2021 Spring MAS 365: Final Exam

Write the following Honor Pledge and sign your name under it.

"I have neither given nor received aid on this examination, nor have I concealed a violation of the Honor Code."

1. [35 (15+10+5+5) points]

x	-1	0	1
$f(x)$	1	5	3

(a) Find a Lagrange polynomial $P(x)$ that passes through the given three data points, using the forward divided difference. Report an upper bound of the maximum error $\max_{x\in[-1,1]}|f(x) - P(x)|$ if $|f^{(3)}(x)| \leq M$ for all x.

(b) Find a clamped cubic spline interpolant with additional information $f'(-1) = 1$ and $f'(1) = -3$.

(c) Find approximation of $\int_{-1}^{1} f(x)dx$ using the composite Trapezoidal rule and the given three data points.

(d) Find the degree of precision of the method in (c) for approximating $\int_{-1}^{1} f(x)dx$. Justify your answer.

Solution:

(a) By the forward divided difference below,

$$\begin{array}{r|cccl}
-1 & 1 & & & \text{(2 points)} \\
 & & 4 & & \text{(2 points)} \\
0 & 5 & & -3 & \text{(2 points)} \\
 & & -2 & & \\
1 & 3 & & &
\end{array}$$

we have (3 points)

$$P(x) = 1 + 4(x + 1) - 3(x + 1)x = -3x^2 + x + 5.$$

This has an error (2 points)

$$f(x) - P(x) = \frac{f^{(3)}(\xi(x))}{3!}(x + 1)x(x - 1)$$

for some $\xi(x) \in (-1, 1)$. Let $g(x) = (x + 1)x(x - 1)$. Then, since $g'(x) = 3x^2 - 1$, $g(x)$ has the maximum value $g\left(-\frac{1}{\sqrt{3}}\right) = -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$ on the interval $[-1, 1]$ (2 points). So, the maximum error is upper bounded as (2 points)

$$\max_{x\in[-1,1]} |f(x) - P(x)| \leq \frac{M}{9\sqrt{3}}.$$

(b) Let

$$s(x) = \begin{cases} s_0(x) = a_0 + b_0(x+1) + c_0(x+1)^2 + d_0(x+1)^3, & -1 \le x < 0, \\ s_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, & 0 \le x \le 1. \end{cases}$$

The eight unknowns satisfy the following eight conditions (8 points)

$$\begin{aligned} &1 = f(-1) = s_0(-1) = a_0, \\ &5 = f(0) = s_0(0) = a_0 + b_0 + c_0 + d_0, \\ &5 = f(0) = s_1(0) = a_1, \\ &3 = f(1) = s_1(1) = a_1 + b_1 + c_1 + d_1, \\ &b_0 + 2c_0 + 3d_0 = s_0'(0) = s_1'(0) = b_1, \\ &2c_0 + 6d_0 = s_0''(0) = s_1''(0) = 2c_1, \\ &1 = f'(-1) = s_0'(-1) = b_0, \\ &-3 = f'(1) = s_1'(1) = b_1 + 2c_1 + 3d_1. \end{aligned}$$

Solving the system of equations gives (2 points)

$$s(x) = \begin{cases} 1 + (x+1) + 8(x+1)^2 - 5(x+1)^3, & -1 \le x < 0, \\ 5 + 2x - 7x^2 + 3x^3, & 0 \le x \le 1. \end{cases}$$

(c) Applying the composite Trapezoidal rule yields

$$\begin{aligned} \int_{-1}^{1} f(x)dx &\approx \frac{1}{2}[f(-1) + 2f(0) + f(1)] \quad \text{(3 points)} \\ &= \frac{1}{2}[1 + 10 + 3] = 7 \quad \text{(2 points)}. \end{aligned}$$

(d) Since (3 points, You also get points using the error term $-\frac{1}{6}h^2 f''(\mu)$)

$$\begin{aligned} f(x) = 1 &: \int_{-1}^{1} 1dx = 2 = \frac{1}{2}[f(-1) + 2f(0) + f(1)], \\ f(x) = x &: \int_{-1}^{1} xdx = 0 = \frac{1}{2}[f(-1) + 2f(0) + f(1)], \\ f(x) = x^2 &: \int_{-1}^{1} x^2dx = \frac{2}{3} \neq 1 = \frac{1}{2}[f(-1) + 2f(0) + f(1)], \end{aligned}$$

the degree of precision is 1. (2 points)

2. [30 (15+15) points]

(a) Show that the following difference formula

$$f''(x_0) \approx \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)]$$

is an $O(h^2)$ formula for $f''(x_0)$, where f is assumed to be infinitely differentiable.

(b) Use Richardson's extrapolation to derive an $O(h^4)$ formula. (Do not directly use the formula in the textbook. In other words, derive the $O(h^4)$ formula from (a).)

