2021 Spring MAS 365: Homework 4

posted on Apr 1; due by Apr 8

1. [5+5+10 points] Let A be a nonsingular $n \times n$ matrix, $||\cdot||$ be any natural norm, and $K_p(A) = ||A||_p||A^{-1}||_p$. Let λ_1 be the smallest and λ_n be the largest eigenvalues of the matrix A^tA.
   (a) Show that if λ is an eigenvalue of A^tA, then $0 < \lambda \leq ||A^t||\,||A||$.
   (b) Show that $K_2(A) = \sqrt{\frac{\lambda_n}{\lambda_1}}$. (Hint: $||A|| = \sqrt{\rho(A^tA)} = \sqrt{\rho(AA^t)}$.)
   (c) Show that $K_2(A) \leq \sqrt{K_1(A)K_\infty(A)}$. (Hint: $||A||_1 = \max_{1\leq j\leq n}\sum_{i=1}^n |a_{ij}|$ and $||A||_\infty = \max_{1\leq i\leq n}\sum_{j=1}^n |a_{ij}|$.)
2. [10+10+10 points] Prove Theorem 7.33 in the textbook using mathematical induction as follows:
   (a) Show that $\langle \boldsymbol{r}^{(1)}, \boldsymbol{v}^{(1)}\rangle = 0$.
   (b) Assume that $\langle \boldsymbol{r}^{(k)}, \boldsymbol{v}^{(j)}\rangle = 0$, for each $k \leq l$ and $j = 1, 2, \ldots, k$, and show that this implies that $\langle \boldsymbol{r}^{(l+1)}, \boldsymbol{v}^{(j)}\rangle = 0$, for each $j = 1, 2, \ldots, l$.
   (c) Show that $\langle \boldsymbol{r}^{(l+1)}, \boldsymbol{v}^{(l+1)}\rangle = 0$.
3. [10+10 points]
   (a) Implement the SOR method with the optimal choice of w for the positive definite and tridiagonal matrix via MATLAB grader.
   (b) Implement the steepest descent method via MATLAB grader.