Scale of Fluctuation for Spatially Varying Soils: Estimation Methods and Values

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Abstract

Spatial variability is one of the major sources of uncertainty in geotechnical applications. This variability is primarily characterized by the scale of fluctuation. Scale of fluctuation describes the distance over which the parameters of a soil or rock are similar or correlated. The scale of fluctuation is required in order to best characterize as well as simulate a spatially variable field.

The purpose of this review paper is two-fold. The paper firstly provides an overview of the various methods available for estimating the scale of fluctuation from cone penetration test (CPT) data, along with two examples for comparing the methods. The first part reveals some issues with two popular estimation methods, namely the method of moments and the maximum likelihood method (MLE). The method of moments is less sensitive to the choice of the autocorrelation function (ACF), but it could be less precise and may be based on a correlation estimator that does not produce a positive definite autocorrelation matrix. For MLE, it can be very sensitive to the choice of the classical one-parameter ACF. It is not uncommon to assume such an ACF, rather than to identify the ACF from actual soil data with a more general two-parameter Whittle-Matérn (WM) model. This practice may not be robust. Nonetheless, a literature survey is useful if one bears these caveats in mind. The second part provides a database table of horizontal and vertical scale of fluctuation values in different locations and for different materials, collected from published case studies, which can be used as a reference when field data is not readily available. The probable range of values as a function of soil type is provided to inform sensitivity analysis.

Introduction

Spatial variability is one of the major sources of uncertainty in geotechnical applications. In recent decades the necessity of considering spatial variability in geotechnical applications has been demonstrated in many studies (e.g., Griffiths et al. 1993, 2009; Cho 2010; Soubra and Massih 2010; Hicks and Spencer 2010; Huang et al. 2010, 2013; Stuedlein et al. 2012; Cassidy et al. 2013; Jha and Ching 2013; Javankhoshdel and Bathurst 2014; Jiang et al. 2014; Le 2014; Li et al. 2015a; Javankhoshdel 2016; Xiao et al. 2016; Li et al. 2016; Luo et al. 2016; Javankhoshdel et al. 2017; Papaioannou and Straub 2017). The state-of-the-practice is to characterize this spatial variability using the scale of fluctuation. The scale of fluctuation describes the distance over which the parameters of a soil or rock are similar or correlated; soil properties sampled from adjacent locations in the soil profile tend to have similar values and as the sampling distance increases the correlation decreases. The scale of fluctuation is possibly the minimum information needed to simulate a spatially variable field that bears some semblance to reality. It is used as an input to an autocorrelation function (ACF) model (e.g. Markovian, Gaussian), which is either prescribed or identified from empirical autocorrelation values at discrete lags through some fitting procedures. This ACF model defines the correlation between two points separated by any arbitrary interval and orientation (for 2D and 3D fields). The scale of fluctuation by itself is insufficient – it can be viewed as a coarse descriptor of the spatial correlation structure in this sense. Other parameters are needed to describe the finer features of the spatial correlation structure (Ching and Phoon 2019a).

In design, we are frequently more interested in the scale of fluctuation relative to the characteristic length of the structure (e.g., footing width, slope height, retaining wall height, tunnel). A scale of fluctuation that is much longer than the characteristic length of the structure is practically infinite, in the sense that the volume of soil that influences soil-structure interaction can be regarded as homogeneous. The notion of a “worst-case” scale of fluctuation to be discussed below is also related to the ratio between the scale of fluctuation and the characteristic length of the geotechnical structure.

The concept of a scale of fluctuation, sometimes referred to as a spatial correlation length, originated in geostatistics for geology. This began with the variogram, which describes the amount of spatial dependence between two locations, as will be explained in the next section.

An important application of the variogram is Kriging. Kriging is an interpolation method originally developed by Daniel Krige for predicting ore grades in spatially varying gold mines (Krige, 1951, 1966). It interpolates known points and uses a weighted average of a function of the covariance between them to obtain the average value at an unknown location. It has a sound theoretical basis in the form of minimizing mean square error not entirely different from regression, except the measured points are correlated rather than independent (Brockwell and Davis 1991). This method became quickly popular in geostatistics and is now used in a widely varying array of disciplines.

The application of the random field to model spatial variability in geotechnical engineering was popularized by Vanmarcke (1977). Two concepts distinct from geostatistics became popular: (1) the scale of fluctuation that unifies autocorrelation function models and the implicit assumption that the scale of fluctuation is more important than the detailed mathematical form of the ACF (e.g. Markovian, Gaussian), and (2) spatial averaging and the variance reduction function (also related to the scale of fluctuation). Vanmarcke (1983)’s key premise stated that all measurements involve spatial averaging and, as such, detailed differences in the spatially varying field are averaged out and the variance of the averaged field is smaller than that of the original field. This reduction is quantified by the variance reduction function. Recent literature demonstrated that spatial averaging in the Vanmarcke sense is not always the key mechanism in geotechnical engineering problems. Another important mechanism not discussed in Vanmarcke (1997, 1983) is the "worst-case" scale of fluctuation as briefly detailed below.

A series of random finite element papers, including Fenton and Griffiths (2003), Jaksa et al. (2005), Fenton et al. (2007), Breysse et al. (2007), Ching et al. (2017a), Luo et al. (2016) and Zhu et al. (2018), showed that a critical or "worst-case" scale of fluctuation exists for a variety of problems. Javankhoshdel et al. (2017) and Shahmalekpoor et al. (2019) have reported the worst-case spatial correlation length using random limit equilibrium as well. The "worst-case" scale of fluctuation is defined as the scale of fluctuation value that results in the highest probability of failure. It has also been identified as the case producing the lowest mean response such as the lowest bearing capacity of a shallow foundation installed in a spatially variable soil. If the response were to be equal to the spatial average along a prescribed slip surface in the Vanmarcke (1977)'s sense, it is useful to point out that the mean response will be equal to the mean of the random field. This is a theoretical result arising from Vanmarcke (1977)'s definition of the spatial average as a stochastic line/surface/volume integral of a random field in a *prescribed* domain. In other words, the limits of the integral are constants. It does not depend on the scale of fluctuation and certainly a "worst-case" will not appear under the notion of a spatial average as defined by Vanmarcke (1977, 1983). The "worst-case" scale of fluctuation, whenever it exists, is particularly useful for design when there is not sufficient data available to estimate the scale of fluctuation directly. Ching et al. (2017a) compiled a table of "worst-case" scale of fluctuations reported in previous studies, which is reproduced in Table 1 with minor updates.

The concept of the "worst-case" scale of fluctuation has been explained in a series of papers using the concept of mobilized strength and modulus (Ching and Phoon, 2013a; Ching and Phoon, 2013b; Ching et al., 2014; Hu and Ching, 2015; Ching et al., 2016a; Ching et al., 2016b; Ching et al., 2016c; Ching et al., 2017a; Ching et al., 2017b). The idea of converting a complex spatially heterogeneous medium to an equivalent (in some sense) homogeneous medium is comparable to the classical homogenization theory in micromechanics (Paiboon, et al., 2013), except the equivalency principle is different. This concept is essentially a generalization of the classical spatial average, which was found to be limited to situations where the failure path is constrained (e.g. side resistance of pile). It does not work for a failure path that is "emergent" (solution of a boundary value problem), such as a slope failure. It is evident that this path cannot be represented by a stochastic line integral with constant integration limits. Hicks and co-workers (Hicks 2012; Hicks & Nuttall 2012; Hicks et al. 2019) introduced a similar idea of an "effective" property that can be back-figured numerically from the response of a structure.

Ching and Phoon (2018) and Ching et al. (2019) further noted that the scale of fluctuation is a necessary but not sufficient characterization of the ACF. They proposed a more complete characterization consisting of the scale of fluctuation and a roughness parameter. This requires the adoption of a two-parameter autocorrelation function such as the powered exponential model and the Whittle-Matérn (WM) model. It is worthwhile to note that all classical autocorrelation functions such as those shown in Table 2 are one-parameter models. This review paper primarily focuses on the scale of fluctuation, as there are few papers characterizing the roughness parameter for real soil data.

Due to the importance of the scale of fluctuation briefly reviewed above, various methods have been developed to characterize this parameter from soil data, particularly cone penetration test (CPT) measurements, the most commonly used method of obtaining near continuous field data. The scale of fluctuation can be estimated from CPT data using methods such as the method of moments (e.g., Tang 1979; Lacasse and Nadim 1996; Uzielli et al. 2005; Zhang and Dasaka 2010), maximum likelihood estimation (e.g., DeGroot and Baecher 1993; Fenton 1999; Hicks and Onisiphorou 2005; Jaksa et al. 2005; Lloret-Cabot et al. 2014), and Bayesian analysis (e.g., Wang et al. 2010; Cao and Wang 2012; Tian et al. 2016).

It should be noted that estimating the scale of fluctuation in the most general setting where all parameters are unknown, including the shape of the trend function and the autocorrelation function, and in the presence of limited data (e.g. one CPT sounding) may not be tractable. Ching et al. (2017c) called this the "identifiability problem." The problem is more tractable in the presence of multiple CPT soundings (Ching et al., 2016d, 2016e; Ching and Phoon, 2017; Xiao et al. 2019).

Even in the case of more soundings however, the trend function used to detrend the data and obtain the residuals from which the scale of fluctuation is estimated, influences the estimation (e.g. Wang et al. 2019b; Pieczyńska-Kozłowska, 2015). Further, since the estimation methods discussed in this study require an assumption of the autocorrelation model, the choice of autocorrelation model also affects the estimates, and the magnitude of this change is sensitive to the estimation method. These concepts are further explored in the next section. The effect of autocorrelation model is also explored in the Examples section.

The purpose of this paper is two-fold. The paper firstly provides an overview of the methods available for estimating the scale of fluctuation from CPT data, along with two examples for comparing the methods. Secondly, it provides a database table of horizontal and vertical scale of fluctuation values in different locations and for different geomaterials. This tabulation is important because commercial software such as Rocscience's *Slide2* (Rocscience, 2019) and SoilVision's *SVSlope* (SoilVision Systems, 2019) which can analyze geotechnical problems with 2D spatial variability, as well as GEO-SLOPE's *SLOPE/W* (GEO-SLOPE International, 2018), which can analyze geotechnical problems with 1D spatial variability are increasingly expanding the reach of their analyses from homogeneous (or layered) soils to more realistic spatially varying soils. In cases where field data is insufficient and engineers find the scale of fluctuation difficult to estimate, this table serves as an important reference to provide a sense of the probable range of values.

