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1) a) $f(x)=\dfrac{1}{x^3}$ with $x_0=1,\ x_1=2,\ x_2=3$

We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, $L_2(x)$ $\Rightarrow$ $L_0(x)=\dfrac{(x-2)(x-3)}{(1-2)(1-3)}=\dfrac{1}{2}(x^2-5x+6)$

$$L_1(x)=\frac{(x-1)(x-3)}{(2-1)(2-3)}=-(x^2-4x+3)$$

$$L_2(x)=\frac{(x-1)(x-2)}{(3-1)(3-2)}=\frac{1}{2}(x^2-3x+2)$$

Also, $f(x_0)=f(1)=1$, $f(x_1)=f(2)=\dfrac{1}{8}$; $f(x_2)=f(3)=$
$=\dfrac{1}{27}\Rightarrow P(x)=\displaystyle\sum_{k=0}^{2}f(x_k)L_k(x)=\dfrac{1}{2}(x^2-5x+6)+$
$+\dfrac{1}{8}\cdot(-x^2+4x-3)+\dfrac{1}{27}\cdot\dfrac{1}{2}(x^2-3x+2)=\dfrac{1}{2}x^2-\dfrac{5}{2}x+3+$
$+-\dfrac{1}{8}x^2+\dfrac{1}{2}x-\dfrac{3}{8}+\dfrac{1}{54}x^2-\dfrac{3}{54}x+\dfrac{1}{27}=\dfrac{27}{54}x^2-\dfrac{3}{54}x+$
$+\dfrac{1}{54}x^2-\dfrac{1}{8}x^2-\dfrac{5}{2}x+\dfrac{1}{2}x+3-\dfrac{3}{8}+\dfrac{1}{27}=\dfrac{28x^2}{54}-\dfrac{1}{8}x^2-$
$-2x+\dfrac{21}{8}+\dfrac{1}{27}-\dfrac{1}{18}x=\left(\dfrac{14}{27}x^2-\dfrac{1}{8}x^2\right)-\dfrac{37}{18}x+\left(\dfrac{21}{8}+\dfrac{1}{27}\right)$
$=\dfrac{112}{216}x^2-\dfrac{27}{216}x^2-\dfrac{37}{18}x+\dfrac{567}{216}+\dfrac{8}{216}=\boxed{\dfrac{85x^2}{216}-\dfrac{37}{18}x+\dfrac{575}{216}}$

2nd Lagrange polynomial for $f(x)=x^{-3}$ on $[1,3]$ $\longrightarrow$ $\boxed{\text{V}}$

b) $f(x)=x^{-3} \Rightarrow f'(x)=-3x^{-4}$; $f''(x)=(-3)(-4)x^{-5}=12x^{-5}$
$f'''(x)=12(-5)x^{-6}=-60x^{-6}$, As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)=\frac{-60(\xi(x))^{-6}}{6}(x-1)(x-2)(x-3)$$

$$=\underline{-10(\xi(x))^{-6}(x-1)(x-2)(x-3)}$$ for $\xi(x)$ in $(1,3)$

The $\underline{\text{max value of } (\xi(x))^{-6}}$ on the interval is $\underline{1^{-6}=1}$

We now need to determine the max value on this interval of the absolute value of polynomial $g(x)=(x-1)(x-2)(x-3)$
$=(x^2-3x+2)(x-3)=x^3-3x^2+2x-3x^2+9x-6=x^3-6x^2+$
$+11x-6$. Because $D_x(x^3-6x^2+11x-6)=3x^2-12x+11=$
$=\left(x-\left(2-\frac{1}{\sqrt{3}}\right)\right)\left(x-\left(2+\frac{1}{\sqrt{3}}\right)\right)$, the critical points occur at

$(3-1=2)$

$x=2-\frac{1}{\sqrt{3}}$ with $g\left(2-\frac{1}{\sqrt{3}}\right)=\left(1-\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\left(-1-\frac{1}{\sqrt{3}}\right)=$

$=\frac{(\sqrt{3}-1)(-1)(-\sqrt{3}-1)}{3\sqrt{3}}=\frac{(\sqrt{3}-1)(\sqrt{3}+1)}{3\sqrt{3}}=\boxed{\frac{2}{3\sqrt{3}}}$ and

$2\ \sqrt{3}\cdot\sqrt{3}=3-3=1$

$(1+\sqrt{3})(1-\sqrt{3})=1-3=-2$

$x=2+\frac{1}{\sqrt{3}}$ with $g\left(2+\frac{1}{\sqrt{3}}\right)=\left(1+\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}-1\right)=\frac{(\sqrt{3}+1)(1-\sqrt{3})}{3\sqrt{3}}=$

$=\frac{-2}{3\sqrt{3}}$. Hence, the maximum error is $\left|\frac{f'''(\xi(x))}{3!}\right|\left|(x-x_0)(x-x_1)(x-x_2)\right|$

$\leq 10\cdot 1\cdot\frac{2}{3\sqrt{3}}=\frac{20}{3\sqrt{3}}=\boxed{\frac{20\sqrt{3}}{9}}$ $\boxed{\checkmark}$ ∎.

