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3) (1) Define the function $f(x) = x - \sin x$. Observe that $f(0) = 0$ and $f'(x) = 1 - \cos x \geq 0$. The derivative is equal to 0 only at isolated points, so the function increases in the interval $[0, \infty) \Rightarrow \forall x > 0$, we have $f(x) > f(0) = 0$. Thus, $\boxed{x > \sin x \text{ for all } x > 0}$ ☒

In particular, $\sin x \leq x$ for all $0 \leq x < 1$. Since $0 < \frac{1}{n} \leq 1$ as $n \to \infty \Rightarrow \left|\sin\left(\frac{1}{n}\right)\right| < \frac{1}{n}$, meaning $\left|\sin\left(\frac{1}{n}\right) - 0\right| < \frac{1}{n}$ and $\boxed{\sin\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right)}$ Squaring both sides gives us

$$\left|\left(\sin\left(\frac{1}{n}\right)\right)^2\right| = \left|\left(\sin\left(\frac{1}{n}\right)\right)^2 - 0\right| < \frac{1}{n^2},$$ hence we conclude

$\boxed{\left(\sin\left(\frac{1}{n}\right)\right)^2 = O\left(\frac{1}{n^2}\right) \text{ with } p = 2}$ ☒ $\boxed{\checkmark}$.

(2) The first Taylor Polynomial about $h = 0$ is $\cos h = 1 - \frac{h^2}{2}\cos \xi(h)$, where $0 < \xi(h) < h$ holds

$\frac{1 - \cos h}{h} = \frac{1}{2} h \cos \xi(h)$, hence $\left|\frac{1 - \cos h}{h} - 0\right| = \left|\frac{h}{2}\cos \xi(h)\right|$

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Since $|\cos\xi(h)| \le 1 \Rightarrow \left|\frac{1-\cos h}{h} - 0\right| = \left|\frac{h}{2}\cos\xi(h)\right| \le \frac{1}{2}|h|$

for sufficiently small $h \Rightarrow \boxed{\frac{1-\cos h}{h} = O(h)}$ ✓.

2) We will consider 2 seperate cases for this problem:

(1) $\underbrace{d_{k+1} < 5}$ $\Rightarrow \left|\frac{y - fl(y)}{y}\right| = \left|\frac{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n - 0.d_1\ldots d_k \times 10^n}{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n}\right|$

Because of the definition from $fl(y) \Rightarrow \left|\frac{y-fl(y)}{y}\right| =$

$$= \left|\frac{0.\overbrace{00\ldots0}^{k} d_{k+1}\ldots \times 10^n}{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n}\right| = \left|\frac{0.d_{k+1}d_{k+2}\ldots \times 10^{n-k}}{0.d_1 d_2\ldots \times 10^n}\right| =$$

$$= \left|\frac{0.d_{k+1}d_{k+2}\ldots}{0.d_1 d_2\ldots}\right| \times 10^{-k} \le \left|\frac{0.d_{k+1}d_{k+2}\ldots}{0.1}\right| \times 10^{-k} \le \left|\frac{0.5}{0.1}\right| \times 10^{-k} =$$

$= 5 \times 10^{-k}$, where we had $d_1 \ge 1 \Rightarrow 0.d_1 d_2\ldots \ge 0.1$ and $d_{k+1} < 5 \Rightarrow 0.d_{k+1}d_{k+2}\ldots \le 0.5$; thus, we are done in this case ✓

(2) $\underbrace{d_{k+1} \ge 5}$ $\Rightarrow \left|\frac{y - fl(y)}{y}\right| = \left|\frac{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n - (0.d_1\ldots d_k \cdot 10^n + 10^{-k}\cdot 10^n)}{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n}\right|$