It is worth pointing out that past random finite element studies have demonstrated that the probability of failure is a function of the spatial correlation structure that may include other characteristics of the autocorrelation model beyond the scale of fluctuation such as the roughness, non-monotonicity, and, degree of anisotropy (ratio of vertical to horizontal scale of fluctuations). The sensitivity of the probability of failure or other quantities of interest to the designer (e.g. resistance factor in the Load and Resistance Factor Design or LRFD) to the scale of fluctuation or other characteristics of the autocorrelation model has not been systematically studied. The relation between scale of fluctuations for different soil parameters is currently unknown. Cross-correlated vector fields involving multiple soil parameters are currently simulated assuming that all soil parameters follow a single autocorrelation model in the absence of data. These issues are important to random field applications, but they are outside the scope of this review paper that only focuses on what have been characterized empirically from actual soil data in the literature.

Estimating the scale of fluctuation from CPT data

Spatial variability is generally characterized with traditional methods of time series analysis used in statistics. This means that the value of a parameter at a given location is generally described by the sum of a trend and a zero-mean spatial variability (Equation 1). As with measurements in time, soil property measurements that are closer together in space, are more similar in value. For simplicity, the methods outlined below will apply to the measurements from a single CPT sounding, from which the vertical scale of fluctuation needs to be estimated. The methods can easily be extended to multiple dimensions, if the ACF is assumed to be separable. To the authors' knowledge, this "separability" assumption has not been validated by real data.

$$X_i = X(s_i) = T(s_i) + \epsilon(s_i), i = 1, \cdots, k \quad (1)$$

In the equation above, $X(s_i)$, or X_i is the value of the soil property at location s_i, where s_i is the vertical depth below the ground surface, for example, and k is the total number of measurements. $T(s_i)$ is the trend component and $\epsilon(s_i)$ is the spatial variability component. The scale of fluctuation describes the distance over which the spatial variability components $\epsilon(s_i)$ are correlated amongst themselves. It should be noted that all methods described below assume that the mean and variance of the data have already been estimated using common methods. Our current state-of-the-art is capable of estimating all random field parameters, such as the mean, variance, and scale of fluctuation simultaneously (e.g. Ching and Phoon 2017; Ching et al. 2017c).

The separation between T and ϵ is the famous "detrending" problem. Characterizing ϵ as a stationary random field without making any assumptions (i.e. determining all random field parameters simultaneously without any assumptions) is a difficult problem, particularly for one CPT sounding (Ching et al. 2016d, 2016e, 2017c; Ching and Phoon 2017). Ching et al. (2016d) demonstrated that this difficulty manifests itself as statistical uncertainties in the context of Bayesian estimation, which in turn results in less precise probabilities of failure for limit states involving random fields. This unfavourable impact on reliability-based design is of practical importance. Jiang and Huang (2018) proposed a non-stationary random field model which can account for the uncertainties of the trend component T. The detrending problem can be avoided by characterizing X_i as a non-stationary random field but this is also a hard problem in the absence of data. Geostatistics is now using multiple point method (Mariethoz and Caers 2014) however this method requires image data beyond CPT. Another promising approach is based on Bayesian Compression Sampling (BCS) (Wang and Zhao 2016; Wang and Zhao 2017). Wang et al. (2017) subsequently combined it with the Karhunen–Loève (KL) expansion (called BCS-KL random field generator) to simulate realizations (or sample paths) directly from sparse measurements (Wang et al. 2019a). The sample path is needed, because the failure mechanism at the ultimate limit state is usually restricted to slip curves, rather than mobilizing the entire soil mass. The BCS-KL generator is non-parametric and data-driven. No pre-determined function forms are needed for marginal probability density function or covariance function of the random field. Therefore, the BCS-KL generator is readily applicable to non-Gaussian and non-stationary random fields, including those with non-stationary auto-covariance structure (Montoya-Noguera et al. 2019) and those with unknown trend function without detrending (Wang Y. et al. 2019b). In addition, the BCS-KL generator may be readily extended to simulate cross-correlated bivariate random fields (Zhao and Wang 2018). An important open question is whether the basis functions in BCS (which are prescribed) can reproduce the finer details of some sample paths such as those produced by the Whittle-Matérn ACF correctly when there are sufficient measurements (convergence) (Phoon and Wang 2019). A second related question is whether it can retain its key practical advantage of representing such "rough" sample paths using sparse measurements (rate of convergence) (Phoon and Wang 2019).

The many faces of scale of fluctuation

In this paper, the scale of fluctuation (θ) is defined as the area under the autocorrelation function, $\rho(\tau)$, as shown in Equation (2) (Vanmarcke, 1983):

$$\theta = \int_{-\infty}^{\infty} \rho(\tau) d\tau = 2\int_{0}^{\infty} \rho(\tau) d\tau \qquad (2)$$

What is alternately referred to as *scale of fluctuation, correlation length,* or *autocorrelation distance,* can result from several different definitions, all of which are regularly shuffled in geotechnical literature. Two important notes are warranted here:

Firstly, the factor of 2 in the equation above is often omitted hence resulting in both θ and $\theta/2$ being referred to scale of fluctuation, correlation length, etc.

Secondly, the scale of fluctuation is often taken to be a model parameter, rather than the θ defined in Equation 2. For example, the generic definition of the single exponential autocorrelation model is $\rho(\tau) = exp\{\frac{-|\tau|}{a}\}$. However, notice that in Table 2 it is defined as $\rho(\tau) = exp\{\frac{-2|\tau|}{\theta}\}$. The version in Table 2 is the converted version, as re-defined by Vanmarcke (1983). It ensures that $2\int_0^{\infty} exp\left\{\frac{-2|\tau|}{\theta}\right\} = \theta$. Using the generic version, however, $2\int_0^{\infty} exp\left\{\frac{-|\tau|}{a}\right\} = 2a$, meaning that $\theta = 2a$. This is the difference between solving for the model parameter, a, and the true scale of fluctuation θ. The confusion results from the model parameter often being taken to be the scale of fluctuation. For good measure, the same example can be done with the squared exponential autocorrelation model. The integral of the generic form, $\rho(\tau) = exp\{-\left(\frac{|\tau|}{a}\right)^2\}$, is $\sqrt{\pi}a$, while the version in Table 2 has been converted to integrate to θ, meaning that $\theta = \sqrt{\pi}a$. If a study were to take a to be the scale of fluctuation, in reality it would be different from the scale of fluctuation by a factor of $1/\sqrt{\pi}$.

In this study, the scale of fluctuation refers to θ as defined in Equation 2, and all the autocorrelation models are taken to be those in Table 2, which have been converted to integrate to θ.

The physical meaning of scale of fluctuation

A simple way to visualize the physical meaning of scale of fluctuation is through the rule-of-thumb estimation method shown in Figure 1 (Spry et al., 1988). This method averages out the distances between CPT intersections (called "zero crossings" in signal processing literature) with the trendline. The scale of fluctuation can then be estimated as 80% of this average distance.

This rule-of-thumb method was derived under the assumption of a Gaussian random field governed by the squared exponential autocorrelation function based on signal processing theory (e.g., Rice, 1944; Zhu et al., 2019). This rule-of-thumb method provides a reasonable estimate when the assumptions (i.e., Gaussian marginal distribution and squared exponential autocorrelation function) are valid and the length of measurement profile is sufficiently long. Otherwise, it could produce biased estimates of the scale of fluctuation. It is undoubtedly attractive to practitioners, because the method is straightforward, but the authors recommend using it only for smooth sample paths that may plausibly arise from the squared exponential autocorrelation function.

The most common methods used to estimate the scale of fluctuation can be grouped into three categories: 1) method of moments, 2) maximum likelihood estimation, and 3) Bayesian analysis. The first two categories are frequentist methods, with the method of moments being the most commonly used method.

In order to best understand the differences between the various estimation methods, the scale of fluctuation from a real CPT sounding is evaluated numerically using each method described below. This is also done for an ideal simulated example where the correct scale of fluctuation is known.

Method of moments

The method of moments is a standard method of estimating population parameters in statistics. With the method of moments, the sample or empirical moments are set equal to the theoretical moments, and the parameters of interest are solved for so as to minimize the error between the two.

The method of moments can be further subdivided into two categories: 1) autocorrelation or autocovariance function fitting, and 2) semivariogram or variogram fitting.

The power spectral density (PSD) function is the Fourier transform of the autocorrelation function. The PSD is more commonly encountered in random vibration problems. To the authors' knowledge, it is less common to estimate PSD than the autocorrelation, although there are some interesting attempts in this direction (Phoon and Fenton 2004).

Autocorrelation or autocovariance function fitting

The scale of fluctuation has been estimated by fitting empirical autocorrelation or autocovariance values typically computed at discrete lags to the theoretical models shown in Table 2 (e.g., Vanmarcke 1977, Campanella et al. 1987, DeGroot and Baecher 1993, Fenton 1999, Baecher and Christian 2003, Wackernagel 2003, Uzielli et al. 2005, Lloret-Cabot et al. 2014). With this method, the theoretical autocorrelation model is set equal to the sample autocorrelation function and the scale of fluctuation is determined using a fitting method such as least squares. This can similarly be done with the autocovariance function, which is the product of the autocorrelation model and the variance of the data.

The autocorrelation model describes the correlation between the same soil parameter measured at different locations. The cross-correlation describes the correlation between different soil parameters measured at the same location. Cross-correlation is not covered in this paper, because it has been detailed elsewhere (Ching et al. 2016e). Several commonly used autocorrelation models are summarized in Table 2. In the table $\rho(\tau)$ represents the correlation coefficient at a given lag distance τ between two points. The lag describes a multiple of the measurement interval. For illustration, if the measurement interval in a CPT sounding is 0.1 m, $\rho(\tau_1)$ describes the correlation coefficient between two points that are 0.1 m apart. $\rho(\tau_2)$ describes the correlation between two points that are 0.2 m apart, and so on. Some of the classical autocorrelation models are special cases of the more general two-parameter Whittle-Matérn model (Ching et al. 2019). The autocorrelation function can be obtained by substituting an appropriate smoothness parameter ν into the Whittle-Matérn model as shown in Table 2. There are effectively only 4 such special cases based on special values of $\nu = p + 0.5$, where p = 0, 1, 2, 3. The case of p = 0 produces the Markovian model. The case of p = 1 produces the second-order Markovian model. To the best of the authors' knowledge, the case of p = 2 is entirely new to geotechnical practice. Because this new autocorrelation model constitutes the product of an exponential function and a quadratic function, it is named a third-order Markovian model in this paper. The case of p = 3 and higher are practically indistinguishable from the Gaussian model ($\nu = \infty$) (Rasmussen and Williams 2006). The "frequency of usage" column denotes the percentage of studies in Table A1 adopting a particular model. It is evident that the Markovian autocorrelation model is by far the most popular, covering almost half of studies surveyed in this review.

