2021 Spring MAS 365: Midterm Exam

Write the following Honor Pledge and sign your name under it.

> *"I have neither given nor received aid on this examination, nor have I concealed a violation of the Honor Code."*

1. [35 (5+15+10+5) points] Consider a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}.$$

Suppose that a two-digit rounding was used when storing A and $\boldsymbol{b}$, and let the corresponding matrices to be $\tilde{A}$ and $\tilde{\boldsymbol{b}}$, respectively. Let $\boldsymbol{x} = (x_1, x_2)$ and $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2)$ be the solutions of $A\boldsymbol{x} = \boldsymbol{b}$ and $\tilde{A}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$, respectively. Note that $\boldsymbol{x} = (1.5, -0.5)$. Use l_∞ norm through out this problem, and consider the theorem below.

Theorem 1. *Suppose A is nonsingular and $||\delta A|| < \frac{1}{||A^{-1}||}$. The solution $\tilde{\boldsymbol{x}}$ to $(A + \delta A)\tilde{\boldsymbol{x}} = \boldsymbol{b} + \delta\boldsymbol{b}$ approximates the solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \boldsymbol{b}$ with the error estimate*

$$\frac{||\boldsymbol{x} - \tilde{\boldsymbol{x}}||}{||\boldsymbol{x}||} \le \frac{K(A)||A||}{||A|| - K(A)||\delta A||} \left(\frac{||\delta\boldsymbol{b}||}{||\boldsymbol{b}||} + \frac{||\delta A||}{||A||} \right).$$

(a) Determine $\tilde{A}$ and $\tilde{\boldsymbol{b}}$.

(b) Use Gaussian elimination with a partial pivoting and a two-digit rounding to approximate the solution $\tilde{\boldsymbol{x}}$ of $\tilde{A}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$.

(c) Find an upper bound of $\frac{||\boldsymbol{x} - \tilde{\boldsymbol{x}}||}{||\boldsymbol{x}||}$.

(d) $\tilde{x}_1$ approximates x_1 to t significant digits. Find a tight lower bound of t using (c).

Solution:

(a) Applying a two-digit rounding to A (3 points) and $\boldsymbol{b}$ (2 points) yields

$$\tilde{A} = \begin{bmatrix} 0.33 & 0.67 \\ 0.67 & 0.33 \end{bmatrix} \quad \text{and} \quad \tilde{\boldsymbol{b}} = \begin{bmatrix} 0.17 \\ 0.83 \end{bmatrix}.$$

(b) Since $|\tilde{a}_{21}| = 0.67$ is greater than $|\tilde{a}_{11}| = 0.33$, we interchange rows and the resulting augmented matrix is (5 points)

$$\left[\begin{array}{cc|c} 0.67 & 0.33 & 0.83 \\ 0.33 & 0.67 & 0.17 \end{array}\right]$$

The multiplier of this system is (2 points)

$$m_{21} = \frac{0.33}{0.67} \approx 0.49$$

Performing the operation $(E_2 - m_{21}E_1) \to (E_2)$ gives

$$\left[\begin{array}{cc|c} 0.67 & 0.33 & 0.83 \\ 0.00 & 0.51 & -0.24 \end{array}\right],$$

where $0.67 - 0.49 \times 0.33 \approx 0.67 - 0.16 \approx 0.51$ (2 points) and $0.17 - 0.49 \times 0.83 \approx 0.17 - 0.41 \approx -0.24$ (2 points). Using the backward substitution, we have

$$x_2 \approx \frac{-0.24}{0.51} \approx -0.47, \quad \text{(2 points)}$$
$$x_1 \approx \frac{0.83 - 0.33 \times (-0.47)}{0.67} \approx \frac{0.83 + 0.16}{0.67} \approx 1.5 \quad \text{(2 points)}.$$

(c) Note that (1 point each)

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \quad \delta A = \tilde{A} - A = \begin{bmatrix} -\frac{1}{300} & \frac{1}{300} \\ \frac{1}{300} & -\frac{1}{300} \end{bmatrix}, \quad \text{and} \quad \delta \boldsymbol{b} = \tilde{\boldsymbol{b}} - \boldsymbol{b} = \begin{bmatrix} \frac{1}{300} \\ -\frac{1}{300} \end{bmatrix}$$