Because of $d_{k+1} \ge 5$, $fl(y) = 0.d_1\ldots d_k \cdot 10^n + 0.\overbrace{00\ldots0}^{k-1}1 \times 10^n =$
$= 0.d_1\ldots d_k \times 10^n + 10^{-k} \times 10^n = 0.d_1\ldots d_k \times 10^n + 10^{n-k}$

$$\left|\frac{y-fl(y)}{y}\right| = \left|\frac{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n - 0.d_1\ldots d_k \times 10^n - 10^{-k}\times 10^n}{0.d_1\ldots d_k d_{k+1}\ldots \times 10^n}\right|$$

$$= \left|\frac{0.\overbrace{00\ldots0}^{k} d_{k+1}\ldots \times 10^n - 10^{-k}\times 10^n}{0.d_1 d_2\ldots \times 10^n}\right| = \left|\frac{0.\overbrace{0\ldots0}^{k} d_{k+1}\ldots - 10^{-k}}{0.d_1 d_2\ldots}\right|$$

$$= \left|\frac{0.\overbrace{00\ldots0}^{k} d_{k+1}\ldots - 0.\overbrace{00\ldots0}^{k-1}1}{0.d_1 d_2\ldots}\right| = \frac{\left|0.\overbrace{00\ldots0}^{k-1}1 - 0.\overbrace{00\ldots0}^{k-1}0 d_{k+1}\ldots\right|}{|0.d_1 d_2\ldots|}$$

Since $d_{k+1} \geq 5 \Rightarrow 0.\overbrace{00\ldots0}^{k-1}1 - 0.\overbrace{00\ldots0}^{k-1}0\, d_{k+1}\ldots \leq$
$\leq 0.\overbrace{00\ldots0}^{k-1}1 - 0.\overbrace{00\ldots0}^{k-1}0\,5 = 0.\overbrace{00\ldots0}^{k-1}0\,5 = 0.\overbrace{00\ldots0}^{k}5 =$
$= 0.5\times 10^{-k}$; therefore, using $d_1 \geq 1$ gives $0.d_1 d_2\ldots \geq$
≥ 0.1 and $\dfrac{\left|0.\overbrace{00\ldots0}^{k-1}1 - 0.\overbrace{00\ldots0}^{k-1}0\, d_{k+1}\ldots\right|}{|0.d_1 d_2\ldots|} \leq \dfrac{0.5\times 10^{-k}}{0.1}$

$= 5\times 10^{-k}$, meaning $\left|\dfrac{y-fl(y)}{y}\right| \leq 5\times 10^{-k}$, done $\boxed{\text{V}}$ ■

4) a) Consider the equation $x^2-17=0$, where the positive root is equal to $\sqrt{17} \Rightarrow f(x)=x^2-17$ has positive root $\sqrt{17}$ where we want to find an approximation for it accurate to within 10^{-3} by implementing bisection method

Since $f(2)=4-17<0$ and $f(5)=25-17>0$, from the intermediate value theorem, as $f(x)=x^2-17$ is continuous on the interval $[2,5]$, there exist root r such that $r\in(2,5)$. The bisection method iteratively computes the middle element of the current interval, and then comparing the sign of $f(x)$ value at that point will determine which half-subinterval we have to look for: If root r is located on the (a_i,b_i), and $p_i=\frac{a_i+b_i}{2}=a_i+\frac{b_i-a_i}{2}$, then we look to the sign of $f(p_i)$, and if $f(p_i)f(a_i)>0 \Rightarrow$ set $a_{i+1}=p_i$; $b_{i+1}=b_i$ otherwise, $a_{i+1}=a_i$ and $b_{i+1}=p_i$. Then, reapply same procedure to the interval $(a_{i+1}; b_{i+1})$ (If p_i is the root then just output it)

Moreover, when it comes to the termination of algorithm, we'll apply Theorem 1: Since f is differentiable on $[2,5]$, and $f(2)f(5)<0$, Bisection Method above will generate the sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero r of f with $|p_n-r|\leq\frac{b-a}{2^n}=\frac{5-2}{2^n}=\frac{3}{2^n}$

It's important to notice that the above result gives only a bound for approximation error, and that this bound might be quite conservative.

