2021 Spring MAS 365
Chapter 2: Solutions of Equations in One Variable

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Motivation: Estimating the Growth of Population

- $N(t)$: the number of the population at time t
- λ: the constant birth rate of the population
- v: the constant immigration rate

- Assume that the population satisfies the differential equation

$$\frac{dN(t)}{dt} = \lambda N(t) + v.$$

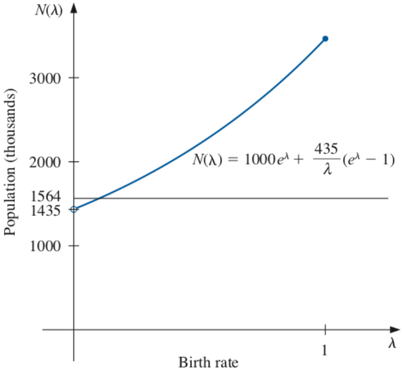
- The solution is

$$N(t) = N_0 e^{\lambda t} + \frac{v}{\lambda}(e^{\lambda t} - 1),$$

where N_0 denotes the initial population.

- Let $N(0) = 1000$, $N(1) = 1564$, and $v = 435$.
- Determine the birth rate λ of this population, we need to solve

$$1,564 = 1000e^{\lambda} + \frac{435}{\lambda}(e^{\lambda} - 1).$$



Root-Finding Problem

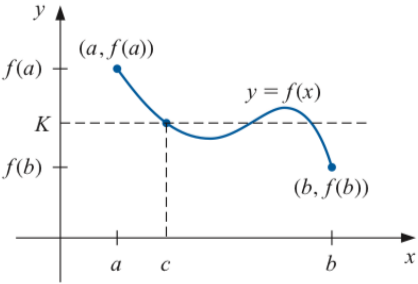
- Find a **root** (zero or solution) of $f(x) = 0$.

The Bisection Method

- The **bisection** (or binary-search) method is based on IVT.

Theorem 1 (Intermediate Value Theorem)

If $f \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c) = K$.



The Bisection Method (cont'd)

- Let $f \in C[a, b]$ with $f(a)$ and $f(b)$ of opposite sign.
- By IVT, a number p exists in (a, b) with $f(p) = 0$.
- Repeatedly halve (or bisect) subintervals of $[a, b]$ and, at each step, locate the half containing p.

The Bisection Method (cont'd)

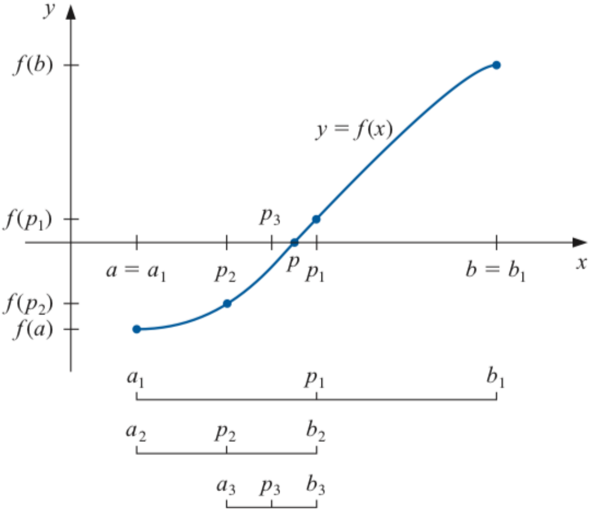
Bisection Method

Initialize $a_1 = a$ and $b_1 = b$ with $f(a) \cdot f(b) < 0$.
For $n = 1, 2, \ldots$
Let p_n be the midpoint of $[a_n, b_n]$; that is

$$p_n = \frac{a_n + b_n}{2}$$

- If $f(p_n) = 0$, then $p = p_n$.
- If $f(p_n) \neq 0$, then
  - if $f(a_n)$ and $f(p_n)$ have the same sign, $p \in (p_n, b_n)$. Set $a_{n+1} = p_n$ and $b_{n+1} = b_n$.
  - otherwise, $p \in (a_n, p_n)$. Set $a_{n+1} = a_n$ and $b_{n+1} = p_n$.

