2021 Spring MAS 365: Homework 9

posted on June 3; due by June 15

1. [10+10 points] Given the multistep method

$$w_{i+1} = -\frac{3}{2}w_i + 3w_{i-1} - \frac{1}{2}w_{i-2} + 3hf(t_i, w_i), \quad \text{for } i = 2, \ldots, N-1,$$

   with starting values w_0, w_1, w_2.

   (a) Find the local truncation error. [Hint: Consider the third-order Taylor polynomials of $y(t_{i+1}), y(t_{i-1})$ and $y(t_{i-2})$ about x_i and their remainder terms.]

   (b) Comment on consistency, stability, and convergence, under the assumptions that $y^{(4)}$ is bounded and w_0, w_1, w_1 are consistent. Justify your answer.

2. [5 points] Show that the fourth-order Runge-Kutta method

$$\begin{aligned}
k_1 &= hf(t_i, w_i), \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right), \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right), \\
k_4 &= hf(t_i + h, w_i + k_3), \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{aligned}$$

   when applied to the differential equation $y' = \lambda y$, can be written in the form

$$w_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right) w_i.$$

3. [10+10 points]

   (a) Use the Gram-Schmidt procedure to calculate L_1 and L_2, where $\{L_0(x), L_1(x), L_2(x)\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ and $L_0(x) = 1$.

   (b) Use the polynomials $\{L_0(x), L_1(x), L_2(x)\}$ to compute the least squares polynomials of degree 2 on the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for the function $f(x) = x^3$.

4. [10+10 points] Consider approximating $f(x) = x^3 - x$ by a polynomial $P_2(x) = a_2x^2 + a_1x + a_0$ that minimizes

$$E_2(a_0, a_1, a_2) = \int_0^1 [f(x) - P_2(x)]^2 dx$$

   (a) Find $P_2(x)$.

(b) Find a polynomial P_1 that solves

$$\min_{P_1 \in \Pi_1} \max_{x \in [0,1]} |P_2(x) - P_1(x)|,$$

where Π_1 denotes set of all polynomials degree at most 1.

5. [10 points] Implement the power method via MATLAB grader.