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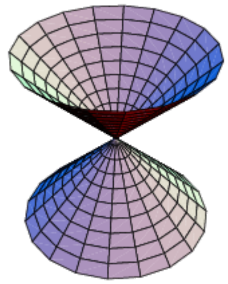
Created, developed, and nurtured by Eric Weisstein at Wolfram Research

Geometry > Solid Geometry > Cones >
Geometry > Surfaces > Surfaces of Revolution >
Geometry > Surfaces > Constant-Curvature Surfaces >
More...

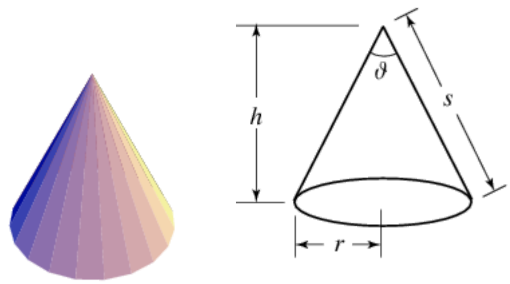
Cone



In general, a cone is a pyramid with a circular cross section. A right cone is a cone with its vertex above the center of its base. However, when used without qualification, the term "cone" often means "right cone."

In discussions of conic sections, the word "cone" is taken to mean "double cone," i.e., two cones placed apex to apex. The double cone is a quadratic surface, and each single cone is called a "nappe." The hyperbola can then be defined as the intersection of a plane with both nappes of the cone.

Cones are implemented in the Wolfram Language as Cone[{{*x*1, *y*1, *z*1}, {*x*2, *y*2, *z*2}}, *r*].



A right cone of height h and base radius r oriented along the z-axis, with vertex pointing up, and with the base located at $z=0$ can be described by the parametric equations

$$x = \frac{h-u}{h} r \cos\theta \quad (1)$$

$$y = \frac{h-u}{h} r \sin\theta \quad (2)$$

$$z = u \quad (3)$$

for $u \in [0, h]$ and $\theta \in [0, 2\pi)$.

The opening angle of a right cone is the vertex angle made by a cross section through the apex and center of the base. For a cone of height h and radius r, it is given by

$$\vartheta = 2\tan^{-1}\left(\frac{r}{h}\right). \quad (4)$$

Adding the squares of (1) and (2) shows that an implicit Cartesian equation for the cone is given by

$$\frac{x^2+y^2}{c^2} = (z-z_0)^2, \quad (5)$$

where

$$c \equiv \frac{r}{h} \quad (6)$$

is the ratio of radius to height at some distance from the vertex, a quantity sometimes called the opening angle, and $z_0 = h$ is the height of the apex above the $z=0$ plane.

The volume of a cone is

$$V = \frac{1}{3} A_b h, \quad (7)$$

where A_b is the base area and h is the height. If the base is circular, then

$$V = \pi \int (x^2+y^2)\, dz \quad (8)$$

$$= \pi \int_0^h \left[\frac{(h-z) r}{h}\right]^2 dz \quad (9)$$

$$= \frac{1}{3} \pi r^2 h. \quad (10)$$

This amazing fact was first discovered by Eudoxus, and other proofs were subsequently found by Archimedes in *On the Sphere and Cylinder* (ca. 225 BC) and Euclid in Proposition XII.10 of his *Elements* (Dunham 1990).

The geometric centroid can be obtained by setting $R_2 = 0$ in the equation for the centroid of the conical frustum,

$$\bar{z} = \frac{\langle z \rangle}{V} = \frac{h\left(R_1^2 + 2 R_1 R_2 + 3 R_2^2\right)}{4\left(R_1^2 + R_1 R_2 + R_2^2\right)}, \quad (11)$$

(Eshbach 1975, p. 453; Beyer 1987, p. 133) yielding

$$\bar{z} = \frac{1}{4} h. \quad (12)$$

The interior of the cone of base radius r, height h, and mass M has moment of inertia tensor about its apex of

$$I = \begin{bmatrix} \frac{1}{20}\left(2 h^2 + 3 r^2\right) M & 0 & 0 \\ 0 & \frac{1}{20}\left(2 h^2 + 3 r^2\right) M & 0 \\ 0 & 0 & \frac{3}{10} M r^2 \end{bmatrix}. \quad (13)$$

For a right circular cone, the slant height s is

$$s = \sqrt{r^2 + h^2} \quad (14)$$

and the surface area (not including the base) is

$$S = \pi r s \quad (15)$$
$$= \pi r \sqrt{r^2 + h^2}. \quad (16)$$

The locus of the apex of a variable cone containing an ellipse fixed in three-space is a hyperbola through the foci of the ellipse. In addition, the locus of the apex of a cone containing that hyperbola is the original ellipse. Furthermore, the eccentricities of the ellipse and hyperbola are reciprocals.

There are three ways in which a grid can be mapped onto a cone so that it forms a cone net (Steinhaus 1999, pp. 225-227).

The equation for a general (infinite, double-napped) cone is given by

$$x = a u \cos v \quad (17)$$
$$y = a u \sin v \quad (18)$$
$$z = u, \quad (19)$$

which gives coefficients of the first fundamental form

$$E = 1 + a^2 \quad (20)$$
$$F = 0 \quad (21)$$
$$G = a^2 u^2, \quad (22)$$

second fundamental form coefficients

$$e = 0 \quad (23)$$
$$f = 0 \quad (24)$$
$$g = \frac{a |u|}{\sqrt{1 + a^2}}, \quad (25)$$

and area element

$$d S = a \sqrt{1 + a^2} \; |u| \, d u \, d v. \quad (26)$$

The Gaussian curvature is

$$K = 0, \quad (27)$$

and the mean curvature is

$$M = \frac{|u|}{2 a \sqrt{1 + a^2} \; u^2}. \quad (28)$$

Note that writing $z = v$ instead of $z = u$ would give a helicoid instead of a cone.

SEE ALSO:
Bicone, Cone Net, Cone Set, Conic Section, Conical Frustum, Cylinder, Double Cone, Generalized Cone, Helicoid, Nappe, Pyramid, Sphere, Sphericon

REFERENCES:

Beyer, W. H. (Ed.). *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 129 and 133, 1987.

Dunham, W. *Journey through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 76-77, 1990.

Eshbach, O. W. *Handbook of Engineering Fundamentals.* New York: Wiley, 1975.

Harris, J. W. and Stocker, H. "Cone." §4.7 in *Handbook of Mathematics and Computational Science.* New York: Springer-Verlag, pp. 104-105, 1998.

Hilbert, D. and Cohn-Vossen, S. "The Cylinder, the Cone, the Conic Sections, and Their Surfaces of Revolution." §2 in *Geometry and the Imagination.* New York: Chelsea, pp. 7-11, 1999.

Kern, W. F. and Bland, J. R. "Cone" and "Right Circular Cone." §24-25 in *Solid Mensuration with Proofs, 2nd ed.* New York: Wiley, pp. 57-64, 1948.

Steinhaus, H. *Mathematical Snapshots, 3rd ed.* New York: Dover, 1999.

Yates, R. C. "Cones." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 34-35, 1952.

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