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2) a) Three-Point Endpoint Formula:

$$f'(x_0)=\frac{1}{2h}\left[-3f(x_0)+4f(x_0+h)-f(x_0+2h)\right]+\frac{h^2}{3}f^{(3)}(\xi_0)$$

where ξ_0 lies between x_0 and x_0+2h

Three-Point Midpoint Formula:

$$f'(x_0)=\frac{1}{2h}\left[f(x_0+h)-f(x_0-h)\right]-\frac{h^2}{6}f^{(3)}(\xi_1),$$ where

ξ_1 lies between x_0-h and x_0+h

While the errors in both equations are $O(h^2)$, the error for "Midpoint Formula" is $\approx \frac{1}{2}$ (the error for "Endpoint Formula")

This occurs because "Midpoint Formula" uses data on both sides of x_0 and "Endpoint Formula" uses data on only 1 side. Note also that f has to be evaluated at only 2 points in "Midpoint Formula", while for the "Endpoint Formula", three evaluations are required

The approximation in "Endpoint Formula" is pretty useful near the ends of an interval, because information about f outside the interval may not be available

Now, regarding the given task, considering aforementioned facts, For the endpoints of the tables, we will use Three-Point Endpoint Formula. The other approximations will come from Three-Point Midpoint Formula:

Taking $\underline{h=0.1}$ for Three-point Endpoint Formula,

$$f'(-0.3) = \frac{1}{2\cdot 0.1}\left[-3f(-0.3) + 4f(-0.3+0.1) - f(-0.3+0.2)\right]$$

$$f'(-0.3) = \frac{1}{0.2}\left[-3(-0.27652) + 4(-0.25074) - (-0.16134)\right]$$

$$= \frac{1}{0.2}\left[0.82956 - 1.00296 + 0.16134\right] = 5(-0.01206)$$

$$= -0.0603 \Rightarrow \boxed{f'(-0.3) \approx -0.0603}$$

Applying Three-Point Midpoint Formula, where $\underline{h=0.1}$

$$f'(-0.2) = \frac{1}{2\cdot 0.1}\left[f(-0.2+0.1) - f(-0.2-0.1)\right]$$

$$f'(-0.2) = \frac{1}{0.2}\left[f(-0.1) - f(-0.3)\right] = 5(-0.16134 + 0.27652)$$

$$= 5(0.11518) = 0.5759 \Rightarrow \boxed{f'(-0.2) \approx 0.5759}$$

Similarly, Three-Point Midpoint Formula with $\underline{h=0.1}$

$$f'(-0.1) = \frac{1}{0.2}\left[f(-0.1+0.1) - f(-0.1-0.1)\right] =$$

$$= 5(f(0) - f(-0.2)) = 5(0 + 0.25074) = 1.2537 \Rightarrow$$

$\boxed{f'(-0.1) \approx 1.2537}$

Finally, using three-point endpoint formula with $h=-0.1$

$$f'(0) = \frac{1}{-0.2}\left[-3f(0)+4f(-0.1)-f(-0.2)\right] =$$

$$=(-5)\left(4(-0.16134)+0.25074\right) = (-5)(-0.64536+0.25074)$$

$$=(-5)(-0.39462) = +1.9731 \Rightarrow \boxed{f'(0) \approx +1.9731}$$

In conclusion we obtain that

$$\boxed{f'(-0.3) \approx -0.0603;\ f'(-0.2) \approx 0.5758;\ f'(-0.1) \approx 1.2537 \quad f'(0) \approx 1.9731}$$

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b) $f(x) = e^{2x} - \cos 2x \Rightarrow f'(x) = 2e^{2x} + 2\sin 2x$ and $f''(x) = 4e^{2x} + 4\cos 2x$, with $f^{(3)}(x) = 8e^{2x} - 8\sin 2x$ in which according to problem statement, $f^{(3)}(x) = 8(e^{2x} - \sin 2x)$ is nonincreasing function on $[-0.3, 0]$. Using the theorems of "Three-Point Endpoint Formula" and "Three-point Midpoint Formula" it is well-known that Error bound for "Endpoint Formula" $\rightarrow \left|\frac{h^2}{3} f^{(3)}(\xi_0)\right|$ where ξ_0 lies between x_0, x_0+2h. In the similar manner, we obviously know that Error bound for "Midpoint Formula" $\rightarrow \left|\frac{h^2}{6} f^{(3)}(\xi_1)\right|$ where ξ_1 lies between x_0-h and x_0+h. Considering the h-value and which formulas we implemented in previous part, we can find that