The experimental autocorrelation function is shown in Equation (3) below, where $\hat{\rho}(\tau_j)$ represents the experimental correlation coefficient between two points separated by τ_j, $\hat{\sigma}^2$ is the sample variance of the data, $\hat{\mu}$ is the sample mean of the data, k is the total number of measurement points, and i counts the total number of pairs of points separated by lag τ_j.

$$\hat{\rho}(\tau_j) = \frac{1}{\hat{\sigma}^2 k}\sum_{i=1}^{k-j}(X_i - \hat{\mu})(X_{i+j} - \hat{\mu}), j = 0, \dots, k-1 \qquad (3)$$

An important note is warranted here. Geotechnical literature often cites the equation above with $(k-j)$ in the denominator instead of k. While this is a natural estimator of the autocorrelation at lag τ_j, it results in a correlation matrix with a negative eigenvalue, hence violating a key requirement of correlation matrices – that they must be positive definite. The reason this requirement is so important, is because a negative eigenvalue is akin to saying that the variance can be negative, which is not possible. Although the revised version (Equation 3) is valid, it is also not well emphasized that it is not robust (Chang and Politis 2016). This is further explored in Example 3 in the Examples section below.

The error between the theoretical and experimental functions can be minimized to find scale of fluctuation (θ) using various methods, the most common of which is least squares. Using least squares, Equation (3) below would be minimized with respect to θ by taking the derivative, setting it equal to zero, and solving for θ:

$$Error = \sum_{j=1}^{k}[\hat{\rho}(\tau_j) - \rho(\tau_j)]^2 \qquad (4)$$

The practical difficulty with this method is that the statistical uncertainty associated with the correlation between two points increases with lag distance. This is because the number of pairs of sampled points decreases with lag distance. For example, if the length of the CPT sounding is L, then there is only one pair of points with lag distance = L. Equation (4) assumes that all estimates from Equation (3) are equally accurate. A weighted regression with less weights assigned to correlations at larger lags to account for larger statistical uncertainties is more logical. Alternately, Equation (4) is only applied to lag distances where the correlations can be more reliably estimated. The rule-of-thumb is to keep lag distance shorter than L/4 (Lumb 1975) or to use correlation coefficients exceeding Bartlett's limits (Uzielli et al. 2005).

Finally, Lloret-Cabot et al. (2014) introduced an autocorrelation function fitting method that uses a conditional random field to calculate the scale of fluctuation. This works by creating a random field from the initial estimates of the scale of fluctuation, and then re-estimating the scale of fluctuation from the conditional random field. The conditional random field is the random field created using the CPT data and the initial scale of fluctuation estimates. The process is repeated for a number of iterations.

Semivariogram or variogram fitting

A semivariogram is a plot of $\gamma(\tau)$ in the y-axis and lag distance on the x-axis, where the theoretical equation follows Equation (5) below:

$$\gamma(\tau) = \frac{1}{2} var(X(s_i) - X(s_i + \tau)) \tag{5}$$

It is related to the autocovariance function by $\gamma(\tau) = sill - cov(\tau)$, where $sill$ is the height that the semivariogram reaches when it levels off. A typical semivariogram is shown in Figure 2. The *range* is the corresponding x-coordinate of the *sill*, or the smallest distance over which two points are no longer correlated ($cov(\tau) = 0$), and hence $\gamma(\tau) = sill$. The *range* is often taken to mean the scale of fluctuation. Although both parameters quantify a similar concept (distance over which points are correlated), they are not always the same. The *range* here is a model parameter, and not the scale of fluctuation as defined in Equation 2. The *nugget* shown in the figure is the quantity at zero lag and is interpreted as measurement noise in geotechnical engineering (Baecher, 1985; ArcGis Desktop Help, 2019).

It is worth noting that although there is a lot of work with the semivariogram in other disciplines such as geology, soil sciences, geophysics, ground water, and reservoir management, among others (Matheron, 1963; Ayyub et al., 1987; White et al., 1990; Cressie, 1992; Chiles and Delfiner, 1999), this review is restricted to geotechnical engineering, where the depth of interest covers only the typical depths of underground structures (Anderson et al. 2008).

The experimental semivariogram follows Equation (6) below:

$$\hat{\gamma}(\tau) = \frac{1}{2k}\sum_{i=1}^{k}[X(s_i) - X(s_i + \tau)]^2 \qquad (6)$$

The experimental semivariogram is then fit to the theoretical one using a fitting method such as least squares to minimize the error between the two. A variogram on the other hand is defined simply as $2\gamma(\tau)$ and the same semivariogram procedure described above is followed.

Maximum likelihood estimation

Maximum likelihood estimation (MLE) is another commonly used method for estimating population parameters in statistics (DeGroot and Baecher, 1993; Liu and Leung 2017; Liu et al 2016). A limitation of this method is that a distribution must be assumed for the random field, such as normal and lognormal. As before, the spatial variability component from Equation (1) is described by an autocovariance or autocorrelation function.

The likelihood is defined as the likelihood of getting a certain distribution parameter, given the known CPT measurements. It would typically be the product of the probabilities for each measurement, assuming that the measurements are independent. Since clearly this is not the case with spatial measurements, the data is put into matrix notation, with **V** representing the variance-covariance matrix. The off-diagonal components of **V** contain the covariance values between multiples of lags. These values are divided by the variance to obtain correlation coefficients, which are then set equal to one of the autocorrelation models in Table 2 in order to solve for θ.

Generally, the log-likelihood is easier to maximize than the likelihood. The log-likelihood equation, assuming a normal distribution is shown in Equation (7) below:

$$L(\boldsymbol{\phi}|\boldsymbol{Y}) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln(|\boldsymbol{V}|) - \frac{1}{2}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{V}^{-1}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) \tag{7}$$

In Equation (7), $\boldsymbol{\phi}$ depicts the vector of unknown parameters which are not yet determined, $\boldsymbol{Y}$ is the vector of CPT measurements, $\boldsymbol{X}\boldsymbol{\beta}$ is the sample mean of $\boldsymbol{Y}$, **V** is the variance-covariance matrix, and |**V**| is the determinant of **V**.

It should be emphasized that the equation above assumes that the data has a normal distribution. If the distribution of the data appears to be significantly different from the normal, then the likelihood should be calculated with the corresponding density function based on some goodness-of-fit criterion. The examples in the next section use Equation (7) and the maximization algorithm outlined by Xiao et al. (2018).

This method was most recently used by Xiao et al. (2019) in a three-dimensional setting. Although the computation of the V^{-1} is potentially costly, Xiao et al. (2019) proposed an efficient scheme for obtaining it under the assumption that the auto-correlation is separable in the vertical and horizontal directions. Ching et al. (2020) fills a gap in efficient conditional simulation of three-dimensional fields that was left unresolved by Xiao et al. (2019). Ching et al. (2019) noted that all classical autocorrelation models shown in Table 2 are one parameter models that can fit the scale of fluctuation alone. These models cannot fit the smoothness parameter, which is distinct from the scale of fluctuation, and can give misleading estimation results for the scale of fluctuation when implemented with MLE. Ching et al. (2019) recommended the use of a two-parameter Whittle-Matérn model to estimate the scale of fluctuation and smoothness parameter at the same time. The maximum likelihood method was found to be necessary in this estimation process.

Bayesian analysis

The methods described above follow frequentist analysis methods. This means that only the known data (values measured at each CPT increment) are used to calculate the scale of fluctuation. With frequentist methods, it is assumed that the scale of fluctuation is an unknown constant which can be solved for using one of the methods above.

Bayesian analysis assumes that the unknown probabilistic model parameters such as the mean, coefficient of variation, and scale of fluctuation are not constants, but rather random variables (possibly correlated). As such, using Bayesian analysis, the scale of fluctuation *distribution* must be solved for. This is done by first defining a prior distribution for the scale of fluctuation, which, if sufficient data is not present, can be set to a relatively uninformative distribution, such as a uniform one between the minimum and maximum values of scale of fluctuation typically found for a given geomaterial. The second part of the paper (literature review) can provide additional guidance on typical minimum and maximum values. The posterior, or refined distribution is then determined from Bayes rule as shown in Equation (8):

$$p(\boldsymbol{\phi}|\boldsymbol{Y}) = \frac{p(\boldsymbol{Y}|\boldsymbol{\phi})p(\boldsymbol{\phi})}{p(\boldsymbol{Y})} \propto p(\boldsymbol{Y}|\boldsymbol{\phi})p(\boldsymbol{\phi}) \tag{8}$$

In the equation above, $\boldsymbol{\phi}$ depicts the vector of unknown probabilistic model parameters which are not yet determined, $\boldsymbol{Y}$ is the vector of CPT measurements, $p(\boldsymbol{\phi})$ is the prior distribution, $p(\boldsymbol{\phi}|\boldsymbol{Y})$ is the posterior distribution, and p depicts the probability distribution. The vertical bar represents the conditional probability of the first term given the second. As such, the result of the above equation, $p(\boldsymbol{\phi}|\boldsymbol{Y})$, is a set of scale of fluctuation values and their corresponding probabilities, or a distribution. The most probable scale of fluctuation value would be the logical choice for the scale of fluctuation to be used in a spatial simulation.

