2021 Spring MAS 365: Homework 1

posted on Mar. 11; due by Mar. 18

1. [5 points each] The Taylor polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{n} \frac{x^i}{i!}$. Use the Taylor polynomial of degree five and three-digit chopping arithmetic to find an approximation to e^{-2} by each of the following methods.

   (1) $e^{-2} \approx \sum_{i=0}^{5} \frac{(-2)^i}{i!}$
   (2) $e^{-2} = \frac{1}{e^2} \approx \frac{1}{\sum_{i=0}^{5} \frac{2^i}{i!}}$

   Solution:

   (1) $e^{-2} \approx 1 + \frac{-2}{1} + \frac{4}{2} + \frac{-8}{6} + \frac{16}{24} + \frac{-32}{120} = 1 - 2 + 2 - 1.33 + 0.666 - 0.266 = 0.07$
   (2) $e^{2} \approx 1 + \frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} = 1 + 2 + 2 + 1.33 + 0.666 + 0.266 = 7.25$, $e^{-2} = \frac{1}{e^2} \approx \frac{1}{7.25} = 0.137$

2. [10 points] Suppose that $fl(y)$ is a k-digit rounding approximation to a positive y. Show that

$$\left| \frac{y - fl(y)}{y} \right| \leq 5 \times 10^{-k},$$

   i.e., $fl(y)$ approximates y to k significant digits.

   Solution: Let $y = 0.d_1 d_2 \ldots \times 10^n$ for $1 \leq d_1 \leq 9$ and $0 \leq d_i \leq 9$ for $i = 2, \ldots$ If $d_{k+1} < 5$, then $fl(y) = 0.d_1 d_2 \ldots d_k \times 10^n$ and thus

$$\left| \frac{y - fl(y)}{y} \right| = \left| \frac{0.d_{k+1} d_{k+2} \ldots \times 10^{n-k}}{0.d_1 d_2 \ldots \times 10^n} \right| \leq 5 \times 10^{-k}.$$

   If $d_{k+1} \geq 5$, then $fl(y) = 0.d_1 d_2 \ldots d_{k-1} \delta_k \times 10^n$, where $\delta_k = d_k + 1$. We then have

$$\left| \frac{y - fl(y)}{y} \right| = \left| \frac{(1 - 0.d_{k+1} d_{k+2} \ldots) \times 10^{n-k}}{0.d_1 d_2 \ldots \times 10^n} \right| \leq 5 \times 10^{-k}.$$

3. [5 points each]

   (1) Determine the rate of convergence of the sequence $\left\{ \left(\sin \frac{1}{n}\right)^2 \right\}_{n=1}^{\infty}$ as $n \to \infty$, using a form $O\left(\frac{1}{n^p}\right)$.
   (2) Determine the rate of convergence of the function $\frac{1 - \cos h}{h}$ as $h \to 0$, using a form $O(h^p)$.

   Solution:

   (1) By MVT, there exists a number c between 0 and x such that $\cos c = \frac{\sin x - \sin 0}{x - 0}$. So we have $|\sin x| = |\cos c|\, |x| \leq |x|$. Then, for $n \geq 1$,

$$\left| \left(\sin \frac{1}{n}\right)^2 \right| = \left| \sin \frac{1}{n} \right|^2 \leq \left| \frac{1}{n^2} \right|.$$

   Therefore, $\left(\sin \frac{1}{n}\right)^2 = 0 + O\left(\frac{1}{n^2}\right)$.

(2) We use the first Taylor polynomial about $h = 0$;

$$\cos h = 1 - \frac{\cos \xi(h)}{6} h^2$$

for some number $\xi(h)$ between 0 and h. Hence,

$$\left| \frac{1 - \cos h}{h} - 0 \right| \leq \left| -\frac{\cos \xi(h)}{6} h \right| \leq \frac{1}{6}|h|,$$

and thus $\frac{1-\cos h}{h} = 0 + O(h)$.

4. [5 points each] Find an approximation to $\sqrt{17}$ (that is between 2 and 5) accurate to within 10^{-3} using the bisection method.

   (1) Briefly describe how one can use the bisection method to approximate $\sqrt{17}$.

   (2) Determine the number of iterations (n) required for the bisection method in (1) to achieve 10^{-3} accuracy of $|p_n - p|$, starting with the interval $[2, 5]$.