• <u>Error bound for $f'(-0.3)$</u> is $\left|\frac{h^2}{3}f^{(3)}(\xi_0)\right| = \left|\frac{0.01}{3}f^{(3)}(\xi_0)\right|$

where we had used Endpoint formula with $h=0.1$ and ξ_0 lying between $-0.3 = x_0$ and $x_0+2h = -0.3+0.2 = -0.1$

Since $f^{(3)}(x)$ is nonincreasing function on $[-0.3, 0]$, we get that $\underbrace{|f^{(3)}(\xi_0)|}_{\xi_0 \text{ lies between } -0.3 \text{ and } -0.1} \leq |f^{(3)}(-0.3)|$ (Notice that $f^{(3)}(x) = 8(e^{2x} - \sin 2x)$)

Notice that $e^{-0.2} - \sin(-0.2) \approx$

($e^{2x} \geq 1 \geq \sin 2x$ for $x \geq 0$) $\approx 1.0174 \Rightarrow \xi_0$ lies between -0.3 and -0.1

$f^{(3)}(-0.3) \geq f^{(3)}(\xi_0) \geq f^{(3)}(-0.1) = 8\left(e^{-0.2} - \sin(-0.2)\right) \approx$

$\approx 8 \times 1.0174 > 0 \Rightarrow$ Error bound is $\frac{0.01}{3}\left|f^{(3)}(\xi_0)\right| =$

$= \frac{0.01}{3}f^{(3)}(\xi_0) \leq \frac{0.01}{3}f^{(3)}(-0.3) = \frac{1}{300}\cdot 8\left(e^{-0.6} - \sin(-0.6)\right)$

$$\boxed{\text{Error bound for } f'(-0.3) \text{ is } \frac{2}{75}\left(e^{-0.6} + \sin(0.6)\right) \approx \approx 0.0296921\underline{1}}$$

• <u>Error bound for $f'(-0.2)$</u> is $\left|\frac{h^2}{6}f^{(3)}(\xi_1)\right| = \left|\frac{0.01}{6}f^{(3)}(\xi_1)\right|$

where we implemented Midpoint formula with $h=0.1$ and ξ_1 lying between $x_0 - h = -0.2-0.1 = -0.3$ and $x_0+h = -0.2+0.1 = -0.1$

Since $f^{(3)}(x)$ - nonincreasing on $[-0.3, 0]$, we can get that $f^{(3)}(-0.3) \geq f^{(3)}(\xi_1) \geq f^{(3)}(-0.1) > 0$ as proved before

So, $\frac{0.01}{6}\left|f^{(3)}(\xi_1)\right| = \frac{0.01}{6}f^{(3)}(\xi_1) \leq \frac{0.01}{6}f^{(3)}(-0.3)$ and

$\frac{0.01}{6} f^{(3)}(-0.3) = \frac{1}{600} \cdot 8\left(e^{-0.6} - \sin(-0.6)\right)$. Henceforth, we've

$$\boxed{\text{Error Bound for } f'(-0.2) \text{ is } \frac{1}{75}\left(e^{-0.6} + \sin(0.6)\right) \approx \approx 0.0148460547}$$

- Error Bound for $f'(-0.1)$ is $\left|\frac{h^2}{6} f^{(3)}(\xi_1)\right| = \left|\frac{0.01}{6} f^{(3)}(\xi_1)\right|$

where midpoint formula was used with $h = 0.1$ and ξ_1 lying between $x_0 - h = -0.2$ and $x_0 + h = 0 \Rightarrow$ Since it's given $f^{(3)}(x)$ - nonincreasing on $[-0.3, 0] \Rightarrow f^{(3)}(-0.2) \geq f^{(3)}(\xi_1)$ $\geq f^{(3)}(0) = 8(e^0 - \sin 0) = 8(1-0) = 8 > 0$; therefore,

$$\frac{0.01}{6}\left|f^{(3)}(\xi_1)\right| = \frac{0.01}{6} f^{(3)}(\xi_1) \leq \frac{1}{600} f^{(3)}(-0.2) =$$

$$= \frac{1}{600}\left(8\left(e^{-0.4} - \sin(-0.4)\right)\right) = \frac{1}{75}\left(e^{-0.4} + \sin(0.4)\right)$$

$$\boxed{\text{Error Bound for } f'(-0.1) \text{ is } \frac{1}{75}\left(e^{-0.4} + \sin(0.4)\right) \approx \approx 0.01413}$$

