1 Introduction

Recall, that 'eigenvalues' distrubution of Gaussian β-ensemble of size N has density

$$\frac{1}{Z_N^{(\beta)}} \exp\left(-\frac{1}{2}\sum_{i=1}^N x_i^2\right) |V_N(x_1,\ldots,x_N)|^\beta,$$

where V_N is a Vandermonde determinant:

$$V_N(x_1,\ldots,x_N) = \prod_{i=1}^N \prod_{j=i+1}^N |x_i - x_j|$$

and $Z_N^{(\beta)}$ is just a normalization coefficient:

$$Z_N^{(\beta)} = \int\limits_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\sum_{i=1}^N x_i^2\right) |V_N(x_1,\ldots,x_N)|^\beta.$$

We denote this density by $g_N^{(\beta)}(x_1,\ldots,x_N)$.

The goal of this paper is, in certain sense, describe the asymptotic behaviour of a random vector

$$(\lambda_1^{(\beta)},\ldots,\lambda_N^{(\beta)}) \propto g_N^{(\beta)},$$

when $N \to +\infty$ while $N \cdot \beta \to 2c$, where c is some constant.

To be more precise, we introduce the random measure

$$L_N^{(\beta)} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_N^{(\beta)}/\sqrt{N}},$$

where δ_a is a Dirac measure.

Then fix any sequence of positive integers $\{N_m\}_{m=1}^\infty$ and positive reals $\{\beta_m\}_{m=1}^\infty$ such that

$$\lim_{m\to\infty} N_m = +\infty \text{ and } \lim_{m\to\infty} N_m\beta_m = 2c,$$

for some constant c and set $L_m \coloneqq L_{N_m}^{(\beta_m)}$. Denote by $\overline{L}_m$ the mean of L_m: $\forall f \in C_b(\mathbb{R})$, $\langle \overline{L}_m, f\rangle = \mathrm{E}\left[\langle L_m, f\rangle\right]$.

We prove that moments of $\overline{L}_m$ converge to the following sequence:

$$\begin{cases} M_{2k+1} = 0 & k \geq 0 \\ M_{2k} = \sum\limits_{r+s=k-1} M_{2r}M_{2s} + c(2k-1)M_{2k-2} & k > 0 \\ M_{2k} = 1 & k = 0. \end{cases}$$

Proposition 1. *For every non-negative integer k:*

$$\lim_{m\to\infty} \langle \overline{L}_m, x^k\rangle = M_k$$

We also prove, that the variance of moments of L_m tends to 0:

Proposition 2. *For every positive integer k:*

$$\lim_{m\to\infty} \mathrm{Var}\left[\left\langle L_m, x^k\right\rangle\right] = 0$$

These two propositions together imply the main result of this paper:

Theorem 1. *There is a unique measure L, whose k-th moments are M_k and L_m converges weakly, in probability to L. Precisely:*

For all $\epsilon > 0$ and $f \in C_b(\mathbb{R})$:

$$\lim_{m\to\infty} \mathrm{P}[|\langle L_m, f\rangle - \langle L, f\rangle| > \epsilon] = 0$$

Coefficients $h^{\tau}_{\mu,\nu}$, in turn, give a connection of Jack polynomials to the maps enumeration.

2 Connections to h-coefficients

First we state the main result of [Oko97], which connects β-distribibution to Jack polynomials.

Denote by $\left[|y|^{|\lambda|}\right] J^{(\alpha)}_{\lambda}(y)$ the coefficient of the polynomial $(\sum x_i^2)^{|\lambda|/2}$ in the power-sum expansion of $J^{(\alpha)}_{\lambda}(y)$.

Then the following holds true:

$$\int\limits_{\mathbb{R}^n} J^{(\alpha)}_{\lambda}(t) g^{(2/\alpha)}_N(t)\, dt = J^{(\alpha)}_{\lambda}(1,\ldots,1)\left(\left[|y|^{|\lambda|}\right] J^{(\alpha)}_{\lambda}(y)\right). \tag{1}$$

We rewrite it as follows:

$$\int\limits_{\mathbb{R}^n} J^{(\alpha)}_{\lambda}(t) g^{(2/\alpha)}_N(t)\, dt = J^{(\alpha)}_{\lambda}(x) J^{(\alpha)}_{\lambda}(y)\Bigg|_{\substack{p_r(x)=N\\ p_r(y)=\delta_{r,2}}}, \tag{2}$$

meaning, that we expand $J^{(\alpha)}_{\lambda}(x)$ and $J^{(\alpha)}_{\lambda}(y)$ in the power-sum basis and for every integer $r \geq 0$ we substitute N instead of $p_r(x)$ and $\delta_{r,2}$ instead of $p_r(y)$, where δ is a Kronecker delta.