We then have (1 point each)

$$||A|| = 1, \quad ||A^{-1}|| = 3, \quad ||\delta A|| = \frac{1}{150}, \quad ||\boldsymbol{b}|| = \frac{5}{6}, \quad ||\delta \boldsymbol{b}|| = \frac{1}{300}, \quad K(A) = ||A|| \cdot ||A^{-1}|| = 3.$$

Therefore, since A is nonsingular (*i.e.*, A has an inverse) and $||\delta A|| = \frac{1}{150} \leq \frac{1}{||A^{-1}||} = \frac{1}{3}$, we have the following upper bound (1 point)

$$\frac{||\boldsymbol{x} - \tilde{\boldsymbol{x}}||}{||\boldsymbol{x}||} \leq \frac{3}{1 - 3 \times \frac{1}{150}} \left(\frac{\frac{1}{300}}{\frac{5}{6}} + \frac{1}{150} \right) = \frac{150}{49} \frac{16}{1500} = \frac{8}{245} \approx 0.0327.$$

(d) Since (3 points)

$$\frac{|x_1 - \tilde{x}_1|}{|x_1|} = \frac{|x_1 - \tilde{x}_1|}{||\boldsymbol{x}||} \leq \frac{||\boldsymbol{x} - \tilde{\boldsymbol{x}}||}{||\boldsymbol{x}||} \approx 0.0327 \leq 5 \times 10^{-2},$$

t is greater than equal to 2. (2 points)

2. [30 (15+15) points] Let $D = \{(x_1, x_2) \ : \ a \le x_1 \le b,\ c \le x_2 \le d\} \subset \mathbb{R}^2$. Suppose that $\boldsymbol{G}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}))^t : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function from D into $\mathbb{R}^2$ with the property that $\boldsymbol{G}(\boldsymbol{x}) \in D$ whenever $\boldsymbol{x} \in D$. Assume that $\boldsymbol{G}(\boldsymbol{x})$ has a unique fixed point $\boldsymbol{p} = (p_1, p_2)^t \in D$.

(a) Show that $||\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{G}(\boldsymbol{y})||_\infty \le K||\boldsymbol{x} - \boldsymbol{y}||_\infty$ for any $\boldsymbol{x}, \boldsymbol{y} \in D$, under the assumption that all the component functions of $\boldsymbol{G}$ have continuous partial derivatives and a constant $K < 1$ exists with, for $i = 1, 2$ and $j = 1, 2$,

$$\left|\frac{\partial g_i(\boldsymbol{x})}{\partial x_j}\right| \le \frac{K}{2}, \quad \text{whenever } \boldsymbol{x} \in D.$$

Consider the following Taylor's Theorem in two variables.

Theorem 2. *Supposed that $g(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ and all its partial derivatives are continuous on $D = \{(x_1, x_2) : a \le x_1 \le b,\ c \le x_2 \le d\}$ and $(\bar{x}_1, \bar{x}_2) \in D$. For every (x_1, x_2), there exists ξ_1 between x_1 and $\bar{x}_1$, and ξ_2 between x_2 and $\bar{x}_2$ with*

$$g(x_1, x_2) = g(\bar{x}_1, \bar{x}_2) + (x_1 - \bar{x}_1)\frac{\partial g(\xi_1, \xi_2)}{\partial x_1} + (x_2 - \bar{x}_2)\frac{\partial g(\xi_1, \xi_2)}{\partial x_2}.$$

(b) Using (a), show that the sequence defined by $\boldsymbol{x}^{(k)} = \boldsymbol{G}(\boldsymbol{x}^{(k-1)})$ for $k \ge 1$ satisfies $||\boldsymbol{x}^{(k)} - \boldsymbol{p}||_\infty \le \frac{K^k}{1-K}||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty$, for any $\boldsymbol{x}^{(0)}$ in D, where $\{\boldsymbol{x}^{(k)}\}$ is assumed to converge to $\boldsymbol{p}$.

