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1) a) Solving the i^{th} equation for x_i gives the fixed point problem $x_1 = \frac{x_1^2 + x_2^2 + 8}{10}$, $x_2 = \frac{x_1 x_2^2 + x_1 + 8}{10}$ (*)

Let $G: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $G(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}))^t$ where $g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 8}{10}$ and $g_2(x_1, x_2) = \frac{x_1 x_2^2 + x_1 + 8}{10}$

Theorems 10.4 and 10.6 will be used to show that G has a unique fixed point in $D = \left\{ (x_1, x_2)^t \,\middle|\, 0 \le x_1, x_2 \le 1.5 \right\}$

For $\mathbf{x} = (x_1, x_2)^t$ in $D \Rightarrow |g_1(x_1, x_2)| = \left| \frac{x_1^2 + x_2^2 + 8}{10} \right| =$

$= \frac{x_1^2 + x_2^2 + 8}{10} \le \frac{1.5^2 + 1.5^2 + 8}{10} = \frac{4.5 + 8}{10} = \frac{12.5}{10} = 1.25 < 1.5$

So, $g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 8}{10} \ge \frac{8}{10} > 0 \Rightarrow \boxed{1.5 > g_1(x_1, x_2) > 0}$

where $0 \le x_1, x_2 \le 1.5$. Similarly, $g_2(x_1, x_2) = \frac{x_1 x_2^2 + x_1 + 8}{10}$ $\ge \frac{8}{10} > 0$ since $x_1, x_2 \ge 0$ and $g_2(x_1, x_2) \le \frac{1.5^3 + 1.5 + 8}{10}$

$= \frac{3.375 + 9.5}{10} = \frac{12.875}{10} = 1.2875 < 1.5$ since $x_1, x_2 \le 1.5$

Hence, $\boxed{1.5 > g_2(x_1, x_2) > 0}$

Thus, $0 < g_1(x_1, x_2),\ g_2(x_1, x_2) < 1.5$ meaning that

$\boxed{G(\mathbf{x}) \in D \text{ whenever } \mathbf{x} \in D}$ Finding bounds for the partial derivatives on D

gives $\left|\dfrac{\partial g_1}{\partial x_1}\right| = \left|\dfrac{2x_1}{10}\right| = \left|\dfrac{x_1}{5}\right| = \dfrac{|x_1|}{5} = \dfrac{x_1}{5} \leq \dfrac{1.5}{5} = \dfrac{3}{10} =$

$= 0.3,\ \left|\dfrac{\partial g_1}{\partial x_2}\right| = \left|\dfrac{2x_2}{10}\right| = \dfrac{|x_2|}{5} = \dfrac{x_2}{5} \leq \dfrac{1.5}{5} = 0.3$ where

$0 \leq x_1, x_2 \leq 1.5 \Rightarrow \boxed{\left|\dfrac{\partial g_1}{\partial x_1}\right|,\ \left|\dfrac{\partial g_1}{\partial x_2}\right| \leq 0.3}$ Similarly, we get $\left|\dfrac{\partial g_2}{\partial x_1}\right| = \left|\dfrac{1}{10} + \dfrac{x_2^2}{10}\right|$

$= \dfrac{x_2^2+1}{10} \leq \dfrac{1.5^2+1}{10} = \dfrac{3.25}{10} = 0.325,\ \left|\dfrac{\partial g_2}{\partial x_2}\right| = \left|\dfrac{x_1}{10} \cdot 2x_2\right| =$

$= \dfrac{|x_1 x_2|}{5} = \dfrac{x_1 x_2}{5} \leq \dfrac{1.5^2}{5} = \dfrac{2.25}{5} = 0.45$ where $0 \leq x_1, x_2 \leq 1.5$

$\boxed{\left|\dfrac{\partial g_2}{\partial x_1}\right| \leq 0.325,\ \left|\dfrac{\partial g_2}{\partial x_2}\right| \leq 0.45}$ The partial derivatives of g_1, g_2 are all bounded on D, so Theorem 10.4 implies that these functions are continuous on D. Consequently, G is continuous on D. Moreover,

$\forall \mathbf{x} \in D,\ \left|\dfrac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \leq 0.45$, for each $i = 1, 2$ and $j = 1, 2$

and the condition in the 2^{nd} part of Theorem 10.6 holds with $K = 2(0.45) = 0.9$

In the same manner, it is shown that $\frac{\partial g_i}{\partial x_j}$ is con-tinuous on D for each $i=1,2$ and $j=1,2$. Therefore, G has a unique fixed point in D, and the nonlinear system has a solution in D ⊠ (Since $x_1, x_2 \geq 0$, the quantities $\left|\frac{\partial g_i}{\partial x_j}\right|$ are indeed equal to $\frac{\partial g_i}{\partial x_j}$ for each $i=1,2$ and $j=1,2$

$$\left(\frac{\partial g_1}{\partial x_1} = \frac{2x_1}{10} = \frac{x_1}{5} = \left|\frac{\partial g_1}{\partial x_1}\right|, \quad \frac{\partial g_2}{\partial x_1} = \frac{x_2^2+1}{10} = \left|\frac{\partial g_2}{\partial x_1}\right|,\right.$$

