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Theory of Collective Modes of Fractional Quantum Hall States

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Contents

Abstract

We reproduce results from the Girvin, MacDonald and Platzman theory of collective excitations in the fractional quantum Hall effect [1]. We make use of the language of guiding-center and cyclotron momenta to perform projections of operators onto the lowest Landau level. We compute numerical results for the collective-mode dispersion and gap energy consistent with [1]. We provide a simple extension of the projections to include projections onto any Landau level. We apply this extension and arrive at numerical results in the first Landau level.

1 Introduction

The fractional quantum Hall effect is one of the most important topics in modern condensed matter physics. Its rich and complex profile renders it an open research problem in physics to this day. Huge efforts were put to understand this complex phenomenon, including one by S. M. Girvin, A. H. MacDonald and P. M. Platzman (GMP for short). Their work built on an idea from Feynman's theory for superfluid helium, namely, *the single-mode approximation*. Based on that, they introduce a variational ansatz for the excited states. Their ansatz avoids the irrelevant, high-energy excitations to higher Landau levels by introducing a projection scheme onto the lowest Landau level. In this work, we reproduce results from their analysis, namely, the algebra of projected density operators they introduce, by means of guiding-center and cyclotron operators. Furthermore, we obtain numerical results for the energy gap and the dispersion of the collective-mode they introduce. We additionally provide a generalization to account for projections onto any Landau level. An outilne of this work is as follows: Section 2 offers a theoretical background on classical motion in magnetic fields, Landau levels, their degenerate states, the fractional quantum Hall effect, and the Laughlin state. Section 3 discusses Feynman's ansatz and the way it is adapted in the GMP analysis. Section 4 introduces the guiding center and cyclotron operators and utilizes them to carry out projections and derive expressions for relevant quantities. In Sections 5 and 6 we derive a formula for the gap energy by means of a Taylor expansion of the dispersion relation and carry out calculations necessary to arrive at numerical results. In Section 7 a discussion on the ground state chosen in the GMP paper is presented and the numerical evaluation of the gap energy is carried out. Section 8 is basically graphing the dispersion relation. In Section 9 we provide a generalization of the projection scheme to readily include projections onto any Landau level and apply our results from this section to the first Landau level. We present our conclusions in Section 10.

2 Theoretical Background

2.1 Classical Charged Particle in a Magnetic Field

Consider a charged particle of mass m and charge q moving in a uniform magnetic field $\mathbf{B}$ which we choose to point in the z-axis $\mathbf{B} = B\hat{z}$. Then, the Lorentz force acting on the particle is given by:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{2.1}$$

This force is always perpendicular to the velocity, so it does no work, and thus the kinetic energy of the particle remains constant. Let $\mathbf{v} = (v_x, v_y)$ be the velocity of the particle, the equations of motion then read:

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \tag{2.2}$$

In components:

$$m\dot{v}_x = qBv_y \text{ and } m\dot{v}_y = -qBv_x$$

These equations describe circular motion in the xy-plane with an angular frequency:

$$\omega_c = \frac{qB}{m} \tag{2.3}$$

which is known as the *cyclotron frequency*.

2.2 Electrons in a Magnetic Field: Landau Levels

The Hamiltonian for a free electron in a magnetic field is given by:

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} + e\hat{\mathbf{A}})^2 \tag{2.4}$$

where $\hat{\mathbf{A}}$ is the vector potential. We choose the magnetic field to be uniform and point in the z-direction $\nabla \times \mathbf{A} = B\mathbf{e}_z$. Under this field, the electron is restricted to the x-y plane, i.e. $\hat{\mathbf{r}} = (\hat{x}, \hat{y})$ and $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y)$. In order to find the eigenvalues and eigenstates of this Hamiltonian, we start by defining the kinetic momentum:

$$\hat{\mathbf{\Pi}} = \hat{\mathbf{p}} + e\hat{\mathbf{A}} \tag{2.5}$$

which has two components $\hat{\Pi}_x$ and $\hat{\Pi}_y$. Their commutator is:

$$\begin{aligned}
[\hat{\Pi}_x, \hat{\Pi}_y] &= [\hat{p}_x + e\hat{A}_x, \hat{p}_y + e\hat{A}_y] &(2.6)\\
&= [\hat{p}_x, \hat{p}_y] + [e\hat{A}_x, \hat{p}_y] + [\hat{p}_x, e\hat{A}_y] + [e\hat{A}_x, e\hat{A}_y]\\
&= [e\hat{A}_x, \hat{p}_y] + [\hat{p}_x, e\hat{A}_y]\\
&= [eA_x, -i\hbar\partial_y] + [-i\hbar\partial_x, eA_y]\\
&= -i\hbar e\left(\partial_x A_y - \partial_y A_x\right)\\
&= -i\hbar eB &(2.7)
\end{aligned}$$

Where we used the canonical commutation relations $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$ and $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. This commutator can be more generally written as:

$$[\hat{\Pi}_i, \hat{\Pi}_j] = -i\hbar eB\epsilon_{ij} \tag{2.8}$$

where $\hat{\Pi}_1 = \hat{\Pi}_x$ and $\hat{\Pi}_2 = \hat{\Pi}_y$ and ϵ_{ij} is the 2-dimensional Levi-Civita symbol which is defined as:

$$\epsilon_{ij} = \epsilon^{ij} = \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{if } i = j \end{cases} \tag{2.9}$$

Now, we define the new operators:

$$\hat{a} = \frac{l}{\hbar\sqrt{2}}\left(\hat{\Pi}_x - i\hat{\Pi}_y\right) \quad \text{and} \quad \hat{a}^\dagger = \frac{l}{\hbar\sqrt{2}}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right) \tag{2.10}$$

where $l = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length and is a natural length scale in quantum systems with magnetic fields. Using Eq. [2.7] we can find their commutator:

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \left[\frac{l}{\hbar\sqrt{2}}\left(\hat{\Pi}_x - i\hat{\Pi}_y\right), \frac{l}{\hbar\sqrt{2}}\left(\hat{\Pi}_x + i\hat{\Pi}_y\right)\right]\\
&= \frac{l^2}{2\hbar^2}[\hat{\Pi}_x - i\hat{\Pi}_y, \hat{\Pi}_x + i\hat{\Pi}_y]\\
&= \frac{l^2}{2\hbar^2}\left([\hat{\Pi}_x, \hat{\Pi}_x] + [\hat{\Pi}_x, i\hat{\Pi}_y] - [i\hat{\Pi}_y, \hat{\Pi}_x] - [i\hat{\Pi}_y, i\hat{\Pi}_y]\right)\\
&= \frac{l^2}{2\hbar^2}2i[\hat{\Pi}_x, \hat{\Pi}_y] = 1 &(2.11)
\end{aligned}$$

it can be easily seen that:

$$\hat{\Pi}_x = \frac{\hbar}{l\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) \quad \text{and} \quad \hat{\Pi}_y = \frac{\hbar}{il\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{2.12}$$

the Hamiltonian now reads:

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}\hat{\mathbf{\Pi}}^2 = \frac{1}{2m}\left(\hat{\Pi}_x^2 + \hat{\Pi}_y^2\right) = \frac{1}{2m}\frac{\hbar^2}{2l^2}\left(\left(\hat{a}^\dagger + \hat{a}\right)^2 - \left(\hat{a}^\dagger - \hat{a}\right)^2\right)\\
&= \frac{1}{2m}\frac{\hbar^2}{l^2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \frac{1}{2m}\frac{\hbar^2}{l^2}\left([\hat{a}, \hat{a}^\dagger] + 2\hat{a}^\dagger\hat{a}\right) = \frac{\hbar^2}{l^2 m}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\\
&= \hbar\omega_c\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) &(2.13)
\end{aligned}$$

which is precisely the Hamiltonian of a quantum harmonic oscillator, where $\omega_c = \frac{eB}{m}$ is the frequency of cyclotron motion. The eigenstates $|n\rangle$ are constructed by applying $\hat{a}^\dagger$ to the ground state $|0\rangle$, defined by $\hat{a}\,|0\rangle = 0$, according to:

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad \text{and} \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{2.14}$$

The energy spectrum is:

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) \tag{2.15}$$

where $n \in \mathbb{N}$. These energy levels are called *Landau levels*. They are equally spaced with the gap proportional to the applied magnetic field B.

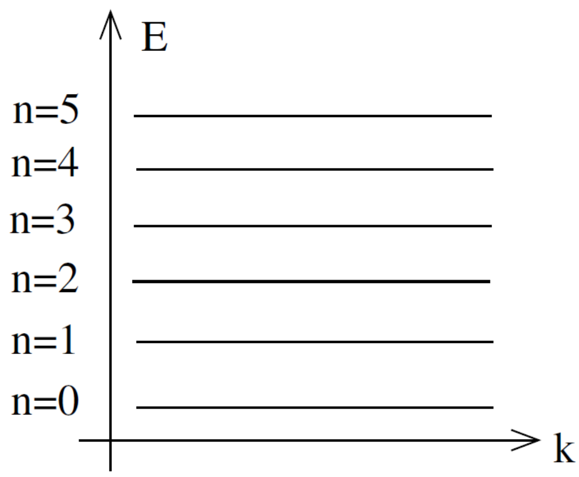

Figure 1: The energy levels of a charged particle in a magnetic field, observe the flat dispersion of the energy with respect to the momentum k [2]

2.3 Degeneracy of Landau Levels

The energy levels we just found are in fact degenerate and do not each correspond to a unique eigenstate. We will demonstrate the source of this degeneracy by finding the wavefunction of an electron in a magnetic field. First, we must pick a gauge for the vector potential. One possible choice is:

$$\mathbf{A} = Bx\mathbf{e}_y \tag{2.16}$$

which is known as the *Landau gauge*. The Hamiltonian reads:

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x^2 + (\hat{p}_y + eBx)^2\right) \tag{2.17}$$

Since $\hat{p}_y$ commutes with the Hamiltonian, it can be replaced by its eigenvalue $p_y = \hbar k$:

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left(\hat{p}_x^2 + (\hbar k + eBx)^2\right) = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega_c^2\left(x + \frac{\hbar k}{m\omega_c}\right)^2 \\ &= \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega_c^2\left(x + kl^2\right)^2\end{aligned} \tag{2.18}$$

which is the Hamiltonain of a harmonic oscillator with the minimum of the potential located at $x = -kl^2$. Again, l here is the magnetic length and ω_c is the cyclotron frequency. The wavefunctions factor into a product of momentum eigenstates in the y-direction and harmonic oscillator eigenstates ψ_n shifted by $x = -kl^2$:

$$\psi_{n,k}(x,y) = e^{iky}\psi_n(x + kl^2) \tag{2.19}$$

The degeneracy manifests itself in the dependence of $\psi_{n,k}$ on two quantum numbers, n and k, while the energy levels depend on n only. A set of wavefunctions that have the same quantum number n is called a Landau level. If the motion along the y-direction is confined, the degeneracy will be finite. If we restrict the system to a rectangle with an area $S = L_x L_y$, k will take the values:

$$k = -\frac{2\pi}{L_y} N \tag{2.20}$$

where N is an integer. We also expect the minimum of the potential at $-kl^2$ to lie within our system (neglecting the radius of the cyclotron motion compared to L_x), i.e. $0 \leq -kl^2 \leq L_x$, and thus we have, $-L_x/l^2 \leq k \leq 0$. Hence, in the area S the number of degenerate states $N_{\max}$ is:

$$0 \leq N \leq \frac{L_x L_y}{2\pi l^2} = \frac{eBS}{2\pi\hbar} = N_{\max} \tag{2.21}$$

which is a very large number (for an electron, the magnetic length at $B = 1$T has approximately the value 2.5×10^{-8}m). $N_{\max}$ can be expressed using the magnetic flux quantum $\Phi_0 = \frac{2\pi\hbar}{e}$:

$$N_{\max} = \frac{BS}{\Phi_0} \tag{2.22}$$

Another important quantity is the filling factor ν, which is defined as [3]:

$$\nu = 2\pi l^2 n \tag{2.23}$$

In a system with many electrons and a strong magnetic field, which is the case we are interested in, all electrons will be in the lowest Landau level and the filling factor becomes $\nu = 2\pi l^2$.

2.4 The Symmetric Gauge

We will study the electron in a magnetic field using a different gauge now, the *symmetric gauge*, whic is given by $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$. This approach will demonstrate the degeneracy of Landau levels purely algebraically. We start by defining the momentum:

$$\hat{\pi} = \hat{\mathbf{p}} - e\hat{\mathbf{A}} \tag{2.24}$$

with the components satisfying $[\hat{\pi}_x, \hat{\pi}_y] = ie\hbar B$. In the symmetric gauge, we have [2]:

$$[\hat{\Pi}_i, \hat{\pi}_j] = 0 \tag{2.25}$$

thus, we can define new creation and annihilation operators that commute with $\hat{a}^\dagger$ and $\hat{a}$:

$$\hat{b} = \frac{l}{\hbar\sqrt{2}} (\hat{\pi}_x - i\hat{\pi}_y) \text{ and } \hat{b}^\dagger = \frac{l}{\hbar\sqrt{2}} (\hat{\pi}_x + i\hat{\pi}_y) \tag{2.26}$$

and similarly, they satisfy [2]:

$$[\hat{b}, \hat{b}^\dagger] = 1 \tag{2.27}$$

Starting from a ground state, $\hat{a}\,|0,0\rangle = \hat{b}\,|0,0\rangle = 0$, we can generate all states by application of the creation operators, in the same manner as Eq. [2.14]:

$$|n, m\rangle = \frac{(\hat{a}^\dagger)^n (\hat{b}^\dagger)^m}{\sqrt{n!m!}} |0,0\rangle \tag{2.28}$$

while

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right) \tag{2.29}$$

The degeneracy is revealed by the second set of creation and annihilation operators, as the energy spectrum is independent of the quantum number m.

