In [1]:
```python
import matplotlib.pyplot as plt
import numpy as np
import math
```

LAB 11 - The Discrete Fourier Transform (DFT)

DUE: 12 April 2024 @ 11.59pm

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The purpose of this lab is to give you practice with the discrete Fourier Transform (DFT). Note: You must show your numerical work, and your computations must be reproducible either as a number of short python codes or (preferrably) in a single Jupyter Notebook! **Please include a PDF version of your assignment to help out the TA with the grading process.**

Q1: Exploring the DFT

This question examines how you approach subsampling of a signal, and what are the consequences of not following the recommended approach.

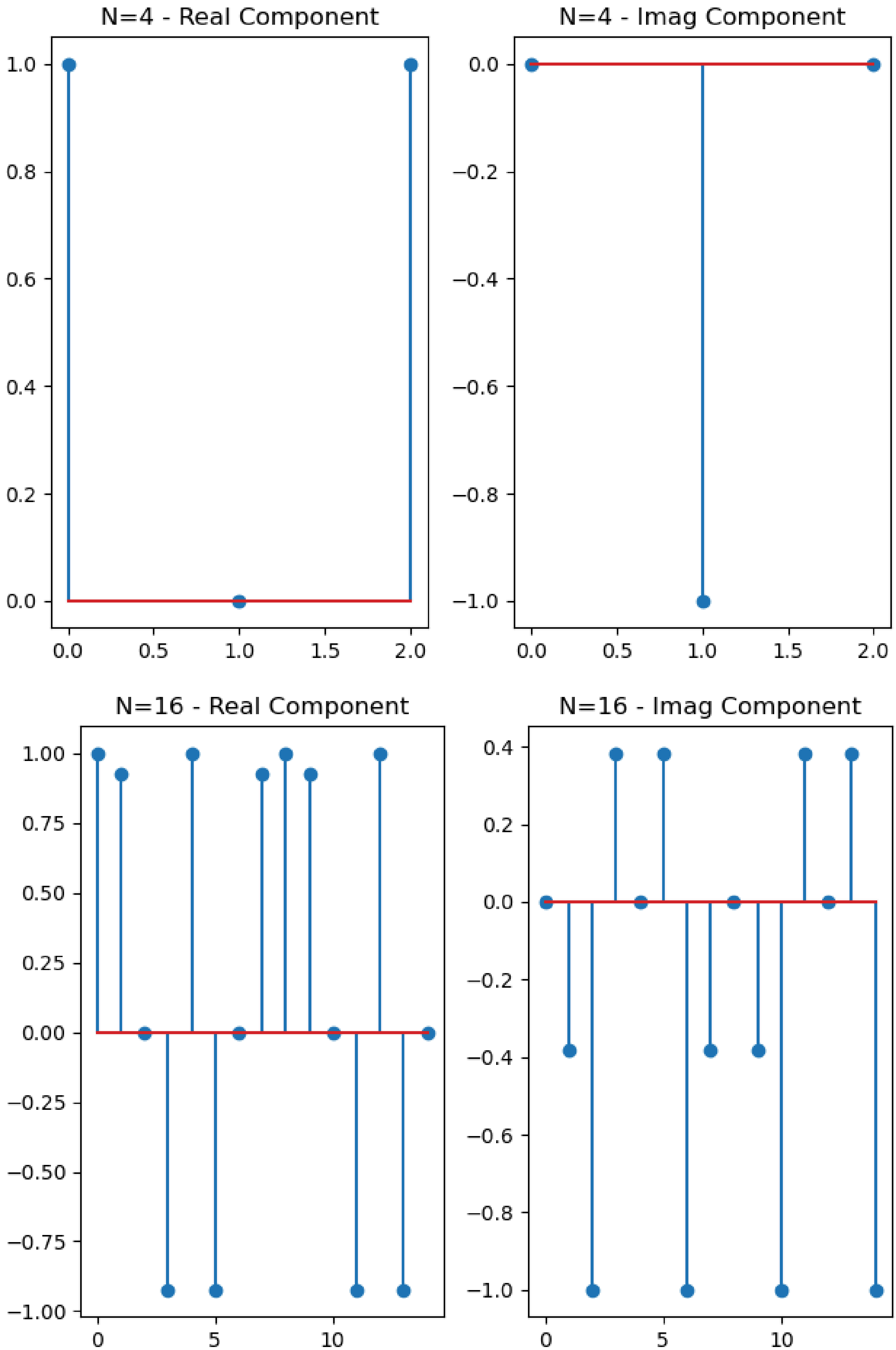
Q1-1: Visualizing the Fourier Matrix: The Fourier Matrix used by the DFT is given by