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2) <u>Taylor's Theorem:</u> Suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. Then, $\forall x \in [a,b]$, there exist $\xi(x)$ between x_0 and x with $f(x) = P_n(x) + R_n(x)$, where:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \rightsquigarrow n^{th} \text{ Taylor Polynomial}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

<u>Proof:</u> Note first that if $x = x_0$, then we know from the definition, that if P - n^{th} Taylor polynomial, then $f(x_0) = P(x_0)$, $f'(x_0) = p'(x_0), \dots, f^{(n)}(x_0) = p^{(n)}(x_0)$ from the Taylor's construction: Let $P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ with

$$P(x_0) = f(x_0), \quad p'(x) = \sum_{j=0+1}^{n} j \cdot \frac{f^{(j)}(x_0)}{j!}(x-x_0)^{j-1} = \sum_{j=0}^{n-1}(j+1)\frac{f^{(j+1)}(x_0)}{(j+1)!} \cdot (x-x_0)^j$$

or just $p'(x) = f'(x_0) + (x-x_0)(\dots)$ For simplicity,

Let $P = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)^n$, where $a_j = \frac{f^{(j)}(x_0)}{j!}$ for each $j = 0, 1, \dots, n \Rightarrow$ If $x = x_0$, $\boxed{P(x_0) = a_0 = f(x_0)}$

Now, differentiating $\rightsquigarrow p'(x) = \sum_{j=0}^{n} j a_j (x-x_0)^{j-1} = \sum_{i=0}^{n-1}(i+1)a_{i+1} \cdot (x-x_0)^i$

Setting $x = x_0$, we observe all terms except the term for $i = 0$ must vanish $\Rightarrow \boxed{p'(x_0) = a_1 = f'(x_0)}$ At the next step, differentiate $p'(x)$ again, shift index, and plug in $x = x_0$

$$p''(x) = \sum_{i=0}^{n-1} (i+1)\, a_{i+1} \cdot i\, (x-x_0)^{i-1} = \sum_{i=0}^{n-2} (i+2)(i+1)\, a_{i+2}\, (x-x_0)^{i}$$

where $x = x_0 \Rightarrow$ only first term will be left and we get

$p''(x_0) = \underbrace{(i+2)(i+1)a_{i+2}}_{\text{for } i=0} = (0+2)(0+1)a_2 = 2a_2 = f''(x_0)$; Thus,

$\boxed{p''(x_0) = f''(x_0)}$ Recursively differentiating, we find that

j^{th} derivative of $P(x) \rightarrow p^{(j)}(x) = \displaystyle\sum_{i=0}^{n-j} \frac{(i+j)!\, a_{i+j}}{i!} (x-x_0)^{i}$.

When $x = x_0$, all terms except for corresponding to $i=0$ must vanish $\Rightarrow p^{(j)}(x_0) = \dfrac{(0+j)!\, a_j}{0!} = j!\, a_j = f^{(j)}(x_0)$

from the definition; Thus, $p^{(j)}(x_0) = f^{(j)}(x_0)$ becomes true

$\boxed{p^{(j)}(x_0) = f^{(j)}(x_0) \text{ for } j = 0, 1, \dots, n}$ Now, construct the following function

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)^{n+1}}{(x-x_0)^{n+1}} \quad \text{where}$$

$$g(x_0) = f(x_0) - P(x_0) - [f(x) - P(x)] \cdot \underbrace{\frac{(x_0-x_0)^{n+1}}{(x-x_0)^{n+1}}}_{\boxed{0}} = 0 \text{ since}$$

$f(x_0) = P(x_0) \Rightarrow \boxed{g(x_0) = 0}$

For definiteness, assume that $a \leq x_0 < x \leq b$, where for each $t \in [x_0, x] \Rightarrow g(t) = f(t) - P(t) - [f(x) - P(x)] \dfrac{(t-x_0)^{n+1}}{(x-x_0)^{n+1}}$

$g(x) = f(x) - P(x) - [f(x) - P(x)] = 0$, $\boxed{g(x) = 0}$

Assume $x \neq x_0$, define function g for t in $[a,b]$ by

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)^{n+1}}{(x-x_0)^{n+1}}$$ Since $f \in C^n[a,b]$