We'll use logarithms to find an integer N such that $|p_N - r| \leq \frac{3}{2^N} < 10^{-3}$ or just $\log_{10}\left(\frac{3}{2^N}\right) = \log_{10} 3 -$

$- \log_{10} 2^N = \log_{10} 3 - N \log_{10} 2 < \log_{10} 10^{-3} = -3 \Rightarrow$

$\log_{10} 3 + 3 < N \log_{10} 2$ or $N > \frac{\log_{10} 3 + 3}{\log_{10} 2} \approx 11.55075 \Rightarrow$

Hence, 12 iterations will ensure an approximation accurate to within 10^{-3}. Now, let's indeed compute numerically the values of sequence $\{p_n\}$

$a_1 = 2,\ b_1 = 5,\ p_1 = 3.5,\ f(p_1) = -4.75$

$a_2 = 3.5,\ b_2 = 5,\ p_2 = 4.25,\ f(p_2) = 1.0625$

$a_3 = 3.5,\ b_3 = 4.25,\ p_3 = 3.875,\ f(p_3) = -1.984375$

$a_4 = 3.875,\ b_4 = 4.25,\ p_4 = 4.0625,\ f(p_4) = -0.496094$

$a_5 = 4.0625,\ b_5 = 4.25,\ p_5 = 4.15625,\ f(p_5) = 0.274414$

$a_6 = 4.0625,\ b_6 = 4.15625,\ p_6 = 4.109375,\ f(p_6) = -0.113037$

$a_7 = 4.109375,\ b_7 = 4.15625,\ p_7 = 4.132813,\ f(p_7) = 0.080139$

$a_8 = 4.109375,\ b_8 = 4.132813,\ p_8 = 4.121094,\ f(p_8) = -0.016586$

$a_8 = 4.121094$, $b_8 = 4.132813$, $p_8 = 4.126953$, $f(p_8) = 0.031742$
$a_{10} = 4.121094$, $b_{10} = 4.126953$, $p_{10} = 4.124023$, $f(p_{10}) = 0.007569$
Since $\sqrt{17} \approx 4.1231056$, we can also check that (p is the root of f)
$|p_7 - p| = |4.132813 - p| < |b_8 - a_8| = 0.023438$
Since $|a_8| < |p|$, we get $\frac{|p-p_7|}{|p|} < \frac{|b_8 - a_8|}{|a_8|} = \frac{0.023438}{4.109375}$
$\approx 0.0057035 < 0.01 = 10^{-2}$, so the approximation is correct to at least within 10^{-2} (It is easy to observe from the first few decimals of $\sqrt{17}$ that p_6 does not satisfy the desired approximation within 10^{-2} and → as well $a_8 \to 10^{-3}$)

B) From the previously mentioned theorem 2.1, we have
$|p_N - p| \leq \frac{b-a}{2^N} = \frac{5-2}{2^N} \leq 10^{-3} \Rightarrow \frac{3}{2^N} \leq \frac{1}{1000}$ or just
$2^N \geq 3000 > 2048 = 2^{11} \Rightarrow N > 11$, equivalently $\boxed{N \geq 12}$
with $2^{12} = 4096 > 3000$. Hence, 12 iterations will ensure an approximation accurate to within 10^{-3}

Note: It's crucial to keep in mind that the error analysis gives only a bound for the # of iterations. In many cases, this bound is much larger than actual number (values have been found in previous part)

$a_{11} = 4.121094$, $b_{11} = 4.124023$, $p_{11} = 4.122559$, $f(p_{11}) = -0.004511$

$a_{12} = 4.122559$, $b_{12} = 4.124023$, $p_{12} = 4.123291$, $f(p_{12}) = 0.001529$

