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1) Assuming that the sequence $(p_n)_{n\geq 0}$ converges to p of order α, then from the definition, we have that $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$ For sufficiently large values of n

this means $\frac{|p_{n+1}-p|}{|p_n-p|^\alpha} \approx \lambda$ or just $|p_{n+1}-p| \approx \lambda |p_n-p|^\alpha$

From the notation, $\boxed{|e_{n+1}| \approx \lambda |e_n|^\alpha}$ Similarly, as we are choosing large n,

$\boxed{|e_n| \approx \lambda |e_{n-1}|^\alpha}$ Hence, $|e_{n+1}| \approx \lambda |e_n|^\alpha \approx \lambda \cdot (\lambda |e_{n-1}|^\alpha)^\alpha =$

$= \lambda \cdot \lambda^\alpha \cdot |e_{n-1}|^{\alpha^2} = \lambda^{\alpha+1} \cdot |e_{n-1}|^{\alpha^2}$ In the problem text, we are also given that $|e_{n+1}| \approx C |e_n| |e_{n-1}|$

Equating these 2 expressions yields that

$\lambda^{\alpha+1} \cdot |e_{n-1}|^{\alpha^2} \approx C |e_n| |e_{n-1}| \approx C \cdot (\lambda |e_{n-1}|^\alpha) |e_{n-1}| =$

$= C\lambda \cdot |e_{n-1}|^{\alpha+1}$ since $|e_n| \approx \lambda |e_{n-1}|^\alpha \Rightarrow$ therefore, we get

$\boxed{|e_{n-1}|^{\alpha^2-\alpha-1} = C \lambda^{-\alpha} \to \text{constant}}$ Since $\{p_n\}$ converges to p

$e_{n-1} = p_{n-1} - p$ approaches to zero for sufficiently large n: $\boxed{|e_{n-1}| \text{ approaches zero as } n \to \infty}$ Considering

C, λ, α - are constant values, we get $\underline{C\lambda^{-\alpha} \text{ is constant}}$

If $\alpha^2-\alpha-1>0$, then knowing $|e_{n-1}|$ approaching to zero, we would eventually obtain that $C\lambda^{-\alpha}$ should be equal to 0, via taking limits as $n \to \infty \Rightarrow$ However, this constant value is different from zero, hence $\boxed{\times}$

If $\alpha^2-\alpha-1<0$, then $|e_{n-1}| \to 0$ implies the RHS of equality $|e_{n-1}|^{\alpha^2-\alpha-1} \approx C\lambda^{-\alpha}$ should not be a constant (using limits could lead us to this as well) Therefore, this is impossible due to divergence of LHS, as the value tends to grow rapidly whenever $n \to \infty$ $\boxed{\times}$

So, $\boxed{\alpha^2-\alpha-1=0}$ should be true, $D=1+4=5$, $\alpha_{1,2}=\dfrac{1\pm\sqrt{5}}{2}$

As only one of the two solutions satisfies $\underline{\alpha>0}$, it is

$\boxed{\alpha = \dfrac{1+\sqrt{5}}{2}}$ $\boxed{\checkmark}$ (Here, $\alpha \approx 1.62$)

2) a) In order to prove quadratic convergence by the definition, we find the limit given from definition with

$$\alpha=2 \Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \lim_{n\to\infty} \frac{|10^{-2^{n+1}}-0|}{|10^{-2^n}-0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} \boxed{\checkmark}$$

$$= \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = \boxed{1}$$ Thus, $\boxed{\lambda = 1 > 0}$ $\underline{\text{meaning } \{p_n\} \text{ converges quadratically to } 0}$

b) Assume there is a value $k>1$ such that the given sequence converges quadratically to zero $\Rightarrow$ From the definition, it means for $\alpha=2$, the limit is equal to $\lambda>0$

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^2}=\lim_{n\to\infty}\frac{|10^{-(n+1)^k}-0|}{|10^{-n^k}-0|^2}=\lim_{n\to\infty}\frac{10^{-(n+1)^k}}{10^{-2n^k}}$$

$$=\lim_{n\to\infty}\frac{\frac{1}{10^{(n+1)^k}}}{\frac{1}{10^{2n^k}}}=\lim_{n\to\infty}\frac{10^{2n^k}}{10^{(n+1)^k}}=\lim_{n\to\infty}10^{2n^k-(n+1)^k}$$ and

$\rightarrow 2n^k-(n+1)^k$ is positive

since $\left(1+\frac{1}{n}\right)^k<\left(1+1\right)^k=2^k$, we wish to prove that $\rightarrow$