Solution:

(1) The function $f(x) = x^2 - 17$ has $\sqrt{17}$ as a root between 2 and 5 (with $f(2) \cdot f(5) < 0$), so using the bisection method on f starting with the interval $[2, 5]$ will approximate $\sqrt{17}$.

(2) We find an integer n that satisfies

$$|p_n - p| \leq \frac{5-2}{2^n} < 10^{-3},$$

which is equivalent to

$$\log_{10} 3 - n \log_{10} 2 < -3 \quad \text{and} \quad n > \frac{\log_{10} 3 + 3}{\log_{10} 2} \approx 11.55.$$

Hence, 12 iterations will ensure the desired accuracy.

5. [5 points each]

   (1) Show that $g(x) = e^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$.

   (2) Estimate the minimum number of iterations required for the fixed-point iteration to achieve 10^{-4} accuracy, with an initial approximation $p_0 = \frac{2}{3}$, considering both bounds (2.5) and (2.6) in the textbook.

Solution:

(1) Since $g'(x) = -e^{-x}$, g is continuous and g' exists on $[\frac{1}{3}, 1]$. g is strictly decreasing since $g'(x) < 0$ for all x, and using $g(\frac{1}{3}) \approx 0.717$ and $g(1) \approx 0.368$ we have that $g(x) \in [\frac{1}{3}, 1]$ for all $x \in [\frac{1}{3}, 1]$. In addition, we have $|g'(x)| \leq |g'(\frac{1}{3})| \approx 0.717$ for all $x \in (\frac{1}{3}, 1)$. Then, Theorem 2.3 implies that g has a unique fixed point in $[\frac{1}{3}, 1]$.

(2) First, using (2.5), we find an integer n that satisfies

$$|p_n - p| \leq 0.717^n \max\left\{p_0 - \frac{1}{3}, 1 - p_0\right\} = 0.717^n \frac{1}{3} < 10^{-4},$$

which is equivalent to

$$n \log_{10}(0.717) - \log_{10} 3 < -4 \quad \text{and} \quad n > \frac{-4 + \log_{10} 3}{\log_{10}(0.717)} \approx 24.4.$$

Using (2.6), we can also find an integer n that satisfies

$$|p_n - p| \le \frac{0.717^n}{1 - 0.717}|p_1 - p_0| = \frac{0.717^n}{0.283}\left|0.513 - \frac{2}{3}\right| = 0.717^n \times 0.544 < 10^{-4},$$

where $p_1 = g(p_0) \approx 0.513$. This can be rewritten as

$$n \log_{10}(0.717) + \log_{10}(0.544) < -4 \quad \text{and} \quad n > \frac{-4 - \log_{10} 0.544}{\log_{10}(0.717)} \approx 25.9.$$

At least 25 iterations will ensure the desired accuracy.

6. [10 points] Show that Theorem 2.3(ii) in the textbook is true if the inequality $|g'(x)| \le k$ is replaced by $g'(x) \le k$, for all $x \in (a, b)$.

   Solution: Suppose p and q are both fixed points in $[a, b]$ with $p \ne q$. MVT implies that a number ξ exists between p and q with

   $$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

   Thus,

   $$p - q = g(p) - g(q) = g'(\xi)(p - q) \le k(p - q) < p - q,$$

   which is a contradiction.

7. [10 points each]

   (1) Implement Newton's method via MATLAB grader.

   (2) Implement the secant method via MATLAB grader.

   Solution:

   (1)
```
function sol = newton(p0, N, eps)
p = p0;
for n=1:N
    f = sin(p) - exp(-p);
    d = cos(p) + exp(-p);
    pold = p;
    p = p - f/d;
    if (abs(p - pold)/abs(p)) < eps
        break;
    end
end
sol = [p; n];
end
```

(2)
```
function sol = secant(p0, p1, N, eps)
pold = p0;
p = p1;
fold = sin(pold) - exp(-pold);
for n=2:N
    f = sin(p) - exp(-p);
    d = (f - fold)/(p - pold);
    pold = p;
    fold = f;
    p = p - f/d;
    if (abs(p - pold)/abs(p)) < eps
        break;
    end
end
sol = [p; n];
end
```