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1) a) $\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$ that has degree of precision three

What we want is to have the formula $\int_{-1}^{1} f(x)dx =$ $= af(-1) + bf(1) + cf'(-1) + df'(1)$ to be held for the polynomials $1, x, x^2, x^3 \Rightarrow$ Setting back these expressions to the given relation, we obtain (must be exact for $f(x) = 1, x, x^2, x^3$)

- $f(x) = x^0 = 1 \Rightarrow \int_{-1}^{1} 1\,dx = x\Big]_{-1}^{1} = \boxed{2 = a + b}$ since $f'(x) = 0$
- $f(x) = x^1 \Rightarrow f'(x) = 1$ and $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x\,dx =$ $= \frac{x^2}{2}\Big]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = \boxed{0 = -a + b + c + d}$
- $f(x) = x^2 \Rightarrow f'(x) = 2x$ and $\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$ $= a + b - 2c + 2d \Rightarrow \boxed{a + b - 2c + 2d = \frac{2}{3}}$
- $f(x) = x^3 \Rightarrow f'(x) = 3x^2$; $\int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0 =$ $= -a + b + 3c + 3d \Rightarrow \boxed{-a + b + 3c + 3d = 0}$ In the end, we obtained 4 equations with the 4 variables a, b, c, d $\Rightarrow$ Combining all these results and solving system'll give

$$\left.\begin{aligned} &a+b=2 \\ &-a+b+c+d=0 \\ &a+b-2c+2d=\frac{2}{3} \\ &-a+b+3c+3d=0 \end{aligned}\right\}$$ $c+d=a-b=3c+3d \Rightarrow \boxed{c+d=0}$ Let $d=-c \Rightarrow -a+b=0$, $b=a \Rightarrow a+b=2a=2$, $\boxed{a=b=1}$ and $d=-c \Rightarrow 2-2c+2d=2-2c-2c=2-4c=\frac{2}{3}$

$\Rightarrow 4c=\frac{4}{3}$ or $\boxed{c=\frac{1}{3}, d=-\frac{1}{3}}$ Then, $\boxed{a=1, b=1, c=\frac{1}{3}, d=-\frac{1}{3}}$

is the resultant conclusion after solving the system

In conclusion, quadrature formula having accuracy $n=3$ is found as

$$\boxed{\int_{-1}^{1} f(x)\,dx = f(-1)+f(1)+\frac{1}{3}f'(-1)-\frac{1}{3}f'(1)}$$

+ $\boxed{V}$

b) Assume $Q(P)=\sum_{i=1}^{n} c_i P(x_i)$ and suppose for the sake ($Q(P)$ has degree of precision $2n$) of contradiction that there exist an integer $M \geq 2n$ and the formula-relationship $Q(P)=\sum_{i=1}^{n} c_i P(x_i)$ s.t. for some $a \neq b$ and for $\forall$ polynomial of deg $\leq M$, we had $\int_a^b P(x)\,dx = Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ (exact for any polynomial with degree $2n$)

(Note that those values of c_i, x_i are fixed as part of the formula Q, along with as "a" and "b" are)

After having this, let's take into account the polynomial

$R(x)=(x-x_1)^2(x-x_2)^2 \dots (x-x_n)^2=\prod_{i=1}^{n}(x-x_i)^2$ where $\deg R=2n$

Note that $R(x_i)=0$ for every $i=1,\dots,n \Rightarrow$ we must obtain that $Q(R)=\sum_{i=1}^{n} c_i R(x_i)=0$ (from the assumptions of Q)

(strictly non-negative)
However, $R(x)\geq 0$ for the whole real-number line, where $R(x)$ is not a zero polynomial $\Rightarrow$ Therefore, when one integrates $R(x)$ over any closed interval, we would obtain sth. strictly positive $\Rightarrow \int_a^b R(x)\,dx>0$
(Since degree of precision was $2n$)