Solution:

(a) By Taylor's theorem, we have (8 points)

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f^{(3)}(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 + \cdots,$$
$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f^{(3)}(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 - \cdots.$$

Adding them and dividing it by h^2 yield (7 points)

$$f''(x_0) = \frac{1}{h^2}[f(x_0-h) - 2f(x_0) + f(x_0+h)] - \frac{1}{12}f^{(4)}(x_0)h^2 + O(h^4).$$

(b) By letting $N_1(h) = \frac{1}{h^2}[f(x_0-h) - 2f(x_0) + f(x_0+h)]$. we have

$$f''(x_0) = N_1(h) - \frac{f^{(4)}(x_0)}{12}h^2 + O(h^4).$$

Another $O(h^2)$ formula is (5 points)

$$f''(x_0) = N_1\left(\frac{h}{2}\right) - \frac{f^{(4)}(x_0)}{48}h^2 + O(h^4),$$

and subtracting the first approximation from twice the second approximation yields (5 points)

$$3f''(x_0) = \left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] + O(h^4).$$

Therefore, the following formula is $O(h^4)$ by extrapolation.

$$\begin{aligned} N_2(h) &= \frac{1}{3}\left[4N_1\left(\frac{h}{2}\right) - N_1(h)\right] \qquad \text{(3 points)} \\ &= \frac{1}{3}\left[\frac{16}{h^2}\left(f\left(x_0-\frac{h}{2}\right) - 2f(x_0) + f\left(x_0+\frac{h}{2}\right)\right) - \frac{1}{h^2}(f(x_0-h) - 2f(x_0) + f(x_0+h))\right] \\ &= \frac{1}{3h^2}\left[-f(x_0-h) + 16f\left(x_0-\frac{h}{2}\right) - 30f(x_0) + 16f\left(x_0+\frac{h}{2}\right) - f(x_0+h)\right] \qquad \text{(2 points)} \end{aligned}$$

3. [35 (5+5+5+5+5+5+5) points] For a given function g, consider the first-order initial value problem

$$\frac{dy}{dt} = -g'(y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{IVP1}$$

where $g'(y) = \frac{dg}{dy}$, and the second-order initial value problem

$$\frac{d^2y}{dt^2} = -\gamma\frac{dy}{dt} - g'(y), \quad a \le t \le b, \quad y(a) = \alpha_1, \quad y'(a) = \alpha_2 \tag{IVP2}$$

for some $\gamma > 0$.

(a) State conditions for the first-order initial value problem (IVP1) to be well posed.

(b) State Euler's method with the step size $h = \frac{b-a}{N}$ for (IVP1), and its local truncation error.

(c) For the case $g(y) = \frac{1}{2}\beta y^2$, where $\beta > 0$, state the condition on h for the Euler's method in (a) to be stable.

(d) Transform the second-order initial value problem (IVP2) into a system of first-order initial value problem.

(e) State Euler's method for the system in (d).

(f) Show that the Euler's method in (e) simplifies into the following two-step multistep method (approximating y):

$$\begin{aligned} w_{1,0} &= \alpha_1, \\ w_{1,1} &= \alpha_1 + h\alpha_2, \\ w_{1,i+1} &= w_{1,i} + (1 - h\gamma)(w_{1,i} - w_{1,i-1}) - h^2 g'(w_{1,i-1}), \quad \text{for each } i = 1, 2, \ldots, N-1. \end{aligned}$$

(g) State the condition on h for the method in (f) to be strongly stable.

Solution:

(a) (IVP1) is well-posed if $g'(y)$ is continuous (2 points) and satisfies a Lipschitz condition in the variable y on $\mathbb{R}$ (3 points).

(b) The Euler's method is (3 points)

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i - hg'(w_i), \quad \text{for each } i = 0, 1, \ldots, N-1, \end{aligned}$$

and its local truncation error is $\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} + g'(y_i) = \frac{h}{2} y''(\xi_i)$ for some ξ_i in (t_i, t_{i+1}), where $t_i = a + ih$. (2 points)

(c) The Euler's method reduces to $w_{i+1} = w_i - h\beta w_i = (1 - h\beta)w_i$. (2 points) This is stable only if $|1 - h\beta| < 1$, which is equivalent to $h < \frac{2}{\beta}$. (3 points) (You get points if you used Theorem 5.20 in the textbook appropriately.)

(d) Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$. (2 points) We then have the equivalent system of first-order initial value problem: (3 points)

$$\begin{aligned} \frac{du_1}{dt} &= u_2, \\ \frac{du_2}{dt} &= \frac{dy'}{dt} = -\gamma u_2 - g'(u_1) \end{aligned}$$

for $a \le t \le b$ with initial conditions $u_1(a) = y(a) = \alpha_1$ and $u_2(a) = y'(a) = \alpha_2$.