Solution:

(a) Applying the theorem to g_i (3 points) for $i = 1, 2$ and for any $\boldsymbol{x}, \boldsymbol{y} \in D$, we have

$$\begin{aligned} |g_i(\boldsymbol{x}) - g_i(\boldsymbol{y})| &= \left|\sum_{j=1}^{2} \frac{\partial g_i(\xi_1, \xi_2)}{\partial x_j}(x_j - y_j)\right| \text{ (2 points)} \\ &\le \sum_{j=1}^{2} \left|\frac{\partial g_i(\xi_1, \xi_2)}{\partial x_j}\right| \ |x_j - y_j| \le ||\boldsymbol{x} - \boldsymbol{y}||_\infty \sum_{j=1}^{2} \left|\frac{\partial g_i(\xi_1, \xi_2)}{\partial x_j}\right| \le K||\boldsymbol{x} - \boldsymbol{y}||_\infty \text{(5 points)}, \end{aligned}$$

where ξ_1 is a number between x_1 and y_1, and ξ_2 is a number between x_2 and y_2. Then, (5 points)

$$||\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{G}(\boldsymbol{y})||_\infty = \max_{i=1,2} |g_i(\boldsymbol{x}) - g_i(\boldsymbol{y})| \le K||\boldsymbol{x} - \boldsymbol{y}||_\infty$$

for any $\boldsymbol{x}, \boldsymbol{y} \in D$.

(b) For $k \ge 1$, we have (5 points)

$$||\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}||_\infty = ||G(\boldsymbol{x}^{(k)}) - G(\boldsymbol{x}^{(k-1)})||_\infty \le K||\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}||_\infty \le \cdots \le K^k||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty$$

Thus for $m > k \ge 1$, (5 points)

$$\begin{aligned} ||\boldsymbol{x}^{(m)} - \boldsymbol{x}^{(k)}||_\infty &= ||\boldsymbol{x}^{(m)} - \boldsymbol{x}^{(m-1)} + \boldsymbol{x}^{(m-1)} - \cdots + \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}||_\infty \\ &\le ||\boldsymbol{x}^{(m)} - \boldsymbol{x}^{(m-1)}||_\infty + \cdots + ||\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}||_\infty \\ &\le K^{m-1}||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty + \cdots + K^k||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty \\ &= K^k||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty(1 + K + K^2 + \cdots + K^{m-k-1}). \end{aligned}$$

Then, (5 points)

$$\begin{aligned} ||\boldsymbol{p} - \boldsymbol{x}^{(k)}|| &= \lim_{m\to\infty} ||\boldsymbol{x}^{(m)} - \boldsymbol{x}^{(k)}||_\infty = \lim_{m\to\infty} K^k||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty \sum_{i=0}^{m-k-1} K^i \\ &= K^k||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty \sum_{i=0}^{\infty} K^i = \frac{K^k}{1-K}||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty. \end{aligned}$$

3. [25 (5+10+10) points] Consider solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A is a $n \times n$ matrix, by a method $\mathbf{R}$:

$$\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + w(\boldsymbol{b} - A\boldsymbol{x}^{(k-1)})$$

for $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ and for some positive w.

(a) State a condition on w that makes the sequence $\{\boldsymbol{x}^{(k)}\}$ of the method $\mathbf{R}$ converge to the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$.

(b) Assume that A is a diagonal matrix with diagonal elements $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Simplify the condition on w in (a) into a form $c < w < d$. (Specify c and d.)

(c) Considering the relationship between the (simultaneous) Jacobi method and the (sequential) Gauss-Seidel Method, derive a (sequential) Gauss-Seidel-like method of the (simultaneous) method $\mathbf{R}$ in a form $\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$.

Solution:

(a) The iteration can be rewritten as

$$\boldsymbol{x}^{(k)} = (I - wA)\boldsymbol{x}^{(k-1)} + w\boldsymbol{b}.$$

By Theorem 7.19, if $\rho(I - wA) < 1$, the sequence converges to the unique solution of $\boldsymbol{x} = (I - wA)\boldsymbol{x} + w\boldsymbol{b}$, which is the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$. Therefore, the condition we are looking for is $\rho(I - wA) < 1$.

