2019 Spring MAS 365: Midterm Exam

Student Number and Name:

Problem (Points)	1 (20)	2(30)	3(20)	4(30)	5(20)	6(30)	Total (150)
Score							

1. [20 (5+5+5+5) points]

   (a) State the definition of the absolute error, the relative error, and the number of significant digits.

   (b) Describe Newton's method and the Secant method for solving $f(x) = 0$. State the advantage and disadvantage of the Secant method over the Newton's method.

   (c) State the difference between Taylor, Lagrange and Hermite polynomials with order $2n + 1$, approximating a function f.

   (d) Let $S(x)$ be a cubic spline interpolant for f on the nodes $x_0 < x_1 < \cdots < x_n$. State the difference between natural and clamped cubic splines.

2. [30 (15+15) points]

(a) Complete the theorem and its proof.

Theorem 1. *(Fixed-Point Theorem) Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant k s.t.* __________ *exists with*

$$\_\_\_\_\_\_\_\_\_\_, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$.

Proof. The assumptions imply that a unique point p exists in $[a,b]$ with $g(p) = p$. (No need to prove this statement.)

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(b) Let $g(x)$ generate a sequence defined by $p_n = g(p_{n-1})$. Construct an appropriate $g(x)$ that converges to the root of $f(x) = x^2 - 2$ on the interval $[1,2]$. Justify your answer.

3. [20 (10 + 10) points]

(a) Suppose that $f \in C[a, b]$ and $f(a)f(b) < 0$. Show that the following variant of the Bisection method

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Set $a_1 = a$ and $b_1 = b$ and let $p_1 = (a_1 + 2b_1)/3$.

If $f(p_1) = 0$, then $p = p_1$, and we are done.

Else, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.

  If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.

  Else, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Reapply the process to the interval $[a_2, b_2]$.

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gives a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \le \alpha^n (b - a), \quad \text{when } n \ge 1.$$

Specify α s.t. $0 < \alpha < 1$.

(b) Show that $f(x) = x^3 - 6x^2 + 13x - 6 = 0$ has a root in $[0, 2]$. Determine the number of iterations necessary to solve $f(x) = 0$ with accuracy 0.4, by the method in (a) using $a_1 = 0$ and $b_1 = 2$. Compare it with the required number of iterations for the (standard) Bisection method.

4. [30 (10 + 10 + 10) points] Consider $f(x) = x^4 - 2x^3 + x^2 - 3$ and the nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

   (a) Find the Hermite interpolating polynomial using x_1, x_2.

   (b) Find the Hermite interpolating polynomial using x_0, x_1, x_2.

   (c) Find the natural cubic spline using x_0, x_1, x_2.

5. [20 (15 + 5) points]

(a) For $h > 0$, the following is the forward-difference formula

$$f'(x_0) \approx \frac{1}{h}[f(x_0 + h) - f(x_0)],$$

which is an $O(h)$ formula for $f'(x_0)$. Use extrapolation to derive an $O(h^2)$ formula for $f'(x_0)$.

(b) Approximate $f'(2)$ using both the forward difference formula and the $O(h^2)$ formula derived in (a). [Note: If you were not able to find the answer to (a), use any $O(h^2)$ formula.] Then, provide actual errors using the fact that the data are from the function $f(x) = x^3 - 2x^2 + 3x$.

x	-2	-1	0	1	2	3	4	5	6
$f(x)$	-22	-6	0	2	6	18	44	90	162

6. [30 (15 + 15) points]

(a) Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the $(n+1)$-point open Newton-Cotes formula with $x_{-1} = a$, $x_{n+1} = b$ and $h = (b-a)/(n+2)$. There exists $\xi \in (a, b)$ for which

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_{-1}^{n+1} t^2(t-1)\cdots(t-n)dt,$$

if n is even and $f \in C^{n+2}[a, b]$. Determine a_0, a_1, a_2 when $n = 2$.

(b) Determine constants a, b, c and x_0 that will produce a quadrature formula

$$\int_2^6 f(x)dx = af(2) + bf(6) + cf'(x_0)$$

that has degree of precision 3. Choose $x_0 \geq 4$.