1 Introduction

Recall, that 'eigenvalues' distrubution of Gaussian β-ensemble of size N has density

$$\frac{1}{Z_N^{(\beta)}} \exp\left(-\frac{1}{2}\sum_{i=1}^N x_i^2\right)|V_N(x_1,\ldots,x_N)|^\beta,$$

where V_N is a Vandermonde determinant:

$$V_N(x_1,\ldots,x_N) = \prod_{i=1}^N \prod_{j=i+1}^N |x_i - x_j|$$

and $Z_N^{(\beta)}$ is just a normalization coefficient:

$$Z_N^{(\beta)} = \int\limits_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\sum_{i=1}^N x_i^2\right)|V_N(x_1,\ldots,x_N)|^\beta.$$

We denote this density by $g_N^{(\beta)}(x_1,\ldots,x_N)$.

The goal of this paper is, in certain sense, describe the asymptotic behaviour of a random vector

$$(\lambda_1^{(\beta)},\ldots,\lambda_N^{(\beta)}) \propto g_N^{(\beta)},$$

when $N \to +\infty$ while $N\cdot\beta \to 2c$, where c is some constant.

To be more precise, we introduce the random measure

$$L_N^{(\beta)} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_N^{(\beta)}/\sqrt{N}},$$

where δ_a is a Dirac measure.

Then fix any sequence of positive integers $\{N_m\}_{m=1}^\infty$ and positive reals $\{\beta_m\}_{m=1}^\infty$ such that

$$\lim_{m\to\infty} N_m = +\infty \text{ and } \lim_{m\to\infty} N_m\beta_m = 2c,$$

for some constant c and set $L_m := L_{N_m}^{(\beta_m)}$. Denote by $\overline{L}_m$ the mean of L_m: $\forall f \in C_b(\mathbb{R})$, $\langle \overline{L}_m, f\rangle = \mathrm{E}\left[\langle L_m, f\rangle\right]$.

We prove that moments of $\overline{L}_m$ converge to the following sequence:

$$\begin{cases} M_{2k+1} = 0 & k \geq 0 \\ M_{2k} = \sum\limits_{r+s=k-1} M_{2r}M_{2s} + c(2k-1)M_{2k-2} & k > 0 \\ M_{2k} = 1 & k = 0. \end{cases}$$

Proposition 1. *For every non-negative integer k:*

$$\lim_{m\to\infty} \langle \overline{L}_m, x^k\rangle = M_k$$

We also prove, that the variance of moments of L_m tends to 0:

Proposition 2. *For every positive integer k:*

$$\lim_{m\to\infty} \mathrm{Var}\left[\left\langle L_m, x^k \right\rangle\right] = 0$$

These two propositions together imply the main result of this paper:

Theorem 1. *There is a unique measure L, whose k-th moments are M_k and L_m converges weakly, in probability to L. Precisely:*

For all $\epsilon > 0$ and $f \in C_b(\mathbb{R})$:

$$\lim_{m\to\infty} \mathrm{P}[|\langle L_m, f\rangle - \langle L, f\rangle| > \epsilon] = 0$$

2 Connections to h-coefficients

First we state the main result of [Oko97], which connects β-distribition to Jack polynomials.

Denote by $\left[|y|^{|\lambda|}\right] J_\lambda^{(\alpha)}(y)$ the coefficient of the polynomial $\left(\sum x_i^2\right)^{|\lambda|/2}$ in the power-sum expansion of $J_\lambda^{(\alpha)}(y)$.

Then the following holds true:

$$\int\limits_{\mathbb{R}^n} J_\lambda^{(\alpha)}(t) g_N^{(2/\alpha)}(t)\, dt = J_\lambda^{(\alpha)}(1, \ldots, 1) \left(\left[|y|^{|\lambda|}\right] J_\lambda^{(\alpha)}(y)\right). \tag{1}$$

We rewrite it as follows:

$$\int\limits_{\mathbb{R}^n} J_\lambda^{(\alpha)}(t) g_N^{(2/\alpha)}(t)\, dt = J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y)\Bigg|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}}, \tag{2}$$

meaning, that we expand $J_\lambda^{(\alpha)}(x)$ and $J_\lambda^{(\alpha)}(y)$ in the power-sum basis and for every integer $r \geq 0$ we substitute N instead of $p_r(x)$ and $\delta_{r,2}$ instead of $p_r(y)$, where δ is a Kronecker delta.