5) a) Since $g'(x) = -e^{-x}$, the function g is continuous and $g'(x)$ exists on $\left[\frac{1}{3}, 1\right]$. Moreover, $g'(x) = -\frac{1}{e^x} < 0$ on the interval $\left[\frac{1}{3}, 1\right]$, implying that $g(x)$ is a decreasing function on this interval: $g\left(\frac{1}{3}\right) \geq g(x) \geq g(1)$ for $\forall x \in \left[\frac{1}{3}, 1\right]$, or just $\boxed{e^{-\frac{1}{3}} \geq g(x) \geq e^{-1}}$ and

$0.7165313 < 1$, meaning $1 > 0.71654 > e^{-\frac{1}{3}} \geq g(x) \geq \frac{1}{e} \approx 0.36787944117 > 0.333...3... = \frac{1}{3}$, with inequality $3 > e \approx 2.7$

$\boxed{\forall x \in \left[\frac{1}{3}, 1\right] \Rightarrow g(x) \in \left[\frac{1}{3}, 1\right]}$ Hence, (i) of Theorem 2.3 is satisfied. To confirm (ii) of Theorem 2.3, notice that $g'(x) = -e^{-x}$ where for $x \in \left[\frac{1}{3}, 1\right] \Rightarrow g''(x) = e^{-x} = \frac{1}{e^x} > 0$, meaning that $\boxed{g'(x) \text{ exists on } \left(\frac{1}{3}, 1\right)}$

$g'(x)$ is an increasing function on the interval $\left[\frac{1}{3}, 1\right] \Rightarrow |g'(x)| \leq \left|g'\left(\frac{1}{3}\right)\right|$ for $\forall x \in \left[\frac{1}{3}, 1\right]$, meaning that

$g'(x) = -e^{-x} : \left(\frac{1}{3}, 1\right) \longrightarrow (-1) \cdot \left(g(1), g\left(\frac{1}{3}\right)\right) = \left(-g\left(\frac{1}{3}\right), -g(1)\right)$

We found $g'(x): \left(\frac{1}{3}, 1\right) \longrightarrow \left(-g\left(\frac{1}{3}\right), -g(1)\right) =$

$= \left(-e^{-\frac{1}{3}}, -e^{-1}\right)$ and $\max_{x\in\left(\frac{1}{3},1\right)} |g'(x)| < e^{-\frac{1}{3}} < 1$ since

$e > 2 > 1$, $e^{\frac{1}{3}} > 1$ or just $e^{-\frac{1}{3}} \approx 0.71653131 1 < 1$ ✓

$\boxed{\max_{x\in\left(\frac{1}{3},1\right)} |g'(x)| < e^{-\frac{1}{3}} < 1 \quad \text{where } k = e^{-\frac{1}{3}} > 0}$ So, (ii) of Theorem 2.3 is satisfied and $g(x)$ must have a unique fixed point on $\left[\frac{1}{3}, 1\right]$ $\boxed{V}$

b) Since we proved in the previous part that $\max_{x\in\left(\frac{1}{3},1\right)} |g'(x)| < e^{-\frac{1}{3}}$, we can take $\boxed{k = e^{-\frac{1}{3}}}$ in Corollary 2.5

As we want $|p_n - p| < 10^{-4}$, this will be true when $k^n \cdot \max\{p_0 - a, b - p_0\} \leq 10^{-4}$, where $p_0 = \frac{2}{3}$, $a = \frac{1}{3}$, $b = 1$

$e^{-\frac{n}{3}} \cdot \max\left\{\frac{1}{3}, \frac{1}{3}\right\} = \frac{1}{3} e^{-\frac{n}{3}} \leq 10^{-4} \Rightarrow \frac{1}{3} \cdot \frac{1}{e^{\frac{n}{3}}} \leq \frac{1}{10^4}$