2.5 Lowest Landau Level States

The ladder operators we have introduced may be written in complex notation as [2]:

$$\hat{a} = -\frac{i}{\sqrt{2}}\left(2l\frac{\partial}{\partial \bar{z}} + \frac{z}{2l}\right) \text{ and } \hat{a}^{\dagger} = -\frac{i}{\sqrt{2}}\left(2l\frac{\partial}{\partial z} - \frac{\bar{z}}{2l}\right) \tag{2.30}$$

$$\hat{b} = -\frac{i}{\sqrt{2}}\left(2l\frac{\partial}{\partial z} + \frac{\bar{z}}{2l}\right) \text{ and } \hat{b}^{\dagger} = -\frac{i}{\sqrt{2}}\left(2l\frac{\partial}{\partial \bar{z}} - \frac{z}{2l}\right) \tag{2.31}$$

where $z = x - iy$ and $\bar{z} = x + iy$ which follows from our choice of the direction of the magnetic field ($B \geq 0$), since we want to construct a holomorphic ground state (if we picked $z = x - iy$, the resulting ground state would be anti-holomorphic). The wavefunction annihilated by both $\hat{a}$ and $\hat{b}$ is given by:

$$\psi_{0,0}(z, \bar{z}) = \frac{1}{\sqrt{2\pi l^2}} e^{-z\bar{z}/4l^2} \tag{2.32}$$

The states in the lowest Landau level can be constructed by applying $\hat{b}^{\dagger}$:

$$\psi_{0,m}(z, \bar{z}) = \frac{z^m}{\sqrt{2\pi l^2 2^m m!}} e^{-z\bar{z}/4l^2} \tag{2.33}$$

2.6 The Fractional Quantum Hall Effect

The integer quantum Hall effect (IQHE) occurs when non-interacting electrons in a two-dimensional electron gas, subject to a strong perpendicular magnetic field. The Hall resistivity exhibits plateaus at integer filling factors $\nu \in \mathbb{Z}$:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \tag{2.34}$$

instead of a linear function of B. Furthermore, the dissipative resistivity ρ_{xx} drops drastically in regions of these plateaus, enabling dissipationless current flow. The fractional quantum Hall effect was discovered in experiments by Tsui, Störmer, and Gossard [4] in 1982 which revealed additional plateaus at fractional values of ν, such as $\nu = 1/3$, as can be seen in Fig. [??], which could not be explained using the same single-particle picture. Unlike the IQHE, which can be understood in terms of filled Landau levels of independent electrons, the FQHE arises due to strong electron-electron interactions, as electrons arrange themselves into a highly correlated state to minimize Coulomb repulsion while (they undergo the effects of) the magnetic field. The physics of this highly correlated state is that of a quantum fluid rather than a simple band structure. The fractional quantization suggests the existence of **fractionally charged quasiparticles**, which are a hallmark of the FQHE. Experiments confirmed the existence of these quasiparticles with fractional elementary charges [5]. These quasiparticles obey neither fermionic nor bosonic statistics, but rather **anyonic statistics**, meaning their wavefunction acquires a phase factor $e^{i\theta}$ upon exchange, rather than a $+1$ for bosons and a -1 for fermions.

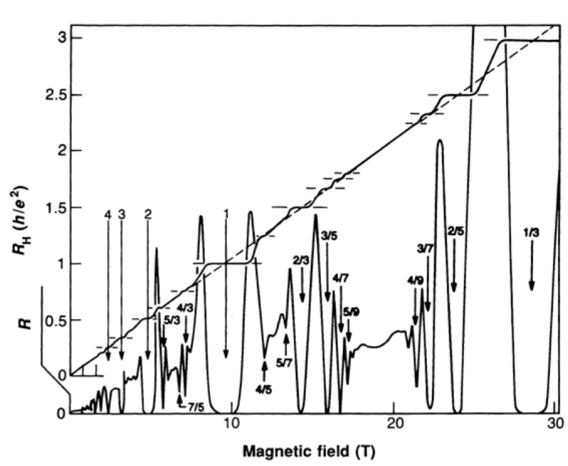

Figure 2: The energy levels of a charged particle in a magnetic field, observe the flat dispersion of the energy with respect to the momentum k [2]

2.7 The Laughlin State

To describe the strongly correlated electron fluid at filling fraction $\nu = 1/m$, Laughlin proposed the wavefunction [6]:

$$\Psi_m(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_k |z_k|^2/4l^2}. \tag{2.35}$$

This wavefunction describes a strongly correlated state of electrons in the lowest Landau level. Despite not being an exact eigenstate, it captures the essential correlations in the FQHE ground state. For fermions, m must be an odd integer, since the wavefunction must be antisymmetric under particle interchange. The prefactor $\prod_{i<j}(z_i - z_j)^m$ ensures that the wavefunction vanishes as $(z_i - z_j)^m$ when any two electrons approach each other, which minimizes the Coulomb repulsion, favoring a correlated liquid state. The integer m controls the strength of this repulsion, with larger m increasing the distance between electrons.

3 An Overview of the GMP Proposal

As we have mentioned previously, the paper by Girvin, MacDonald, and Platzman (GMP for short) [1] presented a theory of the collective excitations in the quantum Hall regime. Their theory is analogous to Feynman's theory of superfluid helium ^{4}He [7]. The core principle in Feynman's theory is the single-mode approximation (SMA). It states that low energy excitations can not be single-particle excitations for a Bose condensate ground state, such as that of superfluid helium. Rather, they are long wavelength density oscillations, a collective excitation. Despite being theorized for a bosonic ground state, Feynman's theory yields excellent results when carefully applied to a Fermi system in a strong magnetic field, as we will see. Feynman's variational ansatz for the density wave can be written as:

$$\phi_{\mathbf{k}}(\mathbf{r}_1, \ldots, \mathbf{r}_N) \;= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{3.1}$$

where $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is the ground state and $\sum_{j=1}^{N} e^{-i\mathbf{k}\cdot\mathbf{r}_j} = \rho_{\mathbf{k}}$ is the Fourier transform of the density operator $\rho(\mathbf{r})$. Note that for each value of k there exists a single density mode, hence the name *single-mode approximation*. This ansatz is orthogonal to the ground state, as we expect for any excited state. This can be easily deduced from the inner product:

$$\langle\psi|\phi_{\mathbf{k}}\rangle = \frac{1}{\sqrt{N}} \int \mathrm{d}\mathbf{r}^2 \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right) \langle\psi|\, \rho(\mathbf{r})\, |\psi\rangle \tag{3.2}$$

the ground state $|\psi\rangle$ has a homogeneous density by definition and thus the inner product is the Fourier transform of a constant, which yields a delta distribution, that is zero for $k \neq 0$. Moreover, the correlations between particles from the ground state are included because of its presence in the expression for $\phi_{\mathbf{k}}$. We can observe key characteristics of a density wave in the definition for $\phi_{\mathbf{k}}$. For more ordered arrangements of particles, where $\rho(\mathbf{r})$ is more or less constant, $\rho_{\mathbf{k}}$ will be close to zero, making such states unlikely to occur. While states with irregular arrangements of particles give rise to finite values of $\rho_{\mathbf{k}}$, making such states more likely to occur, in line with the behavior of a density wave. As usual in variational methods, an upper bound to the lowest excitation energy is obtained:

$$\Delta(k) = \frac{\langle\phi_{\mathbf{k}}|\, \hat{H}\, |\phi_{\mathbf{k}}\rangle}{\langle\phi_{\mathbf{k}}|\phi_{\mathbf{k}}\rangle} - E_0 \tag{3.3}$$

where $\hat{H}$ is the Hamiltonian, E_0 is the ground energy and $\langle\phi_{\mathbf{k}}|\phi_{\mathbf{k}}\rangle = N^{-1}\langle\psi|\, \rho^\dagger_{\mathbf{k}}\rho_{\mathbf{k}}\, |\psi\rangle = s(k)$ is the structure factor for the ground state. We can rewrite the numerator:

$$\begin{aligned}\langle\phi_{\mathbf{k}}|\, \hat{H} - E_0\, |\phi_{\mathbf{k}}\rangle &= \frac{1}{N}\langle\psi|\, \rho^\dagger_{\mathbf{k}}\hat{H}\rho_{\mathbf{k}} - \rho^\dagger_{\mathbf{k}}\rho_{\mathbf{k}}\hat{H}\, |\psi\rangle \\ &= \frac{1}{N}\langle\psi|\, \rho^\dagger_{\mathbf{k}}[\hat{H}, \rho_{\mathbf{k}}]\, |\psi\rangle = f(k) \end{aligned} \tag{3.4}$$

which is the oscillator strength. Since $\bar{\rho}_{-\mathbf{k}} = \bar{\rho}_{\mathbf{k}}$, due to parity symmetry, we can re-derive Eq. [3.4] as:

$$\begin{aligned}
\langle\phi_{\mathbf{k}}| E_0 - \hat{H} |\phi_{\mathbf{k}}\rangle &= \frac{1}{N} \langle\psi| E_0 \rho_{\mathbf{k}}^{\dagger}\rho_{\mathbf{k}} - \rho_{\mathbf{k}}^{\dagger}\hat{H}\rho_{\mathbf{k}} |\psi\rangle \\
&= \frac{1}{N} \langle\psi| \hat{H}\rho_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger} - \rho_{\mathbf{k}}\hat{H}\rho_{\mathbf{k}}^{\dagger} |\psi\rangle \\
&= \frac{1}{N} \langle\psi| [\hat{H}, \rho_{\mathbf{k}}] \rho_{\mathbf{k}}^{\dagger} |\psi\rangle = f(k)
\end{aligned} \tag{3.5}$$

Which makes it possible to write the oscillator strength as a nested commutator:

$$f(k) = \frac{1}{2N} \langle\psi| [\rho_{\mathbf{k}}^{\dagger}, [\hat{H}, \rho_{\mathbf{k}}]] |\psi\rangle \tag{3.6}$$

And finally:

$$\Delta(k) = \frac{f(k)}{s(k)} \tag{3.7}$$

A key assumption in the GMP analysis is that the excited state lies completely within the lowest Landau level. This assumption avoids the high-energy excitations associated with the cyclotron motion (i.e. particles transitioning between Landau levels). To apply this idea, we replace $\rho_{\mathbf{k}}$ in the expression for $|\phi_{\mathbf{k}}\rangle$ by its projection onto the lowest Landau level:

$$\phi_{\mathbf{k}}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{\sqrt{N}} \bar{\rho}_{\mathbf{k}} \psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{3.8}$$

and thus we can write a projected version of Eq. [3.7]:

$$\Delta(k) = \frac{\bar{f}(k)}{\bar{s}(k)} \tag{3.9}$$

where we have replaced $\rho_{\mathbf{k}}$ with $\bar{\rho}_{\mathbf{k}}$ in $f(k)$ and $s(k)$:

$$\bar{f}(k) = \frac{1}{2N} \langle\psi| [\bar{\rho}_{\mathbf{k}}^{\dagger}, [\bar{V}, \bar{\rho}_{\mathbf{k}}]] |\psi\rangle \;\text{ and }\; \bar{s}(k) = \frac{1}{N} \langle\psi| \bar{\rho}_{\mathbf{k}}^{\dagger}\bar{\rho}_{\mathbf{k}} |\psi\rangle \tag{3.10}$$

the reason why the projection of the Hamiltonian contains the interaction potential only (i.e. $\bar{H} = \bar{V}$) is that in a given Landau level, the particles have a flat dispersion (i.e. the kinetic energy is constant). Furthermore, and as we will see in Section 7, Girvin, MacDonald and Platzman choose the Laughlin state to model the ground state $|\psi\rangle$. As we will see, this choice makes it possible to evaluate the structure factor for the ground state and hence the dispersion relation of the collective mode $\Delta(k) = \bar{f}(k)/\bar{s}(k)$.

4 The Guiding Center Approach

We will reproduce results from the GMP paper with a different approach and provide a generalization to any Landau level. We will work with a coordinate system where we decompose the position into two independent degrees of freedom:

$$\hat{\mathbf{R}} = \hat{\mathbf{r}} - \frac{l^2}{\hbar}\hat{\mathbf{z}} \times \hat{\mathbf{\Pi}} \text{ and } \hat{\mathbf{S}} = \frac{l^2}{\hbar}\hat{\mathbf{z}} \times \hat{\mathbf{\Pi}} \tag{4.1}$$

As we can see $\hat{\mathbf{r}} = \hat{\mathbf{R}} + \hat{\mathbf{S}}$. $\hat{\mathbf{R}} = (\hat{R}_1, \hat{R}_2)$ are the guiding center coordinates (or, momenta), they represent the drifting motion of the center of the cyclotron orbit, the $\hat{\mathbf{S}}$ coordinates represent the faster cyclotron motion. An illustration of this coordinate system is shown in the figure below.