$f^{(n+1)}$ exists on $[a,b]$, and $P \in C^\infty[a,b]$, it follows that $g \in C^n[a,b]$ and $g^{(n+1)}$ exists on $[a,b]$. Moreover,

$$g(x) = f(x) - P(x) - [f(x) - P(x)] \frac{(x-x_0)^{n+1}}{(x-x_0)^{n+1}} = 0 \Rightarrow \boxed{g(x) = 0}$$

$$g(x_0) = [f(x_0) - P(x_0)] - [f(x) - P(x)] \cdot \frac{(x_0-x_0)^{n+1}}{(x-x_0)^{n+1}} = 0$$ since $P(x_0) = f(x_0)$

$\boxed{g(x_0) = 0}$ From Generalized Rolle Theorem, we can find that

$$g^{(j)}(t) = f^{(j)}(t) - P^{(j)}(t) - [f(x) - P(x)] \cdot \frac{d^j}{dt^j}\left(\frac{(t-x_0)^{n+1}}{(x-x_0)^{n+1}}\right)$$

$$\frac{d^j}{dt^j}(t-x_0)^{n+1} = (n+1)n(n-1)\dots(n-j+1)(t-x_0)^{n-j+1}$$, meaning that

$$\boxed{\frac{d^j}{dt^j}(t-x_0)^{n+1} = \frac{(n+1)!}{(n-j+1)!}(t-x_0)^{n-j+1} \text{ where } j = 0, 1, \dots, n}$$ From this, we get

$$g^{(j)}(t) = f^{(j)}(t) - P^{(j)}(t) - \frac{f(x) - P(x)}{(x-x_0)^{n+1}} \cdot \frac{(n+1)!}{(n-j+1)!}(t-x_0)^{n+1-j}$$

for $j = 0, 1, \dots, n$ Since $n+1 > n \geq j$, $n+1-j > 0$ or just $n+1-j \geq 1$

If $t = x_0$, then considering $f^{(j)}(x_0) = P^{(j)}(x_0)$ for $j = 0, 1, \dots, n$ and taking into account how $(t-x_0)^{n+1-j}$ becomes zero for $t = x_0$ (because $n+1-j > 0$), we obtain $g^{(j)}(x_0) = f^{(j)}(x_0) - P^{(j)}(x_0) = 0$

So, $g(x_0)=0$ was true and taking derivatives for $j=1,\dots,n$, we proved that $g^{(j)}(x_0)=0 \Rightarrow \boxed{g^{(j)}(x_0)=0 \text{ for } j=0,1,\dots,n}$ ☆

Since $g(x)=g(x_0)=0$, there exist x_1 between x and x_0 such that $g'(x_1)=0$ (from Rolle's Theorem where $n=1$). Similarly, $g'(x_1)=g'(x_0)=0$, where $g\in C^n[a,b]$ and $g^{(n+1)}$ exists on $[a,b]$. Similar approach from Rolle to $g'(x)$ function with $n=1$ yields $\rightsquigarrow \exists x_2$ between x_0 and x_1 such that $g''(x_2)=0$. Assume for some $k\leq n$, $g^{(k-1)}(x_{k-1})=g^{(k-1)}(x_0)=0$ where x_{k-1} lies between x_0 and x_{k-2}. Applying Rolle's Theorem to $g^{(k-1)}(x)$ function with $n=1$ where $g\in C^n[a,b]$ and $g^{(n+1)}$ exists on $[a,b]$, we get that $\exists x_k$ between x_0 and x_{k-1} s.t. $g^{(k)}(x_k)=0$. Constructing such sequence $\{x_k\}$ inductively, we deduce that $g^{(k)}(x_k)=g^{(k)}(x_0)=0$ with x_k lying between x_0 and x_{k-1}. Inductively proceeding yields the existence of x_n between x_0 and x_{n-1} s.t. $g^{(n)}(x_n)=g^{(n)}(x_0)=0$. As $g^{(n+1)}(x)$ exists (on (a,b)), where $g^{(n)}(x)\in C[a,b]$, we deduce the existence of x_{n+1} between x_n and x_0 s.t. $\boxed{g^{(n+1)}(x_{n+1})=0}$. Considering that we had initially x, x_0 were in $[a,b] \longrightarrow x_1\in[a,b] \rightarrow x_2\in[a,b] \rightarrow \dots \rightarrow x_k\in[a,b]$

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If $x_{k-1} \in [a,b]$, then since x_k lies between $x_0, x_{k-1} \Rightarrow x_k \in [a,b]$
Inductively proceeding yields $x_n \in [a,b]$ and as x_{n+1} lies between x_0 and $x_n \Rightarrow \boxed{g^{(n+1)}(x_{n+1}) = 0 \text{ where } x_{n+1} \in [a,b]}$