Application of Bayesian analysis has been attempted within very general and very difficult settings, where the shape of the trend function is unknown and only one CPT sounding is available (Ching et al., 2015, Ching et al. 2016d, 2017c; Ching and Phoon 2017). This Bayesian approach is very powerful and is now further extended to a broader MUSIC-X context (MUSIC = Multivariate, Uncertain and unique, Sparse, Incomplete, and partially Corrupted data and X = spatially variable data). MUSIC-X encapsulates all possible site investigation data in the most realistic setting to date (Ching and Phoon 2019). One clear advantage of the Bayesian approach is that statistical uncertainty is automatically included, because the probabilistic model parameters are considered as random. The frequentist approach requires a separate analysis based on sampling theory to characterize statistical uncertainty. This advantage is critical for geotechnical engineering, because sample sizes are small in many cases (cf. “curse of small sample size” in Phoon, 2017) and one can argue that the current prevalent practice of reporting statistics such as the mean, coefficient of variation, and scale of fluctuation as numbers without the associated statistical uncertainty (which can be significant) is not suitable for geotechnical engineering. The preferred practice is to present the mean, coefficient of variation, scale of fluctuation and others as a 95% confidence interval. This is not carried out in the current paper, because the source data are typically not reported in many papers to perform Bayesian analysis.

Examples

In order to best understand the differences in scale of fluctuation that can result from the frequentist estimation methods, two examples are provided. The first uses a CPT sounding from a study, the scale of fluctuation of which was previously estimated using a method of moments. The second, is a simulated example where the scale of fluctuation is known.

These examples are then used to estimate the scale of fluctuation using the 1) rule of thumb, 2) autocorrelation function fitting with two different autocorrelation models, and 3) maximum likelihood estimation with two different autocorrelation models.

An additional example is presented in order to demonstrate the problem found with the experimental autocorrelation function that is often cited in geotechnical literature.

Example 1: Świebodzice CPT sounding

This example uses a CPT sounding from Świebodzice (Bagińska et al., 2012), the scale of fluctuation of which was estimated by Pieczyńska-Kozłowska (2015). The cone tip resistance q_c for this Świebodzice CPT sounding is shown in Figure 3.

Pieczyńska-Kozłowska (2015) used various autocorrelation models and detrending methods and compared the resulting scale of fluctuations, estimated using the methods of moments. For comparison purposes, the CPT was only linearly detrended and used below. The results from Pieczyńska-Kozłowska (2015) are summarized in Table 3.

The detrended CPT data was then used to estimate the scale of fluctuation using the 1) rule of thumb, 2) autocorrelation function fitting with two types of autocorrelation models, and 3) maximum likelihood estimation with two types of autocorrelation models. The results are summarized in Table 4 below. While the rule of the thumb, method of moments, and Gaussian MLE are in agreement with Pieczyńska-Kozłowska (2015) and with each other, the Markovian MLE estimate is larger than the others. The reason for this is due to the use of the Markovian autocorrelation with MLE. One must be very mindful that MLE is very sensitive to the origin of the classical one-parameter autocorrelation model and hence it is less forgiving of a wrong choice for the autocorrelation model (Ching et al. 2019). For comparison, the scale of fluctuation and the smoothness parameter (ν) of the more general Whittle-Matérn model are estimated using MLE as shown in the last row of Table 4. The ν value of 1.69 indicates that the second-order Markovian model may be a stronger contender ($\nu = 1.5$). Nonetheless, although the Whittle-Matérn model has a better chance of fitting empirical ACFs given the availability of two fitting parameters, there is no guarantee that it is the correct fit. It is safe to say that more robust estimation methods are needed. While the autocorrelation model may be less important than the scale of fluctuation in the estimation of the variance reduction function (Vanmarcke, 1983), it is incorrect to conclude that this is true for other quantities, such as the estimation of the scale of fluctuation itself. Ching and Phoon (2018) presented another example involving the estimation of a probability of failure for a failure mechanism not dominated by spatial averaging. The autocorrelation model is important in this context as well.

Example 2: Simulated data

This example uses data which was simulated to have an isotropic scale of fluctuation of 1 m. This means the vertical scale of fluctuation is equal to the horizontal scale of fluctuation and they are both equal to 1 m. This was done using covariance matrix decomposition with the Cholesky method, as outlined in Li et al. (2019). The simulated field follows a normal distribution with a mean = 10 and a standard deviation = 2. The Markovian and Gaussian autocorrelation models were used to generate the fields. A Markovian random field with mesh size of 0.2 m in a 50 m wide and 50 m high space is shown in Figure 4. The adopted random field parameters are reasonable (Phoon and Kulhawy 1999a). Five equi-spaced vertical samples were taken from the field, at x=5.0 m, x=15.0 m, x=25.0 m, x=35.0 m, and x=45.0 m, as indicated in Figure 4. The sample at 45.0 m is shown in Figure 5 as an example.

The scale of fluctuation was estimated using the same methods described in the previous example and averaged over 20 simulations. Since this data is simulated with a constant mean of 10, detrending was not necessary. The results are summarized in Table 5 and Table 6.

This simulated example is important as a case where the true scale of fluctuation is known, and the various methods can be evaluated based on how close they get to this true value. It is immediately evident that the choice of autocorrelation model used in the estimation has an impact on the estimate; this is especially prominent when using MLE. When the field is generated using the Markovian autocorrelation model (Table 5), the Markovian autocorrelation estimates give better results when compared to the Gaussian model in both the method of moments and MLE cases. It is notable that the MLE Gaussian result is very different from the Markovian – this is due to MLE's sensitivity to the origin of the autocorrelation model as discussed previously (Ching et al. 2019). This is confirmed in Table 6, where a Gaussian field is generated with the same parameters, and the Markovian MLE estimate does much worse than the Gaussian one. The differentiability at the origin of the autocorrelation model is related to the roughness of a profile such as that shown in Fig. 5. This roughness is distinct from the scale of fluctuation. To use MLE correctly, one would need to select an appropriate classical one-parameter autocorrelation model that is consistent with roughness of the observed profile. This caveat is not well recognized in the literature and in practice. Ching et al. (2019) recommended a more robust approach involving applying MLE in conjunction with a two-parameter Whittle-Matérn (WM) model that is flexible enough to capture the roughness and the scale of fluctuation of the observed profile simultaneously. The results are shown in the last rows of Tables 5 and 6. Because the Markovian and Gaussian models are special cases of the Whittle-Matérn model, the MLE is able to identify both the correct model and the correct scale of fluctuation in both tables. Note that the ν values in Table 6 can be regarded as very large, which are consistent with the theoretical $\nu = \infty$ for the Gaussian model. Rasmussen and Williams (2006) pointed out that it is very difficult to distinguish between values of $\nu > 3.5$ from actual finite noisy data and suggested that these ν values are sufficiently large as to be practically indistinguishable from $\nu = \infty$ for the Gaussian model.

Another interesting observation is that the rule of thumb or Rice method is very accurate when the field is generated with the Gaussian autocorrelation model. As previously mentioned, this is on account of this method being derived as a means of estimating a field that follows a Gaussian autocorrelation model.

Considering the fields generated with the Markovian autocorrelation model only, the histograms in Figure 6 show the scale of fluctuation values estimated at each of the five locations, over the 20 simulations (100 values in total), using Markovian method of moments and Markovian MLE, respectively. It is notable that using method of moments leads to a wider range of estimated values with a less distinct mean value, when compared to MLE. This implies that with a small number of samples (as might be found in a real case study) the MLE would provide more accurate results. However, as has been noted, the MLE is highly sensitive to the assumed classical one-parameter autocorrelation model, while method of moments is not. The conventional wisdom that the scale of fluctuation is sufficient for spatial variability analysis is perhaps a bit optimistic when one considers the above characterization problem.

Example 3: The experimental autocorrelation equation

It was explained earlier that geotechnical literature often cites the experimental autocorrelation equation (Equation 3) with $(k-j)$ in the denominator instead of k, as shown in Equation (9) below.

$$\hat{\rho}(\tau_j) = \frac{1}{\hat{\sigma}^2(k-j)} \sum_{i=1}^{k-j} (X_i - \hat{\mu})(X_{i+j} - \hat{\mu}), j = 0, \ \dots, k-1 \tag{9}$$

However, this equation results in a correlation matrix with a negative eigenvalue, hence violating the positive definiteness of correlation matrices. This is explored in this example as follows: a CPT sounding is simulated with five values; the first value is assumed to be measured at ground level, and the others are subsequent measurements into the ground at equal increments, δ. The five points were sampled from a normal distribution with a mean of 10 and variance of 4 and are listed in Table 7. The values are assumed to be detrended.

The variance of the data above is calculated as $\hat{\sigma}^2 =$2.96, and the mean as $\hat{\mu} =$10.26 using standard methods. A normalized variable is defined as $\hat{X}_i = (X_i - \hat{\mu})/\hat{\sigma}$. The corresponding autocorrelation matrix can then be set up as shown below for each τ_j:

$$\begin{bmatrix} \frac{1}{5}\sum_{i=1}^{5}(\hat{X}_i)(\hat{X}_i) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{3}\sum_{i=1}^{3}(\hat{X}_i)(\hat{X}_{i+2}) & \frac{1}{2}\sum_{i=1}^{2}(\hat{X}_i)(\hat{X}_{i+3}) & \frac{1}{1}\sum_{i=1}^{1}(\hat{X}_i)(\hat{X}_{i+4}) \\ \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{5}\sum_{i=1}^{5}(\hat{X}_i)(\hat{X}_i) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{3}\sum_{i=1}^{3}(\hat{X}_i)(\hat{X}_{i+2}) & \frac{1}{2}\sum_{i=1}^{2}(\hat{X}_i)(\hat{X}_{i+3}) \\ \frac{1}{3}\sum_{i=1}^{3}(\hat{X}_i)(\hat{X}_{i+2}) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{5}\sum_{i=1}^{5}(\hat{X}_i)(\hat{X}_i) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{3}\sum_{i=1}^{3}(\hat{X}_i)(\hat{X}_{i+2}) \\ \frac{1}{2}\sum_{i=1}^{2}(\hat{X}_i)(\hat{X}_{i+3}) & \frac{1}{3}\sum_{i=1}^{3}(\hat{X}_i)(\hat{X}_{i+2}) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{5}\sum_{i=1}^{5}(\hat{X}_i)(\hat{X}_i) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) \\ \frac{1}{1}\sum_{i=1}^{1}(\hat{X}_i)(\hat{X}_{i+4}) & \frac{1}{2}\sum_{i=1}^{2}(\hat{X}_i)(\hat{X}_{i+3}) & \frac{1}{3}\sum_{i=1}^{3}(\hat{X}_i)(\hat{X}_{i+2}) & \frac{1}{4}\sum_{i=1}^{4}(\hat{X}_i)(\hat{X}_{i+1}) & \frac{1}{5}\sum_{i=1}^{5}(\hat{X}_i)(\hat{X}_i) \end{bmatrix} \tag{10}$$

The resulting eigenvalues are 2.53, 1.37, 0.69, 0.55, and -0.15. The negative eigenvalue indicates the matrix is not positive definite. On the other hand, if Equation (3) is used for the calculations, the resulting eigenvalues are 1.97, 1.05, 0.83, 0.77, and 0.39. All five eigenvalues are positive, indicating a positive definite matrix. Therefore, only Equation (3) should be used to calculate the experimental autocorrelation. This section reveals that the scales of fluctuation reported in the literature may not be completely accurate. For the method of moments, it is less sensitive to the choice of the autocorrelation function, but it could be less precise and may be based on Eq. (9). For MLE, it can be very sensitive to the choice of the classical one-parameter autocorrelation function, which needs to be identified from actual soil data as well. Nonetheless, a literature survey such as the one conducted below is useful if one bears these caveats in mind.