Lemma 1. *For any partition $\theta \vdash m$:*

$$\int\limits_{\mathbb{R}^N} p_{\theta}(t) g^{(2/\alpha)}_N(t)\, dt = z_{\theta}\alpha^{l(\theta)}[p_{\theta}(z)]\left(\sum_{\lambda} \frac{J^{(\alpha)}_{\lambda}(x) J^{(\alpha)}_{\lambda}(y) J^{(\alpha)}_{\lambda}(z)}{\langle J^{(\alpha)}_{\lambda} J^{(\alpha)}_{\lambda}\rangle_{\alpha}}\right)\Bigg|_{\substack{p_r(x)=N\\ p_r(y)=\delta_{r,2}}} \tag{3}$$

Proof. Note, that:

$$\frac{1}{z_\theta \alpha^{l(\theta)}} \int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\, dt = [p_\theta(z)] \int\limits_{\mathbb{R}^N} \left(\sum_{\lambda \vdash m} \frac{p_\lambda(z) p_\lambda(t)}{z_\lambda \alpha^{l(\lambda)}} \right) g_N^{(2/\alpha)}(t)\, dt\,. \tag{4}$$

Now we use the Cauchy identity:

$$\prod_{i,j} (1 - x_i y_j)^{-\frac{1}{\alpha}} = \sum_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z_\lambda \alpha^{l(\lambda)}} = \sum_\lambda \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha},$$

to switch from power sum symmetric functions to Jack polynomials and apply (2):

$$\int\limits_{\mathbb{R}^N} \left(\sum_{\lambda \vdash m} \frac{p_\lambda(t) p_\lambda(z)}{z_\lambda \alpha^{l(\lambda)}} \right) g_N^{(2/\alpha)}(t)\, dt = \int\limits_{\mathbb{R}^N} \left(\sum_{\lambda \vdash m} \frac{J_\lambda^{(\alpha)}(t) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha} \right) g_N^{(2/\alpha)}(t)\, dt =$$

$$\sum_{\lambda \vdash m} \frac{J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha} \int\limits_{\mathbb{R}^N} J_\lambda^{(\alpha)}(t) g_N^{(2/\alpha)}(t)\, dt =$$

$$\sum_{\lambda \vdash m} \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha} \Bigg|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} \tag{5}$$

Combining (4) and (5) gives the desired result. □

Denote

$$\phi(x, y, z; t, \alpha) = \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha}$$

and

$$\psi(x, y, z; t, \alpha) = \alpha t \frac{\partial}{\partial t} \log(\phi) = \sum_{n \geq 1} t^n \left(\sum_{\mu, \nu, \tau \vdash n} h_{\mu,\nu}^\tau(\alpha - 1) p_\mu(x) p_\nu(y) p_\tau(z) \right). \tag{6}$$

Now we are ready to prove the crucial lemma:

Lemma 2. *For any partition $\theta \vdash m$:*

$$\int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\, dt =$$

$$z_\theta \alpha^{l(\theta)} [t^m p_\theta(z)] \exp\left(\frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu, \tau \vdash 2n} h_{\mu,(2)^n}^\tau (\alpha - 1) N^{l(\mu)} p_\tau(z) \right). \tag{7}$$

In particular

$$\int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\, dt = 0,$$

unless m is even.

Proof. First, we note that:

$$[p_\theta(z)] \left(\sum_\lambda \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)} J_\lambda^{(\alpha)} \rangle_\alpha} \right) \Bigg|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} = \\ = [t^m p_\theta(z)] \phi(x,y,z;t,\alpha) \Big|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} \tag{8}$$

From (6) we deduce:

$$\phi(x,y,z;\alpha) = \exp\left(\int \frac{\psi(x,y,z;\alpha)}{\alpha t}\, dt \right) = \\ \exp\left(\frac{1}{\alpha} \sum_{n \geq 1} \frac{t^n}{n} \left(\sum_{\mu,\nu,\tau \vdash n} h_{\mu,\nu}^{\tau}(\alpha - 1) p_\mu(x) p_\nu(y) p_\tau(z) \right) \right) \tag{9}$$