- Error Bound for $f'(0)$ is $\left|\frac{h^2}{3} f^{(3)}(\xi_0)\right| = \left|\frac{0.01}{3} f^{(3)}(\xi_0)\right|$

where endpoint formula was used with $h = -0.1$ and ξ_0 lying between $x_0 = 0$ and $x_0 + 2h = -0.2 \Rightarrow$ As it was given $f^{(3)}(x)$ - nonincreasing on $[-0.3, 0] \Rightarrow f^{(3)}(-0.2) \geq f^{(3)}(\xi_0)$ $\geq f^{(3)}(0) > 0$ as mentioned before $\Rightarrow \frac{0.01}{3}\left|f^{(3)}(\xi_0)\right| =$

$$= \frac{0.01}{3} f^{(3)}(\xi_0) = \frac{1}{300} f^{(3)}(\xi_0)$$

$$\frac{1}{300} f^{(3)}(\xi_0) \le \frac{1}{300} f^{(3)}(-0.2) = \frac{8}{300}\left(e^{-0.4} - \sin(-0.4)\right) \text{ or}$$

$$\boxed{\text{Error Bound for } f'(0) \text{ is } \frac{2}{75}\left(e^{-0.4} + \sin(0.4)\right) \approx 0.02826}$$

Now, using the approximations from part a), and using for $f'(-0.3), f'(-0.2), f'(-0.1), f'(0)$ the real-valued $f'(x) = 2e^{2x} + 2\sin 2x$ function for $x = -0.3; -0.2; -0.1; 0 \Rightarrow$ we will get that

Actual error for $f'(-0.3)$: $\left|-0.0603 - 2\left(e^{-0.6} + \sin(-0.6)\right)\right|$

$= \left|-0.0603 - 2e^{-0.6} + 2\sin(0.6)\right| \approx \boxed{0.028638832539}$

Actual error for $f'(-0.2)$: $\left|0.5759 - 2\left(e^{-0.4} + \sin(-0.4)\right)\right|$

$= \left|0.5759 - 2e^{-0.4} + 2\sin(0.4)\right| \approx \boxed{0.014097}$

Actual error for $f'(-0.1)$: $\left|1.2537 - 2\left(e^{-0.2} + \sin(-0.2)\right)\right|$

$= \left|1.2537 - 2e^{-0.2} + 2\sin(0.2)\right| \approx \boxed{0.013577155439}$

Actual error for $f'(0)$: $\left|1.9731 - 2\left(e^{0} + \cancel{\sin 0}\right)\right| =$

$= \left|1.9731 - 2\right| = \boxed{0.0269}$ In conclusion, combining gives

X	Actual Error	Error Bound
-0.3	0.028638832539	0.02969211
-0.2	0.014097	0.014846054718
-0.1	0.013577155439	0.01413
0	0.0269	0.02826

2) $N(h)$ is an approximation to M for $\forall h>0$ and $M=N(h)+K_1h^2+K_2h^4+K_3h^6+\dots$ for some constants $K_1, K_2, K_3, \dots \Rightarrow$ Since this's true for $\forall h>0$, substitution $\left(h\to\frac{h}{3} \text{ and } h\to\frac{h}{9}\right)$ gives

$$M=N(h)+K_1h^2+K_2h^4+K_3h^6+\dots \qquad (1)$$

$$M=N\left(\frac{h}{3}\right)+K_1\frac{h^2}{9}+K_2\frac{h^4}{81}+K_3\frac{h^6}{729}+\dots \qquad (2)$$

$$M=N\left(\frac{h}{9}\right)+K_1\frac{h^2}{81}+K_2\frac{h^4}{6561}+K_3\frac{h^6}{531441}+\dots \qquad (3)$$

If we consider $(1)-9\times(2)$ and $(2)-9\times(3)$, one gets

$$-8M=N(h)-9N\left(\frac{h}{3}\right)+K_2h^4-\frac{K_2h^4}{9}+K_3h^6-\frac{K_3h^6}{81}+\dots$$

$$-8M=N\left(\frac{h}{3}\right)-9N\left(\frac{h}{9}\right)+\frac{K_2h^4}{81}-\frac{K_2h^4}{729}+\frac{K_3h^6}{729}-\frac{K_3h^6}{59049}+\dots$$

so, we find

$$-8M=N(h)-9N\left(\frac{h}{3}\right)+\frac{8}{9}K_2h^4+\frac{80}{81}K_3h^6+\dots \longrightarrow (4)$$

$$-8M=N\left(\frac{h}{3}\right)-9N\left(\frac{h}{9}\right)+\frac{8}{729}K_2h^4+\frac{80}{59049}K_3h^6+\dots \longrightarrow (5)$$

