2021 Spring MAS 365: Homework 3

posted on Mar 25; due by Apr 1

1. [10+5 points] The Frobenius norm (which is not a natural norm) is defined for an $n \times n$ matrix A by

$$||A||_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

   (a) Show that $|| \cdot ||_F$ is a matrix norm.

   (b) For any matrix A, show that $||A||_2 \leq ||A||_F$.

Solution:

(a) We show the five properties of the matrix norm in Definition 7.8 of the textbook for all matrices A and B and all real numbers α. First two properties are obvious so we prove the rest three.

(3) We have

$$||\alpha A||_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha a_{ij}|^2 \right)^{1/2} = |\alpha| \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = |\alpha| \, ||A||_F.$$

(4) We have

$$\begin{aligned}
||A+B||_F^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} + b_{ij}|^2 \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_{ij}|^2 + 2|a_{ij}| \, |b_{ij}| + |b_{ij}|^2) \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 + 2 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \right)^{1/2} + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \\
&= ||A||_F^2 + 2||A||_F \, ||B||_F + ||B||_F^2 = (||A||_F + ||B||_F)^2.
\end{aligned}$$

(5) We have

$$\begin{aligned}
||AB||_F^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|^2 \sum_{l=1}^{n} |b_{lj}|^2 \right) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ik}|^2 \sum_{j=1}^{n} \sum_{l=1}^{n} |b_{lj}|^2 = ||A||_F^2 \, ||B||_F^2.
\end{aligned}$$

(b) We have

$$\begin{aligned}||A||_2^2 &= \max_{||\boldsymbol{x}||_2=1} ||A\boldsymbol{x}||_2^2 = \max_{||\boldsymbol{x}||_2=1} \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j \right)^2 \\ &\le \max_{||\boldsymbol{x}||_2=1} \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 \right) \left(\sum_{j=1}^n x_j^2 \right) = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = ||A||_F^2.\end{aligned}$$

2. [5+10 points]

(a) Show that if A is symmetric, then $||A||_2 = \rho(A)$.

(b) Show that if $||\cdot||$ is any natural norm, then $(||A^{-1}||)^{-1} \le |\lambda| \le ||A||$ for any eigenvalue λ of the nonsingular matrix A.

Solution:

(a) Suppose λ is an eigenvalue of A with eigenvector $\boldsymbol{x}$, *i.e.*, $A\boldsymbol{x} = \lambda\boldsymbol{x}$. We then have $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$, implying that λ^2 is an eigenvalue of A^2. Thus,

$$||A||_2 = [\rho(A^tA)]^{1/2} = [\rho(A^2)]^{1/2} = [[\rho(A)]^2]^{1/2} = \rho(A).$$

(b) Suppose λ is an eigenvalue of A with eigenvector $\boldsymbol{x}$, *i.e.*, $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A^{-1}\boldsymbol{x} = \lambda^{-1}\boldsymbol{x}$. Then, we have

$$|\lambda|\,||\boldsymbol{x}|| = ||\lambda\boldsymbol{x}|| = ||A\boldsymbol{x}|| \le ||A||\,||\boldsymbol{x}||,$$

implying that $|\lambda| \le ||A||$. Similarly, we have

$$|\lambda|^{-1}\,||\boldsymbol{x}|| = ||\lambda^{-1}\boldsymbol{x}|| = ||A^{-1}\boldsymbol{x}|| \le ||A^{-1}||\,||\boldsymbol{x}||,$$

implying that $|\lambda|^{-1} \le ||A^{-1}||$.

3. [5+5 points] The linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7, \\ x_1 + x_2 + x_3 &= 2, \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

has the solution $(1, 2, -1)^t$.

(a) Find T_j of Jacobi method, and compute $\rho(T_j)$. Report an approximation after two iterations of Jacobi method using $\boldsymbol{x}^{(0)} = \mathbf{0}$.

(b) Find T_g of Gauss-Seidel method, and compute $\rho(T_g)$. Report an approximation after two iterations of Gauss-Seidel method using $\boldsymbol{x}^{(0)} = \mathbf{0}$.

Solution:

(a) $T_j = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$, and since $\det\{\lambda I - T_j\} = \lambda(\lambda^2 - 2) - 2(\lambda - 2) - 2(2 - 2\lambda) = \lambda^3$, we have $\rho(T_j) = 0$. Since $\boldsymbol{c}_j = (7, 2, 5)^t$, Jacobi method updates as

$$\boldsymbol{x}^{(1)} = \boldsymbol{c}_j, \quad \boldsymbol{x}^{(2)} = T_j\boldsymbol{c}_j + \boldsymbol{c}_j = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} = (13, -10, -13)^t$$

(b) $T_g = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$, and since $\mathsf{det}\{\lambda I - T_g\} = \lambda(\lambda-2)^2$, we have $\rho(T_g) = 2$. Since $\boldsymbol{c}_g = (7, -5, 1)^t$, Gauss-Seidel method updates as

$$\boldsymbol{x}^{(1)} = \boldsymbol{c}_g, \quad \boldsymbol{x}^{(2)} = T_g \boldsymbol{c}_g + \boldsymbol{c}_g = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix} = (19, -18, 3)^t$$

4. [10+10+10 points]

(a) Implement Gauss-Seidel methods via MATLAB grader.

(b) Implement Gauss-Seidel methods via MATLAB grader.

(c) Implement *Randomized* Gauss-Seidel methods via MATLAB grader.

Solution:

(a)
```
function [xj Nj] = Jacobi(A, b, x0, epsilon, N)
    D = diag(diag(A));
    Tj = inv(D) * (D-A);
    cj = inv(D) * b;
    xj = x0;
    for k=1:N
        xj_prev = xj;
        xj = Tj*xj + cj;
        if (norm(xj - xj_prev, Inf)/norm(xj, Inf) < epsilon)
            break;
        end
    end
    Nj = k;
end
```

(a)
```
function [xg Ng] = GS(A, b, x0, epsilon, N)
    n = size(A,1);
    xg = x0;
    for k=1:N
        xg_prev = xg;
        for i=1:n
            xg(i) = 1/A(i,i)*(-A(i,[1:i-1 i+1:n])*xg([1:i-1 i+1:n]) + b(i));
        end
        if (norm(xg - xg_prev, Inf)/norm(xg, Inf) < epsilon)
            break;
        end
    end
    Ng = k;
end
```

(c)
```
function [xg Ng] = Randomized_GS(A, b, x0, epsilon, N)
    n = size(A,1);
    xg = x0;
    for k=1:N
```

```
        xg_prev = xg;
        nn = randperm(n);
        for i=nn
            xg(i) = 1/A(i,i)*(-A(i,[1:i-1 i+1:n])*xg([1:i-1 i+1:n]) + b(i));
        end
        if (norm(xg - xg_prev, Inf)/norm(xg, Inf) < epsilon)
            break;
        end
    end
    Ng = k;
end
```