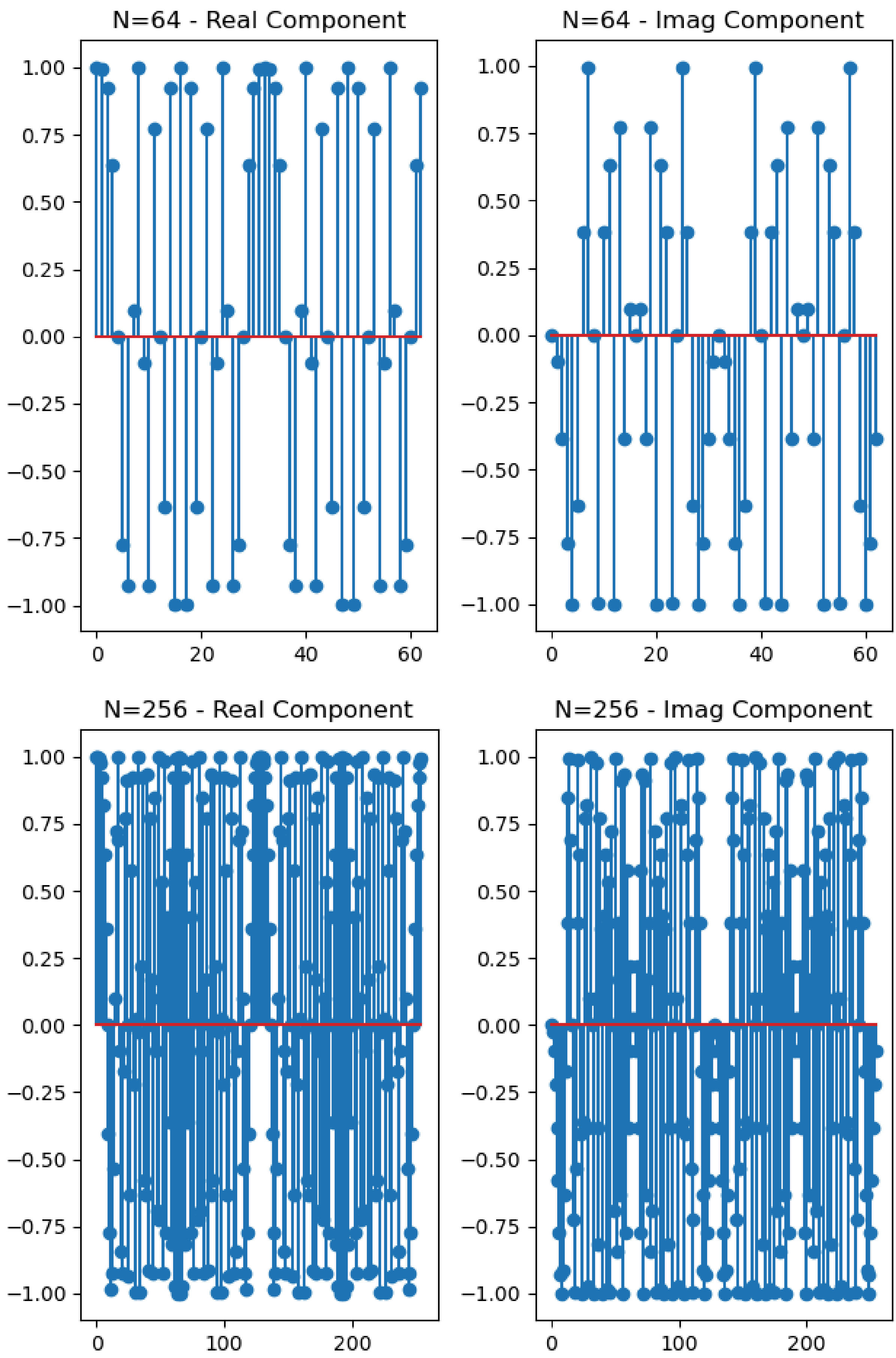
$$[W_N^{kn}] = [e^{-i2\pi kn/N}], \quad \text{where} \quad k,n = 0, N-1$$

Plot the real and imaginary components of W_N^{kn} for N=4, 16, 64 and 256 points.