Meaning that $Q(R)\neq \int_a^b R(x)\,dx$, which contradicts the assumption about $Q \Rightarrow$ We conclude that **such Q can't exist**
(Degree of precision $\leq 2n-1$)

2) Let's denote $D=\{(t,y): 0\leq t\leq 1, -\infty<y<\infty\}$

and we had $f(t,y)=t^2y+1 \Rightarrow$ It is clear that the function f satisfies Lipschitz condition, because for $\forall y_1,y_2\in \mathbb{R} \Rightarrow |f(t,y_1)-f(t,y_2)|=|t^2y_1+1-t^2y_2-1|=$
$=t^2|y_1-y_2|\leq |y_2-y_1|$

Considering that f is a continuous function on D, we would conclude that **The problem is well-posed**

(f satisfies Lipschitz condition on D in the variable y with Lipschitz constant 1, therefore we applied Theorem 5.6 to derive that it's well-posed)
$\exists$ unique solution $y(t)$ for $0\leq t\leq 1$

3) a) $y' = \frac{1+t}{1+y}$, $1 \le t \le 2$, $y(1) = 2$, Euler's Method with $h=0.5$

As we know $w_0 = y(t_0) = y(a) = y(1) = 2$ holds true,

using the step size $\Rightarrow h = \frac{b-a}{N} = \frac{t_N - t_0}{N} = \frac{2-1}{N} = \frac{1}{N}$

So, $N = \frac{1}{h} = \frac{1}{0.5} = 2$ where $t_i = t_0 + ih = 1 + 0.5i$

becomes satisfied $\Rightarrow \boxed{t_i = 1 + 0.5i}$ $f(t,y) = \frac{1+t}{1+y}$ with

$f(t_i, y_i) = \frac{1+t_i}{1+y_i}$ and $f(t_i, w_i) = \frac{1+t_i}{1+w_i}$

If we apply the first result, one obtains the result

$\boxed{f(t_i, w_i) = \frac{2+0.5i}{1+w_i}}$ Subsequently, we will apply Euler's method with the difference equation $w_{i+1} = w_i + h f(t_i, w_i)$ where $i = 0, 1, \dots, N-1$

$w_{i+1} = w_i + \frac{0.5(2+0.5i)}{1+w_i}$ for $i = 0, 1, \dots, N-1 \Rightarrow$ since $N = 2$,

$\boxed{w_{i+1} = w_i + \frac{1+0.25i}{1+w_i} \text{ for } i = 0, 1}$ Since $w_0 = 2$, $i = 0 \Rightarrow w_1 = w_0 + \frac{1}{1+w_0} = 2 + \frac{1}{3} = \frac{7}{3} = 2.333\ldots3\ldots \Rightarrow$

$\boxed{w_1 = \frac{7}{3}}$ $i = 1 \Rightarrow w_2 = w_1 + \frac{1+0.25}{1+w_1} = \frac{7}{3} + \frac{1.25}{\frac{10}{3}} = \frac{7}{3} + \frac{3.75}{10} = \frac{7}{3} + \frac{375}{1000} = \frac{7}{3} + \frac{3}{8} = \frac{56}{24} + \frac{9}{24} = \frac{65}{24}$, $\boxed{w_2 = \frac{65}{24}}$

$$\boxed{w_0=2,\ w_1=\frac{7}{3}=2.333\ldots3\ldots \text{ and } w_2=\frac{65}{24}=2.70833\ldots3\ldots}$$ Thus,

$i=0 \Rightarrow t_0=1$ and $w_0=y(t_0)=y(1)=2$ ✓

$i=1 \Rightarrow t_1=1+0.5=1.5$ and $w_1\approx y(t_1)\approx y(1.5)\approx \frac{7}{3}=2.33\ldots3\ldots$ ✓