The Bisection Method (cont'd)



Stopping Criteria

- Set maximum number of iterations.
- Set a tolerance $\epsilon > 0$ and stop when one of the followings is met:

1. $|p_N - p_{N-1}| < \epsilon$
2. $\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, \quad p_N \neq 0$
3. $|f(p_N)| < \epsilon$

Q. Which one should we use?

Initialization of Bisection Method

- An interval $[a, b]$ must be found with $f(a) \cdot f(b) < 0$.
- At each step the length of the interval known to contain a zero of f is reduced by a factor of 2; hence the smaller the better.

Ex. $f(x) = 2x^3 - x^2 + x - 1$, we have

$$f(-4) \cdot f(4) < 0 \quad \text{and} \quad f(0) \cdot f(1) < 0.$$

- How fast can the bisection method find the solution?

Convergence Rate of Bisection Method

Theorem 2

Suppose that $f \in C[a,b]$ *and* $f(a) \cdot f(b) < 0$*. The Bisection method generates a sequence* $\{p_n\}_{n=1}^{\infty}$ *approximating a zero* p *of* f *with*

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \textit{when } n \geq 1.$$

Convergence Rate of Bisection Method (cont'd)

Ex. Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Numerical Issues in Bisection Method

- Round-off error
- Overflow or underflow

Fixed-Point Finding Problem

Definition 1

The number p is a **fixed point** *for a given function g if $g(p) = p$.*

- Fixed point
- Root

Fixed-Point Finding and Root-Finding Problems

- If the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

  has a zero at p.

- Given $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example as

- Fixed-point form is easier to analyze. Certain fixed-point choices lead to powerful root-finding techniques.

Fixed-Point Finding Problem (cont'd)

Ex. Determine any fixed points of the function $g(x) = x^2 - 2$.

Existence and Uniqueness of A Fixed Point

g-continuous on [a,b]
g(x) ∈ [a,b] for ∀x ∈ [a,b]

Theorem 3

(a) *(Existence) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.*

(b) *(Uniqueness) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k<1$ exists with*

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$.

Theorem 4 (Mean Value Theorem)

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Existence and Uniqueness of A Fixed Point (cont'd)

Ex. Show that Theorem 3 does not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval exists.

Existence and Uniqueness of A Fixed Point (cont'd)

- Difficult to explicitly determine the fixed point of $g(x) = 3^{-x}$.

Fixed-Point Iteration (cont'd)

- To approximate the fixed point of a function g, choose an initial p_0, and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1.$$

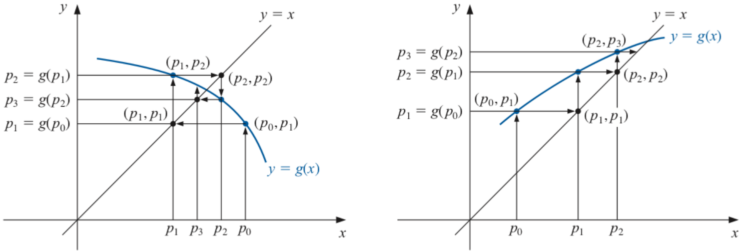
- If the sequence converges to p, and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p),$$

  and a solution to $x = g(x)$ is obtained. This is called **fixed-point iteration**.

Fixed-Point Iteration (cont'd)

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1.$$



Fixed-Point Iteration (cont'd)

Ex. The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. Transform the equation to the fixed-point form $x = g(x)$.

Fixed-Point Iteration (cont'd)

- How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

Fixed-Point Theorem

Theorem 5 (Fixed-Point Theorem)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, the g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \textit{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$.

Proof. Theorem 3 implies that a unique point p exists in $[a,b]$ with $g(p) = p$. Since g maps $[a,b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$, and $p_n \in [a,b]$ for all n. Then,

Fixed-Point Theorem (cont'd)

- Can we get rid of (unknown) $|p_0 - p|$?