Lemma 1. *For any partition $\theta \vdash m$:*

$$\int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\, dt = z_\theta \alpha^{l(\theta)} [p_\theta(z)] \left(\sum_\lambda \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)} J_\lambda^{(\alpha)} \rangle_\alpha}\right)\Bigg|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} \tag{3}$$

Proof. Note, that:

$$\frac{1}{z_\theta \alpha^{l(\theta)}} \int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\, dt = [p_\theta(z)] \int\limits_{\mathbb{R}^N} \left(\sum_{\lambda \vdash m} \frac{p_\lambda(z) p_\lambda(t)}{z_\lambda \alpha^{l(\lambda)}}\right) g_N^{(2/\alpha)}(t)\, dt\,. \tag{4}$$

Now we use the Cauchy identity:

$$\prod_{i,j}(1-x_iy_j)^{-\frac{1}{\alpha}} = \sum_{\lambda} \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda \alpha^{l(\lambda)}} = \sum_{\lambda} \frac{J_\lambda^{(\alpha)}(x)J_\lambda^{(\alpha)}(y)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha},$$

to switch from power sum symmetric functions to Jack polynomials and apply (2):

$$\int\limits_{\mathbb{R}^N} \left(\sum_{\lambda\vdash m} \frac{p_\lambda(t)p_\lambda(z)}{z_\lambda \alpha^{l(\lambda)}}\right) g_N^{(2/\alpha)}(t)\,dt = \int\limits_{\mathbb{R}^N} \left(\sum_{\lambda\vdash m} \frac{J_\lambda^{(\alpha)}(t)J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\right) g_N^{(2/\alpha)}(t)\,dt =$$

$$\sum_{\lambda\vdash m} \frac{J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha} \int\limits_{\mathbb{R}^N} J_\lambda^{(\alpha)}(t) g_N^{(2/\alpha)}(t)\,dt =$$

$$\sum_{\lambda\vdash m} \frac{J_\lambda^{(\alpha)}(x)J_\lambda^{(\alpha)}(y)J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\Bigg|_{\substack{p_r(x)=N\\ p_r(y)=\delta_{r,2}}} \qquad (5)$$

Combining (4) and (5) gives the desired result. □

Denote

$$\phi(x,y,z;t,\alpha) = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{J_\lambda^{(\alpha)}(x)J_\lambda^{(\alpha)}(y)J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}$$

and

$$\psi(x,y,z;t,\alpha) = \alpha t \frac{\partial}{\partial t}\log(\phi) = \sum_{n\geq 1} t^n \left(\sum_{\mu,\nu,\tau\vdash n} h_{\mu,\nu}^{\tau}(\alpha-1) p_\mu(x)p_\nu(y)p_\tau(z)\right). \qquad (6)$$

Now we are ready to prove the crucial lemma:

Lemma 2. *For any partition $\theta\vdash m$:*

$$\int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\,dt =$$

$$z_\theta \alpha^{l(\theta)} [t^m p_\theta(z)] \exp\left(\frac{1}{\alpha}\sum_{n\geq 1} \frac{t^{2n}}{2n} \sum_{\mu,\tau\vdash 2n} h_{\mu,(2)^n}^{\tau}(\alpha-1) N^{l(\mu)} p_\tau(z)\right). \qquad (7)$$

In particular

$$\int\limits_{\mathbb{R}^N} p_\theta(t) g_N^{(2/\alpha)}(t)\,dt = 0,$$

unless m is even.

Proof. First, we note that:

$$[p_\theta(z)] \left(\sum_\lambda \frac{J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(y) J_\lambda^{(\alpha)}(z)}{\langle J_\lambda^{(\alpha)} J_\lambda^{(\alpha)} \rangle_\alpha} \right) \Bigg|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} = \\ = [t^m p_\theta(z)] \phi(x,y,z;t,\alpha) \Big|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} \quad (8)$$

