2021 Spring MAS 365: Homework 1

posted on Mar. 11; due by Mar. 18

1. [5 points each] The Taylor polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{n} \frac{x^i}{i!}$. Use the Taylor polynomial of degree five and three-digit chopping arithmetic to find an approximation to e^{-2} by each of the following methods.

   (1) $e^{-2} \approx \sum_{i=0}^{5} \frac{(-2)^i}{i!}$
   (2) $e^{-2} = \frac{1}{e^2} \approx \frac{1}{\sum_{i=0}^{5} \frac{2^i}{i!}}$

2. [10 points] Suppose that $fl(y)$ is a k-digit rounding approximation to a positive y. Show that

$$\left| \frac{y - fl(y)}{y} \right| \leq 5 \times 10^{-k},$$

   i.e., $fl(y)$ approximates y to k significant digits.

3. [5 points each]

   (1) Determine the rate of convergence of the sequence $\left\{ \left(\sin \frac{1}{n} \right)^2 \right\}_{n=1}^{\infty}$ as $n \to \infty$, using a form $O\left(\frac{1}{n^p}\right)$.
   (2) Determine the rate of convergence of the function $\frac{1-\cos h}{h}$ as $h \to 0$, using a form $O\left(h^p\right)$.

4. [5 points each] Find an approximation to $\sqrt{17}$ (that is between 2 and 5) accurate to within 10^{-3} using the bisection method.

   (1) Briefly describe how one can use the bisection method to approximate $\sqrt{17}$.
   (2) Determine the number of iterations (n) required for the bisection method in (1) to achieve 10^{-3} accuracy of $|p_n - p|$, starting with the interval $[2, 5]$.

5. [5 points each]

   (1) Show that $g(x) = e^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$.
   (2) Estimate the minimum number of iterations required for the fixed-point iteration to achieve 10^{-4} accuracy, with an initial approximation $p_0 = \frac{2}{3}$, considering both bounds (2.5) and (2.6) in the textbook.

6. [10 points] Show that Theorem 2.3(ii) in the textbook is true if the inequality $|g'(x)| \leq k$ is replaced by $g'(x) \leq k$, for all $x \in (a, b)$.

7. [10 points each]

   (1) Implement Newton's method via MATLAB grader.
   (2) Implement the secant method via MATLAB grader.