2021 Spring MAS 365: Homework 9

posted on June 3; due by June 15

1. [10+10 points] Given the multistep method

$$w_{i+1} = -\frac{3}{2}w_i + 3w_{i-1} - \frac{1}{2}w_{i-2} + 3hf(t_i, w_i), \quad \text{for } i = 2, \ldots, N-1,$$

with starting values w_0, w_1, w_2.

(a) Find the local truncation error. [Hint: Consider the third-order Taylor polynomials of $y(t_{i+1}), y(t_{i-1})$ and $y(t_{i-2})$ about x_i and their remainder terms.]

(b) Comment on consistency, stability, and convergence, under the assumptions that $y^{(4)}$ is bounded and w_0, w_1, w_1 are consistent. Justify your answer.

Solution:

(a) The third-order Taylor polynomials of $y(t_{i+1})$, $y(t_{i-1})$ and $y(t_{i-2})$ about x_i and their remainder terms are

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2}h^2 + \frac{y^{(3)}(t_i)}{6}h^3 + \frac{y^{(4)}(\xi_{i+1})}{24}h^4, \\ y(t_{i-1}) &= y(t_i) - y'(t_i)h + \frac{y''(t_i)}{2}h^2 - \frac{y^{(3)}(t_i)}{6}h^3 + \frac{y^{(4)}(\xi_{i-1})}{24}h^4, \\ y(t_{i-2}) &= y(t_i) - y'(t_i)2h + \frac{y''(t_i)}{2}4h^2 - \frac{y^{(3)}(t_i)}{6}8h^3 + \frac{y^{(4)}(\xi_{i-2})}{24}16h^4, \end{aligned}$$

for some $t_i < \xi_{i+1} < t_{i+1}$, $t_{i-1} < \xi_{i-1} < t_i$ and $t_{i-2} < \xi_{i-2} < t_i$.
We then have the local truncation error

$$\begin{aligned} \tau_{i+1}(h) &= \frac{y(t_{i+1}) + \frac{3}{2}y(t_i) - 3y(t_{i-1}) + \frac{1}{2}y(t_{i-2})}{h} - 3f(t_i, y(t_i)) \\ &= \frac{y^{(4)}(\xi_{i+1})}{24}h^3 - \frac{y^{(4)}(\xi_{i-1})}{8}h^3 + \frac{y^{(4)}(\xi_{i-2})}{3}h^3. \end{aligned}$$

(b) Under the assumption that $y^{(4)}$ is bounded by M and w_0, w_1, w_2 are consistent, the method is consistent, since

$$\lim_{h\to 0} |\tau_{i+1}(h)| \leq \lim_{h\to 0} \frac{M}{2}h^3 = 0$$

for all i.
The characteristic equation for this method,

$$0 = P(\lambda) = \lambda^3 + \frac{3}{2}\lambda^2 - 3\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda^2 + \frac{5}{2}\lambda - \frac{1}{2}\right)$$

has three roots $\lambda_1 = 1$, $\lambda_2 = \frac{-5+\sqrt{33}}{4} \approx 0.1861$, $\lambda_3 = \frac{-5-\sqrt{33}}{4} \approx -2.6861$. The method has a root with magnitude greater one, so it is unstable. Then, by Theorem 5.24, it is not convergent.

2. [5 points] Show that the fourth-order Runge-Kutta method

$$\begin{aligned}
k_1 &= hf(t_i, w_i),\\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right),\\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right),\\
k_4 &= hf(t_i + h, w_i + k_3),\\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{aligned}$$

when applied to the differential equation $y' = \lambda y$, can be written in the form

$$w_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right) w_i.$$

Solution: We have

$$\begin{aligned}
k_1 &= h\lambda w_i,\\
k_2 &= h\lambda\left(w_i + \frac{h\lambda w_i}{2}\right),\\
k_3 &= h\lambda\left(w_i + \frac{h\lambda w_i}{2} + \frac{h^2\lambda^2 w_i}{4}\right),\\
k_4 &= h\lambda\left(w_i + h\lambda w_i + \frac{h^2\lambda^2 w_i}{2} + \frac{h^3\lambda^3 w_i}{4}\right),
\end{aligned}$$

and thus

$$\begin{aligned}
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&= w_i + \frac{w_i}{6}\left(h\lambda + 2h\lambda + h^2\lambda^2 + 2h\lambda + h^2\lambda^2 + \frac{h^3\lambda^3}{2} + h\lambda + h^2\lambda^2 + \frac{h^3\lambda^3}{2} + \frac{h^4\lambda^4}{4}\right)\\
&= \left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24}\right) w_i
\end{aligned}$$

3. [10+10 points]

(a) Use the Gram-Schmidt procedure to calculate L_1 and L_2, where $\{L_0(x), L_1(x), L_2(x)\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ and $L_0(x) = 1$.

(b) Use the polynomials $\{L_0(x), L_1(x), L_2(x)\}$ to compute the least squares polynomials of degree 2 on the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for the function $f(x) = x^3$.