Eventually, looking up to $(4)-81\times(5)$ will reveal that

$$(-80)\times(-8M)=N(h)-9N\left(\frac{h}{3}\right)-81N\left(\frac{h}{3}\right)+729N\left(\frac{h}{9}\right)+$$

$$+\frac{8}{9}K_2h^4-\frac{8}{9}K_2h^4+\frac{80}{81}K_3h^6-\frac{80}{729}K_3h^6+\dots$$ and this's

$$640M=N(h)-90N\left(\frac{h}{3}\right)+729N\left(\frac{h}{9}\right)+\frac{640}{729}K_3h^6+\dots$$ which is

$$M=\frac{1}{640}\left(N(h)-90N\left(\frac{h}{3}\right)+729N\left(\frac{h}{9}\right)\right)+\frac{1}{729}K_3h^6+\dots$$ or

$$\boxed{M = \frac{1}{640}\left(N(h) - 90N\left(\frac{h}{3}\right) + 729N\left(\frac{h}{9}\right)\right) + \frac{K_3}{729}h^6 + \ldots}$$

This concludes that

$$\boxed{M = \frac{1}{640}\left(N(h) - 90N\left(\frac{h}{3}\right) + 729N\left(\frac{h}{9}\right)\right) + O(h^6)}$$

Thus, the $O(h^6)$ approximation is given by

$$M = \frac{1}{640}\left(N(h) - 90N\left(\frac{h}{3}\right) + 729N\left(\frac{h}{9}\right)\right) \quad \boxed{V}$$

4) a) We will apply the degree of precision of the given quadrature formula to derive the desired equations based on unknown variables c_0, c_1, and c_2 and solve them. According to statement, the quadrature formula $\int_0^2 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for polynomials of degrees $\rightarrow$ 0, 1, and 2. Let's derive the first relation from "degree 0" polynomials:

$\rightarrow$ Let $f(x) = 1$ be a polynomial of degree zero

$$\int_0^2 f(x)dx = \int_0^2 1\,dx = x\Big]_0^2 = 2 \text{ and } c_0 f(0) + c_1 f(1) + c_2 f(2)$$

$$= c_0 + c_1 + c_2 \Rightarrow \boxed{c_0 + c_1 + c_2 = 2}$$

$\rightarrow$ Let $f(x) = x$ be a polynomial of degree one

$$\int_0^2 f(x)dx = \int_0^2 x\,dx = \frac{x^2}{2}\Big]_0^2 = \frac{2^2}{2} = 2 \text{ and } c_0 f(0) + c_1 f(1) + c_2 f(2) =$$

$$= c_1 + 2c_2 \Rightarrow \boxed{c_1 + 2c_2 = 2}$$

$\rightarrow$ Let $f(x) = x^2$ be a polynomial of degree two

$$\int_0^2 x^2 dx = \frac{x^3}{3}\Big]_0^2 = \frac{8}{3} \Rightarrow \int_0^2 f(x)dx = \frac{8}{3} \text{ and } c_0 f(0) + c_1 f(1) + c_2 f(2) =$$

$$= c_1 + 4c_2 \Rightarrow \boxed{c_1 + 4c_2 = \frac{8}{3}}$$

Combining all these resultant formulas will yield that

$$\left.\begin{aligned} c_0+c_1+c_2&=2\\ c_1+2c_2&=2\\ c_1+4c_2&=\frac{8}{3}\end{aligned}\right\}\quad 2c_2=\frac{8}{3}-2=\frac{2}{3}\Rightarrow \boxed{c_2=\frac{1}{3}}\ \boxed{\star}$$

$$c_1+2c_2=c_1+\frac{2}{3}=2\Rightarrow \boxed{c_1=\frac{4}{3}}\ \boxed{\star}$$

$$c_0+c_1+c_2=c_0+\frac{5}{3}=2\Rightarrow \boxed{c_0=\frac{1}{3}}\ \boxed{\star}$$ Hence, $\boxed{\star}\ \boxed{c_0=\frac{1}{3},\ c_1=\frac{4}{3},\ c_2=\frac{1}{3}}$

and quadrature formula $\rightarrow$ $\boxed{\int_0^2 f(x)\,dx=\frac{1}{3}f(0)+\frac{4}{3}f(1)+\frac{1}{3}f(2)}$ $\boxed{\text{V}}$ $\boxed{\star}$

b) The provided three unknown variables c_0, c_1, and x_1 do indeed guarantee the rule to be exact for at least polynomials $1, x, x^2$ (The degree of precision of quadrature formula is $n \Leftrightarrow$ the error $=0$ for $\forall$ polynomials of deg $=0,1,\dots,n$, but $\neq$ zero for some polynomial of deg $=n+1$)