Corollary 1

If g satisfies the hypotheses of Fixed-Point Theorem, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \quad \text{for all} \quad n \geq 1.$$

Fixed-Point Theorem (cont'd)

Proof. For $n \geq 1$, we have

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq k|p_n - p_{n-1}| \leq \cdots \leq k^n|p_1 - p_0|.$$

Thus for $m > n \geq 1$,

$$\begin{aligned} |p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \cdots + p_{n+1} - p_n| \\ &\leq |p_m - p_{m-1}| + \cdots + |p_{n+1} - p_n| \\ &\leq k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \cdots + k^n|p_1 - p_0| \\ &= k^n|p_1 - p_0|(1 + k + k^2 + \cdots + k^{m-n-1}). \end{aligned}$$

Then,

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$$

$$|p - p_n| = \lim_{m\to\infty} |p_m - p_n| \leq \lim_{m\to\infty} k^n|p_1 - p_0| \sum_{i=0}^{m-n-1} k^i$$

$$= k^n|p_1 - p_0| \sum_{i=0}^{\infty} k^i = \frac{k^n}{1-k}|p_1 - p_0|.$$

□

Fixed-Point Theorem (cont'd)

Ex. The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. Apply the fixed-point theorem to the fixed-point problem

$$x = g(x) := \left(\frac{10}{4+x}\right)^{\frac{1}{2}}.$$

Fixed-Point Theorem (cont'd)

- (Recall) How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

Newton's Method

- Newton introduced a method for finding a root of the equation

$$y^3 - 2y - 5 = 0,$$

which generates a sequence of polynomials.

Newton's Method (cont'd)

- Newton's method is based on Taylor polynomials.
- Suppose that $f \in C^2[a,b]$. Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small."
- Consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$.

$$0 = f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p-p_0)^2}{2} f''(\xi(p))$$

where $\xi(p)$ lies between p and p_0.

Newton's Method (cont'd])

- With the assumption that $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0).$$

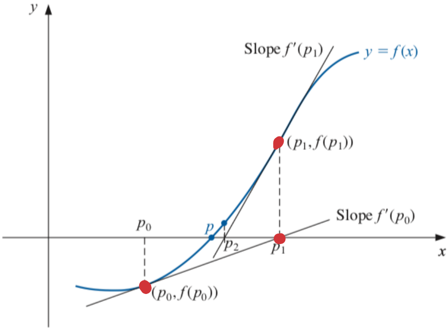
Solving for p gives

$$p \approx$$

Newton's Method (cont'd)

- Starting with p_0, Newton's method generates the sequence $\{p_n\}_{n=0}^{\infty}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$



- Newton's method is a fixed-point iteration with $p_n = g(p_{n-1})$.

Newton's Method (cont'd)

Ex. Approximate a root of $f(x) = \cos x - x = 0$ using (a) a fixed-point method, and (b) Newton's method.

(a) $p_n = \cos(p_{n-1})$

(b) $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}$

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Newton's Method	
n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Q. Is Newton's method effective for all cases?

Convergence of Newton's Method

Theorem 6

Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Proof. Consider Newton's method as a fixed-point iteration $p_n = g(p_{n-1})$, with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Using the Fixed-Point Theorem, it is enough to find an interval $[p - \delta, p + \delta]$ that g maps into itself and for which $|g'(x)| \leq k$, for all $x \in (p - \delta, p + \delta)$, where $k \in (0, 1)$. (Details omitted.) □

Q. How can we determine δ?

Q. How about the rate of convergence?

The Secant Method

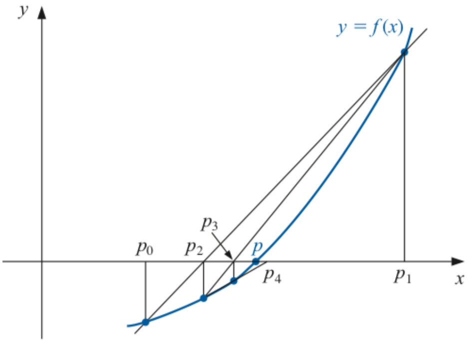
- Newton's method needs to know f' at each p_n.