$10^4 \leq 3e^{\frac{n}{3}}$ or $\frac{10^4}{3} \leq e^{\frac{n}{3}} \Rightarrow \ln\left(\frac{10^4}{3}\right) \leq \frac{n}{3}$, $\boxed{3\ln\left(\frac{10^4}{3}\right) \leq n}$

$3\ln\left(\frac{10000}{3}\right) \approx 24.3352$, we get $n \geq 24.3352$, giving us $n \geq \boxed{25}$ minimum number of iterations to achieve 10^{-4} accuracy bound (2.5)

For Bound (2.6), we want $\frac{k^n}{1-k}|p_1-p_0| \leq 10^{-4}$, where (achieve the required bound with...)

$p_0 = \frac{2}{3} \Rightarrow p_1 = g(p_0) = e^{-p_0} = e^{-\frac{2}{3}}$, since $k = e^{-\frac{1}{3}} \Rightarrow$

$\frac{e^{-\frac{n}{3}}}{1-e^{-\frac{1}{3}}}\left|e^{-\frac{2}{3}} - \frac{2}{3}\right| \leq 10^{-4}$ and $e^{-\frac{2}{3}} = \frac{1}{e^{\frac{2}{3}}} < 1$ where

$e^{-\frac{2}{3}} \approx 0.513417 < 0.666\ldots = \frac{2}{3}$; $\frac{e^{-\frac{n}{3}}\left(\frac{2}{3} - e^{-\frac{2}{3}}\right)}{1-e^{-\frac{1}{3}}} \leq \frac{1}{10^4}$

$\frac{10^4\left(\frac{2}{3} - e^{-\frac{2}{3}}\right)}{1-e^{-\frac{1}{3}}} \leq e^{\frac{n}{3}}$ and $\ln\left(\frac{10000\left(\frac{2}{3} - e^{-\frac{2}{3}}\right)}{1-e^{-\frac{1}{3}}}\right) \leq$

$\leq \frac{n}{3} \Rightarrow \boxed{n \geq 3\ln\left(\frac{10^4\left(\frac{2}{3} - e^{-\frac{2}{3}}\right)}{1-e^{-\frac{1}{3}}}\right) \approx 25.785919}$ So,

Bound (2.6) gives $n \geq \boxed{26}$ minimum # of iterations required to achieve 10^{-4} accuracy $\boxed{V}$ ■ (In practice, fewer iterations are needed, it's just estimate)

1) a) $\frac{(-2)^0}{1} + \frac{(-2)^1}{1} + \frac{(-2)^2}{2} + \frac{(-2)^3}{6} + \frac{(-2)^4}{24} + \frac{(-2)^5}{120}$

Since $\frac{(-2)^3}{6} = -\frac{8}{6} = -1.333\ldots3\ldots = -0.133\ldots3\ldots \times 10^1$, from the 3-digit chopping, we get $-0.133 \times 10^1 = -1.33$ for the term $\frac{(-2)^3}{6}$. Similarly, $\frac{(-2)^4}{24} = \frac{16}{24} = \frac{2}{3} = 0.66...6...$ with 3-digit choopping yields 0.666 for $\frac{(-2)^4}{24}$. Finally, $\frac{(-2)^5}{120} = \frac{-32}{120} = \frac{-4}{15} = -0.266...6...$ where 3-digit chopping results in -0.266 for $\frac{(-2)^5}{120}$ $\Rightarrow$ As we have to sum up the resultant numbers for each of them,

e^{-2} will then be approximated as $1 + \cancel{(-2)} + \cancel{2} +$

$$+ (-1.33) + 0.666 + (-0.266) = 1 - 1.33 + 0.4 =$$

$$= 0.4 - 0.33 = \boxed{0.07}$$

★ Just for note, $\frac{(-2)^0}{1} + \frac{(-2)^1}{1} + \frac{(-2)^2}{2} + \frac{(-2)^3}{6} + \frac{(-2)^4}{24} + \frac{(-2)^5}{120} = 1 - 2 + 2 - \frac{8}{6} + \frac{16}{24} - \frac{32}{120} =$

$$= 1 - \frac{4}{3} + \frac{2}{3} - \frac{4}{15} = 1 - \frac{2}{3} - \frac{4}{15} = \frac{1}{3} - \frac{4}{15} = \frac{1}{15} = 0.0666...6...$$