$\rightarrow$ Let $P_0(x)=1$ be a polynomial of degree zero.

set $f(x)=1 \Rightarrow \int_0^1 f(x)\,dx=\int_0^1 1\,dx=x\Big]_0^1=1$ and

$c_0f(0)+c_1f(x_1)=c_0+c_1\Rightarrow \boxed{c_0+c_1=1}$

$\rightarrow$ Let $P_1(x)=x$ be a polynomial of degree one

plugging $f(x)=x$ yields $\int_0^1 f(x)\,dx=\int_0^1 x\,dx=\frac{x^2}{2}\Big]_0^1=$

$=\frac{1}{2}$ and $c_0f(0)+c_1f(x_1)=c_1x_1\Rightarrow \boxed{c_1x_1=\frac{1}{2}}$

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→ Let $P_2(x) = x^2$ with a polynomial of degree two

Plugging $f(x) = x^2$ reveals $\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 =$

$= \frac{1}{3}$ and $c_0 f(0) + c_1 f(x_1) = c_1 x_1^2 \Rightarrow \boxed{c_1 x_1^2 = \frac{1}{3}}$

Combining the equations $\begin{cases} c_0 + c_1 = 1 \\ c_1 x_1 = \frac{1}{2} \\ c_1 x_1^2 = \frac{1}{3} \end{cases}$ gives $c_1 x_1 \cdot x_1 =$

$= \frac{1}{2} x_1 = \frac{1}{3} \Rightarrow \boxed{x_1 = \frac{2}{3}}$

$c_1 x_1 = c_1 \cdot \frac{2}{3} = \frac{1}{2} \Rightarrow \boxed{c_1 = \frac{3}{4}}$

$c_0 + c_1 = c_0 + \frac{3}{4} = 1 \Rightarrow \boxed{c_0 = \frac{1}{4}}$ Hence, $\boxed{c_0 = \frac{1}{4},\ c_1 = \frac{3}{4},\ x_1 = \frac{2}{3}}$

and $\boxed{\text{quadrature formula} \longrightarrow \int_0^1 f(x)\,dx = \frac{1}{4} f(0) + \frac{3}{4} f\left(\frac{2}{3}\right)}$

Note: Just to ensure that the given quadrature formula does indeed have degree of precision 2 (validation only) we can just test the resultant formula for polynomial of degree three (since we proved rule's exact for $\forall P$ with $\deg P \leq 2$) by setting $f(x) = x^3$: $\int_0^1 f(x)\,dx =$

$= \int_0^1 x^3\,dx = \frac{x^4}{4}\Big]_0^1 = \frac{1}{4}$, but quadrature formula reveals

$\frac{1}{4} f(0) + \frac{3}{4} f\left(\frac{2}{3}\right) = \frac{3}{4} \cdot \frac{8}{27} = \frac{2}{9} \neq \frac{1}{4}$ ⇝ Therefore, as the problem initially suggested, the highest degree of precision $= 2$ should hold true

(The last note was just to verify the correctness of given precision)

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1) b) As there exist 2 unknown variables, we should have 2 equations, and indeed → those equations will be derived from the clamped conditions of spline: $\boxed{\begin{matrix} S_0(1)=S_1(1) \\ S_0'(1)=S_1'(1) \end{matrix}}$

$S_0(1)=S_1(1) \Rightarrow$

$$\left.\begin{matrix} \left[-2x^3+2x^2+\beta x+1\right]_{x=1} = -2+2+\beta+1 = \beta+1 \\ \left[7(x-1)^3-4(x-1)^2+b(x-1)+1\right]_{x=1} = 1 \end{matrix}\right\} \boxed{\beta=0} \star$$

$$S_0'(1)=S_1'(1) \Rightarrow \left.\begin{matrix} \left[-6x^2+4x+\beta\right]_{x=1} = -6+4+\beta = -2 \\ \left[21(x-1)^2-8(x-1)+b\right]_{x=1} = b \end{matrix}\right\} \boxed{b=-2} \star$$