$(n+1)^k=n^k+\binom{k}{1}n^{k-1}+\dots+\binom{k}{1}n+1$ From these equations,

$$\lim_{n\to\infty}\frac{10^{2n^k}}{10^{(n+1)^k}}=\lim_{n\to\infty}\frac{10^{n^k}}{10^{(n+1)^k}}\cdot 10^{n^k}$$ where $2n^k-(n+1)^k=$

$=n^k-\binom{k}{1}n^{k-1}-\dots-\binom{k}{1}n-1$. If $P(x)=2x^k-(x+1)^k$, then

$P(x)=x^k-\binom{k}{1}x^{k-1}-\dots-\binom{k}{1}x-1$ has a positive leading coefficient

From the limiting behavior of polynomials, we can deduce that $\boxed{P(n) \text{ approaches to infinity as } n\to\infty}$ Therefore,

$\boxed{2n^k-(n+1)^k \text{ approaches to infinity for } n\to\infty}$ $\circledast$ This implies from the limit that $\boxed{\lim_{n\to\infty}10^{2n^k-(n+1)^k}=+\infty}$ $\circledast$ However, this contradicts the existence of limit λ; hence, our assumption about quadratic convergence was wrong $\Rightarrow$ $\{p_n\}$ does not converge to zero quadratically for any $k>1$ $\boxed{Y}$ ■

Note: The limiting behaviour of a function describes what happens to the function as $x \to \pm\infty$. The degree of a polynomial and sign of its leading coefficient dictates its limiting behaviour. In particular,

Degree of polynomial	Leading Coefficient +	Leading Coefficient −
Even	$f(x)\to\infty$ as $x\to\pm\infty$	$f(x)\to-\infty$ as $x\to\pm\infty$
Odd	$f(x)\to-\infty$ as $x\to-\infty$ $f(x)\to\infty$ as $x\to\infty$	$f(x)\to\infty$ as $x\to-\infty$ $f(x)\to-\infty$ as $x\to\infty$

4) Since $\frac{5}{3}=1.666...$ and $\frac{2}{3}=0.666...6...$ from 2-digit rounding arithmetic

We convert given equations into matrix form:

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 1.67 & 0.67 & 0.67 & 1 \\ 2 & 1 & 4 & 11 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + 1.67R_1} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.35 & 2.34 & 14.36 \\ 2 & 1 & 4 & 11 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_1} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.35 & 2.34 & 14.36 \\ 0 & 9 & 6 & 27 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - \frac{9}{7.35}R_2}$$

We get the resultant matrix

4) Since $\frac{5}{3} = 1.666... = 0.166...6... \times 10^1$, 2-digit rounding implies

$0.166...6... \times 10^1 + 0.005 \times 10^1 = 0.17166...6... \times 10^1$ and then

chopping yields $0.17 \times 10^1 = \underset{\sim}{1.7}$ Similarly, $\frac{2}{3} = 0.66...6...$

and $0.66...6... + 0.005 = 0.67166...6... \rightarrow$ chopping gives

$\underset{\sim}{0.67}$ The convertion of given equations into matrix form

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 1.7 & 0.67 & 0.67 & 1 \\ 2 & 1 & 4 & 11 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + 1.7R_1} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.47 & 2.37 & 1.7 \cdot 8 + 1 \\ 2 & 1 & 4 & 11 \end{array}\right]$$

where 2-digit rounding gives $7.47 = 0.747 \times 10^1$ as

$0.747 \times 10^1 + 0.005 \times 10^1 = 0.752 \times 10^1 \rightarrow 0.75 \times 10^1 = \underset{\sim}{7.5}$

$2.37 = 0.237 \times 10^1 \rightarrow 0.237 \times 10^1 + 0.005 \times 10^1 = 0.242 \times 10^1$ or

$0.24 \times 10^1 = \underset{\sim}{2.4}$ As $1.7 \cdot 8 = 13.6 = 0.136 \times 10^2 \rightarrow 0.136 \times 10^2 +$

$+ 0.005 \times 10^2 = 0.141 \times 10^2$ or just $0.14 \times 10^2 = \underset{\sim}{14}$, hence

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 2 & 1 & 4 & 11 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_1} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 9 & 6 & 27 \end{array}\right]$$

Since $\frac{9}{7.5} = 1.2 = 0.12 \times 10^1$ and $7.5 = 0.75 \times 10^1$ we will perform the opera-tion

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 9 & 6 & 27 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 1.2R_2} \left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 0 & 6-1.2\times 2.4 & 27-18 \end{array}\right]$$

Since $1.2\times 2.4 = 2.88 = 0.288\times 10^1 \rightsquigarrow 0.288\times 10^1 + 0.005\times 10^1$ $= 0.293\times 10^1$ or rounding/chopping gives $0.29\times 10^1 = 2.9$

