^\perp$ may not be a subspace of $V^\perp$. For example if W is the x-axis and V is the xy-plane, then $W^\perp$ is the yz-plane and $V^\perp$ is the z-axis.

(c) (F). If $A = \begin{bmatrix} 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \end{bmatrix}$, then each row of a matrix A is a linear combination of the rows of a matrix B, but A has the null space $\mathbb{R}$ which is not same as the null space of B, which is $\{0\}$. Generally, $\mathrm{row}(A) \subseteq \mathrm{row}(B)$ implies $\mathrm{null}(A) = \mathrm{row}(A)^\perp \supseteq \mathrm{row}(B)^\perp = \mathrm{null}(B)$ by Theorem 7.3.5 and (b).

(d) (F). If $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then A and B have the same row space but different column spaces.

(e) (T). For elementary matrix E, EA is nothing but apply an elementary row operation. Hence, by Theorem 7.3.7.(a), A and EA have the same row space. Also, note that for any matrix A, B, $\mathrm{row}(BA) \subseteq \mathrm{row}(A)$ since row vectors of $\mathrm{row}(BA)$ are all linear combination of row vectors of A. Hence, if B is invertible, then $\mathrm{row}(A) = \mathrm{row}(B^{-1}BA) \subseteq \mathrm{row}(BA) \subseteq \mathrm{row}(A)$, so $\mathrm{row}(A) = \mathrm{row}(BA)$.

7.3.P3

Since P is invertible, $PA\mathbf{v} = \mathbf{0}$ if and only if $A\mathbf{v} = P^{-1}\mathbf{0} = \mathbf{0}$. Hence $\text{null}(A) = \text{null}(PA)$. Thus, A and PA have same nullity. Now, by Theorem 7.3.8, A and PA have the same row space, so they have same rank, which is the dimension of the row space.

7.3.P4

By Theorem 7.3.4.(b) and (c), $(S^\perp)^\perp = (\text{span}(S)^\perp)^\perp = \text{span}(S)$. The first equality is from (b) and the second one is from (c).