(b) The condition in (a) reduces to $\rho(I - wA) = \max_{i=1,\ldots,n} |1 - w\lambda_i| < 1$, (4 points) which is equivalent to $0 < w < \frac{2}{\lambda_i}$ for all i. (3 points) This is exactly the condition $0 < w < \frac{2}{\lambda_1}$. (3 points)

(c) The method R can be rewritten as (2 points)

$$x_i^{(k)} = x_i^{(k-1)} + w\left[b_i - \sum_{j=1}^{n} a_{ij}x_j^{(k-1)}\right]$$

for each $i = 1, \ldots, n$. Its Gauss-Seidel-like modification yields (4 points)

$$x_i^{(k)} = x_i^{(k-1)} + w\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i}^{n} a_{ij}x_j^{(k-1)}\right]$$

for each $i = 1, \ldots, n$. This is equivalent to

$$x_i^{(k)} + w\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} = x_i^{(k-1)} + w\left[b_i - \sum_{j=i}^{n} a_{ij}x_j^{(k-1)}\right]$$

for each $i = 1, \ldots, n$. Let D be the diagonal matrix whose diagonal entries are those of A, $-L$ be the strictly lower-triangular part of A, and $-U$ be the strictly upper-triangular part of A. Then, we can rewrite the method into a matrix-vector form as (4 points)

$$\begin{aligned}(I - wL)\boldsymbol{x}^{(k)} &= (I - w(D - U))\boldsymbol{x}^{(k-1)} + w\boldsymbol{b},\\ \boldsymbol{x}^{(k)} &= (I - wL)^{-1}(I - w(D - U))\boldsymbol{x}^{(k-1)} + w(I - wL)^{-1}\boldsymbol{b}.\end{aligned}$$

4. [30 (10+10+10) points] Consider a system of nonlinear equations

$$\begin{aligned} f_1(x_1, x_2) &= x_1^2 + x_2^2 - 2 = 0, \\ f_2(x_1, x_2) &= x_1 - x_2 = 0. \end{aligned}$$

(a) Show that one iteration of Newton's method gives $\boldsymbol{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})^t$ with

$$x_1^{(1)} = x_2^{(1)} = \frac{(x_1^{(0)})^2 + (x_2^{(0)})^2 + 2}{2(x_1^{(0)} + x_2^{(0)})},$$

starting from $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)})^t$.

(b) Show that the iteration converges to a fixed point $(1, 1)^t$, if $1 \le x_1^{(0)} + x_2^{(0)} \le M$, possibly using the theorem below.

Theorem 3. *Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with*

$$|g'(x)| \le k, \quad \text{for all } x \in (a, b).$$

Then, for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

converges to the unique fixed point p in $[a, b]$.

(c) Verify that the convergence of $\{x_1^{(n)}\}$ is quadratic, if $1 \le x_1^{(0)} + x_2^{(0)} \le M$. (Do not directly use any theorem in the textbook.)

Solution:

(a) Let $\boldsymbol{F}(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))^t$. The Jacobian matrix $J(\boldsymbol{x})$ for this system is (3 points)

$$J(x_1, x_2) = \begin{bmatrix} 2x_1 & 2x_2 \\ 1 & -1 \end{bmatrix},$$

and its inverse is (3 points)

$$[J(x_1, x_2)]^{-1} = \frac{1}{-2x_1 - 2x_2} \begin{bmatrix} -1 & -2x_2 \\ -1 & 2x_1 \end{bmatrix} = \frac{1}{2(x_1 + x_2)} \begin{bmatrix} 1 & 2x_2 \\ 1 & -2x_1 \end{bmatrix}.$$

So one iteration of Newton's method is (4 points)

$$\begin{aligned} \boldsymbol{x}^{(1)} &= \boldsymbol{x}^{(0)} - [J(\boldsymbol{x}^{(0)})]^{-1} \boldsymbol{F}(\boldsymbol{x}^{(0)}) \\ &= \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} - \frac{1}{2(x_1^{(0)} + x_2^{(0)})} \begin{bmatrix} (x_1^{(0)})^2 + (x_2^{(0)})^2 - 2 + 2x_1^{(0)} x_2^{(0)} - 2(x_2^{(0)})^2 \\ (x_1^{(0)})^2 + (x_2^{(0)})^2 - 2 - 2(x_1^{(0)})^2 + 2x_1^{(0)} x_2^{(0)} \end{bmatrix} \\ &= \frac{1}{2(x_1^{(0)} + x_2^{(0)})} \begin{bmatrix} (x_1^{(0)})^2 + (x_2^{(0)})^2 + 2 \\ (x_1^{(0)})^2 + (x_2^{(0)})^2 + 2 \end{bmatrix} \end{aligned}$$