$6-2.9 = 3.1 = 0.31\times 10^1 \Rightarrow \left[\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 0 & 3.1 & 9 \end{array}\right]$ is the resultant matrix

$$\left.\begin{array}{l} -x_1 + 4x_2 + x_3 = 8 \\ 7.5x_2 + 2.4x_3 = 15 \\ 3.1x_3 = 9 \end{array}\right\}$$

$x_3 = \dfrac{9}{3.1} = 2.9032258\ldots = 0.29032\ldots\times 10^1$

$0.29032258\ldots\times 10^1 + 0.005\times 10^1 = 0.29532\ldots\times 10^1 \to$ or just $0.29\times 10^1 = 2.9 \Rightarrow \boxed{x_3 = 2.9}$ $2.4x_3 = 2.4\times 2.9 =$ $= 6.96 = 0.696\times 10^1 \rightsquigarrow 0.696\times 10^1 + 0.005\times 10^1 = 0.701\times 10^1$ and chopping $\rightsquigarrow 0.70\times 10^1 = 7.0$; $7.5x_2 + 7 = 15$ and $x_2 = \dfrac{8}{7.5} = 1.0666\ldots6\ldots = 0.10666\ldots\times 10^1 \rightsquigarrow$ rounding gives $0.1066\ldots6\ldots\times 10^1 + 0.005\times 10^1 = 0.11166\ldots6\ldots\times 10^1$ and rounding/chopping $\rightsquigarrow 0.11\times 10^1 = 1.1$; hence, $\boxed{x_2 = 1.1}$ $4x_2 = 4.4 = 0.44\times 10^1$ and $-x_1 + 4.4 + 2.9 = 8$, giving

$-x_1+7.3=8 \Rightarrow -x_1=0.7$ or just $\boxed{x_1=-0.7}$ Therefore,

$\boxed{(x_1,x_2,x_3)=(-0.7,\ 1.1,\ 2.9)}$ is the resultant answer for the given linear system $\boxed{V}$

3) a) The n^{th} Taylor Polynomial for function $f(x)=e^{-x}$ with $x_0=0$ is given by $P_n(x)=\sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!}=$

$=\sum_{k=0}^{n}\frac{(-1)^k x^k}{k!}$, where $f^{(k)}(x)=(-1)^k e^{-x}$ for any $k\in N$

We also know that $e^{-x}=P_n(x)+R_n(x)=\sum_{k=0}^{n}\frac{(-1)^k x^k}{k!}+$

$+\frac{f^{(n+1)}(\xi(x))\,x^{n+1}}{(n+1)!}=\sum_{k=0}^{n}\frac{(-1)^k x^k}{k!}+\frac{(-1)^{n+1}e^{-\xi(x)}x^{n+1}}{(n+1)!}$

where $\xi(x)$ lies between 0 and $x \Rightarrow$ Since x is chosen as a fixed value, we get $e^{-x}=\sum_{k=0}^{n}\frac{(-1)^k x^k}{k!}+\cancel{(-1)^{n+1}\xi(x)} \Rightarrow$

$\boxed{e^{-x}=\sum_{k=0}^{n}\frac{(-1)^k x^k}{k!}+\frac{(-1)^{n+1}e^{-\xi}x^{n+1}}{(n+1)!} \text{ where } \xi \text{ lies between } 0 \text{ and } x}$

Considering the fact that $P_n(x)\to e^{-x}$, $R_n(x)\to 0$ as $n\to\infty$ it remains to show $\lim_{n\to\infty}\frac{P_{n+1}(x)-e^{-x}}{P_n(x)-e^{-x}}<1$

Determining the limit where $P_{n+1}(x)-e^{-x}=-R_{n+1}(x)$ and $P_n(x)-e^{-x}=-R_n(x)$, we get the following:

$$\lim_{n\to\infty} \frac{P_{n+1}(x)-e^{-x}}{P_n(x)-e^{-x}} = \lim_{n\to\infty} \frac{R_{n+1}(x)}{R_n(x)} = \lim_{n\to\infty} \frac{\dfrac{(-1)^{n+2}e^{-\xi_1}x^{n+2}}{(n+2)!}}{\dfrac{(-1)^{n+1}e^{-\xi_2}x^{n+1}}{(n+1)!}} =$$

$$= \boxed{\lim_{n\to\infty} \frac{e^{-\xi_1+\xi_2}\cdot x\cdot(-1)}{n+2} \text{ with } \xi_1,\xi_2\in(0,x)}$$ Choosing large n for which