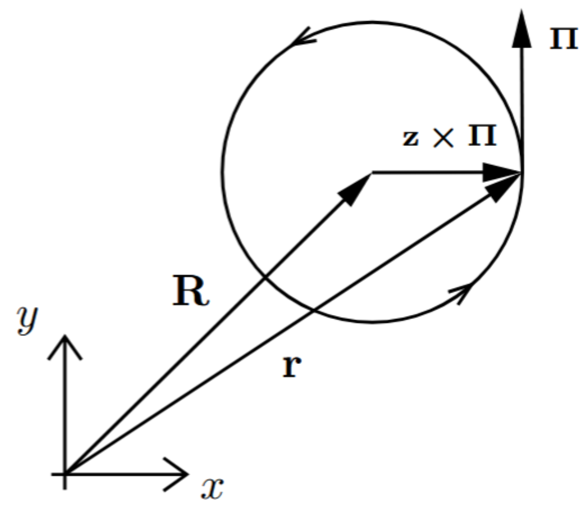

Figure 3: Guiding center and cyclotron momenta

In terms of components, we have:

$$\hat{R}^a = \hat{r}^a - \frac{l^2}{\hbar}\epsilon^{ab}\hat{\Pi}_b \tag{4.2}$$

$$\hat{S}^a = \frac{l^2}{\hbar}\epsilon^{ab}\hat{\Pi}_b \tag{4.3}$$

Now, we can find their commutators:

$$\begin{aligned}
[\hat{R}^a, \hat{S}^b] &= [\hat{r}^a - \frac{l^2}{\hbar}\epsilon^{ac}\hat{\Pi}_c, \frac{l^2}{\hbar}\epsilon^{bd}\hat{\Pi}_d] \\
&= [\hat{r}^a, \frac{l^2}{\hbar}\epsilon^{bd}\hat{\Pi}_d] - [\frac{l^2}{\hbar}\epsilon^{ac}\hat{\Pi}_c, \frac{l^2}{\hbar}\epsilon^{bd}\hat{\Pi}_d] \\
&= \frac{l^2}{\hbar}\epsilon^{bd}[\hat{r}^a, \hat{\Pi}_d] - \frac{l^4}{\hbar^2}\epsilon^{ac}\epsilon^{bd}[\hat{\Pi}_c, \hat{\Pi}_d]
\end{aligned} \tag{4.4}$$

The first commutator is:

$$[\hat{r}^a, \hat{\Pi}_b] = [\hat{r}^a, \hat{p}_b + e\hat{A}_b] = [\hat{r}^a, \hat{p}_b] = i\hbar\delta^a_b \tag{4.5}$$

while the second was previously found. Plugging these results in Eq. 4.4 we get:

$$\begin{aligned}
[\hat{R}^a, \hat{S}^b] &= \frac{l^2}{\hbar}\epsilon^{bd}i\hbar\delta^a_d + \frac{l^4}{\hbar^2}\epsilon^{ac}\epsilon^{bd}i\hbar eB\epsilon_{cd} \\
&= il^2\epsilon^{ba} + i\frac{l^4}{\hbar^2}eB\epsilon^{ac}\delta^b_c \\
&= -il^2\epsilon^{ab} + il^2\epsilon^{ab} \\
&= 0
\end{aligned} \tag{4.6}$$

so $\hat{\mathbf{R}}$ and $\hat{\mathbf{S}}$ are independent. However, for each operator, the components do not commute. For the components of $\hat{\mathbf{S}}$ we have:

$$\begin{aligned}
[\hat{S}^a, \hat{S}^b] &= [\frac{l^2}{\hbar}\epsilon^{ac}\hat{\Pi}_c, \frac{l^2}{\hbar^2}\epsilon^{bd}\hat{\Pi}_d] \\
&= \frac{l^4}{\hbar^2}\epsilon^{ac}\epsilon^{bd}[\hat{\Pi}_c, \hat{\Pi}_d] \\
&= -\frac{l^4}{\hbar}\epsilon^{ac}\epsilon^{bd}\hbar eB\epsilon_{cd} \\
&= -il^2\epsilon^{ac}\delta^b_c \\
&= -il^2\epsilon^{ab}
\end{aligned} \tag{4.7}$$

And for the components of $\hat{\mathbf{R}}$ we have:

$$\begin{aligned}
[\hat{R}^a, \hat{R}^b] &= [\hat{r}^a - \frac{l^2}{\hbar}\epsilon^{ac}\hat{\Pi}_c, \hat{r}^b - \frac{l^2}{\hbar}\epsilon^{bd}\hat{\Pi}_d] \\
&= [\hat{r}^a, \hat{r}^b] - [\hat{r}^a, \frac{l^2}{\hbar}\epsilon^{bd}\hat{\Pi}_d] - [\frac{l^2}{\hbar}\epsilon^{ac}\hat{\Pi}_c, \hat{r}^b] + [\frac{l^2}{\hbar}\epsilon^{ac}\hat{\Pi}_c, \frac{l^2}{\hbar}\epsilon^{bd}\hat{\Pi}_d] \\
&= -\frac{l^2}{\hbar}\epsilon^{bd}[\hat{r}^a, \hat{\Pi}_d] - \frac{l^2}{\hbar}\epsilon^{ac}[\hat{\Pi}_c, \hat{r}^b] + \frac{l^4}{\hbar^2}\epsilon^{ac}\epsilon^{bd}[\hat{\Pi}_c, \hat{\Pi}_d] \\
&= -\frac{l^2}{\hbar}\epsilon^{bd}i\hbar\delta^a_d + \frac{l^2}{\hbar}\epsilon^{ac}i\hbar\delta^b_c - \frac{l^4}{\hbar^2}\epsilon^{ac}\epsilon^{bd}i\hbar eB\epsilon_{cd} \\
&= -il^2\epsilon^{ba} + il^2\epsilon^{ab} - il^2\epsilon^{ac}\delta^b_c = il^2\epsilon^{ab}
\end{aligned} \tag{4.8}$$

The Hamiltonian (without the interaction term) may be rewritten as:

$$\hat{H} = \sum_{i=1}^{N} \frac{\hbar^2}{2ml^4}\hat{\mathbf{S}}_i^2 \tag{4.9}$$

where the index i is a label for particles. We know that this Hamiltonian has the spectrum $E_n = \hbar\omega_c(n+\frac{1}{2})$ and since $[\hat{R}^a, \hat{S}^b] = 0$ we can deduce that projections onto a certain Landau level do not affect $\hat{\mathbf{R}}$. Thus, projections of expressions such as $e^{-i\mathbf{q}\cdot\hat{r}}$ are simplified:

$$\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{r}}}\right]_P = \left[e^{-i\mathbf{q}\cdot\left(\hat{\mathbf{R}}+\hat{\mathbf{S}}\right)}\right]_P = \left[e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}}e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}}\right]_P = e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}}\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}}\right]_P \tag{4.10}$$

Where $e^{-i\mathbf{q}\cdot\left(\hat{\mathbf{R}}+\hat{\mathbf{S}}\right)} = e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}}$ because $[\hat{R}^a, \hat{S}^b] = 0$. To find $\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}}\right]_P$ we write $\hat{S}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{S}_1 = \frac{l^2}{\hbar}\hat{\Pi}_2 = \frac{l}{i\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{4.11}$$

$$\hat{S}_2 = \frac{-l^2}{\hbar}\hat{\Pi}_1 = \frac{-l}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) \tag{4.12}$$

The projection of an operator onto the nth Landau level is basically its expectation value in the harmonic oscillator eigenstates of the cyclotron motion, as we derived in Section 2.2:

$$\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}}\right]_P = \langle n|\exp\left(-i\mathbf{q}\cdot\hat{\mathbf{S}}\right)|n\rangle = \langle n|\exp\left(-i\left(q_1\hat{S}_1 + q_2\hat{S}_2\right)\right)|n\rangle \tag{4.13}$$

Since $\hat{S}_1$ and $\hat{S}_2$ commute with their commutator, we can apply the Baker-Campbell-Hausdorff formula (see Appendix B):

$$\begin{aligned}
\exp\left(-i\mathbf{q}\cdot\hat{\mathbf{S}}\right) &= \exp\left(-iq_1\hat{S}_1\right)\exp\left(-iq_2\hat{S}_2\right)\exp\left(-\frac{1}{2}[iq_1\hat{S}_1, iq_2\hat{S}_2]\right) \\
&= \exp\left(-i\frac{l^2}{2}q_1q_2\right)\exp\left(-iq_1\hat{S}_1\right)\exp\left(-iq_2\hat{S}_2\right) \\
&= \exp\left(-i\frac{l^2}{2}q_1q_2\right)\exp\left(-q_1\frac{l}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right)\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right)\right)
\end{aligned} \tag{4.14}$$

We can use the BCH formula again to separate all the exponentials:

$$\begin{aligned}
\exp\left(-q_1\frac{l}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right)\right) &= \exp\left(-q_1\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(q_1\frac{l}{\sqrt{2}}\hat{a}\right)\exp\left(-\frac{1}{2}[q_1\frac{l}{\sqrt{2}}\hat{a}^\dagger, -q_1\frac{l}{\sqrt{2}}\hat{a}]\right) \\
&= \exp\left(-q_1\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(q_1\frac{l}{\sqrt{2}}\hat{a}\right)\exp\left(-\frac{1}{4}q_1^2l^2\right)
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\exp\left(iq_2\frac{l}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right)\right) &= \exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}\right)\exp\left(-\frac{1}{2}[iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger, iq_2\frac{l}{\sqrt{2}}\hat{a}]\right) \\
&= \exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}\right)\exp\left(-\frac{1}{4}q_2^2l^2\right)
\end{aligned} \tag{4.16}$$

Substituting in Eq. [4.14] we get:

$$\exp\left(-i\mathbf{q}\cdot\hat{\mathbf{S}}\right) = \exp\left(-i\frac{l^2}{2}q_1q_2\right)\exp\left(-\frac{1}{4}q^2l^2\right)\exp\left(-q_1\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(q_1\frac{l}{\sqrt{2}}\hat{a}\right)$$
$$\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}\right) \tag{4.17}$$

Since we are calculating expectation values, we should normal-order our operators:

$$\begin{aligned}\exp\left(-i\mathbf{q}\cdot\hat{\mathbf{S}}\right) &= \exp\left(-i\frac{l^2}{2}q_1q_2\right)\exp\left(-\frac{1}{4}q^2l^2\right)\exp\left(-q_1\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\\ &\quad \exp\left(q_1\frac{l}{\sqrt{2}}\hat{a}\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}\right)\exp\left([q_1\frac{l}{\sqrt{2}}\hat{a}, iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger]\right)\\ &= \exp\left(-\frac{1}{4}q^2l^2\right)\exp\left(-q_1\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(q_1\frac{l}{\sqrt{2}}\hat{a}\right)\\ &\quad \exp\left(iq_2\frac{l}{\sqrt{2}}\hat{a}\right)\\ &= \exp\left(-\frac{1}{4}q^2l^2\right)\exp\left(-\frac{l\,(q_1-iq_2)}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(\frac{l\,(q_1+iq_2)}{\sqrt{2}}\hat{a}\right)\end{aligned} \tag{4.18}$$

where we used the formula $e^Ae^B = e^Be^Ae^{[A,B]}$ (see Appendix B). Substituting in Eq. [4.10] we get:

$$\begin{aligned}\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{r}}}\right]_P &= e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}}\exp\left(-\frac{1}{4}q^2l^2\right)\langle n|\exp\left(-\frac{l\,(q_1-iq_2)}{\sqrt{2}}\hat{a}^\dagger\right)\exp\left(\frac{l\,(q_1+iq_2)}{\sqrt{2}}\hat{a}\right)|n\rangle\\ &= e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}}F(\mathbf{q})\end{aligned} \tag{4.19}$$

where $F(\mathbf{q})$ is the form factor. It follows that $F(\mathbf{q}) = F(q)$ and that $F(-q) = F(q)$ (see Appendix C). For the lowest Landau level (LLL) we have $F(q) = e^{-q^2l^2/4}$. We can project more operators, for example, the density operator in momentum space:

$$\begin{aligned}\bar{\rho}_{\mathbf{q}} &= \sum_{i=1}^{N}\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{r}}_i}\right]_P = \sum_{i=1}^{N}\left[e^{-i\mathbf{q}\cdot\left(\hat{\mathbf{R}}_i+\hat{\mathbf{S}}_i\right)}\right]_P = \sum_{i=1}^{N}\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}\right]_P = \sum_{i=1}^{N}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}\langle n|\,e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}\,|n\rangle\\ &= \sum_{i=1}^{N}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}F(q)\end{aligned} \tag{4.20}$$

which is possible since projections are linear. Multiplying two projected density operators results in:

$$\begin{aligned}\bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}'} &= \sum_{i=1}^{N}\sum_{j=1}^{N}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}F(q)F(q')\\ &= \sum_{i\neq j}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}F(q)F(q') + \sum_{i=1}^{N}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_i}F(q)F(q')\end{aligned} \tag{4.21}$$

The expression $e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_i}$ can be simplified using the BCH formula:

$$e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_i} = e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i+\frac{1}{2}[i\mathbf{q}\cdot\hat{\mathbf{R}}_i, i\mathbf{q}'\cdot\hat{\mathbf{R}}_i]} \tag{4.22}$$