(b) Let $p_n = x_1^{(n+1)} = x_2^{(n+1)}$ for $n \ge 0$. Then, for $n \ge 1$, (2 points)

$$p_n = x_1^{(n+1)} = x_2^{(n+1)} = g(p_{n-1}) = g(x_1^{(n)}) = g(x_2^{(n)}), \quad \text{where } g(x) = \frac{x^2 + x^2 + 2}{4x} = \frac{1}{2}\left(x + \frac{1}{x}\right).$$

p_0 is lower bounded by 1, so let $a = 1$ in the theorem (1 point), since $p_0 = \frac{t^2+(m-t)^2+2}{2m}$ is minimized when $t = \frac{m}{2}$, and $p_0 = \frac{m^2+4}{4m}$ is minimized with a value 1 when $m = 2$. Let $b = p_0$ in the theorem (1 point). g is continuous on $[1, p_0]$ (1 point), since it is differentiable on $[1, p_0]$ with its derivative $g'(x) = \frac{1}{2}\left(1 - \frac{1}{x^2}\right)$. Since $g'(x) \geq 0$ for $x \geq 1$, g is a nondecreasing function on an interval $[1, p_0]$ (1 point), where $g(1) = 1$ and $g(p_0) = \frac{1}{2}\left(p_0 + \frac{1}{p_0}\right) \leq p_0$. Therefore, $g(x) \in [1, p_0]$, for all x in $[1, p_0]$ (1 point). In addition, we have that $|g'(x)| \leq \frac{1}{2}$ for all $x \in (1, p_0)$, (2 points) so, by the theorem (1 points), the sequence defined by $p_n = g(p_{n-1})$ for any number $p_0 \geq 1$ converges to the unique fixed point $p = 1$ in $[1, p_0]$.

(c) Since, for $n \geq 0$, (5 points)

$$\frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{\left|\frac{p_n^2+1}{2p_n} - 1\right|}{|p_n - 1|^2} = \frac{\frac{|p_n-1|^2}{2p_n}}{|p_n - 1|^2} = \frac{1}{2|p_n|},$$

we have that (5 points)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{1}{2|p_n|} = \frac{1}{2}.$$

Therefore, the sequence $\{x_1^{(n)}\}$ converges quadratically to 1.

5. [30 (15+5+10) points] The directional derivative of g at $\boldsymbol{x}$ in the direction of $\boldsymbol{v}$ is defined by

$$D_{\boldsymbol{v}} g(\boldsymbol{x}) = \lim_{h \to 0} \frac{1}{h}[g(\boldsymbol{x} + h\boldsymbol{v}) - g(\boldsymbol{x})] = \boldsymbol{v}^t \nabla g(\boldsymbol{x}).$$

The steepest direction with respect to ℓ_2 norm is found by solving

$$\min_{\boldsymbol{v} \,:\, ||\boldsymbol{v}||_2 = 1} D_{\boldsymbol{v}} g(\boldsymbol{x}).$$

(a) Determine the steepest direction with respect to $||D^{1/2} \cdot ||_2$ by solving

$$\min_{\boldsymbol{v} \,:\, ||D^{1/2}\boldsymbol{v}||_2 = 1} D_{\boldsymbol{v}} g(\boldsymbol{x}),$$

where D is a diagonal matrix.

(b) State a condition for a nonzero vector $\boldsymbol{v}$ to be called a descent direction of g at $\boldsymbol{x}$.

(c) Show that a direction $\boldsymbol{v}^{(k+1)}$ of a conjugate gradient method

