# 2019 Spring MAS 365: Final Exam

| Problem (Points) | 1 (20) | 2(30) | 3(30) | 4(30) | 5(30) | 6(20) | Total (160) |
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1. [20 (5+5+5+5) points]

   (a) Let $y_i$ and $y_{i+1}$ denote the solution at $t_i$ and $t_{i+1}$, respectively. State the definition of the local truncation error $\tau_{i+1}(h)$ of the difference method

   $$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1. \end{aligned}$$

   Also, state the condition of $\tau_i(h)$ that makes the difference method consistent with the differential equation it approximates.

   (b) In Gaussian elimination, pivoting is useful when the pivot element $a_{kk}^{(k)}$ is small relative to the entries $a_{ij}^{(k)}$ for $k \le i \le n$ and $k \le j \le n$. Describe how the partial pivoting works.

   (c) State when a $n \times n$ matrix $A$ is said to be strictly diagonally dominant matrix. Also, state when a $n \times n$ symmetric matrix $A$ is said to be positive definite.

   (d) Describe the definition of the condition number $K(A)$ of the nonsingular matrix $A$ relative to a norm $||\cdot||$. Also, using $K(A)$, state when $A$ is said to be well conditioned and when $A$ is said to be ill conditioned.

2. [30 (10+5+15) points] For the initial value problem

   $$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

   consider the explicit method given by

   $$\begin{aligned} w_0 &= \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \\ w_{i+1} &= w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})] \end{aligned}$$

   and the implicit method given by

   $$\begin{aligned} w_0 &= \alpha, \quad w_1 = \alpha_1, \\ w_{i+1} &= w_{i-1} + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \end{aligned}$$

   (a) Determine whether each of the above two formulas is strongly stable, weakly stable or unstable. Justify your answer.

   (b) Given $w_0$, $w_1$, and $w_2$, describe a predictor-corrector formula using the above two formulas that gives $w_3$.

   (c) Transform the second-order initial-value problem

   $$y'' - y = -t^2 + 2, \quad \text{for } 0 \le t \le 3, \quad y(0) = 0,\ y'(0) = 0,\ y(1) = 1,\ y'(1) = 2,\ y(2) = 4,\ y'(2) = 4,$$

   into a system of first order initial-value problems and use the above explicit method with $h = 1$ to approximate $y(3)$ and $y'(3)$.

3. [30 (20+10) points] Consider solving a linear sytem $A\boldsymbol{x} = \boldsymbol{b}$. Let $D$ be the diagonal matrix whose diagonal entries are those of $A$, $-L$ be the strictly lower-triangular part of $A$, and $-U$ be the strictly upper-triangular part of $A$. The weighted Jacobi method updates as

$$\boldsymbol{x}^{(k)} = w(D^{-1}(L+U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b}) + (1-w)\boldsymbol{x}^{(k-1)}$$

which reduces to the standard Jacobi method when $w = 1$.

   (a) Suppose that $A$ is symmetric and positive definite, and $D = aI$ for a positive constant $a$ and an identity matrix $I$. Find the condition of $w$, in a form $\alpha < w < \beta$, that guarantees the convergence of the weighted Jacobi method. [Note: A symmetric matrix $A$ can be factorized into $V\Lambda V^t$, where $\Lambda$ is a diagonal matrix whose diagonal consists of the eigenvalues of $A$ and $V$ is an orthogonal matrix whose $i$th column is an eigenvector with $l_2$ norm 1 corresponding to the eigenvalue on the $i$th diagonal of $\Lambda$.]

   (b) Determine the choice of $w$ that guarantees the fastest convergence rate for a matrix

$$A = \begin{bmatrix} a & \frac{1}{2}a \\ \frac{1}{2}a & a \end{bmatrix}$$

   with a positive constant $a$.

4. [30 (15+15) points] The monic Chebyshev polynomials $\tilde{T}_n(x)$ have a recurrence relationship

$$\tilde{T}_0(x) = 1, \quad \tilde{T}_1(x) = x, \quad \tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2}\tilde{T}_0(x),$$
$$\tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \frac{1}{4}\tilde{T}_{n-1}(x), \quad \text{for each } n \geq 2.$$

Such polynomials have the property that

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)| \leq \max_{x\in[-1,1]} |P_n(x)|, \quad \text{for all } P_n(x) \in \tilde{\prod}_n,$$

where $\tilde{\prod}_n$ denotes the set of all monic polynomials of degree $n$. Let $\prod_n$ denote the set of all polynomials of degree $n$.