The commutator can be evaluated as follows:

$$\begin{aligned}[i\mathbf{q}\cdot\hat{\mathbf{R}}_i, i\mathbf{q}'\cdot\hat{\mathbf{R}}_i] &= -[q_1\hat{R}_{ix}+q_2\hat{R}_{iy}, q_1'\hat{R}_{ix}+q_2'\hat{R}_{iy}]\\ &= -q_1q_1'[\hat{R}_{ix},\hat{R}_{ix}] - q_1q_2'[\hat{R}_{ix},\hat{R}_{iy}] - q_1'q_2[\hat{R}_{iy},\hat{R}_{ix}] - q_2q_2'[\hat{R}_{iy},\hat{R}_{iy}]\\ &= il^2\left(q_1'q_2 - q_1q_2'\right)\\ &= -il^2\mathbf{q}\times\mathbf{q}'\end{aligned} \tag{4.23}$$

and thus

$$e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_i} = e^{-i\frac{l^2}{2}\mathbf{q}\times\mathbf{q}'}e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i} \equiv f\left(\mathbf{q},\mathbf{q}'\right)e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i} \tag{4.24}$$

Substituting this in Eq. [4.21] we get:

$$\begin{aligned}\bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}'} &= \sum_{i\neq j} e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i} e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j} F(q)F(q') + \sum_{i=1}^{N} e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i} f\left(\mathbf{q},\mathbf{q}'\right) F(q)F(q') \\ &= \sum_{i\neq j} e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i} e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j} F(q)F(q') + f\left(\mathbf{q},\mathbf{q}'\right) \frac{F(q)F(q')}{F(q+q')}\bar{\rho}_{\mathbf{q}+\mathbf{q}'} \end{aligned} \tag{4.25}$$

where $F(q+q')$ is a shorthand notation for $F(|\mathbf{q}+\mathbf{q}'|)$, and where we used the following definition for $\bar{\rho}_{\mathbf{q}+\mathbf{q}'}$:

$$\bar{\rho}_{\mathbf{q}+\mathbf{q}'} = \sum_{i=1}^{N} e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i} F(q+q') \tag{4.26}$$

First, we will project the interaction potential from [1] onto the lowest Landau level. The interaction is given by:

$$V(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \upsilon(q) \sum_{i\neq j} \exp\left[i\mathbf{q}\cdot(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)\right] \tag{4.27}$$

where $\upsilon(q)$ is the Fourier transform of the interaction potential. Only the sum will be affected by the projection. If we insert $\mathbf{q} = -\mathbf{q}'$ and relabel $\mathbf{q}'$ as $\mathbf{q}$ in Eq. [4.25] we get:

$$\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} = \sum_{i\neq j} e^{i\mathbf{q}\cdot\hat{\mathbf{R}}_i} e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_j} F(-q)F(q) + f\left(-\mathbf{q},\mathbf{q}\right) \frac{F(-q)F(q)}{F\left(0\right)}\bar{\rho}_0 \tag{4.28}$$

thus giving

$$\left[\sum_{i\neq j} \exp\left[i\mathbf{q}\cdot(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)\right]\right]_P = \bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} - NF^2(q) \tag{4.29}$$

where $\bar{\rho}_0 = N$ and $f\left(-\mathbf{q},\mathbf{q}\right) = 1$. This result holds for projections P onto any Landau level. The proof that $\sum_{i\neq j} e^{i\mathbf{q}\cdot\hat{R}_i} e^{-i\mathbf{q}\cdot\hat{R}_j} F(-q)F(q)$ is the projection of $\sum_{i\neq j}\exp\left[i\mathbf{q}\cdot(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)\right]$ is given in Appendix A. Substitiuting $F(q) = e^{-q^2l^2/4}$ we get the projection of the interaction onto the lowest Landau level:

$$\begin{aligned}\bar{V} &= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\upsilon(q)\left[\sum_{i\neq j}\exp\left[i\mathbf{q}\cdot(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)\right]\right]_P \\ &= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\upsilon(q)\left(\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} - NF^2(q)\right) \qquad (4.30) \\ &= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\upsilon(q)\left(\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} - Ne^{-q^2l^2/2}\right) \qquad (4.31)\end{aligned}$$

The projected oscillator strength is:

$$\bar{f}(k) = \frac{1}{2N}\left\langle\psi\right|\left[\bar{\rho}^{\dagger}_{\mathbf{k}},\left[\bar{V},\bar{\rho}_{\mathbf{k}}\right]\right]\left|\psi\right\rangle \tag{4.32}$$

$|\psi\rangle$ is, again, the ground state. In order to evaluate $\bar{f}(k)$, it is useful to calculate the commutator of two projected density operators:

$$\begin{aligned}\left[\bar{\rho}_{\mathbf{k}},\bar{\rho}_{\mathbf{q}}\right] &= \bar{\rho}_{\mathbf{k}}\bar{\rho}_{\mathbf{q}} - \bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{k}} \\ &= \sum_{i\neq j} e^{-i\mathbf{k}\cdot\hat{R}_i} e^{-i\mathbf{q}\cdot\hat{R}_j} F(k)F(q) + f\left(\mathbf{k},\mathbf{q}\right)\frac{F(k)F(q)}{F(k+q)}\bar{\rho}_{\mathbf{k}+\mathbf{q}} \\ &- \sum_{i\neq j} e^{-i\mathbf{q}\cdot\hat{R}_i} e^{-i\mathbf{k}\cdot\hat{R}_j} F(q)F(k) - f\left(\mathbf{q},\mathbf{k}\right)\frac{F(q)F(k)}{F(k+q)}\bar{\rho}_{\mathbf{k}+\mathbf{q}} \\ &= \left[f\left(\mathbf{k},\mathbf{q}\right) - f\left(\mathbf{q},\mathbf{k}\right)\right]\frac{F(q)F(k)}{F(k+q)}\bar{\rho}_{\mathbf{k}+\mathbf{q}} \\ &= \left[e^{il^2\mathbf{k}\times\mathbf{q}} - e^{-il^2\mathbf{k}\times\mathbf{q}}\right]\frac{F(q)F(k)}{F(k+q)}\bar{\rho}_{\mathbf{k}+\mathbf{q}} \\ &= \left(e^{\mathrm{k}^*\mathrm{q}/2} - e^{\mathrm{kq}^*/2}\right)\bar{\rho}_{\mathbf{k}+\mathbf{q}} \end{aligned} \tag{4.33}$$

where $\mathrm{k} = k_1 + ik_2$ and $\mathrm{q} = q_1 + iq_2$. Now we can evaluate $\bar{f}(k)$:

$$\begin{aligned}
\bar{f}(k) &= \frac{1}{2N}\langle\psi|\,[\bar{\rho}^{\dagger}_{\mathbf{k}},[\bar{V},\bar{\rho}_{\mathbf{k}}]]\,|\psi\rangle = \frac{1}{2N}\langle\psi|\,[\bar{\rho}^{\dagger}_{\mathbf{k}},[\frac{1}{2}\sum_{\mathbf{q}}\upsilon(q)\left(\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} - Ne^{-q^2l^2/2}\right),\bar{\rho}_{\mathbf{k}}]]\,|\psi\rangle \\
&= \frac{1}{2N}\langle\psi|\,[\bar{\rho}^{\dagger}_{\mathbf{k}},\frac{1}{2}\sum_{\mathbf{q}}\upsilon(q)[\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}} - Ne^{-q^2l^2/2},\bar{\rho}_{\mathbf{k}}]]\,|\psi\rangle \\
&= \frac{1}{4N}\sum_{\mathbf{q}}\upsilon(q)\,\langle\psi|\,[\bar{\rho}^{\dagger}_{\mathbf{k}},[\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}},\bar{\rho}_{\mathbf{k}}] - [Ne^{-q^2l^2/2},\bar{\rho}_{\mathbf{k}}]]\,|\psi\rangle \\
&= \frac{1}{4N}\sum_{\mathbf{q}}\upsilon(q)\,\langle\psi|\,[\bar{\rho}^{\dagger}_{\mathbf{k}},\bar{\rho}_{-\mathbf{q}}[\bar{\rho}_{\mathbf{q}},\bar{\rho}_{\mathbf{k}}] + [\bar{\rho}_{-\mathbf{q}},\bar{\rho}_{\mathbf{k}}]\,\bar{\rho}_{\mathbf{q}}]\,|\psi\rangle \\
&= \frac{1}{4N}\sum_{\mathbf{q}}\upsilon(q)\,\langle\psi|\,[\bar{\rho}_{-\mathbf{k}},\left(e^{\mathrm{k}^*\mathrm{q}/2} - e^{\mathrm{k}\mathrm{q}^*/2}\right)\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{k}+\mathbf{q}} + \left(e^{-\mathrm{k}^*\mathrm{q}/2} - e^{-\mathrm{k}\mathrm{q}^*/2}\right)\bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{k}-\mathbf{q}}]\,|\psi\rangle \\
&= \frac{1}{4N}\sum_{\mathbf{q}}\upsilon(q)\,\langle\psi|\left(e^{\mathrm{k}^*\mathrm{q}/2} - e^{\mathrm{k}\mathrm{q}^*/2}\right)[\bar{\rho}_{-\mathbf{k}},\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{k}+\mathbf{q}}] + \left(e^{-\mathrm{k}^*\mathrm{q}/2} - e^{-\mathrm{k}\mathrm{q}^*/2}\right)[\bar{\rho}_{-\mathbf{k}},\bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{k}-\mathbf{q}}]\,|\psi\rangle \\
&= \frac{1}{2N}\sum_{\mathbf{q}}\upsilon(q)\,\langle\psi|\left(e^{\mathrm{k}^*\mathrm{q}/2} - e^{\mathrm{k}\mathrm{q}^*/2}\right)[\bar{\rho}_{-\mathbf{k}},\bar{\rho}_{-\mathbf{q}}\bar{\rho}_{\mathbf{k}+\mathbf{q}}]\,|\psi\rangle \qquad (4.34)
\end{aligned}$$

where in the last line we sent $\mathbf{q}$ to $-\mathbf{q}$ and then used parity symmetry (i.e. $\bar{\rho}_{-\mathbf{q}} = \bar{\rho}_{\mathbf{q}}$) and that $\bar{\rho}^{\dagger}_{\mathbf{q}} = \bar{\rho}_{-\mathbf{q}}$. This is possible because $s(q)$ and $\upsilon(q)$ depend only on the magnitude of $\mathbf{q}$. Expanding the commutator:

$$\begin{aligned}
\bar{f}(k) &= \frac{1}{2N}\sum_{\mathbf{q}}\upsilon(q)\,\langle\psi|\left(e^{\mathrm{k}^*\mathrm{q}/2} - e^{\mathrm{k}\mathrm{q}^*/2}\right)\Big\{\left(e^{\mathrm{k}^*\mathrm{q}/2} - e^{\mathrm{k}\mathrm{q}^*/2}\right)\bar{\rho}^{\dagger}_{\mathbf{k}+\mathbf{q}}\bar{\rho}_{\mathbf{k}+\mathbf{q}} \\
&\quad + \left(e^{-\mathrm{k}^*(\mathrm{k}+\mathrm{q})/2} - e^{-\mathrm{k}(\mathrm{k}+\mathrm{q})^*/2}\right)\bar{\rho}^{\dagger}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}}\Big\}\,|\psi\rangle \\
&= \frac{1}{2}\sum_{\mathbf{q}}\upsilon(q)\left(e^{k^*q/2} - e^{kq^*/2}\right)\Big\{\bar{s}(q)e^{-|k|^2/2}\left(e^{-k^*q/2} - e^{-kq^*/2}\right) \\
&\quad + \bar{s}(k+q)\left(e^{k^*q/2} - e^{kq^*/2}\right)\Big\} \qquad (4.35)
\end{aligned}$$

where $\bar{s}(q)$ is the projected static structure factor:

$$\bar{s}(q) = \frac{1}{N}\langle\psi|\,\bar{\rho}^{\dagger}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}}\,|\psi\rangle \qquad (4.36)$$

We will show that $\bar{s}(q)$ can be written in terms of $s(q)$ as:

$$\bar{s}(q) = s(q) - \left(1 - e^{-|q|^2/2}\right) \qquad (4.37)$$

We start by deriving a useful relation: $\overline{\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}} = \bar{\rho}^{\dagger}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}} + N\left(1 - e^{-|q|^2/2}\right)$. A projection of the product of two density operators is expressed as:

$$\begin{aligned}
\overline{\rho_{\mathbf{q}}\rho_{\mathbf{q}'}} &= \left[\sum_{ij}e^{-i\mathbf{q}\cdot\hat{\mathbf{r}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{r}}_j}\right]_P = \sum_{ij}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}\left[e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_j}\right]_P \\
&= \sum_{i\neq j}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}\langle n|\,e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_j}\,|n\rangle + \sum_{i}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_i}\langle n|\,e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_i}\,|n\rangle
\end{aligned} \qquad (4.38)$$

Again, $|n\rangle$ here are harmonic oscillator states associated with the cyclotron motion. Analogous to Eqs. [4.22-4.24] we simplify the expression $e^{-i\mathbf{q}\cdot\hat{S}_i}e^{-i\mathbf{q}'\cdot\hat{S}_i}$:

$$e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_i} = e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{S}}_i + \frac{1}{2}[i\mathbf{q}\cdot\hat{\mathbf{S}}_i,i\mathbf{q}'\cdot\hat{\mathbf{S}}_i]} \qquad (4.39)$$