$$\begin{aligned}
&\boldsymbol{r}^{(0)} = \boldsymbol{b} - A\boldsymbol{x}^{(0)}, \quad \boldsymbol{v}^{(1)} = \boldsymbol{r}^{(0)}, \\
&\text{For } k = 1, 2, \ldots, n \\
&t_k = \frac{\langle \boldsymbol{v}^{(k)}, \, \boldsymbol{r}^{(k-1)} \rangle}{\langle \boldsymbol{v}^{(k)}, \, A\boldsymbol{v}^{(k)} \rangle}, \quad \boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + t_k \boldsymbol{v}^{(k)}, \quad \boldsymbol{r}^{(k)} = \boldsymbol{r}^{(k-1)} - t_k A \boldsymbol{v}^{(k)} \\
&s_k = \frac{\langle \boldsymbol{r}^{(k)}, \, \boldsymbol{r}^{(k)} \rangle}{\langle \boldsymbol{r}^{(k-1)}, \, \boldsymbol{r}^{(k-1)} \rangle} \quad \boldsymbol{v}^{(k+1)} = \boldsymbol{r}^{(k)} + s_k \boldsymbol{v}^{(k)},
\end{aligned}$$

for a positive definite matrix A, is a descent direction of some function at some point under some condition. Specify the corresponding function, point and condition. (Do not directly use any theorem in the textbook.)

Solution:

(a) Using the Cauchy-Schwarz inequality (5 points), we have

$$\boldsymbol{v}^t \nabla g(\boldsymbol{x}) = (D^{1/2}\boldsymbol{v})^t (D^{-1/2} \nabla g(\boldsymbol{x})) \geq -||D^{1/2}\boldsymbol{v}||_2 ||D^{-1/2} \nabla g(\boldsymbol{x})||_2 = -||D^{-1/2} \nabla g(\boldsymbol{x})||_2$$

for all $\boldsymbol{v}$ such that $||D^{1/2}\boldsymbol{v}||_2 = 1$. The lower bound is attained for (10 points)

$$\boldsymbol{v} = -\frac{D^{-1} \nabla g(\boldsymbol{x})}{||D^{-1/2} \nabla g(\boldsymbol{x})||_2}.$$

If you discuss about the equality condition of the Cauchy-Schwarz inequality, you will receive 5 points out of 10 points assigned for $\boldsymbol{v}$. The answer $\boldsymbol{v} = \frac{D^{-1} \nabla g(\boldsymbol{x})}{||D^{-1/2} \nabla g(\boldsymbol{x})||_2}$ will receive 7 points.

(b) A nonzero vector $\boldsymbol{v}$ is called a descent direction of g at $\boldsymbol{x}$ if $D_{\boldsymbol{v}} g(\boldsymbol{x}) < 0$.

(c) Let $g(\boldsymbol{x}) = \langle \boldsymbol{x}, \, A\boldsymbol{x} \rangle - 2 \langle \boldsymbol{x}, \, \boldsymbol{b} \rangle$. Then, (2 points)

$$D_{\boldsymbol{v}^{(k+1)}} g(\boldsymbol{x}^{(k)}) = [\boldsymbol{v}^{(k+1)}]^t \nabla g(\boldsymbol{x}^{(k)}) = [\boldsymbol{r}^{(k)} + s_k \boldsymbol{v}^{(k)}]^t (-2\boldsymbol{r}^{(k)})$$

Since $\langle \boldsymbol{r}^{(k)}, \, \boldsymbol{v}^{(k)} \rangle = \langle \boldsymbol{r}^{(k-1)}, \, \boldsymbol{v}^{(k)} \rangle - \frac{\langle \boldsymbol{v}^{(k)}, \boldsymbol{r}^{(k-1)} \rangle}{\langle \boldsymbol{v}^{(k)}, A\boldsymbol{v}^{(k)} \rangle} \langle A\boldsymbol{v}^{(k)}, \, \boldsymbol{v}^{(k)} \rangle = 0$, (1 point) we have (2 points)

$$D_{\boldsymbol{v}^{(k+1)}} g(\boldsymbol{x}^{(k)}) = -2||\boldsymbol{r}^{(k)}||_2^2 < 0.$$

So, $\boldsymbol{v}^{(k+1)}$ is a descent direction of $g(\boldsymbol{x}) = \langle \boldsymbol{x}, \, A\boldsymbol{x} \rangle - 2 \langle \boldsymbol{x}, \, \boldsymbol{b} \rangle$ (2 points) at $\boldsymbol{x}^{(k)}$ (2 points) when $\boldsymbol{r}^{(k)} = \boldsymbol{b} - A\boldsymbol{x}^{(k)}$ is not a zero vector (1 point) (or equivalently, when $\boldsymbol{x}^{(k)}$ is not a solution).