Q. Can we replace f' by some approximation?

The Secant Method (cont'd)

- Starting with two initial p_0 and p_1, the secant method uses the approximation for $f'(p_{n-1})$ as

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$



Order of Convergence

- Let's study a new way of measuring how rapidly a sequence converges.

Definition 2

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=0}^{\infty}$ **converges to p of order α, with asymptotic error constant** *λ.*

- If $\alpha = 1$ (and $\lambda < 1$), the sequence is **linearly convergent**.
- If $\alpha = 2$, the sequence is **quadratically convergent**.

- If $\alpha = 1$ and $\lambda = 0$, the sequence is **superlinearly convergent**.

Order of Convergence (cont'd)

Ex. Suppose that $\{p_n\}_{n=0}^{\infty}$ is linearly convergent to 0 with

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.5$$

and that $\{\tilde{p}_n\}_{n=0}^{\infty}$ is quadratically convergent to 0 with

$$\lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = 0.5.$$

- For simplicity, assume that for each n we have

$$\frac{|p_{n+1}|}{|p_n|} \approx 0.5 \quad \text{and} \quad \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} \approx 0.5.$$

Then, compare the relative speed of convergence of the sequences to 0.

Order of Convergence (cont'd)

- The linearly convergent scheme satisfies

$$|p_n - 0| = |p_n| \approx 0.5|p_{n-1}| \approx (0.5)^2|p_{n-2}| \approx \cdots \approx (0.5)^n|p_0|$$

whereas the quadratically convergent scheme has

$$|\tilde{p}_n - 0| =$$

Order of Convergence (cont'd)

- For $|p_0| = |\tilde{p}_0| = 1$ (why?), compare the relative speed of convergence.

n	Linear Convergence Sequence $\{p_n\}_{n=0}^{\infty}$ $(0.5)^n$	Quadratic Convergence Sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ $(0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

Q. What is the rate of convergence of an arbitrary fixed-point iteration, under our setting? How about Newton's method?

Linear Convergence of Fixed-Point Iteration

Theorem 7

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose, in addition, that g' is continuous on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges only **linearly** *to the unique fixed point p in $[a,b]$.*

Proof. By Fixed-Point Theorem, the sequence converges to p. And ? $\square$

Q. Are there fixed-point methods with faster convergence? If yes, when?

Quadratic Convergence of Fixed-Point Iteration

Theorem 8

Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p.

Then there exists a $\delta > 0$ such that, for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least **quadratically** *to p.*

Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Proof. Choose k in $(0,1)$ and $\delta > 0$ such that on the interval $(p-\delta, p+\delta)$, contained in I, we have $|g'(x)| \leq k$. Then, g maps $[p-\delta, p+\delta]$ into itself. Using the Fixed-Point Theorem, $\{p_n\}_{n=0}^{\infty}$ converges to p.

Quadratic Convergence of Fixed-Point Iteration (cont'd)

Proof. Expanding $g(x)$ in a linear Taylor polynomial for $x \in [p-\delta, p+\delta]$ gives

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(\xi)}{2}(x-p)^2,$$

where ξ lies between x and p. We then have, for $x = p_n$,

$$p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n - p)^2,$$

with ξ_n between p_n and p.

$\{p_n\}_{n=0}^{\infty}$ converges to p, and so is $\{\xi_n\}_{n=0}^{\infty}$, and

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \frac{|g''(\xi_n)|}{2} = \frac{|g''(p)|}{2}.$$

$\square$

Q. How can we construct a fixed-point iteration with quadratic convergence?

Constructing Fixed-Point Iteration w/ Quadratic Conv.

- Consider the sequence $p_n = g(p_{n-1})$ for g in the form

$$g(x) = x - \phi(x)f(x)$$

where ϕ is a differentiable function.

Q. Which ϕ should we choose?

Multiple Roots

- Newton's method converges at least quadratically when $f'(p) \neq 0$.

Q. What should we do when $f'(p) = 0$?

Multiple Roots (cont'd)

Definition 3

A solution p of $f(x) = 0$ is a **zero of multiplicity** m *of f if for $x \neq p$, we can write $f(x) = (x - p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.*

- If $m = 1$, we say that f has a **simple zero**.

