2020 Spring MAS 365: Final Exam

Write the following Honor Pledge and sign your name under it.

> *"I have neither given nor received aid on this examination, nor have I concealed a violation of the Honor Code."*

1. [40 (10+10+5+5+10) points] Consider the initial-value problem:

$$y' = -f(y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

   (a) State the condition on f that the initial-value problem become well-posed. Determine whether or not it is well-posed for $f(y) = 2|y| - 3$.

   (b) Consider $f(y) = \cos(3y)$. Determine the constants C and γ in

$$|y(t_i) - w_i| \leq Ch[e^{\gamma(t_i - a)} - 1]$$

   for $i = 0, 1, 2, \ldots, N$, where $y(t)$ denotes the solution to the initial-value problem and $w_0, w_1, \ldots, w_N$ denote the approximations generated by Euler's method with step size h for some integer N.

   (c) Determine whether Euler' method is strongly stable, weakly stable, or unstable.

   (d) Describe Euler's method with step size h for the mth-order system of the first-order initial-value problem:

$$\begin{aligned}
\frac{du_1}{dt} &= -f_1(u_1, u_2, \ldots, u_m), \\
\frac{du_2}{dt} &= -f_2(u_1, u_2, \ldots, u_m), \\
&\vdots \\
\frac{du_m}{dt} &= -f_m(u_1, u_2, \ldots, u_m),
\end{aligned}$$

   for $a \leq t \leq b$, with the initial conditions

$$u_1(a) = \alpha_1, \; u_2(a) = \alpha_2, \; \ldots, \; u_m(a) = \alpha_m.$$

   (e) Consider the following functions for the mth-order system in (d):

$$f_l(u_1, u_2, \ldots, u_m) = \sum_{j=1}^{m} a_{lj} u_j - b_l,$$

   for $l = 1, \ldots, m$. State the condition of h that guarantees the sequence $\{\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots\}$ generated by Euler's method with step size h to converge to the solution of $f_l(u_1, u_2, \ldots, u_m) = 0$ for $l = 1, \ldots, m$, if it exists.

2. [20 (10+10) points] For the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

consider the Taylor method of order n

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + hT^{(n)}(t_i, w_i, h), \quad \text{for each } i = 0, 1, \ldots, N-1, \end{aligned}$$

where

$$T^{(n)}(t_i, w_i, h) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i).$$

(a) State a condition when this method becomes consistent.

(b) State a condition when this method becomes convergent.

3. [35 (10+15+10) points] Consider approximating $f(x) = x^3 - x$ by a polynomial $P_2(x) = a_2x^2 + a_1x + a_0$ that minimizes

$$E_2(a_0, a_1, a_2) = \int_0^1 [f(x) - P_2(x)]^2 dx$$

(a) Construct the normal equations in a form $H\boldsymbol{a} = \boldsymbol{b}$ to find $P_2(x)$, where H is a 3×3 matrix, $\boldsymbol{b}$ is a 3×1 vector, and $\boldsymbol{a} = (a_0, a_1, a_2)^t$. [Note: List the normal equations in order, so that the problem (c) becomes invariant for every student.]

(b) Apply Gaussian elimination with scaled partial pivoting to the system $H\boldsymbol{a} = \boldsymbol{b}$ to find the solution $\boldsymbol{a}$.

(c) Find a polynomial P_1 that solves

$$\min_{P_1 \in \Pi_1} \max_{x \in [0,1]} |P_2(x) - P_1(x)|,$$

where Π_1 denotes set of all polynomials degree at most 1.

4. [20 (10+10) points] Consider an $n \times n$ matrix

$$A = \begin{bmatrix} 2 & \frac{1}{4} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & 4 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 8 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 2^{n-5} & 0 \\ 0 & 0 & \cdots & 0 & 2^{n-5} & 2^{n-1} & 2^{n-4} \\ 0 & 0 & \cdots & 0 & 0 & 2^{n-4} & 2^n \end{bmatrix}.$$

(a) Describe a strategy that iteratively approximates the ith smallest eigenvalue and its associated eigenvector of the matrix A, without using the deflation technique.

(b) Compute two iterations of the method in (a) with $\boldsymbol{x}^{(0)} = \left(\frac{1}{4}, 1\right)^t$ to approximate the smallest eigenvalue and its approximated eigenvector of A for $n = 2$.

5. [35 (10+15+10) points] Consider the positive definite linear system $A\boldsymbol{x} = \mathbf{0}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ are the eigenvalues of $n \times n$ matrix A, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are associated orthogonal eigenvectors. Conjugate gradient method finds the solution $\boldsymbol{x}^* = \mathbf{0}$ to the system, which produces the minimal value of

$$g(\boldsymbol{x}) = \boldsymbol{x}^t A \boldsymbol{x}.$$

The conjugate gradient method is equivalent to the followings:

$$\begin{aligned}
\boldsymbol{x}^{(k)} &= \arg\min_{\boldsymbol{x}} \left\{ g(\boldsymbol{x}) \ : \ \boldsymbol{x} \in \boldsymbol{x}^{(0)} + \operatorname{span}\{\nabla g(\boldsymbol{x}^{(0)}), \ldots, \nabla g(\boldsymbol{x}^{(k-1)})\} \right\} \\
&= \arg\min_{\boldsymbol{x}} \left\{ g(\boldsymbol{x}) \ : \ \boldsymbol{x} \in \boldsymbol{x}^{(0)} + \operatorname{span}\{A\boldsymbol{x}^{(0)}, \ldots, A^k \boldsymbol{x}^{(0)}\} \right\} \\
&= \arg\min_{\boldsymbol{x}} \left\{ g(\boldsymbol{x}) \ : \ \boldsymbol{x} = P_k(A)\boldsymbol{x}^{(0)}, \ P_k \in \hat{\Pi}_k \right\},
\end{aligned}$$

where $\hat{\Pi}_k$ is defined as

$$\hat{\Pi}_k := \{P_k \in \Pi_k \ : \ P_k(0) = 1\},$$

and Π_k denotes the set of all polynomials degree at most k.

(a) By expressing $\boldsymbol{x}^{(0)}$ as a linear combination of the orthogonal eigenvectors, *i.e.*, $\boldsymbol{x}^{(0)} = \sum_{j=1}^n \beta_j \boldsymbol{v}_j$, show that

$$g(\boldsymbol{x}^{(k)}) \leq \min_{P_k \in \hat{\Pi}_k} \max_{\lambda \in [\lambda_n, \lambda_1]} [P_k(\lambda)]^2 g(\boldsymbol{x}^{(0)}).$$

(b) Show that the following polynomial

$$\hat{T}_k(\lambda) = \frac{T_k\left(\frac{\lambda_1 + \lambda_n - 2\lambda}{\lambda_1 - \lambda_n}\right)}{T_k\left(\frac{\lambda_1 + \lambda_n}{\lambda_1 - \lambda_n}\right)}$$

minimizes the right-hand side of the inequality in (a) among $\hat{\Pi}_k$, where $T_k(x)$ is the Chebyshev polynomial for $x \in [-1, 1]$.

(c) Using the fact that the Chebyshev polynomial can be written as

$$T_k(x) = \frac{1}{2}\left[(x + \sqrt{x^2 - 1})^k + (x - \sqrt{x^2 - 1})^k\right] \quad |x| \geq 1,$$

show that

$$g(\boldsymbol{x}^{(k)}) \leq 4\left(\frac{\sqrt{K} - 1}{\sqrt{K} + 1}\right)^{2k} g(\boldsymbol{x}^{(0)}), \quad \text{where } K = \frac{\lambda_1}{\lambda_n}.$$