Literature review of typical scales of fluctuation

Table A1 in Appendix A summarizes the available data on the horizontal and vertical scale of fluctuation for different soil types collected from published case studies, which can be used as a reference for design when field data is not readily available. Where possible, the test types, methods of calculation, and autocorrelation function used to estimate these parameters are reported. A description of soil type is useful because the site-specific scale of fluctuation values can be extrapolated to other locations, provided the soil deposits are of similar geologic formation and environmental history (Phoon and Kulhawy 1999a). The same logic can be used for test type. The method of calculation and autocorrelation function are also useful for understanding the assumptions used in these estimations. For most of the cases presented in this table, the vertical scale of fluctuation is available, because site data are usually collected at a much higher spatial resolution in the vertical compared to the horizontal direction. The original papers reporting these estimations and the papers citing them are available in the table as well. Table A1 is summarized by the soil type in Table 8 to provide engineers with a sense of the probable range of values. Table 8 contains the minimum, maximum, and average values for horizontal and vertical scale of fluctuation (SF) for each soil type.

Figures 7a and 7b provide histogram plots for horizontal and vertical scale of fluctuations, respectively, using the data extracted from Table A1. For cases where a range was provided, the average was used in the figure. It can be seen in Figure 7a that the horizontal scale of fluctuation ranges from nearly 0 to 100 m with most counts in the 0-60 m range. On the other hand, in Figure 7b, the vertical scale of fluctuation ranges from nearly 0 to 9 m, with most counts in the 0-5 m range. These ranges are in agreement with ranges presented in literature for horizontal and vertical scale of fluctuations (Phoon and Kulhawy, 1999a). Additionally, it can be observed that the horizontal scale of fluctuation data fluctuates more from bin to bin compared to the vertical data, on account of there being more data available in Table A1 (and generally in literature) for vertical scale of fluctuation compared to horizontal scale of fluctuation.

In addition to **Table A1** and summary version (**Table 8**), **Table 9** provides a separate summary of the scale of fluctuation for cement-mixed soils presented by Pan et al (2018; 2019). The scale of fluctuation for cement-mixed soil is interesting as well because spatial variability is a combination of natural variability and variability resulting from the mixing process.

Also, Luo and Bathurst (2018a, b) provided the results of the estimates of the scale of fluctuation for the case of reinforced retaining walls. They showed that the vertical scale of fluctuation for these types of structures is equal or less than the soil compaction length (15 to 17 cm). In theory, the magnitude of vertical soil strength spatial variability is also important when the vertical scale of fluctuation for anisotropic spatially variable soil strength are less than the reinforcement spacing used in reinforced walls, slopes and embankments. However, the soils for these systems are engineered materials that must satisfy specified narrow as-built property ranges. This means that spread (uncertainty) in these material properties is small and hence the influence of spatial variability of soil strength on margins of safety expressed probabilistically is less of a concern. The spatial variability of semi-engineered improved soils, be it through mixing with stabilizing agents such as cement or through compaction, is less well studied at present.

Conclusion

This study serves as a practical reference for engineers performing a spatial variability analysis. The purpose of this study was two-fold. Firstly, it provided a basic summary of the available methods for estimating the scale of fluctuation and compared them with two simple examples.

This part reveals the limitations of two popular methods, namely method of moments and maximum likelihood method. The former is less precise and some studies have adopted a correlation estimator that is not positive definite. The latter can be very sensitive to the choice of the classical one-parameter autocorrelation function. Hence, it should not be assumed, but it should be identified from actual soil data, particularly in conjunction with a more general two-parameter Whittle-Matérn (WM) model. It is safe to say that more robust estimation methods are needed. The second part provided a database table of horizontal and vertical scale of fluctuation values in different locations and for different materials, collected from published case studies, as well as a summary of "worst-case" scale of fluctuations, which can be used as a reference when field data is not readily available. A summary table containing the minimum, maximum and average value as a function of soil type is particularly useful, because it provides engineers with a sense of the probable range of values that can inform a sensitivity analysis.

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Figures

Figure 1. Illustration of the rule-of-thumb method for estimating the scale of fluctuation.

Figure 2. Definition of sill and nugget in a typical semivariogram.

Figure 3. The Świebodzice CPT for q_c.

Figure 4. Normal random field generated with a mean = 10, standard deviation = 2, and Markovian autocorrelation model with an isotropic scale of fluctuation of 1 m (vertical length = horizontal length = 1 m). The vertical sample locations are indicated with white lines.

Figure 5. Vertical sample taken at x=45.0 m from the simulated field shown in Figure 4. The mean of 10 is shown with a vertical line.

Figure 6. Histogram of estimated scale of fluctuation values from random field generated with Markovian autocorrelation model: a) method of moments assuming Markovian autocorrelation; b) MLE assuming Markovian autocorrelation. The histograms display 100 values obtained from 20 simulations, sampled in 5 vertical profiles, as shown in Fig 4. A solid vertical line shows the true value of 1 m. A dashed line is used to show the mean of the histogram (0.98 m and 0.96 m, respectively, per Table 5).

Figure 7. The histogram plot for a) horizontal b) vertical scale of fluctuation extracted from **Table A1**.

Tables

Table 1. "Worst-case" scale of fluctuations reported in previous studies (updated from Ching et al. 2017a).

Study	**Problem type**	**"Worst-case" definition**	**Characteristic length**	**"Worst-case" scale of fluctuation**
Jaksa et al. (2005)	Settlement of a nine-pad footing system	Under-design probability is maximal	Footing spacing (S)	1×S
Fenton and Griffiths (2003)	Bearing capacity of a footing on a c-ϕ soil	Mean bearing capacity is minimal	Footing width (B)	1×B
Soubra et al. (2008)				
Fenton et al. (2005)	Active lateral force for a retaining wall	Under-design probability is maximal	Wall height (H)	0.5~1×H
Fenton and Griffiths (2005)	Differential settlement of footings	Under-design probability is maximal	Footing spacing (S)	1×S
Breysse et al. (2005)	Settlement of a footing system	Footing rotation is Maximal	Footing spacing (S)	0.5×S
		Mean different settlement between footings is maximal	Footing spacing (S) Footing width (B)	f(S,B) (no simple equation)
Griffiths et al. (2006)	Bearing capacity of footing(s) on a ϕ=0 soil	Mean bearing capacity is minimal	Footing width (B)	0.5~2×B
Vessia et al. (2009)	Bearing capacity of footing on c-ϕ soil	Mean bearing capacity is minimal (anisotropic 2D variability)	Footing width (B)	0.3~0.5×B
Ching and Phoon (2013a)	Overall strength of a soil column	Mean strength is Minimal	Column width (W)	1 × W (compression) 0×W (simple shear)
Ching and Phoon (2013b)				
Ahmed and Soubra (2014)	Differential settlement of footings	Under-design probability is maximal	Footing spacing (S)	1×S

Hu and Ching (2015)	Active lateral force for a retaining wall	Mean active lateral force is maximal	Wall height (H)	0.2×H
Stuedlein and Bong (2017)	Differential settlement of footings	Under-design probability is maximal	Footing spacing (S)	1×S
Ali et al. (2014)	Risk of infinite slope	Risk of rainfall induced slope failure is maximal	Slope height (H)	1×H
Pan et al. (2018)	Stress-strain behaviour of cement-treated clay column	Peak global strength	Column diameter (D)	2×D

Table 2. Common autocorrelation models and their frequency of usage in Table A1.

Autocorrelation model	**Correlation as a function of lag τ**	**ν**	**Frequency of usage**
Markovian (single exponential)	$\rho(\tau) = exp\{\frac{-2\|\tau\|}{\theta}\}$	0.5	47%
Second-order Markov	$\rho(\tau) = (1 + 4\frac{\|\tau\|}{\theta})exp\{-4\frac{\|\tau\|}{\theta}\}$	1.5	4%
Third-order Markov	$\rho(\tau) = (1 + \frac{16}{3}\frac{\|\tau\|}{\theta} + \frac{256}{27}(\frac{\|\tau\|}{\theta})^2)exp\{-\frac{16}{3}\frac{\|\tau\|}{\theta}\}$	2.5	New to geotechnical practice
Gaussian (squared exponential)	$\rho(\tau) = exp\{-\pi(\frac{\|\tau\|}{\theta})^2\}$	∞	15%
Spherical			9%

	$\rho(\tau) = \begin{cases} \frac{4}{3} - 2\lvert\frac{\tau}{\theta}\rvert + \frac{2}{3}\lvert\frac{\tau}{\theta}\rvert^3, & if\ \lvert\tau\rvert \le \theta; \\ 0, \text{otherwise} \end{cases}$		
Cosine exponential	$\rho(\tau) = exp\{-\frac{\lvert\tau\rvert}{\theta}\}cos\{\frac{\lvert\tau\rvert}{\theta}\}$		10%
Binary noise	$\rho(\tau) = \begin{cases} 1 - \lvert\tau\rvert/\theta, if\ \lvert\tau\rvert \le \theta \\ 0, \text{otherwise} \end{cases}$		15%
Whittle-Matérn	$\rho(\tau) = \frac{2}{\Gamma(\nu)}\left\{\frac{\sqrt{\pi}\,\Gamma(\nu+0.5)\lvert\tau\rvert}{\Gamma(\nu)\theta}\right\}^{\nu} K_{\nu}\left\{\frac{\sqrt{\pi}\,\Gamma(\nu+0.5)\lvert\tau\rvert}{\Gamma(\nu)\theta}\right\}$		

Note: θ = scale of fluctuation; ν = smoothness parameter that reduces the Whittle-Matérn model to a specific one-parameter autocorrelation model (e.g. ν = 0.5 produces the Markovian exponential model); model name "third-order Markov" is coined by this paper; Γ is the Gamma function (Abramowitz and Stegun 1970); K_ν is the modified Bessel function of the second kind with order ν (Abramowitz and Stegun 1970).