From (6) we deduce:

$$\phi(x,y,z;\alpha) = \exp\left(\int \frac{\psi(x,y,z;\alpha)}{\alpha t}\, dt \right) = \\ \exp\left(\frac{1}{\alpha} \sum_{n \geq 1} \frac{t^n}{n} \left(\sum_{\mu,\nu,\tau \vdash n} h_{\mu,\nu}^{\tau}(\alpha - 1) p_\mu(x) p_\nu(y) p_\tau(z) \right) \right) \quad (9)$$

Note, that:

$$p_\mu(x)\big|_{p_r(x)=N} = N^{l(\mu)} \text{ and } p_\nu(y)\big|_{p_r(y)=\delta_{r,2}} = \begin{cases} 1 & \nu = (2^{l(\nu)/2}) \\ 1 & \text{otherwise} \end{cases}.$$

In particular, $p_\nu(y)\big|_{p_r(y)=\delta_{r,2}} = 0$ unless the size of ν is even. Now we are ready to do the corresponding substitutions in (9) to obtain:

$$\phi(x,y,z;t,\alpha)\Big|_{\substack{p_r(x)=N \\ p_r(y)=\delta_{r,2}}} = \exp\left(\frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu,\tau \vdash 2n} h_{\mu,(2)^n}^{\tau}(\alpha - 1) N^{l(\mu)} p_\tau(z) \right). \quad (10)$$

The rest is just combining (8), (10) and **Lemma 1**. $\square$

Proposition 3. *For all positive integers k the following holds:*

$$\begin{cases} \langle \overline{L}_N^{(2/\alpha)}, x^{2k+1} \rangle = 0 \\ \langle \overline{L}_N^{(2/\alpha)}, x^{2k} \rangle = \frac{1}{N^{k+1}} \sum\limits_{\mu \vdash 2k} h_{\mu,(2^k)}^{(2k)}(\alpha - 1) N^{l(\mu)} \end{cases} \quad (11)$$

Proof. Note, that:

$$\langle \overline{L}_N^{(2/\alpha)}, x^r \rangle = \mathrm{E}\left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{\lambda_i}{\sqrt{N}} \right)^r \right] = \frac{1}{N^{1+r/2}} \int\limits_{\mathbb{R}^N} p_{(r)}(t) g_N^{(2/\alpha)}\, dt\,.$$

It follows from **Lemma 2**, that:

$$\langle \overline{L}_N^{(2/\alpha)}, x^{2k+1} \rangle = 0$$

Now for brevity, denote:

$$A(t) = \frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu,\tau \vdash 2n} h^{\tau}_{\mu,(2)^n}(\alpha - 1) N^{l(\mu)} p_\tau(z).$$

Lemma 2 says, that:

$$\int\limits_{\mathbb{R}^N} p_{(r)}(t) g_N^{(2/\alpha)}\, dt = 2k\alpha[t^{2k} p_{(2k)}(z)] \left(1 + \frac{A(t)}{1!} + \frac{A(t)^2}{2!} + \dots\right)$$

Note, that the term $p_{(2k)}(t)$ can not appear in the part:

$$\frac{A(t)^2}{2!} + \frac{A(t)^3}{3!} + \dots,$$

since all power-sum polynomials apearing there are indexed by partitions having length at least 2.

So we have:

$$\int\limits_{\mathbb{R}^N} p_{(r)}(t) g_N^{(2/\alpha)}\, dt = 2k\alpha[t^{2k} p_{(2k)}(z)] A(t) = \sum_{\mu \vdash 2k} h^{(2k)}_{\mu,(2^k)}(\alpha - 1) N^{l(\mu)}. \tag{12}$$

□

Proposition 4. *For all positive integers k the following holds:*

$$\begin{cases} \mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k+1}\rangle] = 0 \\ \mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k}\rangle] = \frac{2k\alpha}{N^{2+2k}} \sum\limits_{\mu \vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha - 1) N^{l(\mu)}. \end{cases} \tag{13}$$

Proof. Similarly to the previous proposition, we note

$$\mathrm{Var}[\langle L_N^{(2/\alpha)}, x^r\rangle] = \mathrm{E}[L_N^{(2/\alpha)}, x^r] = \mathrm{E}[(\langle L_N^{(2/\alpha)}, x^r\rangle)^2] - \Big(\mathrm{E}[\langle L_N^{(2/\alpha)}, x^r\rangle]\Big)^2 =$$

$$= \frac{1}{N^{2+2k}} \left(\int\limits_{\mathbb{R}^N} p_{(r,r)} g_N^{(2/\alpha)}(t)\, dt - \left(\int\limits_{\mathbb{R}^N} p_{(r)} g_N^{(2/\alpha)}(t)\, dt \right)^2 \right). \tag{14}$$