Generally, $x_1 \rightarrow$ lies between x and x_0,
$x_2 \rightarrow$ lies between x and $x_1 \rightarrow$ between x and x_0
If x_{k-1} lies between x and x_0, then since x_k is between x_0 and $x_{k-1} \Rightarrow x_k$ lies between x and x_0 as well. Thus, Induction reveals that $\boxed{x_n \text{ lies between } x \text{ and } x_0}$ and since x_{n+1} lies between $x_0, x_n \Rightarrow \boxed{g^{(n+1)}(x_{n+1}) = 0 \text{ where } x_{n+1} \text{ lies between } x \text{ and } x_0}$
As we knew previously,

$$g^{(n)}(t) = f^{(n)}(t) - p^{(n)}(t) - \frac{f(x) - P(x)}{(x-x_0)^{n+1}} \cdot (n+1)!\,(t - x_0)$$

since p is a polynomial of degree n, $\boxed{p^{(n+1)}(t) = 0}$
given that $f^{(n+1)}(x)$ exists on $[a,b]$, differentiating by t gives

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \frac{f(x) - P(x)}{(x-x_0)^{n+1}} \cdot (n+1)! \text{ and}$$

setting $t = x_{n+1} \Rightarrow g^{(n+1)}(x_{n+1}) = 0 = f^{(n+1)}(x_{n+1}) - \frac{f(x) - P(x)}{(x-x_0)^{n+1}} \cdot (n+1)!$

So, $\dfrac{f(x) - P(x)}{(x-x_0)^{n+1}} = \dfrac{f^{(n+1)}(x_{n+1})}{(n+1)!}$ for some x_{n+1} lying between x and x_0. In conclusion, we obtain

$$f(x) = P(x) + \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}(x-x_0)^{n+1}, \text{ where } x_{n+1} \text{ lies between } x \text{ and } x_0; \; P(x) \rightsquigarrow n^{th} \text{ Taylor polynomial}$$

$\forall x \in [a,b]$

Note: For $x = x_0$, since $f(x_0) = P(x_0)$, we can choose x_{n+1} (or just $\xi(x)$ in the original theorem) arbitrarily in (a,b) (In fact, there is no need to consider this case) (Because there does not exist any value between $x = x_0$ and x_0) ✓

3) $Q_{0,0} = P_0 = f(x_0) = f(0) = 1$, $Q_{1,0} = f(x_1) = f(0.25) = 2$, $Q_{2,0} = f(x_2) = f(0.5) = P_2$, $Q_{3,0} = P_3 = f(x_3) = f(0.75) = 8$

These are the 4 polynomials of degree zero (constants) that approximate $f(0.4)$. Calculating the first-degree approximation $Q_{1,1}(0.4)$ gives $Q_{1,1}(0.4) =$

$$= \frac{(x-x_0)Q_{1,0} - (x-x_1)Q_{0,0}}{x_1 - x_0} = \frac{0.4\, P_1(0.4) - 0.15\, P_0(0.4)}{0.25}$$

$$= \frac{0.4 \times 2 - 0.15 \times 1}{0.25} = \frac{0.65}{0.25} = 2.6 \Rightarrow P_{0,1} = 2.6$$ Similarly,

$$Q_{2,1}(0.4) = \frac{(0.4 - x_1)Q_{2,0} - (0.4 - x_2)Q_{1,0}}{x_2 - x_1} =$$

$$= \frac{0.15 \times P_2 + 0.1 \times P_1}{0.25} = \frac{0.15 P_2 + 0.2}{1/4} = 0.6 P_2 + 0.8 \Rightarrow$$

$\boxed{P_{1,2} = 0.6 P_2 + 0.8}$ $Q_{3,1}(0.4) = \dfrac{(0.4 - x_2)Q_{3,0} - (x - x_3)Q_{2,0}}{x_3 - x_2}$

$$= \frac{(0.4 - 0.5)P_3 - (0.4 - 0.75)P_2}{0.25} = \frac{-0.1 \times 8 + 0.35 \times P_2}{1/4} =$$

$= 1.4 P_2 - 3.2 \Rightarrow \boxed{P_{2,3} = 1.4 P_2 - 3.2}$ Since it was given that $P_{2,3} = 2.4 \Rightarrow$

$1.4 P_2 - 3.2 = 2.4$ and $1.4 P_2 = 5.6$, $\boxed{P_2 = f(0.5) = 4}$ Now,

$\boxed{P_{1,2} = 0.6 \times 4 + 0.8 = 2.4 + 0.8 = 3.2}$

verifying this value to consequent polynomials, in the similar manner approximations using higher-degree polynomials are given by

$$Q_{2,2}(0.4) = \frac{(0.4-x_0)\,Q_{2,1} - (0.4-x_2)\,Q_{1,1}}{x_2 - x_0} =$$

$$= \frac{(0.4-0)\,P_{1,2} - (0.4-0.5)\,P_{0,1}}{0.5} = \frac{0.4(0.6P_2+0.8)+0.1\times 2.6}{0.5}$$