Calculating the commutator:

$$\begin{aligned}
[i\mathbf{q}\cdot\hat{\mathbf{S}}_i, i\mathbf{q}'\cdot\hat{\mathbf{S}}_i] &= -[q_1\hat{S}_{ix} + q_2\hat{S}_{iy}, q_1'\hat{S}_{ix} + q_2'\hat{S}_{iy}] \\
&= -q_1q_1'[\hat{S}_{ix},\hat{S}_{ix}] - q_1q_2'[\hat{S}_{ix},\hat{S}_{iy}] - q_1'q_2[\hat{S}_{iy},\hat{S}_{ix}] - q_2q_2'[\hat{S}_{iy},\hat{S}_{iy}] \\
&= il^2\left(q_1q_2' - q_1'q_2\right) \\
&= il^2\mathbf{q}\times\mathbf{q}'
\end{aligned} \qquad (4.40)$$

thus:

$$e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_i}=e^{i\frac{l^2}{2}\mathbf{q}\times\mathbf{q}'}e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{S}}_i}=f(\mathbf{q}',\mathbf{q})e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{S}}_i} \tag{4.41}$$

Eq. [4.38] now reads:

$$\begin{aligned}\overline{\rho_{\mathbf{q}}\rho_{\mathbf{q}'}}&=\sum_{i\neq j}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}F(q)F(q')+\sum_i f(\mathbf{q},\mathbf{q}')f(\mathbf{q}',\mathbf{q})e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i}\langle n|\,e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{S}}_i}\,|n\rangle\\&=\sum_{i\neq j}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}F(q)F(q')+\sum_i e^{-i(\mathbf{q}+\mathbf{q}')\cdot\hat{\mathbf{R}}_i}F(q+q')\\&=\sum_{i\neq j}e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i}e^{-i\mathbf{q}'\cdot\hat{\mathbf{R}}_j}F(q)F(q')+\bar{\rho}_{\mathbf{q}+\mathbf{q}'}\end{aligned} \tag{4.42}$$

The $i\neq j$ term is the same from Eq. [4.25] and thus:

$$\begin{aligned}\bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}'}&=\overline{\rho_{\mathbf{q}}\rho_{\mathbf{q}'}}-\bar{\rho}_{\mathbf{q}+\mathbf{q}'}+\frac{F(q)F(q')}{F(q+q')}f(\mathbf{q},\mathbf{q}')\bar{\rho}_{\mathbf{q}+\mathbf{q}'}\\&=\overline{\rho_{\mathbf{q}}\rho_{\mathbf{q}'}}-\bar{\rho}_{\mathbf{q}+\mathbf{q}'}\left(1-\frac{F(q)F(q')}{F(q+q')}f(\mathbf{q},\mathbf{q}')\right)\end{aligned} \tag{4.43}$$

Again, $F(q+q')$ is shorthand notation for $F(|\mathbf{q}+\mathbf{q}'|)$. For projections onto the lowest Landau level and $\mathbf{q}=-\mathbf{q}'$:

$$\begin{aligned}\overline{\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}}&=\bar{\rho}^{\dagger}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}}-N\left(1-\frac{F^2(q)}{F(0)}f(-\mathbf{q},\mathbf{q})\right)\\&=\bar{\rho}^{\dagger}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}}-N\left(1-e^{-q^2/2}\right)\end{aligned} \tag{4.44}$$

since $f(-\mathbf{q},\mathbf{q})=1$ and $F(0)=1$ for projections onto the lowest Landau level. Substituting in the definition for $\bar{s}(q)$ yields:

$$\begin{aligned}\bar{s}(q)&=\frac{1}{N}\langle\psi|\,\bar{\rho}^{\dagger}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}}\,|\psi\rangle=\frac{1}{N}\langle\psi|\,\overline{\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}}-N\left(1-e^{-|q|^2/2}\right)|\psi\rangle\\&=\frac{1}{N}\langle\psi|\,\overline{\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}}\,|\psi\rangle-\left(1-e^{-|q|^2/2}\right)\\&=\frac{1}{N}\langle\psi|\,\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}\,|\psi\rangle-\left(1-e^{-|q|^2/2}\right)\\&=s(q)-\left(1-e^{-|q|^2/2}\right)\end{aligned} \tag{4.45}$$

since we assumed that the ground state $|\psi\rangle$ lies entirely within the lowest Landau level (i.e. $\hat{P}\,|\psi\rangle=|\psi\rangle$), we have $\langle\psi|\,\overline{\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}}\,|\psi\rangle=\langle\psi|\,\hat{P}\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}\hat{P}\,|\psi\rangle=\langle\psi|\,\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}\,|\psi\rangle$. The static structure factor is defined as $s(q)=\langle\psi|\,\rho^{\dagger}_{\mathbf{q}}\rho_{\mathbf{q}}\,|\psi\rangle$.

5 Calculation of the Gap Energy

To calculate the gap energy at $k=0$, we need to analyze how $\bar{f}(k)$ and $\bar{s}(k)$ behave for $k\to 0$ by Taylor expanding around $k=0$. We start with $\bar{f}(k)$. We use the result $\bar{f}(k)\sim k^4$ from [1] and carry out a Taylor expansion to calculate the proportionality constant.

Observe that:

$$\mathrm{k}^*\mathrm{q}=k_1q_1+k_2q_2+i\left(k_1q_2-k_2q_1\right)=\mathbf{k}\cdot\mathbf{q}+i\mathbf{k}\times\mathbf{q} \tag{5.1}$$

and thus:

$$\begin{aligned}e^{\mathrm{k}^*\mathrm{q}/2}-e^{\mathrm{k}\mathrm{q}^*/2}&=e^{(\mathbf{k}\cdot\mathbf{q}+i\mathbf{k}\times\mathbf{q})/2}-e^{(\mathbf{k}\cdot\mathbf{q}-i\mathbf{k}\times\mathbf{q})/2}\\&=e^{\mathbf{k}\cdot\mathbf{q}/2}\left(e^{i\mathbf{k}\times\mathbf{q}/2}-e^{-i\mathbf{k}\times\mathbf{q}/2}\right)\\&=2ie^{\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)\end{aligned} \tag{5.2}$$

Similarly:

$$\begin{aligned}
e^{-\mathrm{k}^*\mathrm{q}/2} - e^{-\mathrm{k}\mathrm{q}^*/2} &= e^{-(\mathbf{k}\cdot\mathbf{q}+i\mathbf{k}\times\mathbf{q})/2} - e^{-(\mathbf{k}\cdot\mathbf{q}-i\mathbf{k}\times\mathbf{q})/2} \\
&= -2ie^{-\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right) \qquad (5.3)
\end{aligned}$$

Then, $\bar{f}\,(k)$ reads:

$$\begin{aligned}
\bar{f}\,(k) &= \frac{1}{2}\sum_{\mathbf{q}} \upsilon(q) 2ie^{\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)\bigg\{ -2ie^{-\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)\bar{s}(q)e^{-k^2/2} \\
&\quad + 2ie^{\mathbf{k}\cdot\mathbf{q}/2}\sin\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)\bar{s}(k+q)\bigg\} \\
&= 2\sum_{\mathbf{q}} \upsilon(q)\sin^2\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)\left\{\bar{s}(q)e^{-|k|^2/2} - e^{\mathbf{k}\cdot\mathbf{q}}\bar{s}(k+q)\right\} \qquad (5.4)
\end{aligned}$$

By Taylor expanding the sin function, according to $\sin x \approx x - \frac{x^3}{6}$, the factor $\sin^2\left(\frac{1}{2}\mathbf{k}\times\mathbf{q}\right)$ yields terms of order k^2, k^4 and k^6. Terms of order higher than k^4 are obviously irrelevant, so we drop them. As can be easily seen the expression in curly brackets is zero for $k=0$. Thus, we can neglect the k^4 term in the expansion of $\sin^2\left(\frac{1}{2}\mathbf{k}\times\mathbf{q}\right)$. Moreover, any terms with order higher than k^2 in the curly brackets in $\bar{f}\,(k)$ are ignored in the following calculation, since they only give rise to powers of k higher than fo ur. Let's write the Taylor expansion of $\bar{s}(k+q)$ up to second order:

$$\bar{s}(k+q) \approx \bar{s}(q) + \mathbf{k}\cdot\nabla\bar{s}(q) + \frac{1}{2}\mathbf{k}^T\mathbf{H}(q)\mathbf{k} \qquad (5.5)$$

where $\nabla\bar{s}(q)$ is the gradient of $\bar{s}(k+q)$ (w.r.t $k+q$) evaluated at $k=0$ and $\mathbf{H}(q)$ is the Hessian matrix evaluated at $k=0$. Since $\bar{s}(q)$ has spherical symmetry, we can simplify the gradient:

$$\nabla\bar{s}(q) = \frac{\bar{s}'(q)}{q}\mathbf{q} \qquad (5.6)$$

where $\bar{s}'(q)$ is the derivative of $\bar{s}(k+q)$ (w.r.t $k+q$) evaluated at $k=0$. Now we compute the Hessian matrix:

$$\mathbf{H}(q) = \bar{s}''(q)\frac{\mathbf{q}\mathbf{q}^T}{q^2} + \bar{s}'(q)\frac{\mathbf{I}_2}{q} - \bar{s}'(q)\frac{\mathbf{q}\mathbf{q}^T}{q^3} \qquad (5.7)$$

where $\mathbf{q}\mathbf{q}^T$ is the outer product of $\mathbf{q}$ with itself and $\mathbf{I}_2$ is the 2×2 identity matrix. Now we can plug everything into Eq. [5.5]:

$$\begin{aligned}
\bar{s}(k+q) &\approx \bar{s}(q) + \bar{s}'(q)\frac{\mathbf{k}\cdot\mathbf{q}}{q} + \frac{1}{2}\left(\bar{s}''(q)\frac{(\mathbf{k}\cdot\mathbf{q})^2}{q^2} + \bar{s}'(q)\frac{\mathbf{k}\cdot\mathbf{k}}{q} - \bar{s}'(q)\frac{(\mathbf{k}\cdot\mathbf{q})^2}{q^3}\right) \\
&= \bar{s}(q) + \bar{s}'(q)k\cos\theta + \frac{1}{2}\left(\bar{s}''(q)k^2\cos^2\theta + \bar{s}'(q)\frac{k^2}{q} - \bar{s}'(q)\frac{k^2\cos^2\theta}{q}\right) \\
&= \bar{s}(q) + \bar{s}'(q)k\cos\theta + k^2 Q(q,\theta) \qquad (5.8)
\end{aligned}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. We expand the exponentials according to $e^x \approx 1 + x + \frac{x}{2}$

and drop irrelevant terms as discussed before:

$$\bar{f}(k) = 2\sum_{\mathbf{q}} \upsilon(q) \left(\frac{1}{2}\mathbf{k}\times\mathbf{q} - \frac{1}{6}\frac{(\mathbf{k}\times\mathbf{q})^3}{8} \right)^2 \Bigg\{ \bar{s}(q)\left(1-\frac{k^2}{2}\right) - \left(1+\mathbf{k}\cdot\mathbf{q}+\frac{(\mathbf{k}\cdot\mathbf{q})^2}{2}\right)\Bigg(\bar{s}(q)$$
$$+ \bar{s}'(q)k\cos\theta + k^2 Q(q,\theta)\Bigg)\Bigg\}$$
$$= -\frac{1}{2}\sum_{\mathbf{q}} \upsilon(q)\left(kq\sin\theta\right)^2 \Bigg\{ -\bar{s}(q) + \bar{s}(q)\frac{k^2}{2} + \left(1 + kq\cos\theta + \frac{1}{2}k^2q^2\cos^2\theta\right)\Bigg(\bar{s}(q)$$
$$+ \bar{s}'(q)k\cos\theta + k^2 Q(q,\theta)\Bigg)\Bigg\}$$
$$= -\frac{1}{2}\sum_{\mathbf{q}} \upsilon(q)k^2q^2\sin^2\theta\Bigg\{ -\bar{s}(q) + \bar{s}(q)\frac{k^2}{2} + \bar{s}(q) + \bar{s}(q)kq\cos\theta + \frac{1}{2}\bar{s}(q)k^2q^2\cos^2\theta$$
$$+ \bar{s}'(q)k\cos\theta + \bar{s}'(q)k^2q\cos^2\theta + k^2Q(q,\theta)\Bigg\}$$
$$= -\frac{1}{2}\sum_{\mathbf{q}} \upsilon(q)q^2\sin^2\theta\Bigg\{ \left(\frac{1}{2}\bar{s}(q) + \frac{1}{2}\bar{s}(q)q^2\cos^2\theta + \bar{s}'(q)q\cos^2\theta + Q(q,\theta)\right)k^4$$
$$+ \Bigg(\bar{s}(q)q\cos\theta + \bar{s}'(q)\cos\theta\Bigg)k^3\Bigg\} \tag{5.9}$$

We can show that the k^3 term vanishes by writing the sum over all values of $\mathbf{q}$ as an integral:

$$\sum_{\mathbf{q}} \upsilon(q)q^2\sin^2\theta\Bigg(\bar{s}(q)q\cos\theta + \bar{s}'(q)\cos\theta\Bigg) = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\,\upsilon(q)q^2\sin^2\theta\Bigg(\bar{s}(q)q\cos\theta + \bar{s}'(q)\cos\theta\Bigg) \tag{5.10}$$