$i=2 \Rightarrow t_2=1+0.5\times 2=2,\ w_2\approx y(t_2)\approx y(2)\approx \frac{65}{24}=2.70833\ldots3\ldots$ ✓

where we applied the formula $t_i=1+0.5i$ for $i=0,1,\ldots,N=$ $=2$ which we found at the begining of previous page ☑ ⊕

b) <u>Theorem 5.9:</u> f-continuous and satisfies Lipschitz condition with constant L on $D=\left\{(t,y)\,\middle|\, a\le t\le b, \text{ and } -\infty<y<\infty\right\}$

and that $\exists M$ with $|y''(t)|\le M$ for $\forall t\in[a,b]$ where $y(t)$ denotes unique solution to IVP $y'=f(t,y)$, $a\le t\le b$, $y(a)=\alpha$. Assume $w_0, w_1,\ldots,w_N$ are the approximations generated by Euler's Method for some $N\ge 1$ $\Rightarrow \forall i=0,1,2,\ldots,N,\ |y(t_i)-w_i|\le \frac{hM}{2L}\left[e^{L(t_i-a)}-1\right]$

which reveals the error estimation of the above IVP from the Euler's Method.

$f(t,y)=\frac{1+t}{1+y},\ \frac{\partial f}{\partial y}=\frac{-(t+1)}{(y+1)^2}=\frac{-(t+1)}{t^2+2t+6}$ as $y(t)=\sqrt{t^2+2t+6}-1$

$\left|\frac{\partial f}{\partial y}\right|=\left|\frac{t+1}{(t+1)^2+5}\right|=\frac{t+1}{(t+1)^2+5}\le\frac{t+1}{(1+1)^2+5}\le\frac{2+1}{9}=\frac{1}{3}$, or just alternatively speaking

$b\ge$

$\frac{t+1}{(t+1)^2+5} = \frac{a}{a^2+5}$ is a decreasing function for $1+1 \le 1+t \le 1+2=3$

denote as $f'(a)$-function

$2 \le 1+t = a \le 3 \Rightarrow f'(a) = \frac{a}{a^2+5}$ is a decreasing function

$\frac{d f'(a)}{da} = \frac{a^2+5-a\cdot 2a}{(a^2+5)^2} = \frac{5-a^2}{(a^2+5)^2} < 0$, thus $f'(a)$ will be a decreasing function whenever $a^2>5$

$a>\sqrt{5} \Rightarrow f'(a) = \frac{a}{a^2+5}$ is a decreasing function (strictly)

$a<\sqrt{5} \Rightarrow f'(a) = \frac{a}{a^2+5}$ is an increasing function (strictly)

$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{t+1}{(t+1)^2+5}\right| = \frac{t+1}{(t+1)^2+5} = \frac{a}{a^2+5}$ where $1+1=2 \le 1+t=a \le 1+2=3$

For $[2,\sqrt{5})$ interval, $\frac{a}{a^2+5}$ is an increasing function

For $(\sqrt{5},3]$ interval, $\frac{a}{a^2+5}$ is a decreasing function

Therefore, $\frac{a}{a^2+5}$ will get its maximum when $\frac{5-a^2}{(a^2+5)^2}=0$ or just $\boxed{a=\sqrt{5}}$ (Note for $a=2$, $\frac{2}{2^2+5} = \frac{2}{9} = 0.222...2... <$

$< \frac{\sqrt{5}}{5+5} = \frac{\sqrt{5}}{10} \approx 0.2236067...$ and for $a=3$, $\frac{3}{3^2+5} = \frac{3}{14} \approx$

$\approx 0.214285714... < 0.2236067... = \frac{\sqrt{5}}{10}$) In conclusion, we finalize that

$\frac{a}{a^2+5} \le \frac{\sqrt{5}}{10}$ for $2 \le a \le 3 \Rightarrow \left|\frac{\partial f}{\partial y}\right| = \left|\frac{t+1}{(t+1)^2+5}\right| \le \frac{\sqrt{5}}{10}$ whenever