Note, that:

$$p_\mu(x)\big|_{p_r(x)=N} = N^{l(\mu)} \text{ and } p_\nu(y)\big|_{p_r(y)=\delta_{r,2}} = \begin{cases} 1 & \nu = (2^{l(\nu)/2}) \\ 1 & \text{otherwise} \end{cases}.$$

In particular, $p_\nu(y)\big|_{p_r(y)=\delta_{r,2}} = 0$ unless the size of ν is even. Now we are ready to do the corresponding substitutions in (9) to obtain:

$$\phi(x,y,z;t,\alpha)\Big|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} = \exp\left(\frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu,\tau \vdash 2n} h_{\mu,(2)^n}^{\tau}(\alpha - 1) N^{l(\mu)} p_\tau(z) \right). \tag{10}$$

The rest is just combining (8), (10) and **Lemma 1**. $\square$

Proposition 3. *For all positive integers k the following holds:*

$$\begin{cases} \langle \overline{L}_N^{(2/\alpha)}, x^{2k+1} \rangle = 0 \\ \langle \overline{L}_N^{(2/\alpha)}, x^{2k} \rangle = \frac{1}{N^{k+1}} \sum\limits_{\mu \vdash 2k} h_{\mu,(2^k)}^{(2k)}(\alpha - 1) N^{l(\mu)} \end{cases} \tag{11}$$

Proof. Note, that:

$$\langle \overline{L}_N^{(2/\alpha)}, x^r \rangle = \mathrm{E}\left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{\lambda_i}{\sqrt{N}} \right)^r \right] = \frac{1}{N^{1+r/2}} \int\limits_{\mathbb{R}^N} p_{(r)}(t) g_N^{(2/\alpha)}\, dt\, .$$

It follows from **Lemma 2**, that:

$$\langle \overline{L}_N^{(2/\alpha)}, x^{2k+1} \rangle = 0$$

Now for brevity, denote:

$$A(t) = \frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu, \tau \vdash 2n} h^{\tau}_{\mu,(2)^n}(\alpha - 1) N^{l(\mu)} p_\tau(z).$$

Lemma 2 says, that:

$$\int\limits_{\mathbb{R}^N} p_{(r)}(t) g_N^{(2/\alpha)}\, dt = 2k\alpha [t^{2k} p_{(2k)}(z)] \left(1 + \frac{A(t)}{1!} + \frac{A(t)^2}{2!} + \dots\right)$$

Note, that the term $p_{(2k)}(t)$ can not appear in the part:

$$\frac{A(t)^2}{2!} + \frac{A(t)^3}{3!} + \dots,$$

since all power-sum polynomials apearing there are indexed by partitions having length at least 2.

So we have:

$$\int\limits_{\mathbb{R}^N} p_{(r)}(t) g_N^{(2/\alpha)}\, dt = 2k\alpha [t^{2k} p_{(2k)}(z)] A(t) = \sum_{\mu \vdash 2k} h^{(2k)}_{\mu,(2^k)}(\alpha - 1) N^{l(\mu)}. \tag{12}$$

□

Proposition 4. *For all positive integers k the following holds:*

$$\begin{cases} \mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k+1} \rangle] = 0 \\ \mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k} \rangle] = \frac{2k\alpha}{N^{2+2k}} \sum\limits_{\mu \vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha - 1) N^{l(\mu)}. \end{cases} \tag{13}$$

Proof. Similarly to the previous proposition, we note

$$\mathrm{Var}[\langle L_N^{(2/\alpha)}, x^r \rangle] = \mathrm{E}[L_N^{(2/\alpha)}, x^r] = \mathrm{E}[(\langle L_N^{(2/\alpha)}, x^r \rangle)^2] - \left(\mathrm{E}[\langle L_N^{(2/\alpha)}, x^r \rangle]\right)^2 =$$

$$= \frac{1}{N^{2+2k}} \left(\int\limits_{\mathbb{R}^N} p_{(r,r)} g_N^{(2/\alpha)}(t)\, dt - \left(\int\limits_{\mathbb{R}^N} p_{(r)} g_N^{(2/\alpha)}(t)\, dt \right)^2 \right). \tag{14}$$