Q. Why do we care about simple zero?

Multiple Roots (cont'd)

Theorem 9

The function $f \in C^1[a,b]$ has a **simple zero** *at p in (a,b) if and only if $f(p) = 0$, but $f'(p) \neq 0$.*

Proof. "$\Rightarrow$": Assume that f has a simple zero at p, then $f(p) = 0$ and $f(x) = (x-p)q(x)$, where $\lim_{x \to p} q(x) \neq 0$. Since $f \in C^1[a,b]$,

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p} q(x) \neq 0.$$

"$\Leftarrow$": $\square$

Multiple Roots (cont'd)

Theorem 10

The function $f \in C^m[a,b]$ has a zero of multiplicity m at p in (a,b) if and only if $0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p)$, but $f^{(m)}(p) \neq 0$.

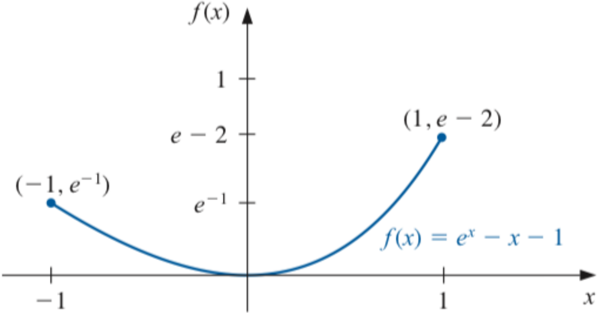
- Newton's method will have a problem when we have a zero of multiplicity higher than 1. Which problem?

Multiple Roots (cont'd)

Ex. Let $f(x) = e^x - x - 1$.

(a) Show that f has a zero of multiplicity 2 at $x = 0$.

(b) Show that Newton's method with $p_0 = 1$ converges to this zero but not quadratically.



n	p_n
0	1.0
1	0.58198
2	0.31906
3	0.16800
4	0.08635
5	0.04380
6	0.02206
7	0.01107
8	0.005545
9	2.7750×10^{-3}
10	1.3881×10^{-3}
11	6.9411×10^{-4}
12	3.4703×10^{-4}
13	1.7416×10^{-4}
14	8.8041×10^{-5}
15	4.2610×10^{-5}
16	1.9142×10^{-6}

Modified Newton's Method for Handling Multiple Roots

- Let p be a zero of multiplicity of m of f with

$$f(x) = (x-p)^m q(x).$$

Q. How can we make Newton's method to rapidly find p for $m > 1$ with a quadratic convergence?

Modified Newton's Method for Handling Multiple Roots

- Consider

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m q(x)}{m(x-p)^{m-1}q(x) + (x-p)^m q'(x)}.$$

- This can be rewritten as

$$\mu(x) = (x-p)\frac{q(x)}{mq(x) + (x-p)q'(x)},$$

which has p as a simple zero.

- Apply Newton's method to $\mu(x)$ as

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}.$$

Q. Any drawback?

Modified Newton's Method (cont'd)

Ex. Recall $f(x) = e^x - x - 1$ that has a zero of multiplicity 2 at $x = 0$.

- Newton's method:

$$g(x) = x - \frac{f(x)}{f'(x)}$$
$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{e-2}{e-1} \approx 0.58$$

- Modified Newton's method:

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$
$$p_1 = p_0 - \frac{f(p_0)f'(p_0)}{f'(p_0)^2 - f(p_0)f''(p_0)} = 1 - \frac{(e-2)(e-1)}{(e-1)^2 - (e-2)e} \approx -0.23$$

Accelerating Convergence

- Quadratic convergence is not easy to achieve.

- Aitken's Δ^2 method
- Steffensen's method: Modified Aitken's Δ^2 method

Aitken's Δ^2 Method

Q. Given a sequence $\{p_n\}_{n=0}^{\infty}$ that linearly converges to p,
can we construct a sequence with faster convergence?