$1+1 \le 1+t \le 1+2$ or $2 \le t+1 \le 3 \Rightarrow$ $\boxed{\left|\frac{\partial f}{\partial y}\right| \le \frac{\sqrt{5}}{10} \approx 0.2236068 \text{ for } 1 \le t \le 2}$

Thus, $\boxed{f(t,y) \text{ satisfies Lipschitz condition for } L = \frac{\sqrt{5}}{10}}$ ⌧

Moreover, $y''(t) = ?$ has to be evaluated in order to proceed the theorem: $y(t) = \sqrt{t^2+2t+6} - 1$, $\frac{d\,y(t)}{dt} = y'(t) =$

$$= \frac{d}{dt}\left[(t^2+2t+6)^{\frac{1}{2}}\right] = \frac{1}{2}(t^2+2t+6)^{-\frac{1}{2}} \cdot (2t+2) = \frac{t+1}{\sqrt{t^2+2t+6}}$$

and $y''(t) = \frac{d}{dt}\,y'(t) = \dfrac{\sqrt{t^2+2t+6} - (t+1)\cdot\frac{1}{2}(t^2+2t+6)^{-\frac{1}{2}}(2t+2)}{t^2+2t+6}$

$$= \frac{\sqrt{t^2+2t+6} - \frac{(t+1)^2}{\sqrt{t^2+2t+6}}}{t^2+2t+6} = \frac{t^2+2t+6-t^2-2t-1}{(t^2+2t+6)^{3/2}} = \frac{5}{(t^2+2t+6)^{3/2}}.$$

$$\boxed{y''(t) = \frac{5}{((t+1)^2+5)^{3/2}} \quad \text{where } 1 \le t \le 2}$$

Let $g(t) = y''(t)$, then $g'(t) = 5\cdot\frac{(-3)}{2}\left((t+1)^2+5\right)^{-\frac{5}{2}} \cdot 2(t+1)$ where we know that

$(t+1)^2 \geq (1+1)^2 = 4$, $(t+1-2)(t+1+2) = (t-1)(t+3) \geq 0$, then:

$$(t+1)^2+5 \geq 4+5 = 9 = 3^2 \Rightarrow y''(t) = \frac{5}{[(t+1)^2+5]^{3/2}} \leq \frac{5}{[3^2]^{3/2}}$$

$= \frac{5}{3^3} = \frac{5}{27}$. Since there is also an equality case when $t = 1 \Rightarrow \boxed{|y''(t)| \leq \frac{5}{27} \text{ for } 1 \leq t \leq 2}$ ⌧.

Hence, $\boxed{M = \frac{5}{27}}$ ⌧ in the theorem 5.9.

After applying the error estimation from the Euler Method mentioned in previous page,

$$|y(t_i)-w_i| \le \frac{0.5 \times \frac{5}{27}}{2 \times \frac{\sqrt{5}}{10}} \left[e^{\frac{\sqrt{5}}{10}(t_i-1)} - 1 \right]$$ where $h=0.5$, $M=\frac{5}{27}$, $L=\frac{\sqrt{5}}{10}$, $a=1$

for $i=0,1,2,\dots,N$ or just $i=0,1,2$ since $N=2$

Using the previous part results for the w_i, we would get

→ for $i=0$, $t_0=a=1$ and $w_0=2$; $y(t_0)=2$ along with

$|y(t_0)-w_0| = \left|\sqrt{t_0^2+2t_0+6}-1-2\right| = \left|\sqrt{9}-1-2\right| = 0$ ⟶

actual error is zero and upper bound is $\frac{\frac{0.25}{27}}{\frac{\sqrt{5}}{5}}\left[e^0-1\right] =$

