2021 Spring MAS 365: Midterm Exam

Write the following Honor Pledge and sign your name under it.

> *"I have neither given nor received aid on this examination, nor have I concealed a violation of the Honor Code."*

1. [35 (5+15+10+5) points] Consider a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}.$$

Suppose that a two-digit rounding was used when storing A and $\boldsymbol{b}$, and let the corresponding matrices to be $\tilde{A}$ and $\tilde{\boldsymbol{b}}$, respectively. Let $\boldsymbol{x} = (x_1, x_2)$ and $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2)$ be the solutions of $A\boldsymbol{x} = \boldsymbol{b}$ and $\tilde{A}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$, respectively. Note that $\boldsymbol{x} = (1.5, -0.5)$. Use l_∞ norm through out this problem, and consider the theorem below.

Theorem 1. *Suppose A is nonsingular and $||\delta A|| < \frac{1}{||A^{-1}||}$. The solution $\tilde{\boldsymbol{x}}$ to $(A + \delta A)\tilde{\boldsymbol{x}} = \boldsymbol{b} + \delta\boldsymbol{b}$ approximates the solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \boldsymbol{b}$ with the error estimate*

$$\frac{||\boldsymbol{x} - \tilde{\boldsymbol{x}}||}{||\boldsymbol{x}||} \le \frac{K(A)||A||}{||A|| - K(A)||\delta A||} \left(\frac{||\delta\boldsymbol{b}||}{||\boldsymbol{b}||} + \frac{||\delta A||}{||A||} \right).$$

(a) Determine $\tilde{A}$ and $\tilde{\boldsymbol{b}}$.

(b) Use Gaussian elimination with a partial pivoting and a two-digit rounding to approximate the solution $\tilde{\boldsymbol{x}}$ of $\tilde{A}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$.

(c) Find an upper bound of $\frac{||\boldsymbol{x} - \tilde{\boldsymbol{x}}||}{||\boldsymbol{x}||}$.

(d) $\tilde{x}_1$ approximates x_1 to t significant digits. Find a tight lower bound of t using (c).

2. [30 (15+15) points] Let $D = \{(x_1, x_2) \;:\; a \le x_1 \le b,\; c \le x_2 \le d\} \subset \mathbb{R}^2$. Suppose that $\boldsymbol{G}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}))^t : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function from D into $\mathbb{R}^2$ with the property that $\boldsymbol{G}(\boldsymbol{x}) \in D$ whenever $\boldsymbol{x} \in D$. Assume that $\boldsymbol{G}(\boldsymbol{x})$ has a unique fixed point $\boldsymbol{p} = (p_1, p_2)^t \in D$.

(a) Show that $||\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{G}(\boldsymbol{y})||_\infty \le K||\boldsymbol{x} - \boldsymbol{y}||_\infty$ for any $\boldsymbol{x}, \boldsymbol{y} \in D$, under the assumption that all the component functions of $\boldsymbol{G}$ have continuous partial derivatives and a constant $K < 1$ exists with, for $i = 1, 2$ and $j = 1, 2$,

$$\left|\frac{\partial g_i(\boldsymbol{x})}{\partial x_j}\right| \le \frac{K}{2}, \quad \text{whenever } \boldsymbol{x} \in D.$$

Consider the following Taylor's Theorem in two variables.

Theorem 2. *Supposed that $g(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ and all its partial derivatives are continuous on $D = \{(x_1, x_2) \;:\; a \le x_1 \le b,\; c \le x_2 \le d\}$ and $(\bar{x}_1, \bar{x}_2) \in D$. For every (x_1, x_2), there exists ξ_1 between x_1 and $\bar{x}_1$, and ξ_2 between x_2 and $\bar{x}_2$ with*

$$g(x_1, x_2) = g(\bar{x}_1, \bar{x}_2) + (x_1 - \bar{x}_1)\frac{\partial g(\xi_1, \xi_2)}{\partial x_1} + (x_2 - \bar{x}_2)\frac{\partial g(\xi_1, \xi_2)}{\partial x_2}.$$

(b) Using (a), show that the sequence defined by $\boldsymbol{x}^{(k)} = \boldsymbol{G}(\boldsymbol{x}^{(k-1)})$ for $k \ge 1$ satisfies $||\boldsymbol{x}^{(k)} - \boldsymbol{p}||_\infty \le \frac{K^k}{1-K}||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||_\infty$, for any $\boldsymbol{x}^{(0)}$ in D.