It follows from **Lemma 2**, that:

$$\mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k+1}\rangle] = 0.$$

Similarly to the proof of the **Prop 3** we denote:

$$A(t) = \frac{1}{\alpha} \sum_{n \geq 1} \frac{t^{2n}}{2n} \sum_{\mu,\tau \vdash 2n} h^{\tau}_{\mu,(2)^n}(\alpha - 1) N^{l(\mu)} p_\tau(z),$$

and, using **Lemma 2**, note, that

$$\int_{\mathbb{R}^N} p_{(2k,2k)} g_N^{(2/\alpha)}(t)\,dt = z_{(2k,2k)}\alpha^2[t^{4k}p_{(2k,2k)}(z)]\left(\frac{A(t)}{1!}+\frac{A(t)}{2!}\right) =$$

$$= 8k^2\alpha^2\left(\frac{1}{4k\alpha}\sum_{\mu\vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha-1)N^{l(\mu)} + \frac{1}{8k^2\alpha^2}\left(\sum_{\mu\vdash 2k} h^{(2k)}_{\mu,(2^k)}(\alpha-1)N^{l(\mu)}\right)^2\right) =$$

$$= 2k\alpha\sum_{\mu\vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha-1)N^{l(\mu)} + \left(\int_{\mathbb{R}^N} p_{(r)} g_N^{(2/\alpha)}(t)\,dt\right)^2. \quad (15)$$

The last equality follows from (12). The rest is just combining (14) and (15). □

3 Proofs of the main results

We borrow the notions of *bridge*, *border*, *twisted edge*, *handle* from [Doł16] and denote a number of corresponding type root edges appearing during the root deletion process of the map by $br(M)$, $bo(M)$, $t(M)$, $h(M)$. Also we'll use $\eta(M)$ – one of the measures of non-orientability, also introduced in [Doł16]. By $F(M)$ we denote the face degree partition of M.

Lemma 3. *The following holds true:*

- $v(M) = br(M) + 1$
- $e(M) = bo(M) + t(M) + h(M) + br(M)$
- $f(M) = bo(M) - h(M) + 1$

Proof. During the root deletion process the number of connected components increases only if the border deleted. Since the initial number of connected components of M is 1 and the final number is $v(M)$. We get $v(M) = br(M)+1$.

The second identity is obvious.

Suppose at some point of the root deletion process there are k connected maps $M_1, \dots, M_k$. We are interested in the characteristic $\left(\sum_{i=1}^{k} f(M_i)\right) - k$. We note, that the deletion of borders and twisted edges doesn't change the characteristic, the deletion of handles – increases by 1, the deletion of borders – decreases by 1. At the beginning it's value is $f(M) - 1$ and in the end – 0, so we get $f(M) - 1 = bo(M) - h(M)$. □

Lemma 4.

$$\eta(M) \le 2g(M),$$

and the equality holds if and only if M is unhandled ($h(M) = 0$).

Proof. We rewrite Euler's identity, using **Lemma 3**:

$$\begin{aligned}
&2g(M) = 2 - v(M) + e(M) - f(M) = \\
&= 2 - (br(M) + 1) + (bo(M) + t(M) + h(M) + br(M)) - (bo(M) - h(M) + 1) = \\
&\qquad = t(M) + 2h(M)
\end{aligned}$$

By the definition of η:

$$t(M) \leq \eta(M) \leq t(M) + h(M) \leq t(M) + 2h(M) = 2g(M),$$

which proves the statement. □

Proof of the ***Proposition 2****.* We begin with the reformulation of **Proposition 4** in terms of maps. The key ingredient for that is [La 09] Corollary 4.17:

$$\sum_{l(\mu)=v} h^{\tau}_{\mu,(2^n)}(b) = \sum_{F(M)=\tau, v(M)=v} b^{\eta(M)},$$