Solution:

(a) We have to compute

$$L_1(x) = x - B_1, \quad \text{where } B_1 = \frac{\int_0^\infty xe^{-x}[L_0(x)]^2 dx}{\int_0^\infty e^{-x}[L_0(x)]^2 dx}$$

$$L_2(x) = (x - B_2)L_1(x) - C_2 L_0(x), \quad \text{where } B_2 = \frac{\int_0^\infty xe^{-x}[L_1(x)]^2 dx}{\int_0^\infty e^{-x}[L_1(x)]^2 dx}, \quad C_2 = \frac{\int_0^\infty xe^{-x}L_1(x)L_0(x) dx}{\int_0^\infty e^{-x}[L_0(x)]^2 dx}.$$

Note that

$$\int_0^\infty e^{-x}dx = \int_0^\infty e^{-x}xdx = 1, \quad \int_0^\infty e^{-x}x^2dx = 2, \quad \int_0^\infty e^{-x}x^3dx = 6,$$
$$\int_0^\infty e^{-x}x^4dx = 24, \quad \text{and} \quad \int_0^\infty e^{-x}x^5dx = 120.$$

Since

$$B_1 = \frac{\int_0^\infty xe^{-x}dx}{\int_0^\infty e^{-x}dx} = \frac{1}{1} = 1,$$

we have

$$L_1(x) = x - 1.$$

And since

$$B_2 = \frac{\int_0^\infty xe^{-x}(x-1)^2dx}{\int_0^\infty e^{-x}(x-1)^2dx} = \frac{3}{1} = 3, \quad \text{and} \quad C_2 = \frac{\int_0^\infty xe^{-x}(x-1)dx}{\int_0^\infty e^{-x}dx} = \frac{1}{1} = 1,$$

we have

$$L_2(x) = (x-3)(x-1) - 1 = x^2 - 4x + 2.$$

(b) The least squares approximation to f on $(0, \infty)$ with respect to $w(x)$ is

$$P(x) = \sum_{j=0}^{2} a_j L_0(x), \quad \text{where } a_j = \frac{\int_0^\infty e^{-x}L_j(x)x^3dx}{\int_0^\infty e^{-x}[L_j(x)]^2dx}.$$

Since

$$a_0 = \frac{\int_0^\infty e^{-x}x^3dx}{\int_0^\infty e^{-x}dx} = \frac{6}{1} = 6, \quad a_1 = \frac{\int_0^\infty e^{-x}(x-1)x^3dx}{\int_0^\infty e^{-x}(x-1)^2dx} = \frac{18}{1} = 18,$$
$$a_2 = \frac{\int_0^\infty e^{-x}(x^2-4x+2)x^3dx}{\int_0^\infty e^{-x}(x^2-4x+2)^2dx} = \frac{36}{4} = 9,$$

we have

$$P(x) = 6L_0(x) + 18L_1(x) + 9L_2(x) = 6 + 18(x-1) + 9(x^2-4x+2) = 9x^2 - 18x + 6.$$

4. [10+10 points] Consider approximating $f(x) = x^3 - x$ by a polynomial $P_2(x) = a_2x^2 + a_1x + a_0$ that minimizes

$$E_2(a_0, a_1, a_2) = \int_0^1 [f(x) - P_2(x)]^2dx$$

(a) Find $P_2(x)$.

(b) Find a polynomial P_1 that solves

$$\min_{P_1 \in \Pi_1} \max_{x \in [0,1]} |P_2(x) - P_1(x)|,$$

where Π_1 denotes set of all polynomials degree at most 1.

Solution:

(a) The normal equations are

$$\sum_{k=0}^{2} a_k \int_0^1 x^k dx = \int_0^1 f(x)dx,$$
$$\sum_{k=0}^{2} a_k \int_0^1 x^{1+k} dx = \int_0^1 xf(x)dx,$$
$$\sum_{k=0}^{2} a_k \int_0^1 x^{2+k} dx = \int_0^1 x^2 f(x)dx,$$

which are equivalent to

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = -\frac{1}{4},$$
$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = -\frac{2}{15},$$
$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = -\frac{1}{12}.$$

The solutions are

$$a_0 = \frac{1}{20} = 0.05, \quad a_1 = -\frac{8}{5} = -1.6, \quad a_2 = \frac{3}{2} = 1.5.$$

(b) Let $P_1(x) = c_0 + c_1 x$. The problem can be rewritten as

$$\begin{aligned}
&\min_{P_1 \in \Pi_1} \max_{x\in[-1,1]} \left| P_2\left(\frac{1}{2}(x+1)\right) - P_1\left(\frac{1}{2}(x+1)\right) \right| \\
&= \min_{c_0,c_1} \max_{x\in[-1,1]} \left| -\frac{3}{8} - \frac{1}{20}x + \frac{3}{8}x^2 - \left(c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_1 x\right) \right| \\
&= \frac{3}{8} \min_{c_0,c_1} \max_{x\in[-1,1]} \left| -1 - \frac{2}{15}x + x^2 - \frac{8}{3}\left(c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_1 x\right) \right| \\
&\geq \frac{3}{8} \max_{x\in[-1,1]} |\tilde{T}_2(x)|,
\end{aligned}$$

where the last inequality follows from the property of the monic Chebyshev polynomial $\tilde{T}_2(x)$. The equality holds when

$$-1 - \frac{2}{15}x + x^2 - \frac{8}{3}\left(c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_1 x\right) = \tilde{T}_2(x) = x^2 - \frac{1}{2},$$

so $c_0 = -\frac{11}{80}$ an $c_1 = -\frac{1}{10}$.

5. [10 points] Implement the power method via MATLAB grader.

Solution:

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function [mu x iter] = power_method(A, x0, epsilon, N)
    x = x0;
    p = find(abs(x) == max(abs(x)), 1);
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    x = x/x(p);
    for i=1:N
        xprev = x;
        y = A*x;
        mu = y(p);
        p = find(abs(y) == max(abs(y)), 1);
        x = y/y(p);
        if norm(x - xprev, Inf)/norm(x, Inf) < epsilon
            break;
        end
    end
    iter = i;
end
```