Now, the desired derivative is $f'(0)=S_0'(0)$ and we get $f'(0)=\left[-6x^2+4x+\beta\right]_{x=0} = \beta = 0 \Rightarrow \boxed{f'(0)=0} \star$ Similarly, we know $f'(2)=S_1'(2)$ or just $f'(2)=\left[21(x-1)^2-8(x-1)+b\right]_{x=2}$ $=21-8+b = b+13 = -2+13 = 11 \Rightarrow \boxed{f'(2)=11} \star$ In the end, we conclude that $\boxed{f'(0)=0,\ f'(2)=11} \star$ $\boxed{\checkmark}$ ∎

<u>Note:</u> $S_0'(x) = -6x^2+4x+\beta$, $S_1'(x) = 21(x-1)^2-8(x-1)+b$

a) Our claim is that $|f(x)-F(x)| \leq \frac{M}{8} \max_{0\leq j\leq n-1} (x_{j+1}-x_j)^2$

where $M = \max_{a\leq x\leq b} |f''(x)|$

From Theorem 3.3, it's obvious that

$f(x) = P_i(x) + \frac{(x-x_{i-1})(x-x_i)}{2} f''(c_x)$ for some c_x between the minimum and maximum of x_0, x_1, and x, and where $P_i(x) \to$ linear polynomial interpolating $f(x)$ at x_{i-1} and x_i (and where $f \in C^2[a,b]$ with nodes $a = x_0 < x_1 < \dots < x_n = b$)

So, $P_i(x) \to$ is usually considered as an approximation for $f(x)$ in $x \in [x_{i-1}, x_i] \Rightarrow |f(x) - P_i(x)| = \left|\frac{(x-x_{i-1})(x-x_i)}{2}\right| |f''(c_x)|$

where this's an error bound $\Rightarrow$

$|f(x)-P_i(x)| = \frac{(x-x_{i-1})(x_i-x)}{2} \cdot |f''(c_x)| \leq \frac{(x-x_{i-1})(x_i-x)}{2} \max_{x_{i-1}\leq x\leq x_i} |f''(x)|$

So, $|f(x)-P_i(x)| \leq \frac{(x-x_{i-1})(x_i-x)}{2} \max_{x_{i-1}\leq x\leq x_i} |f''(x)|$, where $x \in [x_{i-1}, x_i]$

$ab \leq \frac{(a+b)^2}{4} \Leftrightarrow 4ab \leq (a+b)^2 \Leftrightarrow 4ab \leq a^2+b^2+2ab \Leftrightarrow 2ab \leq a^2+b^2 \Leftrightarrow$

$\Leftrightarrow (a-b)^2 \geq 0$ which is true ✓ $\Rightarrow \boxed{ab \leq \frac{(a+b)^2}{4}}$ and taking $a = x - x_{i-1}$, $b = x_i - x$

$a+b = x - x_{i-1} + x_i - x = x_i - x_{i-1}$

we find $|f(x)-P_i(x)| \leq \frac{(x-x_{i-1})(x_i-x)}{2} \max |f''(x)| \leq$

$\leq \frac{(x_i-x_{i-1})^2}{8} \max_{x_{i-1}\leq x\leq x_i} |f''(x)| \Rightarrow \boxed{|f(x)-P_i(x)| \leq \frac{(x_i-x_{i-1})^2}{8} \max_{x_{i-1}\leq x\leq x_i} |f''(x)| \text{ where } x \in [x_{i-1}, x_i]}$

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$$|f(x)-P_i(x)| \le \frac{\max\limits_{x_{i-1}\le x\le x_i}|f''(x)|\cdot(x_i-x_{i-1})^2}{8} \le \frac{M}{8}(x_i-x_{i-1})^2$$

in which $M=\max\limits_{a\le x\le b}|f''(x)| \Rightarrow |f(x)-P_i(x)| \le \frac{M(x_i-x_{i-1})^2}{8}$

and similarly, since $(x_i-x_{i-1})^2 \le \max\limits_{0\le j\le n-1}(x_{j+1}-x_j)^2$, we get

$$\boxed{|f(x)-F(x)| \le \frac{M}{8}\cdot\max_{0\le j\le n-1}(x_{j+1}-x_j)^2 \ \forall x\in[a,b] \text{ where } M=\max_{a\le x\le b}|f''(x)| \text{ and } f\in C^2[a,b] \text{ for piecewise linear interpolating function } F}$$

$\boxed{V} \ \blacksquare \ \oplus$