It's enough to prove the proposition for even moments. We see, that:

$$\begin{aligned}
&\sum_{\mu \vdash 4k} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha - 1) N^{l(\mu)} = \sum_{v=1}^{4k} N^v \sum_{\mu \vdash 4k, l(\mu)=v} h^{(2k,2k)}_{\mu,(2^{2k})}(\alpha - 1) = \\
&\quad = \sum_{v=1}^{4k} N^v \sum_{\substack{F(M)=(2k,2k), \\ v(M)=v}} (\alpha - 1)^{\eta(M)} = \sum_{F(M)=(2k,2k)} (\alpha - 1)^{\eta(M)} N^{v(M)}. \quad (16)
\end{aligned}$$

Now note, that if $F(M) = (2k, 2k)$, then $e(M) = 2k$ and $f(M) = 2$ and hence $v(M) - 2k = -2g(M)$. We get:

$$\begin{aligned}
\mathrm{Var}[\langle L_N^{(2/\alpha)}, x^{2k} \rangle] &= \frac{2k\alpha}{N^{2+2k}} \sum_{F(M)=(2k,2k)} (\alpha - 1)^{\eta(M)} N^{v(M)} = \\
&= \frac{2k\alpha}{N^2} \sum_{F(M)=(2k,2k)} (\alpha - 1)^{\eta(M)} N^{-2g(M)}. \quad (17)
\end{aligned}$$

Note that $L_m = L_{N_m}^{2/(2/\beta_m)}$, hence:

$$\mathrm{Var}[\langle L_m, x^{2k} \rangle] = \frac{4k}{\beta_m N_m^2} \sum_{F(M)=(2k,2k)} \left(\frac{2}{\beta_m} - 1 \right)^{\eta(M)} N_m^{-2g(M)}.$$

Since $\eta(M) \leq 2g(M)$:

$$\lim_{m \to \infty} \left(\frac{2}{\beta_m} - 1 \right)^{\eta(M)} N_m^{-2g(M)} \leq (2c)^{\eta(M)},$$

and:

$$\lim_{m\to\infty} \frac{4k}{\beta_m N_m^2} = 0,$$

we get:

$$\lim_{m\to\infty} \mathrm{Var}[\langle L_m, x^{2k}\rangle] = 0.$$

□

Lemma 5. *Rooted unhandled, unicellular maps with e edges and v vertices are in bijection with rooted oriented maps with the same number of edges and vertices.*

Proof. The root deletion process defines an ordering of edges. Reversing this ordering and adding ribbons one by one, starting from v disconnected vertices, defines the *root adding* process. During the process, added ribbons connect the same corners they connected initially, but we're allowed to twist ribbons, as shown on the picture:

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We describe algorithm of choosing the twists, which will constitute the desired bijection.

We start by exploring how twist affects the type of the root in different settings.

Case 1 *The root connects corners from two disconnected components.*

The root twist results in the isomorphic map. In any case the added root is a **bridge**. Also note (we'll need this fact later), that connecting two unicellular components gives a unicellular component too.

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Case 2 *The root connects corners from the same face.* First, we depict the traversed face before adding the root:

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Then depending on the twist, we either add one more face (in that case the added root is a **border**) or we keep the number of faces unchanged (the added root is a **twist**):

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Note, that the border doesn't change the orientability of the map, while adding a twist always gives a non-orientable map.

Case 3 *The root connects corners from different faces but from the same connected component.*

We diagrammatically depict the traversed faces, containing the added root:

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In both cases we decrease the number of faces by one and the added root is a **handle**. But the twist changes the traversing direction of one of the faces. So if the map without the added root is oriented, only one of the resulting maps is oriented.

First, we show how to make an oriented rooted map from unicellular unhandled one. We start the root adding process from v trivial components and by induction we assume, that on each step all the connected components are orientable.

- If we encounter **Case 1** during the root addition process, we choose any of the twists.
- If we encounter **Case 2**, we twist the ribbon in a way to get a border.
- If we encounter **Case 3**, we twist the ribbon in unique available way to keep the corresponding component orientable.

Now we give an algorithm of getting an unicellular unhandled map from an oriented rooted map. Again, we do the root addition process. Our strategy is to avoid the **Case 3**, since it always results in adding a handle. Inductively we maintain the following property: every connected component is unicellular.