$= 0 \Rightarrow \boxed{\text{error bound} = 0}$ * ✓

→ for $i=1$, $t_1=1.5$ and $w_1=\frac{7}{3}=2.333\dots3$; computing

$y(t_1)=\sqrt{1.5^2+3+6}-1=\sqrt{2.25+9}-1=\sqrt{11.25}-1$, therefore

actual error is $|y(t_1)-w_1| = \left|\sqrt{11.25}-1-\frac{7}{3}\right| = \left|\sqrt{11.25}-\frac{10}{3}\right|$

$\approx 0.0207686329\underline{1}$ and error bound is $|y(t_1)-w_1| \le$

$$\le \frac{\frac{2.5}{27}}{\frac{\sqrt{5}}{5}}\left[e^{\frac{\sqrt{5}}{10}(1.5-1)}-1\right] = \frac{12.5}{27\sqrt{5}}\left[e^{\frac{\sqrt{5}}{10}\times\frac{1}{2}}-1\right] = \frac{1250}{2700\sqrt{5}}\left[\dots\right]$$

$$= \frac{1250\sqrt{5}}{2700\times 5}\left[e^{\frac{\sqrt{5}}{20}}-1\right] = \frac{\sqrt{5}}{10.8}\left[e^{\frac{\sqrt{5}}{20}}-1\right],$$

error bound $= \frac{5\sqrt{5}}{54}\left[e^{\frac{\sqrt{5}}{20}}-1\right] \approx$ * $\approx 0.02449177\underline{3}$

where the actual error is $\approx$

$\approx 0.0207686329\underline{1}$ ✓

$\rightarrow$ for $i=2$, $t_2=2$ and $w_2=\frac{65}{24}=2.70833...3...$; computing

$y(t_2)=\sqrt{2^2+2\cdot 2+6}-1=\sqrt{4+4+6}-1=\sqrt{14}-1$, therefore

actual error is $|y(t_2)-w_2|=\left|\sqrt{14}-1-\frac{65}{24}\right|=\left|\sqrt{14}-\frac{89}{24}\right|$

$\approx 0.0333240534 4 \Rightarrow$ actual error $\approx 0.0333240534 4$

and error bound is $|y(t_2)-w_2|\leq\frac{5\sqrt{5}}{54}\left[e^{\frac{\sqrt{5}}{10}(2-1)}-1\right]=$

$=\frac{5\sqrt{5}}{54}\left[e^{\frac{\sqrt{5}}{10}}-1\right]\approx 0.0518807506 9$ where $\dfrac{\frac{0.5\times 5}{27}}{\frac{\sqrt{5}}{5}}=$

$=\frac{12.5}{27\sqrt{5}}=\frac{12.5\sqrt{5}}{27\times 5}=\frac{125\sqrt{5}}{27\times 50}=\frac{5\sqrt{5}}{27\times 2}=\frac{5\sqrt{5}}{54}$; therefore,

$$\boxed{\text{error bound}=\frac{5\sqrt{5}}{54}\left[e^{\frac{\sqrt{5}}{10}}-1\right]\approx 0.0518807506 9}$$ ✓

In conclusion, we computed all the desired error bounds for the correspondent approximations $\boxed{\checkmark}$ ■ (+)

4) a) Assume the given IVP is $y'(t)=f(t,y)$ with $a\leq t\leq b$, $y(a)=\alpha \Rightarrow$ If $f(t,y(t))$ had been interpolated using $\deg=(m-1)$ polynomial, along with the known points $t_{i+1-m}, t_{i+2-m}, \ldots, t_{i-1}, t_i \Rightarrow$ we will then find

$$f(t,y(t))=P_{m-1}(t)+\frac{f^{(m)}(\xi_i, y(\xi_i))}{m!}(t-t_{i+1-m})\ldots(t-t_{i-1})\cdot(t-t_i)$$

for some $\xi_i \in (t_{i+1-m}, t_i)$. Regarding the Adam-Bashford procedure with order two, one could take points t_i, t_{i-1}. Applying Lagrange interpolation for linear type, one can easily observe that $f(t, y(t)) \approx \frac{t-t_{i-1}}{t_i - t_{i-1}} f(t_i, y(t_i))$