It follows from **Lemma 2**, that:

$$\mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k+1} \rangle] = 0.$$

Similarly to the proof of the **Prop 3** we denote:

$$A(t) = \frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu, \tau \vdash 2n} h^{\tau}_{\mu,(2)^n}(\alpha - 1) N^{l(\mu)} p_\tau(z),$$

and, using **Lemma 2**, note, that

$$\int\limits_{\mathbb{R}^N} p_{(2k,2k)} g_N^{(2/\alpha)}(t)\, dt = z_{(2k,2k)} \alpha^2 [t^{4k} p_{(2k,2k)}(z)] \left(\frac{A(t)}{1!} + \frac{A(t)}{2!} \right) =$$

$$= 8k^2\alpha^2 \left(\frac{1}{4k\alpha} \sum_{\mu \vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha-1) N^{l(\mu)} + \frac{1}{8k^2\alpha^2} \left(\sum_{\mu \vdash 2k} h^{(2k)}_{\mu,(2^k)}(\alpha-1) N^{l(\mu)} \right)^2 \right) =$$

$$= 2k\alpha \sum_{\mu \vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha-1) N^{l(\mu)} + \left(\int\limits_{\mathbb{R}^N} p_{(r)} g_N^{(2/\alpha)}(t)\, dt \right)^2 . \quad (15)$$

The last equality follows from (12). The rest is just combining (14) and (15). □

3 Proofs of the main results

We borrow the notions of *bridge*, *border*, *twisted edge*, *handle* from [Doł16] and denote a number of corresponding type root edges appearing during the root deletion process of the map by $br(M)$, $bo(M)$, $t(M)$, $h(M)$. Also we'll use $\eta(M)$ – one of the measures of non-orientability, also introduced in [Doł16]. By $F(M)$ we denote the face degree partition of M.

Lemma 3. *The following holds true:*

- $v(M) = br(M) + 1$
- $e(M) = bo(M) + t(M) + h(M) + br(M)$
- $f(M) = bo(M) - h(M) + 1$

Proof. During the root deletion process the number of connected components increases only if the border deleted. Since the initial number of connected components of M is 1 and the final number is $v(M)$. We get $v(M) = br(M)+1$.

The second identity is obvious.

Suppose at some point of the root deletion process there are k connected maps $M_1, \dots, M_k$. We are interested in the characteristic $\left(\sum_{i=1}^{k} f(M_i)\right) - k$. We note, that the deletion of borders and twisted edges doesn't change the characteristic, the deletion of handles – increases by 1, the deletion of borders – decreases by 1. At the beginning it's value is $f(M) - 1$ and in the end – 0, so we get $f(M) - 1 = bo(M) - h(M)$. □

Lemma 4.

$$\eta(M) \le 2g(M),$$

and the equality holds if and only if M is unhandled ($h(M) = 0$).

Proof. We rewrite Euler's identity, using **Lemma 3**:

$$
\begin{aligned}
&2g(M) = 2 - v(M) + e(M) - f(M) = \\
&= 2 - (br(M)+1) + (bo(M) + t(M) + h(M) + br(M)) - (bo(M) - h(M) + 1) = \\
&\qquad = t(M) + 2h(M)
\end{aligned}
$$

By the definition of η:

$$t(M) \leq \eta(M) \leq t(M) + h(M) \leq t(M) + 2h(M) = 2g(M),$$

which proves the statement. □

*Proof of the **Proposition 2**.* We begin with the reformulation of **Proposition 4** in terms of maps. The key ingredient for that is [La 09] Corollary 4.17:

$$\sum_{l(\mu)=v} h^{\tau}_{\mu,(2^n)}(b) = \sum_{F(M)=\tau, v(M)=v} b^{\eta(M)},$$

It's enough to prove the proposition for even moments. We see, that:

$$
\begin{aligned}
&\sum_{\mu \vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha-1) N^{l(\mu)} = \sum_{v=1}^{4k} N^v \sum_{\mu \vdash 4k, l(\mu)=v} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha-1) = \\
&\quad = \sum_{v=1}^{4k} N^v \sum_{\substack{F(M)=(2k,2k),\\ v(M)=v}} (\alpha-1)^{\eta(M)} = \sum_{F(M)=(2k,2k)} (\alpha-1)^{\eta(M)} N^{v(M)}. \qquad (16)
\end{aligned}
$$

Now note, that if $F(M) = (2k, 2k)$, then $e(M) = 2k$ and $f(M) = 2$ and hence $v(M) - 2k = -2g(M)$. We get:

$$
\begin{aligned}
\mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k} \rangle] &= \frac{2k\alpha}{N^{2+2k}} \sum_{F(M)=(2k,2k)} (\alpha-1)^{\eta(M)} N^{v(M)} = \\
&= \frac{2k\alpha}{N^2} \sum_{F(M)=(2k,2k)} (\alpha-1)^{\eta(M)} N^{-2g(M)}. \qquad (17)
\end{aligned}
$$