$\boxed{0.58}$

$$= \frac{0.24P_2 + 0.32 + 0.26}{1/2} = 0.48P_2 + 1.16,$$ and since we found $P_2 = 4$, plugging in gives $Q_{2,2}(0.4) = P_{0,1,2} = 0.48\times 4 + 1.16 = 1.92 + 1.16 = 3.08$

$\boxed{P_{0,1,2} = 3.08}$ $$Q_{3,2}(0.4) = \frac{(0.4-x_1)\,Q_{3,1} - (0.4-x_3)\,Q_{2,1}}{x_3 - x_1}$$

$$= \frac{0.15\times P_{2,3} + 0.35\,P_{1,2}}{1/2} = \frac{0.15\times 2.4 + 0.35(0.6P_2+0.8)}{1/2}$$

$\boxed{0.64}$

$$= \frac{0.36 + 0.21P_2 + 0.28}{1/2} = 0.42P_2 + 1.28,$$ and since we had $P_2 = 4$, setting in gives $Q_{3,2}(0.4) = P_{1,2,3} = 0.42\times 4 + 1.28 = 1.68 + 1.28 = 2.96$ which indeed verifies the value written in the given table ✓ $P_{1,2,3} = 2.96$ $Q_{3,3}(0.4) =$

$$= \frac{(0.4-x_0)\,Q_{3,2} - (0.4-x_3)\,Q_{2,2}}{x_3 - x_0} = \frac{0.4\,P_{1,2,3} + 0.35\,P_{0,1,2}}{0.75} =$$

(A) (Backwardly finding P_2 from $P_{0,1,2,3} \to P_{0,1,2} \to P_{1,2}$ could also be identically same method)

$$= \frac{0.4\times 2.96 + 0.35\times 3.08}{0.75} = \frac{1.184 + 1.078}{0.75} = \frac{2.262}{0.75} = 3.016$$

$= 3.016$

which again verifies the approximation $P_{0,1,2,3} = Q_{3,3}(0.4) =$ written on the table $\Rightarrow$ Thus, $\boxed{P_2 = f(0.5) = 4}$ is correct value for approximating $f(0.4)$

V +

4) Assume that there exist another polynomial $P(x)$ such that $P(x_i) = f(x_i)$, $P'(x_i) = f'(x_i)$ for each $i = 0, 1, 2, \dots, n$ where $\deg P \leq 2n+1$

Consider $D(x) = H_{2n+1}(x) - P(x)$ where we know initially $\deg H_{2n+1} \leq 2n+1 \Rightarrow \boxed{\deg D \leq 2n+1}$ Moreover, since it is well-known that Hermite polynomial agrees with f and f' at $x_0, x_1, \dots, x_n$ we get $H_{2n+1}(x_j) = f(x_j)$, $H'_{2n+1}(x_j) = f'(x_j)$ for $\forall j \in [0,n] \Rightarrow$

$D(x_j) = H_{2n+1}(x_j) - P(x_j) = f(x_j) - f(x_j) = 0$ for $\forall j \in [0,n]$

$\boxed{D(x_j) = 0, \forall j \in [0,n]}$ Similarly, since $D'(x) = H'_{2n+1}(x) - P'(x)$

and $D'(x_j) = H'_{2n+1}(x_j) - P'(x_j) = f'(x_j) - f'(x_j) = 0 \Rightarrow$

$\boxed{D'(x_j) = 0, \forall j \in [0,n]}$ From Theorem 2.11 (as well as Theorem 2.12) since $D(x_j) = D'(x_j) = 0$, polynomial $D(x)$ will have at least "multiplicity of 2" zeros for each $x_j \Rightarrow$ Otherwise, if there would exist x_t which is not at least "multiplicity of 2" zero, then it would become a simple zero (only $D(x_j) = 0$), but then from theorem $\Rightarrow D'(x_j) \neq 0$ which is a contradiction $\boxed{\times}$ Thus, $D(x)$ will contain x_j as at least multiplicity of 2 zeros (it can be possible that x_j is exactly multiplicity of 2, but we mention "at least" to guarantee other cases.)