(e) The corresponding Euler's method is

$$\begin{aligned} w_{1,0} &= \alpha_1, \quad w_{2,0} = \alpha_2, \\ w_{1,i+1} &= w_{1,i} + hw_{2,i}, \\ w_{2,i+1} &= w_{2,i} - h(\gamma w_{2,i} + g'(w_{1,i})), \quad \text{for each } i = 0, 1, \ldots, N-1. \end{aligned}$$

(f) It is obvious that $w_{1,1} = w_{1,0} + hw_{2,0} = \alpha_1 + h\alpha_2$. (1 point) Adding the equations for $w_{1,i+1}$ and $w_{1,i}$ with multipliers 1 and $-(1 - h\gamma)$, respectively, we have (4 points)

$$\begin{aligned} w_{1,i+1} &= w_{1,i} + (1 - h\gamma)(w_{1,i} - w_{1,i-1}) + h(w_{2,i} - (1 - h\gamma)w_{2,i-1}) \\ &= w_{1,i} + (1 - h\gamma)(w_{1,i} - w_{1,i-1}) + h^2 g'(w_{1,i-1}), \end{aligned}$$

for each $i = 1, 2, \ldots, N-1$.

(g) The associated characteristic equation is (2 points)

$$P(\lambda) = \lambda^2 - (2 - h\gamma)\lambda + (1 - h\gamma) = 0,$$

where the roots of $P(\lambda)$ are (1 points)

$$\lambda = 1 \quad \text{and} \quad \lambda = \frac{(2 - h\gamma) - \sqrt{(2 - h\gamma)^2 - 4(1 - h\gamma)}}{2} = 1 - h\gamma.$$

Therefore, if $|1 - h\gamma| < 1$, *i.e.*, $h < \frac{2}{\gamma}$, the method is strongly stable. (2 points)

4. [25 (10+15) points] Let $x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ for $i = 1, \ldots, n$, and $c = \frac{\pi}{n}$.

   (a) Show that

$$c = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

   for all $i = 1, \ldots, n$. [Hint: $T_n'(\theta) = \frac{n\sin(n\theta)}{\sin\theta}$, where $T_n(x) = \cos(n \arccos x)$ and $\theta = \arccos x$, and $\int_0^\pi \frac{\cos(n\theta) - \cos(n\phi)}{\cos\theta - \cos\phi} d\theta = \pi \frac{\sin n\phi}{\sin\phi}$]

   (b) Show that

$$\int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}} dx = c \sum_{i=1}^{n} P(x_i)$$

   for any polynomial $P(x)$ of degree less than $2n$.

Solution:

(a) Note that $x_1, x_2, \ldots, x_n$ are the zeros of the nth Chebyshev polynomial $T_n(x)$. (2 points) Using $\theta = \arccos x$ and $\phi = \arccos x_i$, we have

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_n(x) - T_n(x_i)}{(x - x_i)T_n'(x_i)} dx \quad \text{(3 points)} \\
&= -\frac{1}{T_n'(\phi)} \int_{\pi}^{0} \frac{\cos(n\theta) - \cos(n\phi)}{\cos\theta - \cos\phi} d\theta \quad \text{(3 points)} \\
&= \frac{\pi}{n} \quad \text{(2 points)}
\end{aligned}$$

(b) First, assume that the degree of $P(x)$ is less than n. Then, $P(x)$ can be written in terms of $(n-1)$th Lagrange coefficient polynomials as (3 points)

$$P(x) = \sum_{i=1}^{n} P(x_i) L_i(x) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} P(x_i).$$

Then, (3 points)

$$\int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \left[\frac{1}{\sqrt{1-x^2}} \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} P(x_i) \right] dx = \sum_{i=1}^{n} \left[\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx \right] P(x_i).$$

Next, assume that the degree of $P(x)$ is at least n but less than $2n$. Divide $P(x)$ by the n Chebyshev polynomial $T_n(x)$ as (3 points)

$$P(x) = Q(x)T_n(x) + R(x),$$

where both $Q(x)$ and $R(x)$ are of degree less than n. We then have (3 points)

$$\int_{-1}^{1} \frac{Q(x)T_n(x)}{\sqrt{1-x^2}} dx = 0, \quad \text{and} \quad \int_{-1}^{1} \frac{R(x)}{\sqrt{1-x^2}} dx = c \sum_{i=1}^{n} R(x_i).$$

Since $P(x_i) = R(x_i)$, we have (3 points)

$$\int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}} = \int_{-1}^{1} \frac{R(x)}{\sqrt{1-x^2}} = c\sum_{i=1}^{n} R(x_i) = c\sum_{i=1}^{n} P(x_i).$$

5. [25 (10+15) points] Let $P(x) = 4x^3 + x^2$.