Note that $L_m = L_{N_m}^{2/(2/\beta_m)}$, hence:

$$\mathrm{Var}[\langle L_m, x^{2k} \rangle] = \frac{4k}{\beta_m N_m^2} \sum_{F(M)=(2k,2k)} \left(\frac{2}{\beta_m} - 1\right)^{\eta(M)} N_m^{-2g(M)}.$$

Since $\eta(M) \leq 2g(M)$:

$$\lim_{m\to\infty} \left(\frac{2}{\beta_m} - 1\right)^{\eta(M)} N_m^{-2g(M)} \leq (2c)^{\eta(M)},$$

and:

$$\lim_{m\to\infty} \frac{4k}{\beta_m N_m^2} = 0,$$

we get:

$$\lim_{m\to\infty} \mathrm{Var}[\langle L_m, x^{2k}\rangle] = 0.$$

$\square$

Lemma 5. *Rooted unhandled, unicellular maps with e edges and v vertices are in bijection with rooted oriented maps with the same number of edges and vertices.*

Lemma 6.

$$\lim_{m\to\infty} \langle \overline{L}_m, x^{2k}\rangle = c^{k+1} \sum_{\substack{M\ -\ rooted,\\ oriented, e(M)=k}} c^{-v(M)} \tag{18}$$

Proof. Similarly to the proof of **Proposition 2**, we use [La 09] **Corollary 4.17** to reformulate **Proposition 3** in terms of maps:

$$\begin{aligned} &\langle \overline{L}_N^{(2/\alpha)}, x^{2k}\rangle = \frac{1}{N^{k+1}} \sum_{\mu\vdash 2k} h^{(2k)}_{\mu,(2^k)}(\alpha-1) N^{l(\mu)} = \\ &= \frac{1}{N^{k+1}} \sum_{F(M)=(2k)} (\alpha-1)^{\eta(M)} N^{v(M)} = \sum_{F(M)=(2k)} (\alpha-1)^{\eta(M)} N^{-2g(M)} \end{aligned} \tag{19}$$

And hence:

$$\langle \overline{L}_m, x^{2k}\rangle = \sum_{F(M)=(2k)} \left(\frac{2}{\beta_m} - 1\right)^{\eta(M)} N_m^{-2g(M)}.$$

From **Lemma 4** it follows, that:

$$\begin{cases} \lim\limits_{m\to\infty} \left(\frac{2}{\beta_m} - 1\right)^{\eta(M)} N_m^{-2g(M)} = c^{-2g(M)} & \text{if } M \text{ is unhandled} \\ \lim\limits_{m\to\infty} \left(\frac{2}{\beta_m} - 1\right)^{\eta(M)} N_m^{-2g(M)} = 0 & \text{otherwise} \end{cases}$$

Using **Lemma 5**:

$$\begin{aligned} \lim_{m\to\infty} \langle \overline{L}_m, x^{2k}\rangle &= \sum_{\substack{M\ -\ \text{unhandled},\\ \text{unicellular}, e(M)=k}} c^{-2g(M)} = \\ &= \sum_{\substack{M\ -\ \text{rooted},\\ \text{unicellular},\\ \text{unhandled}, e(M)=k}} c^{-k+v(M)-1} = c^{-k-1} \sum_{\substack{M\ -\ \text{rooted},\\ \text{oriented}, e(M)=k}} c^{v(M)}. \end{aligned}$$

$\square$

Proof of the ***Proposition 1****.* Let $M(k)$ denote the set of all rooted, oriented, connected maps with e edges. By deleting the root edge, we see, for $k \geq 1$:

$$M(k) \equiv \bigsqcup_{r+s=k-1} M(r) \times M(s) \sqcup (2k-1)M(k-1). \tag{20}$$

Now denote:

$$a_k = c^{k+1} \sum_{\substack{M\text{ – rooted,}\\ \text{oriented,} e(M)=k}} c^{-v(M)}.$$

and combining it with (20), we get the recurrence relation for $k > 1$:

$$\begin{cases} a_k = \sum\limits_{r+s=k-1} a_r a_s + c(2k-1)a_{k-1} & k \geq 1 \\ a_k = 1 & k = 0 \end{cases}, \tag{21}$$

which coincides with the reccurence for M_{2k}. $\square$

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