Problem Statement:

1. (a) Write an R function to sample from the Wishart distribution. The only inputs to your function should be a positive definite matrix Σ and the degrees of freedom m.

(b) Using your function, generate 100 matrices Σ from the inverse-Wishart prior distribution, so that $\Sigma^{-1} \sim \text{Wishart}(p + 1, I_{pxp})$. For each Σ, compute C, the associated correlation matrix. Store the off-diagonal elements of your simulated C-matrices, and then make a histogram of them. Comment on it.

Solution:

Relevant Theoretical Understanding:

Bartlett decomposition of Wishart matrices. In Statistics, the classical Bartlett decomposition provides the distribution of the factor in the Cholesky decomposition of an empirical covariance matrix following the Wishart distribution.

Let us denote by $\mathcal{S}_m^+$ the set of $m \times m$ symmetric matrices with non-negative spectrum. Let $X_1, \ldots, X_n$ be iid random vectors of $\mathbb{R}^m$ following the Gaussian law $\mathcal{N}(\mu, \Sigma)$ with mean $\mu \in \mathbb{R}^m$ and covariance matrix $\Sigma \in \mathcal{S}_m^+$:

$$\mu := \mathbb{E}(X_1) \in \mathbb{R}^m$$

and

$$\Sigma := \mathbb{E}(X_1 \otimes X_1) - \mu \otimes \mu = \mathbb{E}((X_1 - \mu) \otimes (X_1 - \mu)) \in \mathcal{S}_m^+.$$

The random $m \times m$ matrix

$$S_n := \sum_{k=1}^{n} X_k \otimes X_k$$

takes its values in $\mathcal{S}_m^+$, and its law is known as the **Wishart distribution**

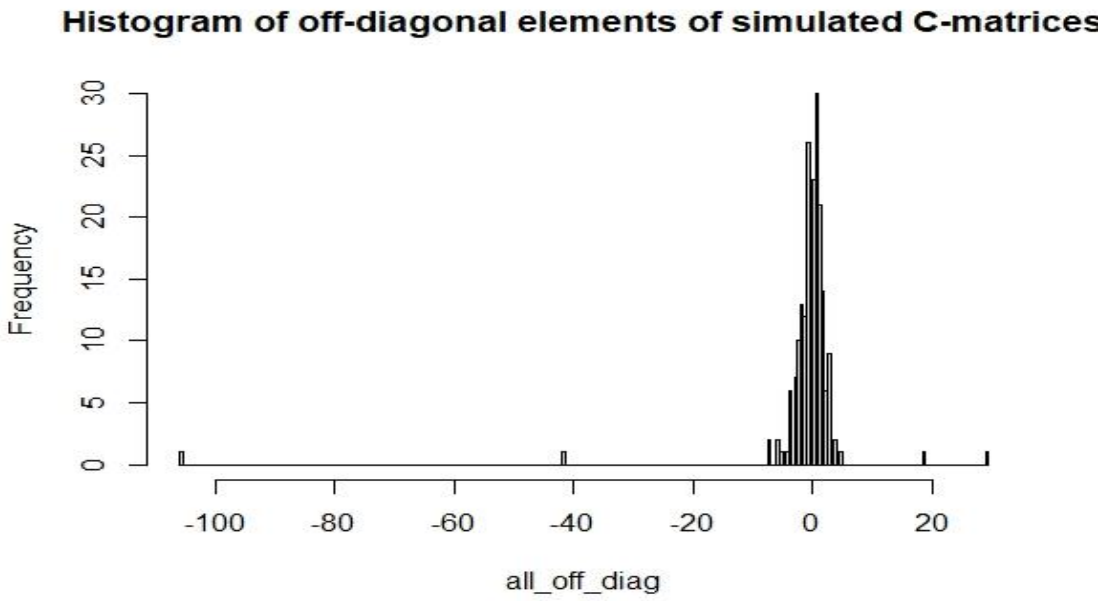
$$\mathcal{W}_m(\Sigma, n),$$

#Creating the function####

```
wishart_sample <- function(Sigma, m) {
  p <- ncol(Sigma)
  t(chol(Sigma))%*%matrix(rnorm(m*p),ncol=p)%*%chol(Sigma)
}
install.packages("corpcor")
library(corpcor)
# Set parameters
p <- 5
m <- 100
df <- p + 1
Sigma_inv <- diag(p)

# Generate matrices Σ from inverse-Wishart prior distribution
for (i in 1:m) {
```

```
  Sigma <- solve(wishart_sample(Sigma_inv,5))
C <- cov2cor(Sigma)
  off_diag <- C[lower.tri(C)]
  if (i == 1) {
    all_off_diag <- off_diag
  } else {
    all_off_diag <- c(off_diag)
  }
}
hist(off_diag, breaks=200, main="Histogram of off-diagonal elements of simulated C-matrices",)
```



The resulting histogram shows that the off-diagonal elements of the simulated C matrices are centred around 0, with a few positive and negative outliers. This is expected, as the inverse-Wishart prior is a conjugate prior for the covariance matrix of a multivariate normal distribution, and the correlation matrix is a monotonic transformation of the covariance matrix. Therefore, the off-diagonal elements of the correlation matrix are expected to be centred around 0, indicating little correlation between the variables.