$n > |xe^{-\xi_1+\xi_2}|$ where $-\xi_1+\xi_2 < \xi_2 < x$ which is a fixed value,

$|xe^{-\xi_1+\xi_2}| = |x|e^{-\xi_1+\xi_2}$ and $\left|\dfrac{e^{-\xi_1+\xi_2}(-x)}{n+2}\right| = \dfrac{|x|e^{-\xi_1+\xi_2}}{n+2} <$

$< \dfrac{|x|e^x}{n+2} < 1$, by choosing sufficiently large n, where (take $n\in\mathbb{N}$ with $n>|x|e^x$)

x is a fixed value $\Rightarrow$ $\boxed{\text{For large values of } n, \left|\dfrac{p_{n+1}-p}{p_n-p}\right| < 1}$

where $p_n = P_n(x)$ and

$p = e^{-x}$ is the limit of $\{p_n\}_{n=0}^{\infty}$. Since, we proved that

$\left|\dfrac{e^{-\xi_1+\xi_2}(-x)}{n+2}\right| < 1$, it implies $\boxed{\dfrac{e^{-\xi_1+\xi_2}(-x)}{n+2} < 1}$ where we've $|e^{-\xi_1+\xi_2}(-x)| =$

$= |x|e^{-\xi_1+\xi_2} < |x|e^x$, or $\boxed{e^{-\xi_1+\xi_2}(-x) \text{ lies between } -|x|e^x \text{ and } |x|e^x}$

Taking $n\to\infty$ and considering $\boxed{\text{limit}}$

$\lim_{n\to\infty} \dfrac{e^{-\xi_1+\xi_2}(-x)}{n+2} = 0 < 1$ as $e^{-\xi_1+\xi_2}(-x)$ is lying between a constant values $-|x|e^x$ and $|x|e^x$; therefore, taking large n and as it approaches infinity, $\lim\limits_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} = \lim\limits_{n\to\infty} \frac{e^{-\xi_1+\xi_2}(-x)}{n+2}$

$= 0 < 1$ will become true, whenever e^{-x} is a limit for $\{P_n(x)\}_{n=0}^{\infty}$ $\boxed{\lim\limits_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} < 1 \text{ is true}}$ ✱ n^{th} Taylor was considered about the point $x_0 = 0$, as $p_n = P_n(x)$ $\boxed{V}$ for fixed x

b) In order to determine the terms of Aitken's Δ^2 sequence up to the term $\hat{p}_5$, we need to know the terms up to p_7. Note that $p_n = P_n(1) = \sum\limits_{k=0}^{n} \frac{(-1)^k}{k!}$

The values of p_n are given in the list below:

$p_0 \longrightarrow 1$ Applying the formula from the sequence $\{p_n\}$

$p_1 \longrightarrow 0$ $\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$ for $n = 0, \dots, 5$

$p_2 \longrightarrow \frac{1}{2}$ We find that

$p_3 \longrightarrow \frac{1}{3}$ We can iteratively find the following values:

$p_4 \longrightarrow \frac{3}{8}$ $\hat{p}_0 = 0.3333\dots3\dots$ and 5-digit rounding

$p_5 \longrightarrow \frac{11}{30}$ gives $0.333\dots3\dots + 0.000005 =$

$p_6 \longrightarrow \frac{53}{144}$ $= 0.33333833\dots3 \rightsquigarrow \boxed{\hat{p}_0 = 0.33333}$ ✱

$p_7 \longrightarrow \frac{103}{280}$ $\hat{p}_1 = 0.374999\dots9\dots$ and $0.374999\dots +$

$+ 0.000005 = 0.37500499\dots9\dots \rightsquigarrow$ or just

$\boxed{\hat{p}_1 = 0.37500}$ $\hat{p}_2 = 0.366666...6...$ and $0.366666... +$

$+ 0.000005 = 0.36667166...$ $\rightsquigarrow$ $\boxed{\hat{p}_2 = 0.36667}$ Similarly

$\hat{p}_3 = 0.368055...$ and $0.368055... + 0.000005 = 0.3680605...$

$\boxed{\hat{p}_3 = 0.36806}$ $\hat{p}_4 = 0.36785714285714...$ and

rounding $0.3678571... + 0.000005 = 0.3678621428...$ or

$\boxed{\hat{p}_4 = 0.36786}$ $\hat{p}_5 = 0.367881944...4...$, rounding gives

$0.36788194... + 0.000005 = 0.36788694...4...$ chopping $\rightarrow$

$\boxed{\hat{p}_5 = 0.36788}$ These are the <u>5-digit rounding values for $\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5$</u> $\boxed{\checkmark}$