3. [25 (5+10+10) points] Consider solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A is a $n \times n$ matrix, by a method **R**:

$$\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + w(\boldsymbol{b} - A\boldsymbol{x}^{(k-1)})$$

for $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ and for some positive w.

(a) State a condition on w that makes the sequence $\{\boldsymbol{x}^{(k)}\}$ of the method **R** converge to the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$.

(b) Assume that A is a diagonal matrix with diagonal elements $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Simplify the condition on w in (a) into a form $c < w < d$. (Specify c and d.)

(c) Considering the relationship between the (simultaneous) Jacobi method and the (sequential) Gauss-Seidel Method, derive a (sequential) Gauss-Seidel-like method of the (simultaneous) method **R** in a form $\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$.

4. [30 (10+10+10) points] Consider a system of nonlinear equations

$$\begin{aligned} f_1(x_1, x_2) &= x_1^2 + x_2^2 - 2 = 0, \\ f_2(x_1, x_2) &= x_1 - x_2 = 0. \end{aligned}$$

(a) Show that one iteration of Newton's method gives $\boldsymbol{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})^t$ with

$$x_1^{(1)} = x_2^{(1)} = \frac{(x_1^{(0)})^2 + (x_2^{(0)})^2 + 2}{2(x_1^{(0)} + x_2^{(0)})},$$

starting from $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)})^t$.

(b) Show that the iteration converges to a fixed point $(1,1)^t$, if $1 \le x_1^{(0)} + x_2^{(0)} \le M$, possibly using the theorem below.

Theorem 3. *Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with*

$$|g'(x)| \le k, \quad \textit{for all } x \in (a,b).$$

Then, for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

converges to the unique fixed point p in $[a,b]$.

(c) Verify that the convergence of $\{x_1^{(n)}\}$ is quadratic, if $1 \le x_1^{(0)} + x_2^{(0)} \le M$. (Do not directly use any theorem in the textbook.)

5. [30 (15+5+10) points] The directional derivative of g at $\boldsymbol{x}$ in the direction of $\boldsymbol{v}$ is defined by

$$D_{\boldsymbol{v}}g(\boldsymbol{x}) = \lim_{h\to 0}\frac{1}{h}[g(\boldsymbol{x}+h\boldsymbol{v}) - g(\boldsymbol{x})] = \boldsymbol{v}^t\nabla g(\boldsymbol{x}).$$

The steepest direction with respect to ℓ_2 norm is found by solving

$$\min_{\boldsymbol{v}\,:\,||\boldsymbol{v}||_2=1} D_{\boldsymbol{v}}g(\boldsymbol{x}).$$

(a) Determine the steepest direction with respect to $||D^{1/2}\cdot||_2$ by solving

$$\min_{\boldsymbol{v}\,:\,||D^{1/2}\boldsymbol{v}||_2=1} D_{\boldsymbol{v}}g(\boldsymbol{x}),$$

where D is a diagonal matrix.

(b) State a condition for a nonzero vector $\boldsymbol{v}$ to be called a descent direction of g at $\boldsymbol{x}$.

(c) Show that a direction $\boldsymbol{v}^{(k+1)}$ of a conjugate gradient method

$$\boldsymbol{r}^{(0)} = \boldsymbol{b} - A\boldsymbol{x}^{(0)}, \quad \boldsymbol{v}^{(1)} = \boldsymbol{r}^{(0)},$$

For $k = 1, 2, \ldots, n$

$$t_k = \frac{\langle \boldsymbol{v}^{(k)}, \boldsymbol{r}^{(k-1)}\rangle}{\langle \boldsymbol{v}^{(k)}, A\boldsymbol{v}^{(k)}\rangle}, \quad \boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + t_k\boldsymbol{v}^{(k)}, \quad \boldsymbol{r}^{(k)} = \boldsymbol{r}^{(k-1)} - t_kA\boldsymbol{v}^{(k)}$$

$$s_k = \frac{\langle \boldsymbol{r}^{(k)}, \boldsymbol{r}^{(k)}\rangle}{\langle \boldsymbol{r}^{(k-1)}, \boldsymbol{r}^{(k-1)}\rangle} \quad \boldsymbol{v}^{(k+1)} = \boldsymbol{r}^{(k)} + s_k\boldsymbol{v}^{(k)},$$

for a positive definite matrix A, is a descent direction of some function at some point under some condition. Specify the corresponding function, point and condition. (Do not directly use any theorem in the textbook.)