2021 Spring MAS 365: Final Exam

Write the following Honor Pledge and sign your name under it.

> *"I have neither given nor received aid on this examination, nor have I concealed a violation of the Honor Code."*

1. [35 (15+10+5+5) points]

x	-1	0	1
$f(x)$	1	5	3

   (a) Find a Lagrange polynomial $P(x)$ that passes through the given three data points, using the forward divided difference. Report an upper bound of the maximum error $\max_{x\in[-1,1]}|f(x)-P(x)|$ if $|f^{(3)}(x)| \le M$ for all x.

   (b) Find a clamped cubic spline interpolant with additional information $f'(-1) = 1$ and $f'(1) = -3$.

   (c) Find approximation of $\int_{-1}^{1} f(x)dx$ using the composite Trapezoidal rule and the given three data points.

   (d) Find the degree of precision of the method in (c) for approximating $\int_{-1}^{1} f(x)dx$. Justify your answer.

2. [30 (15+15) points]

   (a) Show that the following difference formula

$$f''(x_0) \approx \frac{1}{h^2}[f(x_0-h) - 2f(x_0) + f(x_0+h)]$$

   is an $O(h^2)$ formula for $f''(x_0)$, where f is assumed to be inifinitely differentiable.

   (b) Use Richardson's extrapolation to derive an $O(h^4)$ formula. (Do not directly use the formula in the textbook. In other words, derive the $O(h^4)$ formula from (a).)

3. [35 (5+5+5+5+5+5+5) points] For a given function g, consider the first-order initial value problem

$$\frac{dy}{dt} = -g'(y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{IVP1}$$

   where $g'(y) = \frac{dg}{dy}$, and the second-order initial value problem

$$\frac{d^2y}{dt^2} = -\gamma\frac{dy}{dt} - g'(y), \quad a \le t \le b, \quad y(a) = \alpha_1, \quad y'(a) = \alpha_2 \tag{IVP2}$$

   for some $\gamma > 0$.

   (a) State conditions for the first-order initial value problem (IVP1) to be well posed.

(b) State Euler's method with the step size $h = \frac{b-a}{N}$ for (IVP1), and its local truncation error.

(c) For the case $g(y) = \frac{1}{2}\beta y^2$, where $\beta > 0$, state the condition on h for the Euler's method in (a) to be stable.

(d) Transform the second-order initial value problem (IVP2) into a system of first-order initial value problem.

(e) State Euler's method for the system in (d).

(f) Show that the Euler's method in (e) simplifies into the following two-step multistep method (approximating y):

$$\begin{aligned} w_{1,0} &= \alpha_1, \\ w_{1,1} &= \alpha_1 + h\alpha_2, \\ w_{1,i+1} &= w_{1,i} + (1 - h\gamma)(w_{1,i} - w_{1,i-1}) + h^2 g'(w_{1,i-1}), \quad \text{for each } i = 1, 2, \ldots, N-1. \end{aligned}$$

(g) State the condition on h for the method in (f) to be strongly stable.

4. [25 (10+15) points] Let $x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ for $i = 1, \ldots, n$, and $c = \frac{\pi}{n}$.

(a) Show that

$$c = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

for all $i = 1, \ldots, n$. [Hint: $T_n'(\theta) = \frac{n \sin(n\theta)}{\sin\theta}$, where $T_n(x) = \cos(n \arccos x)$ and $\theta = \arccos x$, and $\int_0^\pi \frac{\cos(n\theta) - \cos(n\phi)}{\cos\theta - \cos\phi} d\theta = \pi \frac{\sin n\phi}{\sin\phi}$]

(b) Show that

$$\int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}} dx = c \sum_{i=1}^{n} P(x_i)$$

for any polynomial $P(x)$ of degree less than $2n$.

5. [25 (10+15) points] Let $P(x) = 4x^3 + x^2$.

(a) Find a polynomial $P_2(x)$ of degree at most 2 that minimizes the following error (in the ℓ_∞-norm sense):

$$\max_{x \in [-1,1]} |f(x) - P_2(x)|.$$

(b) Find a polynomial $Q_2(x)$ of degree at most 2 that minimizes the following error (in the weighted ℓ_2-norm sense):

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} [P(x) - Q_2(x)]^2 dx.$$