Aitken's Δ^2 Method (cont'd)

- Further assume that the signs of $p_n - p$, $p_{n+1} - p$, and $p_{n+2} - p$ agree and that n is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

- We then have

$$(p_{n+1} - p)^2 \approx (p_{n+2} - p)(p_n - p),$$

and this can be rewritten as

$$(p_{n+2} + p_n - 2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2.$$

- We can further reformulate it as

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Aitken's Δ^2 Method (cont'd)

- Aitken's Δ^2 method is based on the assumption that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n},$$

converges more rapidly to p than does the original sequence $\{p_n\}_{n=0}^{\infty}$.

Ex. The sequence $\{p_n\}_{n=1}^{\infty}$, where $p_n = \cos(1/n)$, converges linearly(?) to $p = 1$. (Note: this sequence converges sublinearly with a rate $O(1/n^2)$.) Determine the first five terms of the sequence given by Aitken's Δ^2 method.

Aitken's Δ^2 Method (cont'd)

n	p_n	$\hat{p}_n$
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

Aitken's Δ^2 Method (cont'd)

Q. Why do we call it Δ^2 method?

Definition 4

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the **forward difference** Δp_n *is defined by*

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2.$$

- Aitken's Δ^2 method is equivalent to

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \quad \text{for } n \geq 0.$$

Aitken's Δ^2 Method (cont'd)

Theorem 10

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to the limit p and that

$$\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} < 1.$$

Then the Aitken's Δ^2 sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n-p}{p_n-p} = 0.$$

Steffensen's Method

- Recall Aitken's Δ^2 method:

$$\hat{p}_n = \{\Delta^2\}(p_n) = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Fixed-point iteration	Aitken's Δ^2 method
p_0	
$p_1 = g(p_0)$	
$p_2 = g(p_1)$	$\hat{p}_0 = \{\Delta^2\}(p_0)$
$p_3 = g(p_2)$	$\hat{p}_1 = \{\Delta^2\}(p_1)$
$\vdots$	$\vdots$

Q. Can we do better?

Steffensen's Method (cont'd)

- Apply fixed-point iteration to $\hat{p}_0$ instead of p_2.

Fixed-point	Aitken's	Fixed-point	Steffensen's
p_0		p_0	
$p_1 = g(p_0)$		$p_1 = g(p_0)$	
$p_2 = g(p_1)$	$\hat{p}_0 = \{\Delta^2\}(p_0)$	$p_2 = g(p_1)$	$\hat{p}_0 = \{\Delta^2\}(p_0)$
$p_3 = g(p_2)$	$\hat{p}_1 = \{\Delta^2\}(p_1)$	$p_3 = \hat{p}_0$	
$\vdots$	$\vdots$	$p_4 = g(p_3)$	
		$p_5 = g(p_4)$	$\hat{p}_1 = \{\Delta^2\}(p_3)$
		$\vdots$	$\vdots$

Theorem 11

Suppose that $x = g(x)$ has the solution p with $g'(p) \neq 1$. If there exists a $\delta > 0$ such that $g \in C^3[p-\delta, p+\delta]$, then Steffensen's method gives quadratic convergence for any $p_0 \in [p-\delta, p+\delta]$.

Steffensen's Method (cont'd)

- Steffensen's method:

$$\hat{p}_n = \hat{p}_{n-1} - \frac{(g(\hat{p}_{n-1}) - \hat{p}_{n-1})^2}{g(g(\hat{p}_{n-1})) - 2g(\hat{p}_{n-1}) + \hat{p}_{n-1}}$$

can be interpretated as a fixed-point iteration similar to Newton's method.

Steffensen's Method (cont'd)

- Consider a problem $f(x) = g(x) - x$ and its fixed-point iteration in a form, for some h, similar to Newton's method:

$$s(x) = x - \frac{f(x)}{\frac{f(x+h)-f(x)}{h}}.$$

- Let $h =$, then we have

$$s(x) =$$

Algebraic Polynomials

Definition 5

A **polynomial of degree** n *has the form*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_i*'s, called the* **coefficients** *of* P*, are constants and* $a_n \neq 0$.

