$$b = (r\rho + \rho r^\dagger) - Tr(r\rho + \rho r^\dagger)\rho$$

$$\frac{\partial\, Tr(A)}{\partial A}$$

$$b' = r + r^\dagger - (r + r^\dagger)\rho - Tr(r\rho + \rho r^\dagger)$$

$$\underbrace{\left(Tr(r\rho) + Tr(\rho r^\dagger)\right)}_{r \;\; + r^\dagger}$$

$$\frac{\partial\, Tr(r\rho + \rho r^\dagger)}{\partial (r\rho + \rho r^\dagger)} \frac{\partial\, r\rho + \rho r^\dagger}{\partial \rho}$$

$$u = (r\rho + \rho r^\dagger)$$

$$V = Tr(u)$$

$$w = r + r^\dagger$$

$$b = u - V\rho$$

$$b' = w - w\rho - V$$

$$b = (r\rho + \rho r^\dagger) - Tr(r\rho + \rho r^\dagger)\rho \qquad r = dZ^* C_{out}$$

$$b' = r + r^\dagger - (r + r^\dagger)\rho - Tr(r\rho + \rho r^\dagger)$$

$$r = dZ^* C_{out}$$

$$dZ^* c_{out} \quad \xrightarrow{dag} \quad c_{out}^\dagger dZ$$

Factorization

stochastic Part: $s = (dZ^* c_{out}\rho + \rho\, c_{out}^\dagger dZ) - Tr(dZ^* c_{out}\, \rho + \rho\, c_{out}^\dagger dZ)\rho$

$$s = (dZ^* c_{out}\rho + \rho\, c_{out}^\dagger dZ) - \left(Tr(dZ^* c_{out}\, \rho) + Tr(\rho\, c_{out}^\dagger dZ)\right)\rho$$

$$dZ = a + bi = dW_x + dW_y i$$

$$s = \left((a-bi)\, c_{out}\rho + \rho\, c_{out}^\dagger (a+bi)\right) - \left((a-bi)\, Tr(c_{out}\, \rho) + (a+bi)\, Tr(\rho\, c_{out}^\dagger)\right)\rho$$

$$s = -i\left(b\,Tr(\rho c_{out}^\dagger)\rho\right) - a\,Tr(\rho c_{out}^\dagger)\rho + ib(\rho c_{out}^\dagger) + a(\rho c_{out}^\dagger) + i\left(b\,Tr(c_{out}\rho)\rho\right)$$
$$- a\,Tr(c_{out}\rho)\rho - ib(c_{out}\rho) + a(c_{out}\rho)$$

$$s = s_a + s_b$$

$$s_a = -a\,Tr(\rho c_{out}^\dagger)\rho + a(\rho c_{out}^\dagger) - a\,Tr(c_{out}\rho)\rho + a(c_{out}\rho)$$
$$= a\left(-Tr(\rho c_{out}^\dagger)\rho + \rho c_{out}^\dagger - Tr(c_{out}\rho)\rho + c_{out}\rho\right)$$

$$s_b = -i\left(b\,Tr(\rho c_{out}^\dagger)\rho\right) + ib(\rho c_{out}^\dagger) + i\left(b\,Tr(c_{out}\rho)\rho\right) - ib(c_{out}\rho)$$
$$= ib\left(\underbrace{-Tr(\rho c_{out}^\dagger)\rho + \rho c_{out}^\dagger}_{t_1} + \underbrace{Tr(c_{out}\rho)\rho - c_{out}\rho}_{t_2}\right)$$

$$s = a(t_1 - t_2) + ib(t_1 + t_2) = (t_1 - t_2)dW_x + i(t_1 + t_2)dW_y$$

Optimization

$$t_1 = -\mathrm{Tr}(\rho c_{out}^{+})\rho + \rho c_{out}^{+} \qquad t_2 = \mathrm{Tr}(c_{out}\rho)\rho - c_{out}\rho$$

$$u_1 = \rho c_{out}^{+} \qquad u_2 = c_{out}\rho$$

$$t_1 = -\mathrm{Tr}(u_1)\rho + u_1 \qquad t_2 = \mathrm{Tr}(u_2)\rho + u_2$$

The Milstein b'

$$b_1 = (t_1 - t_2) \qquad b_2 = i(t_1 + t_2)$$

$$t_1' = -c_{out}^{+}\rho - \mathrm{Tr}(\rho c_{out}^{+}) + c_{out}^{+} \qquad t_2' = c_{out}\rho + \mathrm{Tr}(c_{out}\rho) - c_{out}$$

$$b_1' = -c_{out}^{+}\rho - \mathrm{Tr}(\rho c_{out}^{+}) + c_{out}^{+} - c_{out}\rho - \mathrm{Tr}(c_{out}\rho) + c_{out}$$

$$b_2' = i\left(-c_{out}^{+}\rho - \mathrm{Tr}(\rho c_{out}^{+}) + c_{out}^{+}\right) + i\left(c_{out}\rho + \mathrm{Tr}(c_{out}\rho) - c_{out}\right)$$

Optimization

$$u_1 = \rho c_{out}^{+} \qquad u_2 = c_{out}\rho$$

$$v_1 = -\mathrm{Tr}(u_1) \qquad v_2 = -\mathrm{Tr}(u_2)$$

$$t_1 = v_1\rho + u_1 \qquad t_2 = v_2\rho + u_2$$

$$t_1' = -c_{out}^{+}\rho + v_1 + c_{out}^{+} \qquad t_2' = u_2 + v_2 - c_{out}$$

worse unless custom X−I operator

$$= c_{out}^{+}(I-\rho) + v_1 = v_1 - c_{out}^{+}(\rho - I)$$

finally

$$b_1 = t_1 - t_2 \qquad b_2 = i(t_1 + t_2) \qquad b_1' = t_1' - t_2' \qquad b_2' = i(t_1' + t_2')$$