- If we encounter **Case 1**, we choose any of the twists. Note, that connecting to unicellular components results in an unicellular component too.
- If we encounter **Case 2**, we twist the ribbon i a way that gives a twisted root, thus keeping the number of faces of the component equal to 1.
- We can not encounter **Case 3**, since it implies, that there is a component with at least 2 faces, contradicting our induction hypothesis.

Careful analysis of both algorithms shows, that these transformations are inverse to each other. □

Lemma 6.

$$\lim_{m\to\infty}\langle \overline{L}_m, x^{2k}\rangle = c^{k+1} \sum_{\substack{M\ -\ rooted,\\ oriented, e(M)=k}} c^{-v(M)} \tag{18}$$

Proof. Similarly to the proof of **Proposition 2**, we use [La 09] **Corollary 4.17** to reformulate **Proposition 3** in terms of maps:

$$\langle \overline{L}_N^{(2/\alpha)}, x^{2k}\rangle = \frac{1}{N^{k+1}} \sum_{\mu\vdash 2k} h^{(2k)}_{\mu,(2^k)}(\alpha-1)N^{l(\mu)} =$$

$$= \frac{1}{N^{k+1}} \sum_{F(M)=(2k)} (\alpha-1)^{\eta(M)} N^{v(M)} = \sum_{F(M)=(2k)} (\alpha-1)^{\eta(M)} N^{-2g(M)} \tag{19}$$

And hence:

$$\langle \overline{L}_m, x^{2k} \rangle = \sum_{F(M)=(2k)} \left(\frac{2}{\beta_m} - 1 \right)^{\eta(M)} N_m^{-2g(M)}.$$

From **Lemma 4** it follows, that:

$$\begin{cases} \lim_{m\to\infty} \left(\frac{2}{\beta_m} - 1 \right)^{\eta(M)} N_m^{-2g(M)} = c^{-2g(M)} & \text{if } M \text{ is unhandled} \\ \lim_{m\to\infty} \left(\frac{2}{\beta_m} - 1 \right)^{\eta(M)} N_m^{-2g(M)} = 0 & \text{otherwise} \end{cases}$$

Using **Lemma 5**:

$$\lim_{m\to\infty} \langle \overline{L}_m, x^{2k} \rangle = \sum_{\substack{M \text{ – unhandled,} \\ \text{unicellular}, e(M)=k}} c^{-2g(M)} =$$

$$= \sum_{\substack{M \text{ – rooted,} \\ \text{unicellular,} \\ \text{unhandled}, e(M)=k}} c^{-k+v(M)-1} = c^{-k-1} \sum_{\substack{M \text{ – rooted,} \\ \text{oriented}, e(M)=k}} c^{v(M)}.$$

□

*Proof of the **Proposition 1**.* We show, that a quantity obtained in the **Lemma 6** satisfies a recurrence given in the statement of the proposition. All maps in this proof are oriented.

Denote the set of oriented maps with k edges by $\mathrm{M}(k)$. To derive a recurrence relation, we do one step of the root deletion process for a map $M \in \mathrm{M}(k)$.

Case 1 *M splits into two connected components M_1 and M_2. We denote the set of such maps by* $\mathrm{M}_1(k)$.

Conversely, we can obtain a unique map $\tilde{M}$ from two smaller maps $\tilde{M}_1$ and $\tilde{M}_2$ in a way reverse to the root deletion process.

We connect the root corners of the map $\tilde{M}_1$ and $\tilde{M}_2$ by an edge. It splits the root corner of $\tilde{M}_1$ into two. Only one of them can be chosen as a root such that the deletion of the new root edge will give M_1 and M_2 back.

Case 2 *The deletion of the root results in a connected map M_1. We denote the set of such maps by* $\mathrm{M}_2(k)$

Conversely, we can obtain $2k-1$ maps $\tilde{M}$, $e(M) = k$ from a map $\tilde{M}_1$ in a way reverse to the root deletion process.

Note, that $\tilde{M}_1$ contains $2k-3$ corners other than a root corner. We start by drawing a half-edge from a root corner, which splits it into two new ones. So we can connect the other half-edge to one of the $(2k-3)+2 = 2k-1$. Again, the choice of the new root is uniquely determined.