Table 3. Pieczyńska-Kozłowska (2015) linearly detrended scale of fluctuation results.

	Markov Autocorrelation	**Gaussian Autocorrelation**
Vanmarcke Method (Autocorrelation function fitting)	0.28 m	0.22 m
Rice Method (Rule-of-thumb method based on mean crossing distance)	0.23 m	0.29 m

Table 4. Scale of fluctuation for q_c from Świebodzice CPT soundings, obtained using three different estimation methods.

Estimation Method	**Scale of fluctuation (m)**
Rule of thumb	0.30
Method of moments fitting:	

Markovian model	0.29
Gaussian model	0.26
Maximum likelihood estimation*:	
Markovian model (theoretical ν=0.5)	0.52
Second-order Markov model (theoretical ν=1.5)	0.40
Third-order Markov model (theoretical ν=2.5)	0.37
Gaussian model (theoretical ν=∞)	0.32
Whittle-Matérn model (theoretical ν unknown)	0.39 (ν = 1.69)

*MLE requires the inversion of the correlation matrix. CPT measurements that are very closely spaced often result in a singular correlation matrix. Hence the MLE estimates were computed with a CPT spacing of 0.15 m, while the rule of thumb and method of moments estimates were computed with a 0.05 m CPT spacing.

Table 5. Scales of fluctuation estimated using methods of moments and MLE for data simulated with the Markovian autocorrelation model. Values averaged over 100 values from 20 simulations sampled over 5 vertical profiles as shown in Fig. 4.

		Measurement Location (m)				
Estimation Method	**Avg. Scale of fluctuation (m)**	**5**	**15**	**25**	**35**	**45**
Rule of thumb	**0.60**	0.60	0.63	0.58	0.58	0.59
Method of moments fitting:						
Markovian model	**0.98**	1.07	0.98	0.94	0.90	1.00
Gaussian model	**0.89**	0.94	0.89	0.84	0.82	0.94
Maximum likelihood estimation:						
Markovian model (theoretical ν=0.5)	**0.97**	0.97	1.00	0.93	0.95	1.00
Second-order Markov model (theoretical ν=1.5)	**0.58**	0.58	0.59	0.60	0.57	0.58
Third-order Markov model (theoretical ν=2.5)	**0.51**	0.51	0.52	0.52	0.50	0.51

Gaussian model	**0.42**	0.42	0.42	0.41	0.42	0.42
(theoretical ν=∞)						
Whittle-Matérn model	**1.04**	0.99	1.05	1.13	1.06	0.97
(theoretical ν unknown)	(ν=**0.51**)	(0.52)	(0.52)	(0.49)	(0.48)	(0.52)

Table 6. Scales of fluctuation estimated using methods of moments and MLE for data simulated with the Gaussian autocorrelation model. Values averaged over 100 values from 20 simulations sampled over 5 vertical profiles as shown in Fig. 4.

		Measurement Location (m)				
Estimation Method	**Avg. Scale of fluctuation (m)**	**5**	**15**	**25**	**35**	**45**
Rule of thumb	**1.03**	1.04	1.00	0.99	1.07	1.06
Method of moments fitting:						
Markovian autocorrelation model	**1.06**	1.11	1.10	0.98	1.03	1.06
Gaussian autocorrelation model	**1.01**	1.04	1.03	0.97	0.98	1.03
Maximum likelihood estimation:						
Markovian model	**3.11**	3.13	3.18	2.97	3.02	3.25
(theoretical ν=0.5)						
Second-order Markov model	**2.36**	2.35	2.37	2.36	2.33	2.37
(theoretical ν=1.5)						
Third-order Markov model	**1.97**	1.96	1.95	1.99	1.97	1.98
(theoretical ν=2.5)						
Gaussian model	**0.94**	0.94	0.95	0.94	0.94	0.96
(theoretical ν=∞)						
Whittle-Matérn model	**1.02**	1.02	1.00	1.04	1.02	1.02
(theoretical ν unknown)	(ν=**72.4**)	(66.3)	(75.7)	(80.7)	(67.2)	(72.2)

Table 7. Five simulated parameter values from a CPT sounding, sampled from a normal distribution, $N(10, 2^2)$.

Depth	Parameter Value
0	8.75
$-\delta$	10.37
-2δ	8.33
-3δ	13.19
-4δ	10.66

Table 8. Summary of literature review (Table A1) by soil type. Min, max, and average scale of fluctuation are displayed.

Soil type	Horizontal SF (m)				Vertical SF (m)			
	Num. studies	Min	Max	Average	Num. Studies	Min	Max	Average
Alluvial	9	1.1	49	14.8	11	0.07	2.53	0.66
Ankara Clay	-	-	-	-	1	1	6.2	3.63
Chicago Clay	-	-	-	-	2	0.4	1.25	0.72
Clay	17	0.14	92.4	24.43	24	0.06	12.7	2.47
Clay, Sand, Silt mix	13	1	1546	152.38	28	0.07	21	1.65
Hangzhou Clay	-	-	-	-	1	0.5	0.77	0.65
Marine Clay	6	2	60	31.3	7	0.11	6	1.85
Marine Sand	1	55	55	55	4	0.08	7.2	1.77
Offshore Soil	5	14	67	34.71	9	0.05	9.1	2.37
Over Consolidated Clay	-	-	-	-	2	0.6	2.55	1.38
Sand	8	1.7	75	11.29	12	0.1	4	1.14
Sensitive Clay	2	30	46	38	3	2	4	3
Silt	3	12.7	45.5	33.22	5	0.14	7.19	2.08

Silty Clay	6	5	45.4	30.26	13	0.095	6.47	1.58
Soft Clay	4	22.1	80	41.1	11	0.14	6	1.76
Undrained Engineered soil	-	-	-	-	23	0.3	2.7	1.43
Water Content	9	2	60	18.5	5	0.2	3	1.22

Table 9. Statistical characteristics of cement-admixed soils (Source: Liu et al., 2015; Pan et al. 2018, 2019)

References	Test (Result)	Scale of fluctuation[*] (m)	
		Vertical	Horizontal
Honjo(1982)	Unconfined Compressive Test (UCS)	0.8-8.0	-
Babasaki et al., (1996)	Unconfined Compressive Test (UCS)	-	-
Hedman and Kuokkanen (2003)	Hand-operated penetrometer test (c_u)	0.38-1.12	0.07-0.33[‡]
Navin and Filz (2005)	Unconfined Compressive Test (UCS)	-	Approximate 24.0
Larsson et al. (2005) [‡]	Hand-operated penetrometer test (c_u)	-	Radial:<0.13 Orthogonal:<0.32[‡]
Larsson and Nilsson (2009)	Cone penetration test (Tip resistance)	-	1.8-3.6
Chen et al.(2011) (MFBC)	Unconfined Compressive Test (UCS)	-	-
Chen et al.(2011) (NCHS)	Unconfined Compressive Test (UCS)	-	-
Al-Naqshabandy et al. (2012)	Cone penetration test (Tip resistance)	0.2-0.7	2.0-3.0
Namikawa and Koseki (2013)	Unconfined compressive test (UCS)	-	-
Bruce et al., (2013)	Unconfined Compressive Test (UCS)	-	-
Chen et al., (2016)	Binder concentration	-	-
Liu et al. (2016)[†] (MFBC)	Unconfined Compressive Test (UCS)	-	-

	(Marina One)	Unconfined Compressive Test (UCS)	-	-
Liu et al. (2018)		Centrifuge test (binder concentration***)	-	Intra-column**: Radial: 0.12-0.28D Circumferential : 67-133° Inter-column**: 100-200m

Notes:

*The concept “auto-correlation distance” used in some studies (e.g. Namikawa and Koseki, 2013) is converted to “scale of fluctuation” Vanmarcke (1983) by multiplying 2.0;

**The intra-column SOF refers to the variation led by the insufficient mixing; the inter-column SOF refers to the variation led by the variation of insitu water content.

†Liu et al. (2016) normalized the strength to 28-day equivalent strength to eliminate the effect of curing period.