$+ \frac{t-t_i}{t_{i-1}-t_i} f(t_{i-1}, y(t_{i-1}))$. However, we also know from the truncation error properties

that $\int_{t_i}^{t_{i+1}} (t-t_i)(t-t_{i-1}) \frac{f''(\xi_i, y(\xi_i))}{2!} dt =$

$= \int_{t_i}^{t_{i+1}} (t-t_{i-1})(t-t_i) dt \cdot \frac{f''(\xi_i, y(\xi_i))}{2} =$

$= \int_{t_i}^{t_{i+1}} (t-t_{i-1})(t-t_i) dt \cdot \frac{f''(\xi_i, y(\xi_i))}{2} = \frac{5h^3}{12} f''(\xi_i, y(\xi_i))$

where $\int_{t_i}^{t_{i+1}} (t-t_{i-1})(t-t_i) dt = \frac{5h^3}{6}$ and ξ_i lies between t_{i-1} and t_i. About the approximation, one can obtain $y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t)) dt \approx$

$$\approx y(t_i) + \frac{1}{h}\left[\int_{t_i}^{t_{i+1}} (t-t_{i-1}) dt \cdot f(t_i, y(t_i)) - \int_{t_i}^{t_{i+1}} (t-t_i) dt \cdot f(t_{i-1}, y(t_{i-1}))\right]$$

where indeed, the RHS becomes equal to

$y(t_i) + \frac{1}{h}\left[\frac{3h^2}{2} \times f(t_i, y(t_i)) - \frac{h^2}{2} \times f(t_{i-1}, y(t_{i-1}))\right]$ with

$\int_{t_i}^{t_{i+1}} (t - t_{i-1})\,dt = \frac{3h^2}{2}$ and $\int_{t_i}^{t_{i+1}} (t - t_i)\,dt = \frac{h^2}{2}$ $\Rightarrow$

Then, RHS becomes $y(t_i) + \frac{h}{2}\left[3f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))\right]$

$y(t_{i+1}) \approx y(t_i) + \frac{h}{2}\left[3f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))\right]$ and

plugging in $y(t_i) = w_i$, one can conclude that

$$\boxed{w_{i+1} = w_i + \frac{h}{2}\left[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})\right]}$$

which in fact reveals the desired

Adam-Bashford procedure with order two ✓ (+)

b) Suppose given IVP is as follows: $y'(t) = f(t, y(t))$ where $a \leq t \leq b$, $y(a) = \alpha$

Applying the approximation $y(t_{i+1}) - y(t_{i-3}) = \int_{t_{i-3}}^{t_{i+1}} f(t, y(t))\,dt$ along with <u>Newton-Cotes formula</u> (for $n=2$), one easily observes that $\int_{t_{i-3}}^{t_{i+1}} f(t, y(t))\,dt = \frac{4h}{3}\left[2f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1})) + 2f(t_{i-2}, y(t_{i-2}))\right] + \frac{14h^5}{45} f^{(4)}(\xi)$ where ξ lies between t_{i-3} and t_{i+1}

Using the previous result, one can deduce the following approximation:

$$y(t_{i+1}) \approx y(t_{i-3}) + \frac{4h}{3}\left[2f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1})) + 2f(t_{i-2}, y(t_{i-2}))\right]$$

and using the fact that $y(t_j) = w_j$, setting this will yield

$$\boxed{w_{i+1} = w_{i-3} + \frac{4h}{3}\left[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})\right]}$$

where the local truncation error would be reported as

$\frac{14h^5 f^{(4)}(\xi)}{45}$ or just ⇒ Local Truncation Error is reported as $\frac{14h^5 y^{(5)}(\xi)}{45}$ since $y'(t) = f(t, y(t))$

or $f^{(4)}(\xi)$ will be same with $y^{(5)}(\xi)$ $\boxed{V}$ ■ +