Q. Why polynomials?

Q. Is there any benefit working on polynomials?

Algebraic Polynomials (cont'd)

Theorem 12

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly complex) root.

Algebraic Polynomials (cont'd)

Corollary 2

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, \ldots, x_k$, possibly complex, and unique positive integers $m_1, m_2, \ldots, m_k$, such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k}.$$

Corollary 3

Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all values of x.

- In other words, if two polynomials of degree n agree at least $(n + 1)$ distinct points, then they must be the same.

Horner's Method

Q. How can we compute $P(x)$ and $P'(x)$ efficiently?

Horner's Method (cont'd)

Theorem 13 (Horner's Method or Synthetic Division)

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. *Define* $b_n = a_n$ *and*

$$b_k = a_k + b_{k+1} x_0, \quad \text{for } k = n-1, n-2, \ldots, 1, 0.$$

Then $b_0 = P(x_0)$. *Moreover, if*

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1.$$

then

$$P(x) = (x - x_0) Q(x) + b_0.$$

- Q. What is the number of arithmetic operations needed to compute $P(x_0)$?
- Q. When does x_0 becomes a root of $P(x)$?
- Q. What do we additionally have from using the Horner's method?

Horner's Method (cont'd)

- Since $P(x) = (x - x_0)Q(x) + b_0$, where

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1,$$

differentiating with respect to x gives

$$P'(x) = Q(x) + (x - x_0)Q'(x) \quad \text{and} \quad P'(x_0) = Q(x_0).$$

- Computing $P(x)$ and $P'(x)$ in this efficient way will be useful in Newton's method.

Horner's Method (cont'd)

Ex. Use Horner's method to evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$.

Newton's Method Using Horner's Method

Ex. Find an approximation to a zero of $P(x) = 2x^4 - 3x^2 + 3x - 4$ using Newton's method with $x_0 = -2$ and synthetic division to evaluate $P(x_0)$ and $P'(x_0)$.

Deflation: Repeating Newton's Method

- The Nth iterate, x_N, of Newton's method is an approximate zero of P, so

$$P(x) = (x - x_N)Q(x) + b_0 = (x - x_N)Q(x) + P(x_N) \approx (x - x_N)Q(x).$$

- Let $\hat{x}_1 = x_N$ and $Q_1(x) \equiv Q(x)$, *i.e.*

$$P(x) \approx (x - \hat{x}_1)Q_1(x).$$

- Apply Newton's method to $Q_1(x)$, and so on.

Deflation: Repeating Newton's Method (cont'd)

- Newton's method is used on the reduced polynomial $Q_k(x)$, where

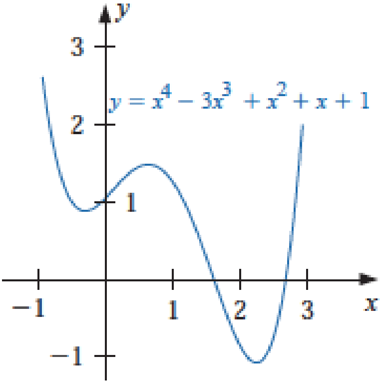
  $$P(x) \approx (x - \hat{x}_1)(x - \hat{x}_2) \cdots (x - \hat{x}_k) Q_k(x).$$

  So, a zero of $Q_k(x)$ may not generally approximate a zero of $P(x)$ well, especially as k increases.

- One could improve the approximations by applying Newton's method to the original $P(x)$, starting from $\hat{x}_k$.

Complex Zeros

Ex. Consider the polynomial $f(x) = x^4 - 3x^3 + x^2 + x + 1$.



Q. How can we find complex zeros?

Complex Zeros (cont'd)

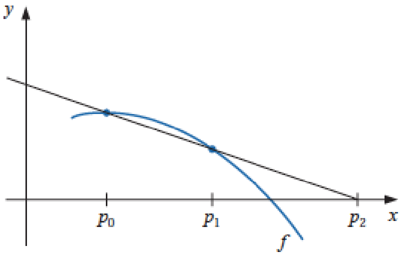
Theorem 14

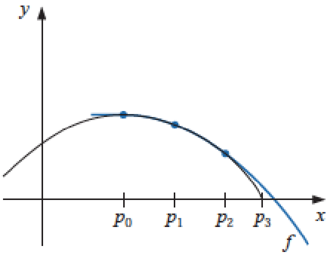
If $x = a + bi$ is a complex zero of multiplicity m of the polynomial $P(x)$ with real coefficients, then $\bar{z} = a - bi$ is also a zero of multiplicity m of the polynomial $P(x)$, and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$.