So for $k \geq 1$ we get a bijection:

$$\mathrm{M}(k) = \mathrm{M}_1(k) \sqcup \mathrm{M}_2(k) \longleftrightarrow$$
$$\longleftrightarrow \left(\bigsqcup_{r+s=k-1} \mathrm{M}(r) \times \mathrm{M}(s) \right) \sqcup (\{1,\, \ldots,\, 2k-1\} \times \mathrm{M}(k-1)) \,. \quad (20)$$

Now denote:

$$a_k = c^{k+1} \sum_{M \in \mathrm{M}(k)} c^{-v(M)}.$$

Clearly, $a_0 = 1$, and, using (20), for $k \geq 1$ we get

$$a_k = \sum_{M \in \mathrm{M}_1(k)} c^{k+1-v(M)} + \sum_{N \in \mathrm{M}_2(k)} c^{k+1-v(N)} =$$
$$= \sum_{\substack{r+s=k-1 \\ M_1 \in \mathrm{M}(r) \\ M_2 \in \mathrm{M}(s)}} c^{r+1+v(M_1)} c^{s+1\, v(M_2)} + \sum_{N_1 \in \mathrm{M}(k-1)} (2k-1)c \cdot c^{(k-1)+1+v(N_1)} =$$
$$= \sum_{r+s=k-1} a_r a_s + c(2k-1)a_{k-1}, \quad (21)$$

which coincides with the reccurence for M_{2k}. $\square$

Proof. It's a corollary of **Proposition 3** (shows, that all moments of $\overline{L}_m$ exist), **Proposition 1** and **Theorem 3.5** from [Mic22]. $\square$

*Proof of the **Theorem 1**.* The brief plan of the proofs is the following:

- Show, that numbers M_k satisfy the Carleman's condition.
- It follows from **Theorem 3.5** of [Mic22], that there exits a unique measure L, such that $\langle L, x^k \rangle = M_k$ and $\overline{L}_m$ weakly converges L.
- Show, that $\langle L_m, x^k \rangle$ converges in probability to $M_k = \langle L, x^k \rangle$.
- Theorem statement follows from **Theorem 3.7**, part *ii)* of [Mic22].

For brevity, denote $a_k \coloneqq M_{2k}$. Note, that for $k \geq 1$:

$$a_k = \left(\sum_{\substack{r+s=k-1 \\ r,s \geq 1}} a_r a_s \right) + 2a_0 a_k + c(2k-1)a_{k-1} = \left(\sum_{\substack{r+s=k-1 \\ r,s \geq 1}} a_r a_s \right) + (c(2k-1)+2)a_{k-1} \leq$$
$$\leq (c + (2k-1) + 2) \sum_{r+s=k-1} a_r a_s \quad (22)$$

Denote $C = c + (2k-1) + 2$. Define a sequence:

$$\begin{cases} b_k = C \sum_{r+s=k-1} b_r b_s q & k \geq 1 \\ b_k = 1 & k = 0 \end{cases}$$

Clearly, $a_k \leq b_k$ and a sequence $\frac{b_k}{C^k}$ satisfies Catalan's numbers recurrence, so using the Stirling's approximation:

$$a_k = C^k \frac{1}{k+1}\binom{2k}{k} C^k \sim \frac{C^k 2^{2k}}{\sqrt{2\pi k}} :- c_k,$$

Clearly:

$$\sum_{k=1}^{\infty} c_k^{-1/2k} = +\infty,$$

so M_k satisfies Carleman's condition.

To show, that $\langle L, x^k \rangle$ converges in probability to M_k we note that:

$$\mathrm{P}\left[|\langle L_m, x^k\rangle - M_k| > \epsilon\right] \leq \\ \leq \mathrm{P}\left[\left|\langle L_m, x^k\rangle - \langle \overline{L}_m, x^k\rangle\right| > \frac{\epsilon}{2}\right] + \mathrm{P}\left[\left|\langle \overline{L}_m, x^k\rangle - M_k\right| > \frac{\epsilon}{2}\right].$$

The latter summand is eventually 0 by **Proposition 1**. To estimate the first summand we use the Tchebysheff inequality and **Proposition 2**:

$$\mathrm{P}\left[\left|\langle L_m, x^k\rangle - \langle \overline{L}_m, x^k\rangle\right|\right] \leq \frac{\mathrm{Var}\langle L_m, x^k\rangle}{(\epsilon/2)^2} \longrightarrow 0.$$

□

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