‡SOF within the column cross-section

Appendix A

Table A1 Summary of scales of fluctuation reported in the literature

Soil type	Method of Measurement	Parameter Measured	SFX (m)	SFY (m)	Method of Calculation	Autocorrelation function	Cited by	Reference
Ankara Clay		Liquid Limit, wL		4-6.2			Oguz et al. (2019)	Akbas and Kulhawy (2010)
Ankara Clay		Natural water content, wn		2.5-5.5				
Ankara Clay		Undrained shear strength, Su		3-Jan				
Ankara Clay	SPT	N value		3-3.8				
Sand	CPT	qc, fs	2-25, 7-19	0.26-3.14	MMBF (CF)		Xiao et al (2018), Oguz et al (2019)	Akkaya and Vanmarcke (2003)
Clay	CPT	qc, fs	2.5–30, 2–14	0.3-3.62	MMBF (CF)		Xiao et al (2018)	
-		su	2-10				Salgado and Kim (2014)	Al-Homoud and Tanash (2001)
Clean sand and sand fill	SPT	N value		0.3-4			Oguz et al. (2019)	Alonson and Krizek (1975), Lumb (1975), reported by Huber (2013)
Organic soft clay	VST	cu	-	1.2			Ahmed (2012), El-Ramly (2003)	Asaoka and A-Grivas 1982
Organic soft clay	VST	cu	-	3.1				
Soft clay, New York	VST	su		2.5-6		Exponential	JCSS-Cl (2006) Salgado and Kim (2014)	Asoaka et al (1981)
Sensitive clay	VST	su	46	2			Salgado and Kim (2014)	Baecher (1982)
Lignite mine waste dump	CPTu	qc	0.8-3.5	0.15-0.22	CF (Lloret-Cabot et al. (2014))	Markov (Exponential)	Baginska et al. (2018)	Baginska et al. (2018)

	CPTu	qc	Isotropic	0.36-0.56		Markovian, Gaussian, CSX	Baginska et al. (2016)	Baginska et al. (2016)
Very soft clay	VST	cu	22.1	1.1			Ahmed (2012), El-Ramly (2003)	Bergado et al. (1994)
Sand and clay	CPT	qn	12.2-16.1	0.07-0.78	MMBF (CF)	ECF, ACF, GCF	Xiao et al (2018)	Bombasaro and Kasper (2016)
Marine Clay	CPTu	qc		0.78		ECF	Bombasaro and Kasper (2016)	Bombasaro and Kasper (2016)
Marine Sand	CPTu	qc		0.08		ACF		
Continental Clay	CPTu	qc		0.21		ACF		
Marine Alluvial Clay	CPTu	qc	12.15	0.5		ECF		
Marine Alluvial clay with sand laminae	CPTu	qc	15.67	0.29		ECF		
Marine Alluvial sand	CPTu	qc	15	0.07		ECF		
Fluvial alluvial clay	CPTu	qc	15.06	0.08		GCF(ECF for horiz.)		
Fluvial alluvial sand	CPTu	qc	16.11	0.38		ECF		
Onshore sandy soils (loose to medium dense sands, dense fine sands and silty sands)	UC tests and light dynamic probing (DPL) In-situ test			0.32-1.32 (0.78)			Oguz et al. (2019)	Bouayad (2017)
Taranto clay	CPT	qc		0.195-0.72		CSX	Li et al. (2015), Nie et al (2015), Oguz et al. (2019)	Cafaro and Cherubini (2002)
Alluvial deposit	CPT	qc1N	1.1-1.5	0.2-0.29	MMVSM	VXP, VRF, BLM, and AMF	Xiao et al (2018)	Cai et al. (2017)
Sand	CPT	qc		0.13-0.71			Nie et al (2015)	Campanella et al. (1987)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.04	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Chai et al. (2002)

	CPT, VST			1.5		variogram, spherical	JCSS-Cl (2006) (Baker and Calle 2006)	Chiasson et al. (1995)
Sandy soil	CPT	qc		0.1-1.0		SQX, CSX, BIN	Li et al. (2015)	Cheng et al. (2000)
Clay	CPT	qc		0.1-1.8		SQX, CSX, BIN		
Soft clay	CPT	qc		0.2-2.0		SQX, CSX, BIN		
Offshore soils	Undrained Shear Strength		9000	7.1 - 9.1				Cheon and Gilbert (2014)
Clay	CPTu			042-0.96		OLS	Cherubini et al. (2016)	Cherubini et al. (2016)
Sensitiv clay	VST	su		4			Salgado and Kim (2014), Hicks and Sami (2002), Oguz et al. (2019), Ahmed (2012), El-Ramly (2003)	Chiasson et al. (1995)
Clay	Vane shear test		46–60	2.0–6.2		SNX	Li et al. (2015)	Ching et al. (2011)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		2.3	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Dascal and Tournier (1975)
Sensitive clay	VST	su	va				Salgado and Kim (2014)	DeGroot and Baecher (1993)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.2	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Eide and Holmberg (1977)
Clayey silty sand	CPT	Normalized tip resistance		0.2-0.5	MMBF (CF)		Oguz et al. (2019), Xiao et al. (2018)	Firouzianbandpey et al. (2014)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.62	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Flaate and Preber (1974)

Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1	Approximate method (Vanmarcke 1977)			
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.6	Approximate method (Vanmarcke 1977)			
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.1	Approximate method (Vanmarcke 1977)			
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1	Approximate method (Vanmarcke 1977)			
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.8	Approximate method (Vanmarcke 1977)			
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.25	Approximate method (Vanmarcke 1977)			Flaate and Preber (1974)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.72	Approximate method (Vanmarcke 1977)			Flaate and Preber(1974), La Rochelle et al.(1974),
Keswick Clay-Over consolidated clay	CPT	undrained shear strength		0.6-1.75		Semivariogram	Jaksa et al. (1993)	from the Probabilisic methods in Geotechnical
	CPT	qt		26-31		linear trend		

	CPT	fs		35-38		linear trend	Wickremesinghe and Campanella (1993)	Engineering, 1993 Balkema, Rotterdam book
	CPT	pore pressure		13-30		linear trend		
Shanghai silty clay				0.31-0.42		SNX,CSX	Li et al. (2015)	Gao (1996)
Silty clay	CPT	qc		0.8–6.1		SNX		Haldar and Sivakumar Babu (2009)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		2.5	Approximate method (Vanmarcke 1977)			Hanzawa (1983) and Kishida et al. (1983)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		2.5	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Hanzawa et al. (1980) and Hanzawa (1983)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.57	Approximate method (Vanmarcke 1977)			Hanzawa et al. (1994)
			2-10	0.2-1		Exponential, variogram	JCSS-Cl (2006) (Baker and Calle 2006)	Hess et al. (1992)
Clay				0.25–2.5		SNX	Li et al. (2015)	Hicks and Samy (2002)
Offshore soils	CPT	qc	30				Nie et al (2015)	Hoeg (1977); Tang (1979)
Marine clay (different levels)	CPT	qc	35-60				Zhang et al. (2016)	Hoeg (1977); Tang (1979)
Soft clay	Qu	cu	40	2			Ahmed (2012), El-Ramly (2003)	Honjo and Kuroda (1991)
			80	4		Exponential	JCSS-Cl (2006) (Baker and Calle 2006)	Honjo and Kuroda (1991)
Clay			1.22	1.22		SNX	Li et al. (2015)	Hsu and Nelson (2006)

		k	15-20	0.63			Hess et al. (1992)	Hufschmied (1986), Berdat et al. (1986)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.6	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Ireland (1954)
Clay	CPT	qc	0.14	0.06-0.25 (mean 0.148)	MMBF (SF)	Exponential (Vanmarcke)	Xiao et al (2018)	Jaksa (1995)
Relatively homogenous, stiff, overconsolidated clay known as Keswich Clay	CPT	Detrended residuals of cone tip resistance measurements		0.63-2.55			Oguz et al. (2019)	Jaksa et al. (1999)
clean sand, clay	CPTu	qc,fs		0.21-2.33		Markovian	Jamshidi Chenari and Farahbakhsh (2015)	Jamshidi Chenari and Farahbakhsh (2015)
In situ soils			30–60	1.0–6.0		SNX	Li et al. (2015)	Ji et al. (2012)
Offshore sand	CPT	qc	14-38	0.66-0.99			Nie et al (2015)	Keaveny et al. (1989)
Offshore soils	CPT		24.6-66.5				Li et al. (2016)	
Offshore soils	Undrained Shear Strength			0.48-7.14				
Offshore soils	CPT	qc	14-38				Zhang et al. (2016), Ahmed (2012), El-Ramly (2003)	Keaveny et al. (1989)
Offshore cohesive soil	Undrained Shear Strength	CU triaxial		0.66-.99			Zhang et al. (2016), Oguz et al. (2019)	
Offshore soil	UU	cu	-	3.6			Ahmed (2012), El-Ramly (2003)	Keavy et al. 1989
Offshore soil	DSS	cu	-	1.4				Keavy et al. 1989
Clean sand	CPT	qc	-	1.6				Kulatilake and Ghosh (1988)
Silty Clay	CPT		5-12	1.4-2			Li et al. (2016), Nie et al. (2015),	Lacasse and de Lamballerie (1995)

							Ahmed (2012), El-Ramly (2003)	
Offshore sand	CPT	qc	25-67c		MMBF (CF)	Exponential	Xiao et al (2018)	Lacasse and Nadim (1996)
Laminated clay	CPT	qc	9.6	-			Ahmed (2012), El-Ramly (2003)	Lacasse and Nadim (1996)
Dense sand	CPT	qc	37.5	-				Lacasse and Nadim (1996)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.94	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Ladd (1972)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		2.5	Approximate method (Vanmarcke 1977)			Lafleur et al.(1988)
Taiyuan silty clay	Direct shear test		36.2–41.7	0.37–0.58		BIN	Li et al. (2015)	Li et al. (2003)
Taiyuan silty clay	Direct shear test		36–41.4	0.35–0.49		BIN		
Taiyuan silt	Direct shear test		41.5–45.1	0.6–0.84		BIN		
Taiyuan silt	Direct shear test		41.8–45.5	0.54–0.92		BIN		
Hangzhou silty clay	Direct shear test		40.5–45.4	0.52–0.75		BIN		
Hangzhou silty clay	Direct shear test		40.4–45.2	0.49–0.71		BIN		
Hangzhou clay	Direct shear test			0.5–0.77		BIN		
Hangzhou clay	Direct shear test			0.59–0.73		BIN		Li et al. (2003)
Clay-bound sand	CPT	qa	13-Jun	0.34-1.7	MMBF (CF)	ACF (Exp), ACF (Sph), Bartlet	Xiao et al (2018)	Lingwanda et al. (2017)
Sand, silt, and clay	CPT	qc, fs	126.9-163.9		MMBF (CF)	Exponential		Liu and Chen (2006)
Sand, silt, and clay	CPT	qc, fs	66-1546	0.18-1.96	MMBF (CF)	Exponential		

onshore alluvial deposits (loose sandy soils, cohesive soils, medium dense to dense sands and clay layers)	CPT	Cone tip resistance	62-2000	1.72-2.53			Oguz et al. (2019)	Liu and Chen (2010)
Offshore clays	CPTU	Cone tip resistance		0.05-1				Liu et al. (2015)
Filled sand in artificial Island	CPT	qc	1.7–15.9	0.4	MMBF (CF)		Xiao et al (2018), Zhang et al. (2016), Oguz et al. (2019)	Lloret-Cabot et al. (2014)
Marine clay, Japan				1.3-2.7		SNX	Li et al. (2015)	Matsuo (1976)
			50-70			Variogram, spherical	JCSS-Cl (2006) (Baker and Calle 2006)	Mulla (1988)
			40-60			Variogram, spherical		
			40-70			Variogram, spherical		
			60-80			Variogram, spherical		
			40-60			Variogram, spherical		
Clay	CPT	qnet	20	0.4	MMBF (CF)		Xiao et al (2018)	Müller et al. (2013)
Different soil units	CPT	Cone tip resistance		0.18-0.39			Oguz et al. (2019)	Nadim (2015)
Clay and silt clay	CPT	qc	283, 225		MMBF (VF)		Xiao et al (2018)	Ng and Zhou (2010)
Yan'an silty clay				1.47			Li et al. (2015)	Ni et al. (2002)
Yan'an silty clay				1.44				
Jiangzhang silty clay				6.47				
Jiangzhang silty clay				2.96				
Tongguan silt				7.19				
Tongguan silt				1.2				