Complex Zeros: Müller's Method

- Recall: Given p_0 and p_1, Secant method determines p_2 as the intersection of the x-axis with the line through $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

- Müller method: Given p_0, p_1 and p_2, determines p_3 by considering the intersection of the x-axis with the parabola through $(p_0, f(p_0))$, $(p_1, f(p_1))$ and $(p_2, f(p_2))$.





Complex Zeros: Müller's Method (cont'd)

- In specific, consider the quadratic polynomial

$$P(x) = a(x - p_2)^2 + b(x - p_2) + c$$

that passes through $(p_0, f(p_0))$, $(p_1, f(p_1))$ and $(p_2, f(p_2))$, where the constants a, b and c by fitting

$$\begin{aligned} f(p_0) &= a(p_0 - p_2)^2 + b(p_0 - p_2) + c, \\ f(p_1) &= a(p_1 - p_2)^2 + b(p_1 - p_2) + c, \\ f(p_2) &= c. \end{aligned}$$

Complex Zeros: Müller's Method (cont'd)

- To determine p_3, we apply the quadratic formula to $P(x) = 0$, that is

$$p_3 - p_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}},$$

instead of $p_3 - p_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Complex Zeros: Müller's Method (cont'd)

- Among two choices, Müller's method chooses the one closer to p_2:

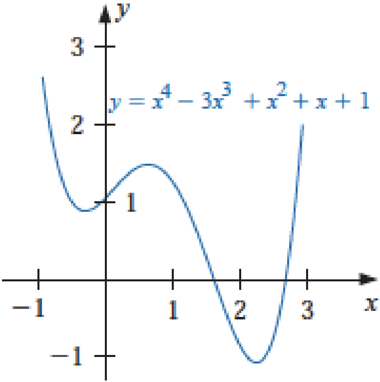
$$p_3 = p_2 - \frac{2c}{b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}}.$$

- Since p_n can be complex, so is a, b and c. Therefore, the use of $\operatorname{sgn}(b)$ in the textbook is incorrect, which does not consider the fact that b can be complex. Note that the signum function is defined as

$$\operatorname{sgn}(b) = \begin{cases} b/|b|, & b \neq 0, \\ 0, & b = 0. \end{cases}$$

Complex Zeros: Müller's Method (cont'd)

Ex. Consider the polynomial $f(x) = x^4 - 3x^3 + x^2 + x + 1$.



Complex Zeros: Müller's Method (cont'd)

$p_0 = 0.5,\ p_1 = -0.5,\ p_2 = 0$		
i	p_i	$f(p_i)$
3	$-0.100000 + 0.888819i$	$-0.01120000 + 3.014875548i$
4	$-0.492146 + 0.447031i$	$-0.1691201 - 0.7367331502i$
5	$-0.352226 + 0.484132i$	$-0.1786004 + 0.0181872213i$
6	$-0.340229 + 0.443036i$	$0.01197670 - 0.0105562185i$
7	$-0.339095 + 0.446656i$	$-0.0010550 + 0.000387261i$
8	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$
9	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$

$p_0 = 0.5,\ p_1 = 1.0,\ p_2 = 1.5$			$p_0 = 1.5,\ p_1 = 2.0,\ p_2 = 2.5$		
i	p_i	$f(p_i)$	i	p_i	$f(p_i)$
3	1.40637	−0.04851	3	2.24733	−0.24507
4	1.38878	0.00174	4	2.28652	−0.01446
5	1.38939	0.00000	5	2.28878	−0.00012
6	1.38939	0.00000	6	2.28880	0.00000
			7	2.28879	0.00000