Performing the integration in polar coordinates (q,θ), the Jacobian is q:

$$\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\,\upsilon(q)q^2\Bigg(\bar{s}(q)q + \bar{s}'(q)\Bigg)\sin^2\theta\cos\theta = \int_0^\infty \mathrm{d}q\, W(q)\int_0^{2\pi}\mathrm{d}\theta\,\sin^2\theta\cos\theta = 0 \tag{5.11}$$

since $\int_0^{2\pi} d\theta\sin^2\theta\cos\theta = 0$. Thus, after substituting $Q(q,\theta)$, the projected oscillator strength near $k=0$ yields:

$$\bar{f}(k) = -\frac{1}{2}\sum_{\mathbf{q}} \upsilon(q)\sin^2\theta\Bigg\{\frac{1}{2}\bar{s}(q)\left(1+q^2\cos^2\theta\right)q^2 + \frac{1}{2}\bar{s}'(q)\left(2q^3\cos^2\theta + q\left(1-\cos^2\theta\right)\right)$$
$$+ \frac{1}{2}\bar{s}''(q)q^2\cos^2\theta\Bigg\}k^4$$
$$= \frac{1}{2}\sum_{\mathbf{q}} \upsilon(q)\Bigg\{\sin^2\theta\left(\frac{1}{2}\bar{s}(q)q^2 + \frac{1}{2}\bar{s}'(q)q\right) + \sin^2\theta\cos^2\theta\Bigg(\frac{1}{2}\bar{s}(q)q^4 + \bar{s}'(q)q^3 - \frac{1}{2}\bar{s}'(q)q$$
$$+ \frac{1}{2}\bar{s}''(q)q^2\Bigg)\Bigg\}k^4 \tag{5.12}$$

Now we can eliminate the θ dependence by carrying out the angular integrals. Using that $\int_0^{2\pi} d\theta\sin^2\theta\cos^2\theta = \frac{\pi}{4}$ and $\int_0^{2\pi} d\theta\sin^2\theta = \pi$ we obtain:

$$\bar{f}(k) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}q}{(2\pi)^2}\,q\upsilon(q)\Bigg\{\pi\left(\frac{1}{2}\bar{s}(q)q^2 + \frac{1}{2}\bar{s}'(q)q\right) + \frac{\pi}{4}\Bigg(\frac{1}{2}\bar{s}(q)q^4 + \bar{s}'(q)q^3 - \frac{1}{2}\bar{s}'(q)q$$
$$+ \frac{1}{2}\bar{s}''(q)q^2\Bigg)\Bigg\}k^4$$
$$= \frac{1}{8\pi}\int_0^\infty \mathrm{d}q\, q\upsilon(q)\Bigg\{\frac{1}{2}\bar{s}(q)q^2 + \frac{1}{2}\bar{s}'(q)q + \frac{1}{4}\Bigg(\frac{1}{2}\bar{s}(q)q^4 + \bar{s}'(q)q^3 - \frac{1}{2}\bar{s}'(q)q$$
$$+ \frac{1}{2}\bar{s}''(q)q^2\Bigg)\Bigg\}k^4$$
$$= \frac{1}{64\pi}\int_0^\infty \mathrm{d}q\,\upsilon(q)\Bigg\{\bar{s}(q)q^5 + 2\bar{s}'(q)q^4 + \left(4\bar{s}(q) + \bar{s}''(q)\right)q^3 + 3\bar{s}'(q)q^2\Bigg\}k^4 \tag{5.13}$$

Hence, in order to evaluate $\bar{f}(k)$ we need to find the derivatives of $\bar{s}(q)$. Moreover, the derivatives are needed to perform a Taylor expansion of $\bar{s}(k)$ in the denominator of $\Delta(k)$.

6 Evaluating derivatives of $\bar{s}(k)$

The static structure factor can be written as a Fourier transform of the radial distribution function of the ground state $g(r)$ [1]:

$$s(k) = 1 + \rho \int \mathrm{d}^2\mathbf{r}\ \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)[g(r) - 1] + \rho(2\pi)^2\delta^2(\mathbf{k}) \tag{6.1}$$

where ρ is the average density. $g(r)$ is independent of the direction of $\mathbf{r}$ since we are dealing with an isotropic liquid ground state. Writing the integral in polar coordinates:

$$s(k) = 1 + \rho \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ re^{-ikr\cos\theta}\,[g(r) - 1] = 1 + \rho \int_0^\infty \mathrm{d}r\ rJ_0(kr)\,[g(r) - 1] \tag{6.2}$$

where

$$J_0(kr) = \int_0^{2\pi} \mathrm{d}\theta\ e^{-ikr\cos\theta} \tag{6.3}$$

is the zeroth-order Bessel function of the first kind. The analytic form of $g(r)$ is known for any liquid ground state, namely [1]:

$$g(r) = 1 - e^{-r^2/2} + e^{-r^2/4}\sum_{m=1}^{\infty}{}' \frac{2}{m!}c_m\left(r^2/4\right)^m \tag{6.4}$$

where c_m are coefficients that distinguish different ground states and the summation index m takes only odd values, indicated by a prime on the sum. Substituting this in Eq. [6.2] yields:

$$\begin{aligned} s(k) &= 1 - \nu \int_0^\infty \mathrm{d}r\ rJ_0(kr)e^{-r^2/2} + \nu \int_0^\infty \mathrm{d}r\ rJ_0(kr)e^{-r^2/4}\sum_{m=1}^{\infty}{}' \frac{2}{m!}c_m\left(r^2/4\right)^m \\ &= 1 - \nu \int_0^\infty \mathrm{d}r\ rJ_0(kr)e^{-r^2/2} + \nu\sum_{m=1}^{\infty}{}' \frac{2}{4^m m!}c_m \int_0^\infty \mathrm{d}r\ r^{2m+1}J_0(kr)e^{-r^2/4} \end{aligned} \tag{6.5}$$

Note the two Bessel function integral identities:

$$\int_0^\infty \mathrm{d}t\ J_\alpha(bt)e^{-p^2t^2}t^{\alpha+1} = \frac{b^\alpha}{(2p^2)^{\alpha+1}}e^{-b^2/(4p^2)} \tag{6.6}$$

$$\int_0^\infty \mathrm{d}t\ J_\alpha(bt)e^{-p^2t^2}t^{\beta-1} = \frac{\left(\frac{1}{2}b/p\right)^\alpha\,\Gamma\!\left(\frac{\alpha+\beta}{2}\right)}{2p^\beta\Gamma(\alpha+1)}e^{-b^2/(4p^2)}M\left(\frac{\alpha-\beta}{2}+1, \alpha+1, \frac{b^2}{4p^2}\right) \tag{6.7}$$

where $\Gamma(z)$ is the Gamma function and $M(a, b, z)$ is the confluent hypergeometric function of the first kind. Making use of these identities, $s(k)$ reads:

$$s(k) = 1 - \nu e^{-k^2/2} + 4\nu e^{-k^2}\sum_{m=1}^{\infty}{}' c_m M(-m, 1, k^2) \tag{6.8}$$

where we set the magnetic length equal to one $l = 1$ and thus the first Landau level filling factor is $\nu = 2\pi\rho$. In the special case where the first argument of $M(a, b, z)$ is a non-positive integer, the confluent hypergeometric function is proportional to the generalized Laguerre polynomials. Moreover, when the second argument is exactly equal to one, it holds that [8]:

$$M(-m, 1, k^2) = L_m^{(0)}(k^2) \tag{6.9}$$

where $L_m^{(\alpha)}(z)$ is the mth generalized Laguerre polynomial. So we can write:

$$s(k) = 1 - \nu e^{-k^2/2} + 4\nu e^{-k^2}\sum_{m=1}^{\infty}{}' c_m L_m^{(0)}(k^2) \tag{6.10}$$

The derivatives of $s(k)$ are now readily obtained using differentiation rules of Laguerre polynomials [9]:

$$\frac{d^n}{dz^n} L_m^{(\alpha)}(x) = \begin{cases} (-1)^n L_{m-n}^{(\alpha+n)}(x) & \text{if } n \leq m, \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

we will write down the expressions for all derivatives up to the fourth, since we will need their analytic forms later on. The first derivative is:

$$s'(k) = \nu k e^{-k^2/2} - 8\nu k e^{-k^2} \sum_{m=1}^{\infty}{}' c_m \left\{ L_m^{(0)}(k^2) + L_{m-1}^{(1)}(k^2) \right\} \tag{6.12}$$

the second derivative:

$$\begin{aligned} s''(k) = \nu \left(1 - k^2\right) e^{-k^2/2} - 8\nu e^{-k^2} \sum_{m=1}^{\infty}{}' c_m \left\{ \left(1 - 2k^2\right) L_m^{(0)}(k^2) \right. \\ \left. + \left(1 - 4k^2\right) L_{m-1}^{(1)}(k^2) - 2k^2 L_{m-2}^{(2)}(k^2) \right\} \end{aligned} \tag{6.13}$$

the third derivative:

$$\begin{aligned} s^{(3)}(k) = -\nu k \left(3 - k^2\right) e^{-k^2/2} + 16\nu k e^{-k^2} \sum_{m=1}^{\infty}{}' c_m \left\{ \left(3 - 2k^2\right) L_m^{(0)}(k^2) \right. \\ \left. + 6\left(1 - k^2\right) L_{m-1}^{(1)}(k^2) + 3\left(1 - 2k^2\right) L_{m-2}^{(2)}(k^2) - 2k^2 L_{m-3}^{(3)}(k^2) \right\} \end{aligned} \tag{6.14}$$

and finally, the fourth derivative:

$$\begin{aligned} s^{(4)}(k) = -\nu \left(3 - 6k^2 + k^4\right) e^{-k^2/2} + 16\nu e^{-k^2} \sum_{m=1}^{\infty}{}' c_m \left\{ \left(3 - 12k^2 + 4k^4\right) L_m^{(0)}(k^2) \right. \\ + \left(6 - 36k^2 + 16k^4\right) L_{m-1}^{(1)}(k^2) + \left(3 - 30k^2 + 24k^4\right) L_{m-2}^{(2)}(k^2) \\ \left. + \left(-12k^2 + 16k^4\right) L_{m-3}^{(3)}(k^2) + 4k^5 L_{m-4}^{(4)}(k^2) \right\} \end{aligned} \tag{6.15}$$

Now, we write down $\bar{s}(k)$, $\bar{s}'(k)$ and $\bar{s}''(k)$, using Eq. [6.16]:

$$\bar{s}(k) = (1 - \nu) e^{-k^2/2} + 4\nu e^{-k^2} \sum_{m=1}^{\infty}{}' c_m L_m^{(0)}(k^2) \tag{6.16}$$

$$\bar{s}'(k) = (\nu - 1) k e^{-k^2/2} - 8\nu k e^{-k^2} \sum_{m=1}^{\infty}{}' c_m \left\{ L_m^{(0)}(k^2) + L_{m-1}^{(1)}(k^2) \right\} \tag{6.17}$$

$$\begin{aligned} \bar{s}''(k) = (\nu - 1)\left(1 - k^2\right) e^{-k^2/2} - 8\nu e^{-k^2} \sum_{m=1}^{\infty}{}' c_m \left\{ \left(1 - 2k^2\right) L_m^{(0)}(k^2) \right. \\ \left. + \left(1 - 4k^2\right) L_{m-1}^{(1)}(k^2) - 2k^2 L_{m-2}^{(2)}(k^2) \right\} \end{aligned} \tag{6.18}$$

Now, we are able to write the Taylor expansion of $\bar{s}(k)$ around $k = 0$:

$$\begin{aligned} \bar{s}(k) &= s(k) - \left(1 - e^{-k^2/2}\right) \\ &\approx s(0) + \frac{1}{2}k^2 s''(0) + \frac{1}{4!}k^4 s^{(4)}(0) - \left(1 - 1 + \frac{1}{2}k^2 - \frac{1}{8}k^4\right) \\ &= \frac{1}{8}\left(\frac{1}{3}s^{(4)}(0) + 1\right) k^4 \end{aligned} \tag{6.19}$$

The Taylor expansion of $s(k)$ includes only even powers because $s(-k) = s(k)$. Moreover, $s(0) = 0$ and $s''(0) = 1$. We are interested in the case of the Coulomb interaction, which has the Fourier transform $v(q) = \frac{2\pi}{q}$ (see Appendix D). The ratio $\Delta(k) = \bar{f}(k)/\bar{s}(k)$ for $k \to 0$ will give the value for the gap energy $\Delta(0)$:

$$\Delta(0) = \frac{1}{8\pi} \frac{\int_0^\infty \mathrm{d}q\, v(q) \left\{ \bar{s}(q)q^5 + 2\bar{s}'(q)q^4 + (4\bar{s}(q) + \bar{s}''(q))\, q^3 + 3\bar{s}'(q)q^2 \right\}}{\frac{1}{3}s^{(4)}(0) + 1} \tag{6.20}$$

Evaluating the denominator:

$$\begin{aligned} s^{(4)}(0) &= -3\nu + 48\nu \sum_{m=1}^{\infty}{}' c_m \left\{ L_m^{(0)}(0) + 2L_{m-1}^{(1)}(0) + L_{m-2}^{(2)}(0) \right\} \\ &= -3\nu + 24\nu \sum_{m=1}^{\infty}{}' c_m \left(m^2 + 3m + 2 \right) \end{aligned} \tag{6.21}$$

using $L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ and thus:

$$\frac{1}{3}s^{(4)}(0) + 1 = (1-\nu) + 8\nu \sum_{m=1}^{\infty}{}' c_m \left(m^2 + 3m + 2 \right) \tag{6.22}$$

In order to evaluate the sum, and ultimately, the gap energy, we need to know the coefficients c_m which are determined by the ground state. We discuss this in the next section.