In [2]:
```python
# . . Your answer goes here
N=np.array([4, 16, 64, 256])
for p in N:
    k=np.arange(0,p-1)
    n=np.arange(0,p-1)
    W= np.exp((-1j*2 * np.pi * k * n ) / p)
    plt.subplot(121);
    plt.stem(k,W.real);
    plt.title("N="+ str(p) +' - Real Component')

    plt.subplot(122);
    plt.stem(k,W.imag);
    plt.title('N='+ str(p) + ' - Imag Component')
    plt.tight_layout(); plt.show()
```


N=4 - Real Component
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.5 1.0 1.5 2.0
N=4 - Imag Component
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.0 0.5 1.0 1.5 2.0
N=16 - Real Component
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0 5 10
N=16 - Imag Component
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0 5 10



Q1-2: Find the **analytic** 16-point DFT of the following sequence:

$$x[n] = \delta[n] + \delta[n-4]$$

Illustrate your answer with a numerical example **without using the np.fft.fft() function**. (Hint: Think matrix multiplication using one of your results from **Q1-1**.)

Solution:

$$X[k] = \sum_{n=0}^{15} x[n] e^{-i2\pi kn/16} \tag{1}$$

We see that $\delta[n-4] = 0$, except when $n = 4$, and $\delta[n] = 0$, except when $n = 0$. Thus $x[n] = 0$, except when $n = 0, 4$, where it equals 1. Thus the equation above becomes:

$$X[k] = 1 + e^{-i2\pi k4/16}. \tag{2}$$

In [3]:

```
# . . Your answer goes here
#x[n] is non zero only at idx 0 & 4

#fourier matrix W_16^kn
N=16
x=[1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0]
k=np.arange(0,N)
n=np.arange(0,N)
W= np.exp((-1j*2 * np.pi * np.outer(k, n) ) / N)
X=np.dot(x, W)
print(X)
```

```
[2.+0.00000000e+00j 1.-1.00000000e+00j 0.-1.22464680e-16j
 1.+1.00000000e+00j 2.+2.44929360e-16j 1.-1.00000000e+00j
 0.-3.67394040e-16j 1.+1.00000000e+00j 2.+4.89858720e-16j
 1.-1.00000000e+00j 0.-6.12323400e-16j 1.+1.00000000e+00j
 2.+7.34788079e-16j 1.-1.00000000e+00j 0.-8.57252759e-16j
 1.+1.00000000e+00j]
```

Q1-3: Find the analytic 8-point DFT of the following sequence:

$$x[n] = \cos\left(\frac{\pi n}{4}\right) \sin\left(\frac{3\pi n}{4}\right)$$

Illustrate your answer with a numerical example **without using the np.fft.fft() function**.

Solution:

$$X[k] = \sum_{n=0}^{7} x[n] e^{-i2\pi kn/8} \tag{3}$$

We see that for $n = 0, 4$ our sine term equals 0 and for $n = 2, 6$ our cosine term equals 0. So $x[n] = 0$ except for when $n = 1, 3, 5, 7$. We can evaluate x at these points and rewrite the above equation to get our solution:

$$X[k] = 0.5e^{-i2\pi k/8} - 0.5e^{-i2\pi k3/8} + 0.5e^{-i2\pi k5/8} - 0.5e^{-i2\pi k7/8} \tag{4}$$

In [4]:
```python
# . . Your answer goes here
N=8
x=[0, 0.5, 0, -0.5, 0, 0.5, 0, -0.5]
k=np.arange(0,N)
n=np.arange(0,N)
W= np.exp((-1j*2 * np.pi * np.outer(k, n) ) / N)
X=np.dot(x, W)
print(X)
```

```
[ 0.00000000e+00+0.00000000e+00j -5.55111512e-17-1.11022302e-16j
  4.89858720e-16-2.00000000e+00j -6.10622664e-16+2.77555756e-16j
  0.00000000e+00+2.44929360e-16j -3.33066907e-16+1.66533454e-16j
 -1.46957616e-15+2.00000000e+00j -9.43689571e-16-2.16493490e-15j]
```

Q1-4 Find the 10-point **inverse** DFT of the following sequence :

$$X[k] = \begin{cases} 1, & k = 0 \\ -1, & else \end{cases}$$

Illustrate your answer with a numerical example **without using the np.fft.fft() function**. (Hint: think linearity and how you can define two useful function from the one below!)