   (a) Find a polynomial $P_2(x)$ of degree at most 2 that minimizes the following error (in the ℓ_∞-norm sense):

$$\max_{x\in[-1,1]} |P(x) - P_2(x)|.$$

   (b) Find a polynomial $Q_2(x)$ of degree at most 2 that minimizes the following error (in the weighted ℓ_2-norm sense):

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} [P(x) - Q_2(x)]^2 dx.$$

Solution:

(a) The error (3 points)

$$\max_{x\in[-1,1]} |P(x) - P_2(x)| = 4 \max_{x\in[-1,1]} \left|\frac{1}{4}(P(x) - P_2(x))\right|$$

is minimized when $\frac{1}{4}(P(x) - P_2(x)) = \tilde{T}_3(x) = x^3 - \frac{3}{4}x$. (5 points) Therefore, (2 points)

$$P_2(x) = P(x) - 4\tilde{T}_3(x) = 4x^3 + x^2 - 4\left(x^3 - \frac{3}{4}x\right) = x^2 + 3x.$$

(b) Consider the Chebyshev polynomials $\{T_0(x), T_1(x), T_2(x)\} = \{1, x, 2x^2 - 1\}$ that satisfy (3 points)

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & n = m = 0, \\ \frac{\pi}{2}, & n = m > 0, \\ 0, & n \neq m. \end{cases}$$

Then, $Q_2(x) = \sum_{j=0}^{2} a_j T_j(x)$, where, for $j = 0, 1, 2$, (6 points)

$$a_j = \begin{cases} \frac{1}{\pi} \int_{-1}^{1} \frac{T_j(x)P(x)}{\sqrt{1-x^2}} dx, & j = 0, \\ \frac{2}{\pi} \int_{-1}^{1} \frac{T_j(x)P(x)}{\sqrt{1-x^2}} dx, & j = 1, 2, \end{cases}$$

Since $T_j(x)P(x)$ is a polynomial of degree $3 + j$, using Problem 4 with $n = 3$ and $x_1 = -\frac{\sqrt{3}}{2}$, $x_2 = 0$, $x_3 = \frac{\sqrt{3}}{2}$, we have (3 points)

$$a_0 = \frac{1}{3}\sum_{i=1}^{3} T_0(x_i)P(x_i) = \frac{2}{3}\frac{3}{4} = \frac{1}{2},$$

$$a_1 = \frac{2}{3}\sum_{i=1}^{3} T_1(x_i)P(x_i) = \frac{4}{3}4\frac{9}{16} = 3,$$

$$a_2 = \frac{2}{3}\sum_{i=1}^{3} T_2(x_i)P(x_i) = \frac{4}{3}\left(2\frac{9}{16} - \frac{3}{4}\right) = \frac{1}{2}.$$

Therefore, (3 points)

$$Q_2(x) = \frac{1}{2} + 3x + \frac{1}{2}(2x^2 - 1) = x^2 + 3x.$$

∘ One can instead consider the change of the variable $x = \cos\theta$, (3 points) and let $Q_2(x) = a_0 + a_1 x + a_2 x^2$. Then,

$$\begin{aligned} E = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}[P(x) - Q_2(x)]^2 dx &= \int_0^{\pi} [P(\cos\theta) - Q_2(\cos\theta)]^2 d\theta \\ &= \int_0^{\pi} (4\cos^3\theta + \cos^2\theta - a_2\cos^2\theta - a_1\cos\theta - a_0)^2 d\theta. \end{aligned}$$

We thus choose a_0, a_1, a_2 such that (6 points)

$$\begin{aligned} \frac{\partial E}{\partial a_0} &= \int_0^{\pi} 2(a_2\cos^2\theta + a_1\cos\theta + a_0 - 4\cos^3\theta - \cos^2\theta)d\theta = 2a_0\pi + (a_2 - 1)\pi \\ \frac{\partial E}{\partial a_1} &= \int_0^{\pi} 2\cos\theta(a_2\cos^2\theta + a_1\cos\theta + a_0 - 4\cos^3\theta - \cos^2\theta)d\theta = (a_1 - 3)\pi \\ \frac{\partial E}{\partial a_2} &= \int_0^{\pi} 2\cos\theta^2\theta(a_2\cos^2\theta + a_1\cos\theta + a_0 - 4\cos^3\theta - \cos^2\theta)d\theta = a_0\pi + \frac{3}{4}(a_2 - 1)\pi, \end{aligned}$$

which are $a_0 = 0, a_1 = 3, a_2 = 1$ (3 points). Therefore, (3 points)

$$Q_2(x) = x^2 + 3x.$$