Thus, $\boxed{D(x) = \prod_{j=0}^{n} (x - x_j)^2 R(x)}$ where $R(x)$ - polynomial

From here, we deduce that if $R(x)$ is a nonzero polynomial with $\deg R$ - defined and $\deg R \geq 0$, then from the relation

$$\deg D = \sum_{j=0}^{n} 2 + \deg R = 2\underbrace{(1+1+\dots+1)}_{n+1} + \deg R = 2(n+1) + \deg R \geq 2n+2$$

whereas we initially supposed $\deg D \leq 2n+1$ $\boxed{\times}$ Henceforth, $\boxed{R(x) \equiv 0}$ is identically zero polynomial $\Rightarrow$

So, we find $\boxed{D(x) \equiv 0}$ which contradicts the fact that $P(x)$ was different polynomial from $H_{2n+1}(x)$ satisfying the aforementioned conditions (agreeing with f and f' at $x_0, \dots, x_n$ of least deg) $\boxed{V}$ $\oplus$

Thus, $H_{2n+1}(x)$ is a unique polynomial of least degree agreeing with f and f' at $x_0, \dots, x_n$ where $\deg H_{2n+1} \leq 2n+1$

(differently saying, there does not exist such polynomial P which is different from $H_{2n+1}(x)$ satisfying above conditions with of least degree)

V.

5) a) The endpoint information implies that $x_0=0$, $x_1=5$, $y_0=0$, $y_1=2$, and the guide points at $(1,1)$ and $(6,3)$ imply that $\rightsquigarrow$ The guide point for (x_0, y_0) is given by $(x_0+\alpha_0, y_0+\beta_0)$ and $(1,1)$ is given as a guide point for $(x_0,y_0)=(0,0) \Rightarrow x_0+\alpha_0=\boxed{\alpha_0=1}$, $y_0+\beta_0=\boxed{\beta_0=1}$. Similarly, the guide point for (x_1, y_1) is given by $(x_1-\alpha_1, y_1-\beta_1)$, but it is given that $(6,3)$: guide point for $(x_1,y_1)=(5,2)$

$$x_1-\alpha_1=5-\alpha_1=6, \quad y_1-\beta_1=2-\beta_1=3 \Rightarrow \boxed{\alpha_1=-1, \ \beta_1=-1}$$

The unique cubic polynomial satisfying the given conditions is

$$x(t)=\left[2(x_0-x_1)+(\alpha_0+\alpha_1)\right]t^3+\left[3(x_1-x_0)-(\alpha_1+2\alpha_0)\right]t^2+\alpha_0 t+x_0$$

In the similar manner, the unique cubic polynomial satisfying conditions is

$$y(t)=\left[2(y_0-y_1)+(\beta_0+\beta_1)\right]t^3+\left[3(y_1-y_0)-(\beta_1+2\beta_0)\right]t^2+\beta_0 t+y_0$$

$\Rightarrow$ Plugging in these values reveal

$$x(t)=(2(-5)+0)t^3+(3\cdot5-(-1+2))t^2+t=-10t^3+(15-1)t^2+t \Rightarrow \boxed{x(t)=-10t^3+14t^2+t}$$

Similarly, we get that

$$y(t)=(2(-2)+0)t^3+(3\cdot2-(-1+2))t^2+t=-4t^3+(6-1)t^2+t=-4t^3+5t^2+t \Rightarrow \boxed{y(t)=-4t^3+5t^2+t}$$

Henceforth, parametric cubic Hermite approximations for the given information $\Rightarrow$ $\boxed{\begin{array}{l} x(t)=-10t^3+14t^2+t \\ y(t)=-4t^3+5t^2+t \end{array}}$ ✓