Alluvial deposit	CPT	qc	2-7	1-2	MMBF (SF)	variogram	Xiao et al (2018)	O'Neil and Yoon (2003)
onshore alluvial deposits (loose sandy soils, cohesive soils, medium dense to dense sands and clay layers)	CPT	Sleeve friction		0.18-1.96			Oguz et al. (2019)	Oguz et al. (2019)
Clayey silty sand	CPT	Normalized friction ratio		0.2			Oguz et al. (2019)	Oguz et al. (2019)
Offshore sand and clay sublayers	CPT			0.4-2.9		Maximum Likelihood		Overgard (2015)
Onshore two clay sites				0.11-0.29				Pantelids and Christodoulou (2017)
Clay		Undrained shear strength		0.8-6.1 (2.5)			Oguz et al. (2019), Salgado and Kim (2014), Li et al. (2015)	Phoon and Kulhawy (1999a;1999b)
Sand, Clay	CPT	Cone tip resistance		0.1-2.2 (0.9)				
Clay	CPT	Corrected Cone tip resistance		0.2-0.5 (0.3)				
Clay	Vane shear test	Undrained shear strength	46-60	2-6.2				
Clay, loam		Natural water content		1.6-12.7 (5.7)				
Clay	CPT	Su		0.8-6.1(mean 2,5)			Oguz et al. (2019), Xiao et al (2018)	Phoon et al. (1995)
Clay	CPT	qt	23-66 (mean 44.5)	0.1-2.2 (mean 0.9)				
Sand and clay	CPT	qc	3-80 (mean 47.9)	0.2-0.5 (mean 0.3)				
Clay	VST	Su	46-60(mean 50.7)	2-6.2 (mean 3.8)				

Offshore sediments	CPT, Lab tests	Shear strength		0.38-0.8			Oguz et al. (2019)	Phoon et al. (2003)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.96-2.7	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Pilot (1972), Pilot et al. (1982) and Talesnick and Baker (1984)
Sand	SPT	Nspt	12.1	0.95		Separable Auto-correlation function	Hicks and Sami (2002)	Popescu et al. (1995)
-		su	Isotropic 0.5. rarely more than 10				Salgado and Kim (2014)	Rackwitz (2000)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		2.5	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Ramalho-Ortigão et al. (1983) and and Ferkh and Fell (1994)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.8	Approximate method (Vanmarcke 1977)			
		k	12.7	1.6			Hess et al. (1992)	Rehfeldt et al. (1989a, b), Young and Boggs (1990)
			7.5-22.6	1-2.3		Exponential	JCSS-Cl (2006) (Baker and Calle 2006)	Rehfeldt et al. (1992)
			25-50	1.5-3		Exponential		
Clay	Direct shear test		92.4	1.2-2		SQX	Li et al. (2015), JCSS-Cl (2006) (Baker and Calle 2006)	Ronold (1990)
			750			variogram, spherical	JCSS-Cl (2006) (Baker and Calle 2006)	Rosenbaum (1987)
Clay	CPT	qc	10–62	1.3–4.0		SNX	Li et al. (2015)	

Sand	CPT	qc	35–75	2.2–3.0		SNX		Salgado and Kim (2014)
For materials such as keuper and middle trias formations		reports literature values		10-Jan			Oguz et al. (2019)	Schweiger et al. (2007)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		1.8	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Sevaldson (1956)
Very soft clay (sand inclusion)	Static CPT			0.16-0.32 (0.23)		Space average method	Oguz et al. (2019)	Shuwang and Linping (2015)
Mud and very soft clay	Static CPT			0.14-1 (0.37)		ACF, OLS (exponential, squared exponential, cosine exponential)		
very soft clay and clay	Static CPT			0.16-0.57 (0.37)				
Clay	Static CPT			0.13-0.32 (0.24)				
Silty Clay	CPT	Cone tip resistance		0.1-0.43 (0.23)				
			13-19	3		Exponential, Semivariogram	JCSS-Cl (2006) (Baker and Calle 2006)	Soulie' et al (1990)
Marine Clay		cu		6			Hicks and Sami (2002)	
Sensitive clay	VST	cu	30	3			Ahmed (2012), El-Ramly (2003)	Soulie et al. (1990)

Clay	CPT	qt	3.0–9.9	1.16-1.17	MMVSM	SNX, BIN, CSX, SOM, SQX	Xiao et al (2018)	Stuedlein et al. (2012)
		k	2.8	0.12			Hess et al. (1992)	Sudicky (1986), Freyberg (1986)
	CPT		55			Gaussian	JCSS-Cl (2006) (Baker and Calle 2006), Ahmed (2012), El-Ramly (2003), Li et al. (2016), Xiao et al (2018)	Tang (1979)
	CPT		35-60			Gaussian	JCSS-Cl (2006) (Baker and Calle 2006), Ahmed (2012), El-Ramly (2003), Li et al. (2016), Xiao et al (2018)	
			5.68-9.27			Semivariogram (OLS, ML, RML)	JCSS-Cl (2006) (Baker and Calle 2006)	Unlu et al (1990)
			8.89-20			Semivariogram (OLS, ML, RML)		
			4.02-7.5			Semivariogram (OLS, ML, RML)		
Sand, Clay, Silt (Mixture)	CPT	Cone tip resistance		0.13-1.11 (0.7)			Oguz et al. (2019), Li et al. (2015), Nie et al (2015), Li et al. (2016)	Uzielli et al. (2005)
Sand, Clay, Silt (Mixture)	CPT	CPT friction ratio		0.12-0.6 (0.36)				

Superficial soft clay		Soil layer thickness	22.2				Zhang et al. (2016)	Valdez-Llamas et al. (2003)
Superficial soft clay		Natural water content		0.8-2.0			Zhang et al. (2016), Oguz et al. (2019)	
Deep deposits with alternating clayey and sandy soils		Natural water content	1000	21				
New Liskeard varved clay		su, N value	46	5			Li et al. (2015), Oguz et al. (2019), Salgado and Kim (2014)	Vanmarcke (1977)
Sand	CPT	qa	22-34		Bayesian	Exponential	Xiao et al (2018), JCSS-Cl (2006) (Baker and Calle 2006)	Vrouwenvelder and Calle (2003)
Sand	CPT	qc		0.3			Hicks and Sami (2002)	Wickremesinghe and Campanella (1993)
Undrained engineered slope	unconfined compression (UC) or field vane (FV) shear tests	undrained shear strength		0.3	Approximate method (Vanmarcke 1977)		Jha and Ching (2013)	Wilkes (1972)
Chicago clay	Qu	cu	-	0.4			Ahmed (2012), El-Ramly (2003)	Wu (1974)
Fine sand	CPT	qc	26	0.4	MMBF (VF)	Exponential	Xiao et al (2018), Li et al. (2016)	Wu et al. (1987)
Clay			10–40	0.5–3.0		SNX	Li et al. (2015)	Wu et al. (2011)
Alluvial soil			30–49	0.2–0.9		SNX		
Ocean and lake sedimentary soils			40–80	1.3–8.0				
Moraine soil				2				
Aeolian soil				1.2–7.2				
Undrained engineered slope	unconfined compression	undrained shear strength		1.5	Approximate method		Jha and Ching (2013)	Wu et al. (1977)

	(UC) or field vane (FV) shear tests				(Vanmarcke 1977)			
Chicago clay				0.79–1.25		Exponential	Li et al. (2015)	Xie (2009)
Saturated clay, Japan				1.25–2.86				Xie (2009)
Tianjin port clay	CPT	qc	8.37	0.132–0.322				Yan et al. (2009)
Tianjin port silty clay	CPT	qc	9.65	0.095–0.426				
Tianjin port silt	CPT	qc	12.7	0.140–1.0				
Sandy	SPT	N value		1.36-3.01		Exponential-Squared Exponential	Oguz et al. (2019)	Zhang and Chen (2012)
Sand, Clay, Silt (Mixture)	CPT	qc	-	0.36–4.92	SAI	-	-	Sasanian et al. (2019)
Sand, Clay, Silt (Mixture)	CPT	qc	-	0.16	VRF	-	-	Pishgah and Jamshidi Chenari (2013)
Sand, Clay, Silt (Mixture)	CPT	qc	-	0.44-1.52	VXP, SAI, AMF, BLM and VRF	Single Exponential	-	Eslami Kenarsari et al. (2013)
Sand, Clay, Silt (Mixture)	CPT	qc	-	0.5	AMF	Single Exponential	-	Jamshidi et al. (2018)
(0-3 m below sea bottom)	CPT	qc	55				Zhang et al. (2016)	Zhang et al. (2016)
Fine to medium grained kaolinic sandstones overlain by consolidated clay-bound sands	CPT	qc		0.6-1.4	Bartlett's limits			Prastings (2019)
	CPT	qc	7	0.59-1.44	Approximate Method			
	CPT	qc		0.4-1.7	Bartlett's limits			
	CPT	qc		0.35-1.5	ACF	Spherical Method		
	CPT	qc		0.34-1.4	ACF	Exponential Method		
	CPT	qc	10		ACF	Triangular Model		

	DPL	N_{10}		0.3-1.8	Bartlett's limits			
	DPL	N_{10}	5	0.54-0.98	Approximate Method			
	DPL	N_{10}		0.38-1.5	ACF	Spherical Method		
	DPL	N_{10}		0.37-1.59	ACF	Exponential Method		
	SPT	N_{60}	7	1.7-4.2	Approximate Method			
	Oedometer	M	7	2.5-2.9	Approximate Method			

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