   (a) Consider approximating

$$P_3(x) = 4x^3 + 5x^2 - 3x$$

   with a polynomial of degree at most 2. In specific, among the polynomials in $\prod_2$, find $P_2(x)$ that minimizes

$$\max_{x\in[-1,1]} |P_3(x) - P_2(x)|.$$

   (b) Given a function $f(x) = x\sqrt{1-x^2}$ and a weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, find a linear combination

$$P(x) = \sum_{k=0}^{2} a_k \tilde{T}_k(x)$$

   that minimizes the error

$$E = E(a_0, a_1, a_2) = \int_{-1}^{1} w(x) \left[ f(x) - \sum_{k=0}^{2} a_k \tilde{T}_k(x) \right]^2 dx.$$

   [Note: $\int_{-1}^{1} w(x)[\tilde{T}_0(x)]^2 dx = \pi$ and $\int_{-1}^{1} w(x)[\tilde{T}_n(x)]^2 dx = \frac{1}{4^{n-1}}\frac{\pi}{2}$ for $n \geq 1$.]

5. [30 (5+10+10+5) points]

   (a) Specify $\alpha_i$ and $\beta_i$ in the Geršgorin Circle Theorem.

   **Theorem 1.** *Let $A$ be an $n \times n$ matrix and $R_i$ denote the circle in the complex plane with center $\alpha_i$ and radius $\beta_i$; that is,*

$$R_i = \left\{ z \in \mathcal{C} \;\middle|\; |z - \alpha_i| \leq \beta_i \right\}$$

   *where $\mathcal{C}$ denotes the complex plane. The eigenvalues of $A$ are contained within the union of these circles, $R = \cup_{i=1}^{n} R_i$. Moreover, the union of any $k$ of the circles that do not intersect the remaining $(n-k)$ contains precisely $k$ (counting multiplicities) of the eigenvalues.*

   (b) Use the Geršgorin Circle Theorem to compute the interval that contains all (real) singular values $s_1$, $s_2$, $s_3$ of a matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   in a form $s_1, s_2, s_3 \in [\delta_1, \delta_2]$. (Specify $\delta_1$ and $\delta_2$.)

   (c) Given a $n \times n$ matrix $A$ and a unit vector $\boldsymbol{x}_0$ relative to $||\cdot||_\infty$ with $p_0$ denoting the smallest integer for which $|x_{p_0}^{(0)}| = ||\boldsymbol{x}^{(0)}||_\infty$, the Power method generates sequences of vectors $\{\boldsymbol{x}^{(m)}\}_{m=0}^{\infty}$ and $\{\boldsymbol{y}^{(m)}\}_{m=1}^{\infty}$ and a sequence of scalars $\{\mu^{(m)}\}_{m=1}^{\infty}$ by

$$\boldsymbol{y}^{(m)} = A\boldsymbol{x}^{(m-1)}, \quad \mu^{(m)} = y_{p_{m-1}}^{(m)}, \quad \boldsymbol{x}^{(m)} = \frac{\boldsymbol{y}^{(m)}}{y_{p_m}^{(m)}},$$

   where at each step $p_m$ is used to represent the smallest integer for which $|y_{p_m}^{(m)}| = ||\boldsymbol{y}^{(m)}||_\infty$. Modify the Power method so that it finds the smallest singular value of a $m \times n$ matrix $B$ whose singular values are all nonzero and $m \geq n$.

   (d) Compute one iteration of the modified Power method in (c) on a matrix $B$ in (b) with $\boldsymbol{x}^{(0)} = (1, 1, 1)^t$.

6. [20 points] Given a matrix $A$ that has a singular value decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

   and $\boldsymbol{b} = (2, \sqrt{2}, 2\sqrt{2})^t$, find $\boldsymbol{x}$ that minimizes $||A\boldsymbol{x} - \boldsymbol{b}||_2$ by exploiting the singular value decomposition structure of $A$. Also, report the least squares error.