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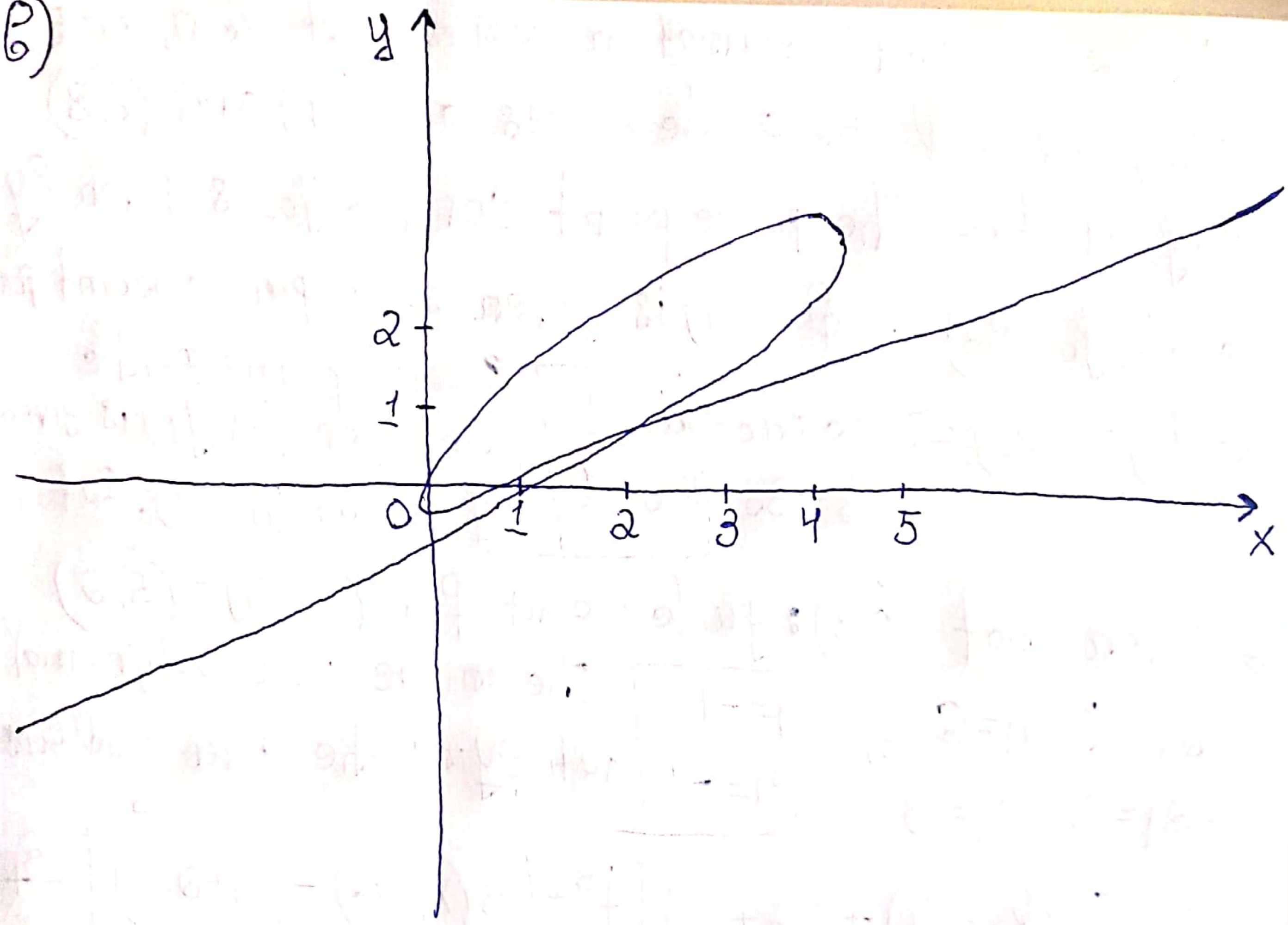
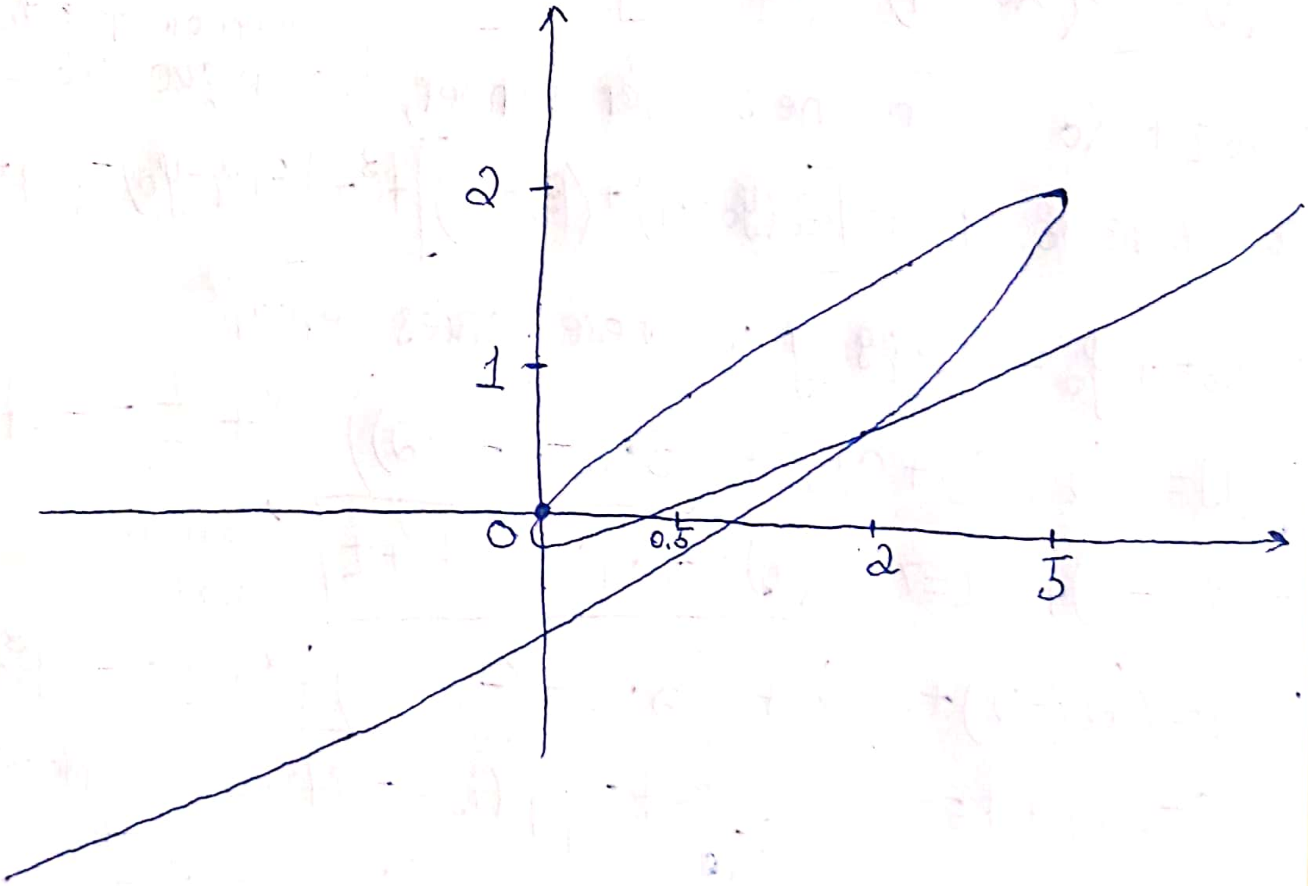