7 Numerical Evaluation of the Gap Energy

In their analysis, Girvin, MacDonald, and Platzman, chose the Laughlin state as a ground state due to the lack of experimental data from which $s(k)$ can be found and given the high accuracy of the Laughlin state as a ground state in the LLL [10]. Earlier, we wrote the static structure in terms of the radial distribution function $g(r)$ (Eq. [6.4]). In [1], $g(r)$ was fitted to Monte Carlo data for $\nu = 1/3$ and $\nu = 1/5$ (the fitting parameters being the coefficients c_m). The generated data satisfies constraints which manifest, for the Laughlin state with a filling factor ν, as the following equations [11]:

$$\sum_{m=1}^{\infty}{}' c_m = \frac{1-M}{4} \tag{7.1}$$

$$\sum_{m=1}^{\infty}{}' (m+1)c_m = \frac{1-M}{8} \tag{7.2}$$

$$\sum_{m=1}^{\infty}{}' (m+2)(m+1)c_m = \frac{(1-M)^2}{8} \tag{7.3}$$

where $M = 1/\nu$. The coefficients are computed up to $m = 27$ and shown in Table [1]:

m	c_m for $\nu = 1/3$	c_m for $\nu = 1/5$
1	-1.00000	-1.0000
3	0.51053	-1.0000
5	-0.02056	0.6765
7	0.31003	0.3130
9	-0.49050	-0.1055
11	0.20102	0.8910
13	-0.00904	-0.3750
15	-0.00148	-0.7750
17	0.00000	0.3700
19	0.00120	0.0100
21	0.00060	-0.0050
23	-0.00180	-0.0000
25	0.00000	-0.1000
27	0.00000	0.1000

Table 1: Coefficients c_m for different values of ν [1]

With these coefficients, and the expressions for $\bar{s}(k)$ and its derivatives, we can evaluate Eq. [6.20] and arrive at the values [12]:

$$\Delta_{1/3}(0) = 0.150811 \text{ and } \Delta_{1/5}(0) = 0.0495146 \tag{7.4}$$

which match the results from [1].

8 Graphing the Dispersion Relation $\Delta(k)$

To graph the dispersion relation $\Delta(k)$, we start from Eq. [5.4]. We write the sum as an integral and expand the cross and dot products:

$$\begin{aligned}\bar{f}(k) &= 2\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\, \upsilon(q)\sin^2\left(\frac{\mathbf{k}\times\mathbf{q}}{2}\right)\left\{\bar{s}(q)e^{-k^2/2} - e^{\mathbf{k}\cdot\mathbf{q}}\bar{s}(k+q)\right\} \\ &= 2\int_0^\infty \frac{\mathrm{d}q}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\; q\upsilon(q)\sin^2\left(\frac{1}{2}kq\sin(\theta)\right)\left\{\bar{s}(q)e^{-k^2/2} - e^{kq\cos(\theta)}\bar{s}(|\mathbf{k}+\mathbf{q}|)\right\}\end{aligned} \tag{8.1}$$

making use of Table 1 and Eq. [6.16], we can graph $\Delta(k) = \bar{f}(k)/\bar{s}(k)$ by numerical integration [12]:

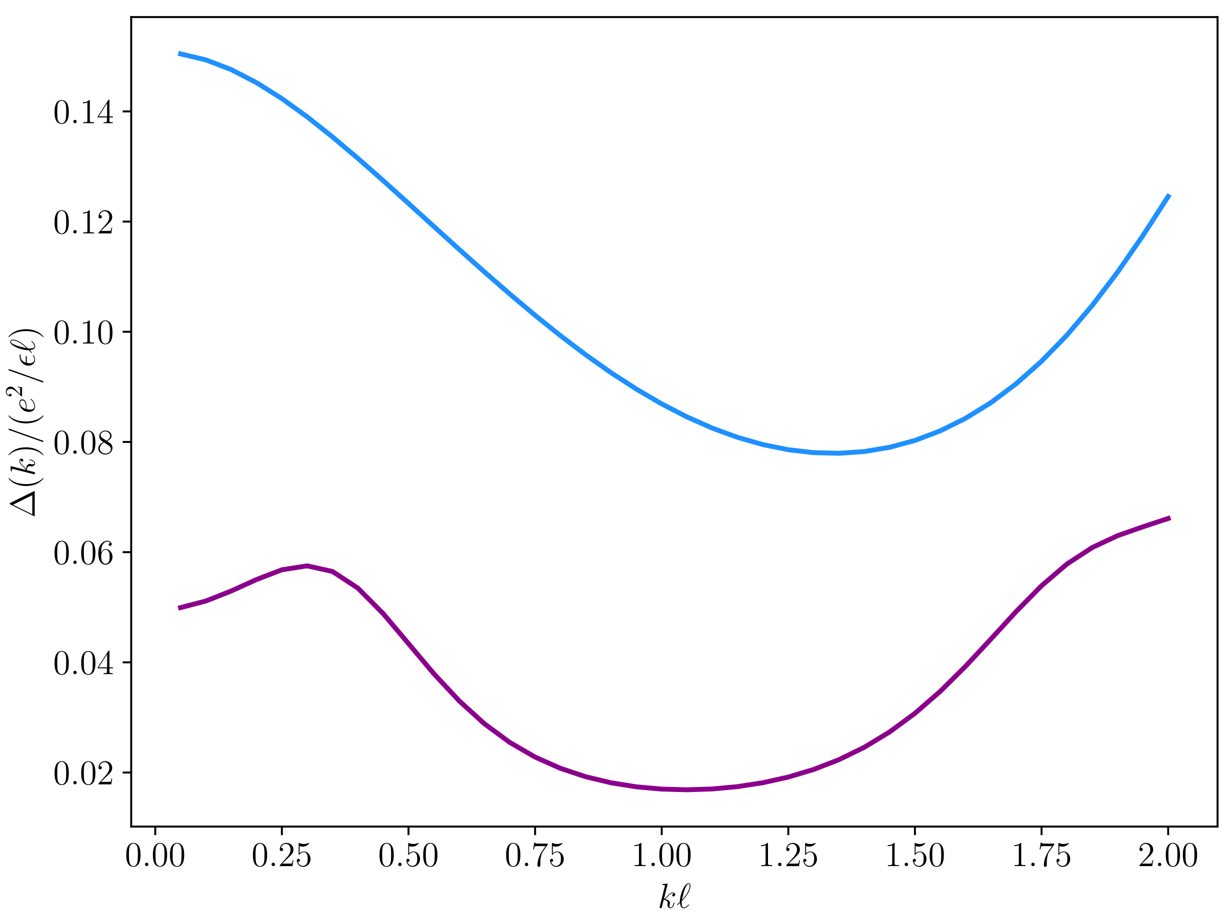

Figure 4: The dispersion curve $\Delta(k)$ of the collective mode in the LLL for $\nu = 1/3$ (blue) and $\nu = 1/5$ (purple)

where we set the magnetic length $l = 1$ and used a unit of energy $e^2/(\epsilon l)$ where ϵ is the dielectric constant of the background medium (see Appendix D). We see that the curves match the results from [13].

9 Generalization to the n-th Landau level

It turns out that this procedure is essentially the same for projections onto any Landau level. Namely, the only difference when going beyond the LLL is the factor $\upsilon(q)$ in the interaction potential, as we will demonstrate in this section.

Let's use the notation $\bar{\rho}_{\mathbf{q}}^{(n)}$ and $F_n(q)$ for $\bar{\rho}_{\mathbf{q}}$ and $F(q)$ in the nth Landau level, respectively. With this notation, we can write:

$$\bar{\rho}_{\mathbf{q}}^{(n)} = F_n(q) \sum_{i=1}^{N} e^{-i\mathbf{q}\cdot\hat{\mathbf{R}}_i} \equiv F_n(q)\bar{\rho}_{\mathbf{q}}^{\mathrm{R}} \tag{9.1}$$

the superscript R refers to the guiding center momentum $\mathbf{R}$. We can now write:

$$\bar{\rho}_{\mathbf{q}}^{(0)} = F_0(q)\bar{\rho}_{\mathbf{q}}^{\mathrm{R}} \tag{9.2}$$

and thus for any Landau level

$$\bar{\rho}_{\mathbf{q}}^{(n)} = \frac{F_n(q)}{F_0(q)}\bar{\rho}_{\mathbf{q}}^{(0)} \tag{9.3}$$

We can use this relation to project the interaction potential onto a general Landau level. We write Eq. [4.30] in terms of the notation we have just introduced and substitute:

$$\begin{aligned}
\bar{V}_{(n)} &= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \upsilon(q) \left(\bar{\rho}_{-\mathbf{q}}^{(n)}\bar{\rho}_{\mathbf{q}}^{(n)} - NF_n^2(q)\right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \upsilon(q) \left(\left(\frac{F_n(q)}{F_0(q)}\right)^2 \bar{\rho}_{-\mathbf{q}}^{(0)}\bar{\rho}_{\mathbf{q}}^{(0)} - NF_n^2(q)\right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \upsilon(q) \left(\frac{F_n(q)}{F_0(q)}\right)^2 \left(\bar{\rho}_{-\mathbf{q}}^{(0)}\bar{\rho}_{\mathbf{q}}^{(0)} - NF_0^2(q)\right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \tilde{\upsilon}(q) \left(\bar{\rho}_{-\mathbf{q}}^{(0)}\bar{\rho}_{\mathbf{q}}^{(0)} - Ne^{-q^2l^2/2}\right)
\end{aligned} \tag{9.4}$$

where we used $F_0^2(q) = e^{-q^2l^2/2}$ and absorbed the ratio of form factors into the effective interaction $\tilde{\upsilon}(q)$. We see that $\bar{V}_{(n)}$ has exactly the same form as $\bar{V}_{(0)}$ from Eq. [4.31], just with a modified interaction:

$$\bar{V}_{(n)}[\upsilon(q)] = \bar{V}_{(0)}[\tilde{\upsilon}(q)] \tag{9.5}$$

This key result will greatly simplify the calculation of $\Delta(k)$. Let's project the oscillator strength onto the nth Landau level:

$$\begin{aligned}
\bar{f}_{(n)}(k) &= \frac{1}{2N}\langle\psi|\,[\bar{\rho}_{\mathbf{k}}^{\dagger(n)}, [\bar{V}_{(n)}, \bar{\rho}_{\mathbf{k}}^{(n)}]]\,|\psi\rangle = \frac{1}{2N}\langle\psi|\left[\frac{F_n(k)}{F_0(k)}\bar{\rho}_{\mathbf{k}}^{\dagger(0)}, \left[\bar{V}_{(n)}, \frac{F_n(k)}{F_0(k)}\bar{\rho}_{\mathbf{k}}^{(0)}\right]\right]|\psi\rangle \\
&= \left(\frac{F_n(k)}{F_0(k)}\right)^2 \frac{1}{2N}\langle\psi|\,[\bar{\rho}_{\mathbf{k}}^{\dagger(0)}, [\bar{V}_{(0)}[\tilde{\upsilon}(q)], \bar{\rho}_{\mathbf{k}}^{(0)}]]\,|\psi\rangle
\end{aligned} \tag{9.6}$$

It is clear that $\bar{f}_{(n)}(k)$ has the same form as $\bar{f}_{(0)}(k)$ with a scaling factor $\left(\frac{F_n(k)}{F_0(k)}\right)^2$. Finally, we compute the projection of the static structure factor onto the nth Landau level:

$$\bar{s}_{(n)}(k) = \frac{1}{N}\langle\psi|\,\bar{\rho}_{\mathbf{k}}^{\dagger(n)}\bar{\rho}_{\mathbf{k}}^{(n)}\,|\psi\rangle = \left(\frac{F_n(k)}{F_0(k)}\right)^2 \frac{1}{N}\langle\psi|\,\bar{\rho}_{\mathbf{k}}^{\dagger(0)}\bar{\rho}_{\mathbf{k}}^{(0)}\,|\psi\rangle = \left(\frac{F_n(k)}{F_0(k)}\right)^2 \bar{s}_{(0)}(k) \tag{9.7}$$

which is the same scaling factor in $\bar{f}_{(n)}(k)$. Thus, the dispersion curve in the nth Landau level can be calculated in a similar manner to the LLL dispersion, by substituting the correct modified interaction $\tilde{\upsilon}(q)$. This shows that, algebraically, the operator $\bar{\rho}_{\mathbf{q}}^{\mathrm{R}}$ is more fundamental than $\bar{\rho}_{\mathbf{q}}^{(n)}$ since it contains no information about the Landau level. We would like to graph the dispersion curve in the first Landau level. As discussed, all we need is to find $\tilde{\upsilon}(q)$:

$$\tilde{\upsilon}(q) = \upsilon(q)\left(\frac{F_n(q)}{F_0(q)}\right)^2 = \upsilon(q)(1-q^2/2)^2 = \frac{2\pi}{q}(1-q^2/2)^2 \tag{9.8}$$

We substitute this in Eq. [8.1] and integrate numerically [12] to get:

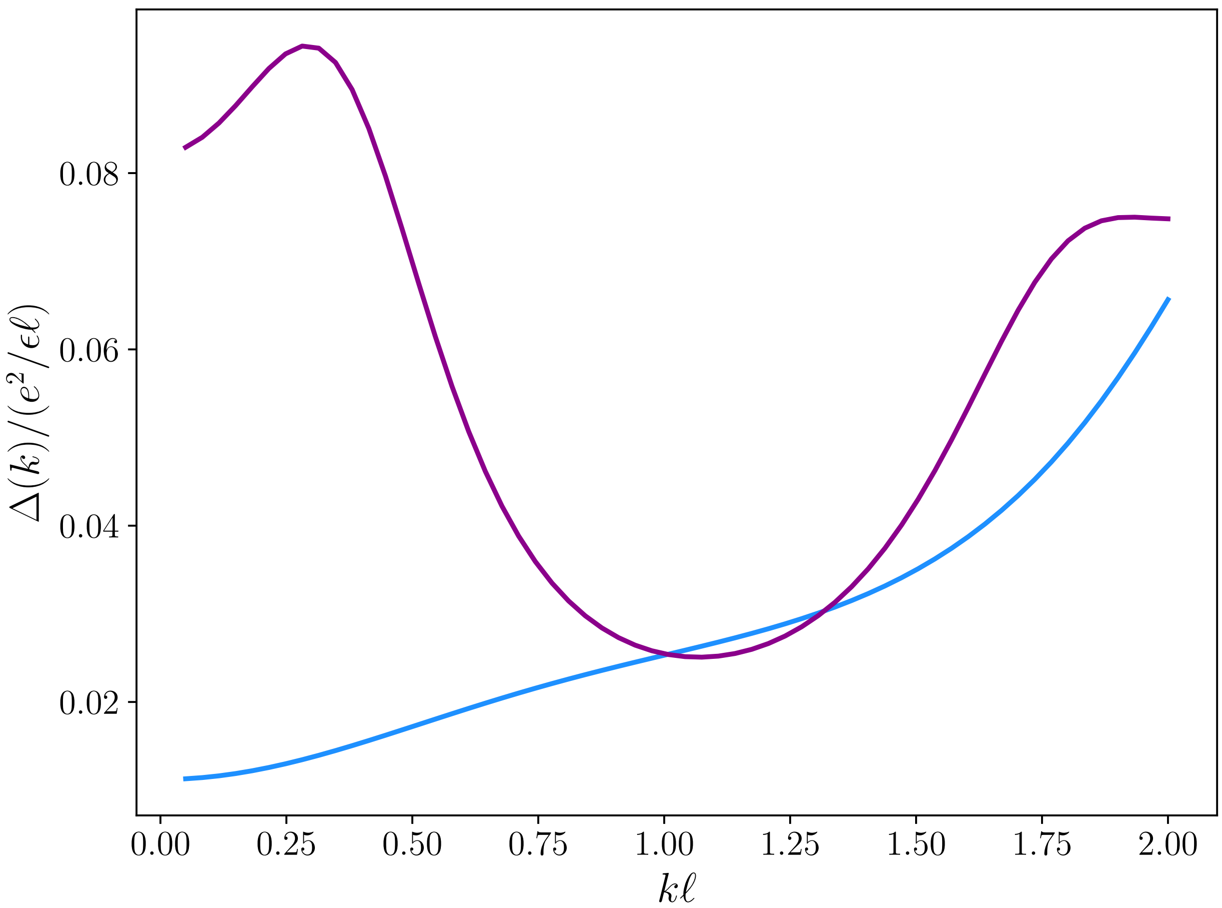

Figure 5: The dispersion curve $\Delta(k)$ of the collective mode in the first Landau level for $\nu = 1/3$ (blue) and $\nu = 1/5$ (purple)

The same energy unit of $e^2/(\epsilon l)$ is used. The values for the gap energy in the first Landau level are obtained in the same manner as the LLL case and by modifying the interaction $v(q)$ in Eq. [6.20]. The values are [12]:

$$\Delta_{1/3}^{(1)} = 0.0112105 \text{ and } \Delta_{1/5}^{(1)} = 0.0822622 \tag{9.9}$$

For $\nu = 1/3$, the gap energy is significantly lower in the first Landau level compared to the LLL (in fact, one order of magnitude lower). This means that the Laughlin state with a filling factor of 1/3 is less stable in the first Landau level compared to the LLL. However, we find a higher gap energy in the first Landau level for the Laughlin state with $\nu = 1/5$.

10 Conclusion

We reproduced important results from the Girvin, MacDonald and Platzman theory of collective excitations in the fractional quantum Hall regime. We employed the concept of guiding center coordinates to perform projections onto the lowest Landau level and derive the algebra of projected density operators. We arrived at numerical results for the gap energy and the collective-mode dispersion consistent with [1]. We generalized our results to account for projections onto the nth Landau level, demonstrating that higher Landau levels are equivalent to the lowest Landau level with a modified interaction. We argued for a more fundamental definition for the density operator, expressed only in terms of the guiding-center momentum and thus independent of Landau levels. In the first Landau level and for a filling factor equal to 1/3, we found a gap energy one order of magnitude lower, compared to the Lowest landau level. This is an indication of the reduced stability of the Laughlin state in higher Landau levels. While the $\nu = 1/5$ Laughlin state has a higher gap energy in the first Landau level.

Appendix

A Multi-particle states

Our harmonic oscillator multi-particle states in a given Landau level can be expressed as:

$$|n\rangle = \bigotimes_{i=1}^{N} |n_i\rangle \tag{A.1}$$

where $|n_i\rangle$ are single-particle states. An expression such as $\langle n| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n\rangle$ can be written more accurately as:

$$\begin{aligned}
\langle n| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n\rangle &:= \langle n| \bigotimes_{k=1}^{N} e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_k \delta_{ki}} |n\rangle \\
&= \langle n| \left(\mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) |n\rangle \\
&= \bigotimes_{m=1}^{N} \langle n_m| \left(\mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) \bigotimes_{l=1}^{N} |n_l\rangle \\
&= \bigotimes_{m=1}^{N} \langle n_m| \left(|n_1\rangle \otimes \ldots \otimes |n_{i-1}\rangle \otimes \left(e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n_i\rangle\right) \otimes |n_{i+1}\rangle \otimes \ldots \otimes |n_N\rangle\right)
\end{aligned} \tag{A.2}$$

using the rules of inner products over tensor product spaces [14]:

$$\begin{aligned}
\langle n| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n\rangle &= \prod_{k=1}^{N} \langle n_k| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_k \delta_{ik}} |n_k\rangle \\
&= \langle n_1|n_1\rangle \ldots \langle n_{i-1}|n_{i-1}\rangle \langle n_i| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n_i\rangle \langle n_{i+1}|n_{i+1}\rangle \ldots \langle n_N|n_N\rangle \\
&= \langle n_i| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n_i\rangle \\
&= F(q)
\end{aligned} \tag{A.3}$$

Similarly, the product of the two operators, $\langle n| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_j} |n\rangle$, is:

$$\begin{aligned}
\langle n| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_j} |n\rangle &:= \langle n| \bigotimes_{k=1}^{N} e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_k \delta_{ki}} e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_k \delta_{kj}} |n\rangle \\
&= \langle n| \left(\mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_j} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) |n\rangle
\end{aligned} \tag{A.4}$$

Resulting in:

$$\begin{aligned}
\langle n| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_j} |n\rangle &= \bigotimes_{m=1}^{N} \langle n_m| \Big(\mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_j} \otimes \\
&\quad \mathbb{I} \otimes \ldots \otimes \mathbb{I}\Big) \bigotimes_{l=1}^{N} |n_l\rangle \\
&= \bigotimes_{m=1}^{N} \langle n_m| \Big(|n_1\rangle \otimes \ldots \otimes |n_{i-1}\rangle \otimes \left(e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n_i\rangle\right) \otimes |n_{i+1}\rangle \otimes \ldots \\
&\quad \otimes |n_{j-1}\rangle \otimes \left(e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_j} |n_j\rangle\right) \otimes |n_{j+1}\rangle \otimes \ldots \otimes |n_N\rangle\Big) \\
&= \langle n_i| e^{-i\mathbf{q}\cdot\hat{\mathbf{S}}_i} |n_i\rangle \langle n_j| e^{-i\mathbf{q}'\cdot\hat{\mathbf{S}}_j} |n_j\rangle \\
&= F(q)F(q')
\end{aligned} \tag{A.5}$$

B The Baker-Campbell-Hausdorff Formula

The Baker–Campbell–Hausdorff (BCH) formula provides an expression for Z in the equation:

$$e^X e^Y = e^Z \tag{B.1}$$

where X and Y belong to a possibly noncommutative Lie algebra. The formula expresses Z as a formal series in terms of X, Y, and their iterated commutators [15]:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \tag{B.2}$$

the next terms are higher-order commutator terms. If X and Y are sufficiently small elements of a Lie algebra, the series converges. If X and Y commute with their commutator $[X, Y]$ we will have:

$$Z = X + Y + \frac{1}{2}[X, Y] \tag{B.3}$$

and thus

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} \tag{B.4}$$

and if $[X, Y]$ is a constant:

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]} \tag{B.5}$$

and thus

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]} \tag{B.6}$$

which results in

$$e^X e^Y = e^Y e^X e^{[X,Y]} \tag{B.7}$$

C Evaluation of the Form Factor $F(\mathbf{q})$

We defined the form factor $F(\mathbf{q})$ as:

$$F(\mathbf{q}) = \exp\left(-\frac{1}{4}q^2 l^2\right) \langle n| \exp\left(-\frac{l\,(q_1 - iq_2)}{\sqrt{2}}\hat{a}^\dagger\right) \exp\left(\frac{l\,(q_1 + iq_2)}{\sqrt{2}}\hat{a}\right) |n\rangle \tag{C.1}$$

Let's abbreviate the exponents by defining $\alpha = \frac{l(q_1+iq_2)}{\sqrt{2}}$ and use the exponential series:

$$F(\mathbf{q}) = e^{-q^2l^2/4} \langle n| e^{-\alpha^* \hat{a}^\dagger} e^{\alpha \hat{a}} |n\rangle = e^{-q^2l^2/4} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\alpha^*)^k \alpha^m}{k!m!} \langle n| (a^\dagger)^k a^m |n\rangle \tag{C.2}$$

Since the states $|n\rangle$ are orthogonal and $\hat{a}^k |n\rangle = 0$ for $k > n$, all terms vanish except for the $k = m \leq n$ terms, leaving:

$$F(\mathbf{q}) = e^{-q^2l^2/4} \sum_{k=0}^{n} \frac{(-|\alpha|^2)^k}{(k!)^2} \langle n| (a^\dagger)^k a^k |n\rangle = e^{-q^2l^2/4} \sum_{k=0}^{n} \frac{n!}{(n-k)!(k!)^2} (-|\alpha|^2)^k \tag{C.3}$$

Recall the series for Laguerre polynomials [9]:

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!(k!)^2} (-x)^k \tag{C.4}$$

we can easily see that

$$F(\mathbf{q}) = e^{-q^2l^2/4} L_n(|\alpha|^2) = e^{-q^2l^2/4} L_n(|\mathbf{q}|^2/2) = e^{-q^2l^2/4} L_n(q^2/2) = F(q) \tag{C.5}$$

It holds that [9]:

$$L_0(q^2/2) = 1 \;\text{ and }\; L_1(q^2/2) = 1 - q^2/2 \tag{C.6}$$

and thus for the lowest and first Landau levels, we have:

$$F(q) = e^{-q^2l^2/4} \;\text{ and }\; F(q) = e^{-q^2l^2/4}\left(1 - q^2/2\right) \tag{C.7}$$

respectively.

D 2D Fourier Transform of the 3D Coulomb Potential

We derive the 2D Fourier transform of the 3D Coulomb potential restricted to the $z = 0$ plane. The Coulomb potential takes the form $V(r) = \frac{1}{r}$ if we take the unit of energy to be $e^2/(\epsilon l)$ where e is the electron charge, ϵ the dielectric constant of the background medium and l the magnetic length. The 3D Coulomb potential restricted to the $z = 0$ plane is:

$$V(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \tag{D.1}$$

The 2D Fourier transform is:

$$\upsilon(q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i(q_1 x + q_2 y)}}{\sqrt{x^2 + y^2}} \, \mathrm{d}x \, \mathrm{d}y \tag{D.2}$$

Switch to polar coordinates (r, θ). We choose an orientation of the x and y axes such that the angle between $\mathbf{r}$ and $\mathbf{q}$ is θ. The exponent becomes:

$$q_1 x + q_2 y = qr \cos(\theta) \tag{D.3}$$

The Fourier transform simplifies to:

$$\upsilon(q) = \int_0^{\infty} \int_0^{2\pi} \frac{e^{-iqr\cos(\theta)}}{r} r \, \mathrm{d}r \, \mathrm{d}\theta \tag{D.4}$$

The angular integral over θ is:

$$\int_0^{2\pi} e^{-iqr\cos(\theta)} \mathrm{d}\theta = 2\pi J_0(qr) \tag{D.5}$$

where $J_0(qr)$ is the zero-th order Bessel function of the first kind. The Fourier transform now reduces to:

$$\upsilon(q) = 2\pi \int_0^{\infty} J_0(qr) \, \mathrm{d}r \tag{D.6}$$

substituting $\tilde{r} = qr$ and using $\int_0^{\infty} J_0(r) \, \mathrm{d}r = 1$ [16] we get:

$$\upsilon(q) = 2\pi \cdot \frac{1}{q} = \frac{2\pi}{q} \tag{D.7}$$

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