$$\left.\frac{\partial g_1}{\partial x_2} = \frac{x_2}{5} = \left|\frac{\partial g_1}{\partial x_2}\right|, \quad \frac{\partial g_2}{\partial x_2} = \frac{2x_1x_2}{10} = \frac{x_1x_2}{5} = \left|\frac{\partial g_2}{\partial x_2}\right|\right)$$

b) To approximate the fixed point p, we chose $x^{(0)} = (0,0)^t$. The sequence of vectors generated by

$$x_1^{(k)} = \frac{\left(x_1^{(k-1)}\right)^2 + \left(x_2^{(k-1)}\right)^2 + 8}{10}, \quad x_2^{(k)} = \frac{x_1^{(k-1)}\left(x_2^{(k-1)}\right)^2 + x_1^{(k-1)} + 8}{10}$$

converges to the unique solution of the system in ⊛ of 1^{st} page. The results should be generated until $\|x^{(k)} - p\|_\infty < 10^{-8}$

In fact, we could use the error bound (10.3) on pg. 633

$$\|x^{(k)}-p\|_\infty \le \frac{K^k}{1-K}\|x^{(1)}-x^{(0)}\|_\infty$$ with $K=0.9$ in $x^{(0)}=(0,0)^t$

the previous part $\Rightarrow x^{(1)}=\left(x_1^{(1)}, x_2^{(1)}\right)^t$ where using the

gives $x_1^{(1)}=\frac{8}{10}=0.8$, $x_2^{(1)}=\frac{8}{10}=0.8$ or $\underline{x^{(1)}=(0.8, 0.8)^t}$

$x^{(1)}-x^{(0)}=(0.8,0.8)^t \Rightarrow \|x^{(1)}-x^{(0)}\|_\infty=0.8$ where

$\|x^{(k)}-p\|_\infty \le \frac{0.9^k}{0.1}\cdot 0.8 = 8\cdot 0.9^k < 10^{-3}$ reveals that

$0.9^k < \frac{10^{-3}}{8} = \frac{1}{8\cdot 10^3}$ or $8\cdot\frac{9^k}{10^k} < 10^{-3}$, $8\cdot 9^k < 10^{k-3}$ and

$\boxed{8 < \frac{10^{k-3}}{9^k}}$ $0.9^k < \frac{10^{-3}}{8}$ implies $k\cdot \ln(0.9) < \ln\left(\frac{1}{8}\right) - 3\ln(10)$

$\log_{10}(0.9^k) = k\log_{10}(0.9) < \log_{10}\left(\frac{1}{8}\right) + \log_{10}(10^{-3}) =$

$= \log_{10}(2^{-3}) - 3 = -3\log_{10}(2) - 3$, $\log_{10}\left(\frac{9}{10}\right) = \log_{10}\left(\frac{1}{10}\right) +$

$+ \log_{10}(9) = -1 + \log_{10}(3^2) = 2\log_{10}(3) - 1$, $k\log_{10}(0.9) =$

$= 2k\log_{10}(3) - k < -3\log_{10}(2) - 3$, $k\left(2\log_{10}(3) - 1\right) <$

$< -3\log_{10}(2) - 3$, $\underline{k\left(1 - 2\log_{10}(3)\right) > 3\log_{10}(2) + 3}$, so

$$\boxed{k > \frac{3(\log_{10}(2)+1)}{1-2\log_{10}(3)} \approx 85.2994764172}$$ Thus, $\boxed{k \geq 86}$ will ensure that fixed-point iteration will achieve 10^{-3} accuracy of $\|x^{(k)}-p\|_\infty$ with $x^{(0)}=(0,0)^t \Rightarrow$ $\boxed{\#\text{ of iterations} = 86}$

2) Since $f_j(x_1,\dots,x_n) = a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n - b_j$, we get $\frac{\partial f_j}{\partial x_i} = a_{ji} \Rightarrow$ Hence,

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = A, \quad \boxed{J(x) = A}$$

Furthermore, we get that

$$F(x^{(0)}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = J(x^{(0)})x^{(0)} - b$$

since $F(x) = (f_1(x), f_2(x), \dots, f_n(x))^t \Rightarrow \boxed{F(x^{(0)}) = J(x^{(0)})x^{(0)} - b}$

Thus, given $x^{(0)}$, we have $x^{(1)} = x^{(0)} - J(x^{(0)})^{-1} \cdot F(x^{(0)}) =$
$= x^{(0)} - J(x^{(0)})^{-1} \cdot (J(x^{(0)})x^{(0)} - b)$, where $A(x^{(0)})^{-1} = J(x^{(0)})^{-1}$

Hence, $x^{(1)} = x^{(0)} - J(x^{(0)})^{-1}\left(J(x^{(0)})x^{(0)} - b\right) = x^{(0)} -$

$- J(x^{(0)})^{-1} J(x^{(0)}) x^{(0)} + J(x^{(0)})^{-1} b = x^{(0)} - x^{(0)} + J(x^{(0)})^{-1} b =$

$= J(x^{(0)})^{-1} b = A^{-1} b \Rightarrow \boxed{x^{(1)} = A^{-1} b}$ So, given any $x^{(0)}$,

(considering A^{-1} exists, or nonsingularity) $\boxed{V}$

the solution to the linear system is $x^{(1)} = A^{-1} b$ ⊕