(3-digit chopping reveals 0.066)

and $e^{-2} \approx 0.13533528323$, which is different ☑ from our 3-digit chopping approximation 0.07

b) $\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!}$ can be approximated by 3-digit chopping using the values we found from previous exercise: $\frac{8}{6} = \frac{4}{3} = \frac{2^3}{3!}$ can be approximated as 1.33; similarly

$\frac{2^4}{4!}$ as 0.666 and $\frac{2^5}{5!}$ as 0.266 (calculations have been done in previous part)

Thus, $\frac{2^0}{0!}+\frac{2^1}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}$ can be approximated

via 3-digit chopping as $1+2+2+1.33+0.666+$

$+0.266=6.33+0.666+0.266=6.996+0.266=$

$=\underline{7.262} \Rightarrow e^{-2} \approx \frac{1}{\sum_{i=0}^{5}\frac{2^i}{i!}}$ can be approximated as

$\frac{1}{7.262} \approx 0.1377031209$ After we found that

the resultant value for $\sum_{j=0}^{5}\frac{2^j}{j!}$ after 3-digit chopping

is 7.262, we again chop it $\Rightarrow 0.7262 \times 10^1$ is approximated as

$0.726 \times 10^1 = 7.26$ Since the expression $\frac{1}{\sum_{j=0}^{5}\frac{2^j}{j!}}$ is needed

$\frac{1}{7.26} \approx 0.1377410468$ can be applied to the

3-digit chopping and we get $\boxed{0.137}$ as the approximation for e^{-2}

In fact, $\sum_{j=0}^{5}\frac{2^j}{j!}=1+2+2+\frac{4}{3}+\frac{2}{3}+\frac{32}{120}=5+2+\frac{4}{15}=$

$=7+\frac{4}{15}=\frac{109}{15}$ or $\frac{1}{\sum_{j=0}^{5}\frac{2^j}{j!}}=\frac{15}{109}\approx 0.1376146789$ (wherein 3-digit chopping would yield 0.137) and $e^{-2} \approx 0.13533528323$. Thus, the new resultant approximation 0.137 for e^{-2} via 3-digit chopping applied to $\sum_{j=0}^{5} \frac{2^j}{j!}$ gives much better accuracy for e^{-2}

6) The first part of Theorem 2.3 shows that $g \in C[a,b]$ such that $g(x) \in [a,b]$ $(\forall x \in [a,b])$ has at least 1 fixed point. This does not depend on the constraint over the derivative of g; thus, proof of (i) given in the section holds here as well.

It remains to show that the fixed point is unique if $g'(x)$ exists and $\exists K$-constant, $0<K<1$, such that $g'(x) \le K$ for $\forall x \in (a,b)$

As in the original proof of (ii), suppose that there are two different fixed points, p and q, $p \neq q$. From the Mean Value Theorem, there is a number $\xi \in [p,q]$ (between p and q, so in $[a,b]$) for which $\dfrac{g(p)-g(q)}{p-q} = g'(\xi)$

Since $g(p)=p$, $g(q)=q$, and $p-q \neq 0 \Rightarrow g'(\xi) = \dfrac{p-q}{p-q} = 1$. The fact that $g'(\xi)=1$ contradicts the assumption that $g'(\xi) \le K < 1$ in $[a,b]$ $\boxed{X}$ Therefore, there exist exactly 1 fixed point even when we replace the inequality $|g'(x)| \le K$ in the (ii) with $g'(x) \le K$, $\forall x \in (a,b)$