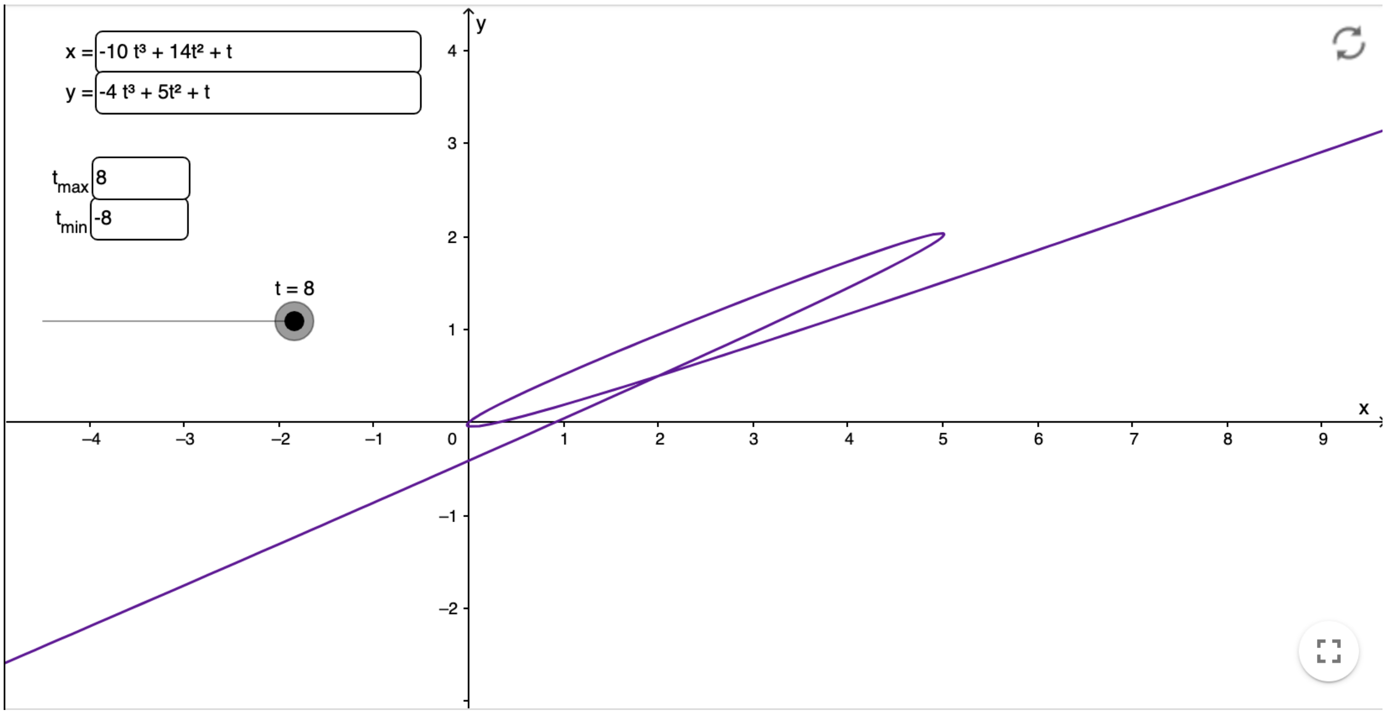




Scanned with CamScanner


x = -10 t³ + 14t² + t
y = -4 t³ + 5t² + t
tmax 8
tmin -8
t = 8
y
x