We can rewrite $X[k]$ as:

$$X[k] = X_1[k] + X_2[k] \tag{5}$$
$$X_1[k] = 1 \quad \text{for } k = 0 \tag{6}$$
$$X_1[k] = -1 \quad \text{for } k \neq 0 \tag{7}$$

Using the linearity property we know that the inverse DFT of $X_1 + X_2$ is $x_1 + x_2$. Thus we can find the inverse DFT of each X_1 and X_2, add them to together to get the full sequence $x[n]$.

We know $X_1 = 0$ except for $k = 0$. Thus,

$$x_1[n] = \frac{1}{10} \tag{8}$$

We see that $X_2 = -1$ except for $k = 0$, which would take forever to type...

In [5]:
```python
# . . Your answer goes here
N=10
X1=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0]
X2=[0, -1, -1, -1, -1, -1, -1, -1, -1, -1]
k=np.arange(0,N)
n=np.arange(0,N)
W= np.exp((1j*2 * np.pi * np.outer(k, n) ) / N)
x1=np.dot(X1, W) / N
x2= np.dot(X2, W) / N
x= x1 + x2
print(x)
```

```
[-0.8+0.00000000e+00j  0.2+0.00000000e+00j  0.2-2.22044605e-17j
  0.2-1.66533454e-16j  0.2-5.55111512e-17j  0.2+4.71674712e-16j
  0.2-1.66533454e-16j  0.2-2.22044605e-16j  0.2-7.77156117e-17j
  0.2+1.22124533e-16j]
```

Q2: Evaluating 1st Derivative FD Operator Accuracy

Background

One of the most commonly used approaches for calculating the numerical derivative of a discrete time series is to use the finite-difference method (or FD). We commonly refer to the "N-th order of accuracy" of the FD approximation as $\mathcal{O}\left(\Delta h^N\right)$. This is commonly represented by a Taylor-series approximation of a function $f(x_0+\Delta h)$ some Δh away from x_0:

$$f(x_0+\Delta h)=f(x_0)+\frac{f'(x_0)}{1!}\Delta h+\frac{f^{(2)}(x_0)}{2!}\Delta h^2+\cdots+\frac{f^{(n)}(x_0)}{n!}\Delta h^n+\mathcal{O}\left(\Delta h^N\right),$$

where $n!$ denotes the factorial of n, and $\mathcal{O}\left(\Delta h^N\right)$ is a remainder term representing the difference between the Taylor polynomial of degree n and the original function. (Note: a larger n will be more accurate than a lower n.) For example, an approximation for the first-order $\mathcal{O}\left(\Delta h\right)$ first derivative approximation is given by:

$$f(x_0+\Delta h)=f(x_0)+f'(x_0)\Delta h+\mathcal{O}\left(\Delta h\right),$$

which can be solved in the following way:

$$f'(x_0)=\frac{f(x_0+\Delta h)-f(x_0)}{\Delta h}+\mathcal{O}\left(\Delta h\right).$$

Or, as an approximation,

$$f'(x_0)\approx\frac{f(x_0+\Delta h)-f(x_0)}{\Delta h}.$$

Generally speaking, a $\mathcal{O}\left(\Delta h^N\right)$ derivative approximation is the equivalent of

$$f'(x_0)\approx\sum_{k=-N+1}^{N-1}c_k\,f(x_0+k\Delta h),$$

where c_k are the FD coefficients.

Q2-1 Let's use the DFT to evaluate the accuracy of some fairly standard approximations for the first derivative.

- 3-point $\mathcal{O}\left(\Delta h^2\right)$ with coefficients $c_k^{(2)}=[-1/2,0,1/2]$
- 5-point $\mathcal{O}\left(\Delta h^4\right)$ with coefficients $c_k^{(4)}=[1/12,-2/3,0,2/3,-1/12]$
- 7-point $\mathcal{O}\left(\Delta h^6\right)$ with coefficients $c_k^{(6)}=[-1/60,3/20,-3/4,0,3/4,-3/20,1/60]$
- 9-point $\mathcal{O}\left(\Delta h^8\right)$ with coefficients $c_k^{(8)}=[1/280,-4/105,1/5,-4/5,0,4/5,-1/5,4/105,-1/280]$

(a) Compute a 64-point DFT for each of the four 1st-derivative approximations listed above assuming $\Delta h=1$. (Note that this requires zero padding 61 points for the 1st case, 59 points for the 2nd case, 57 for the 3rd case, and 55 for the 4th case.) Use

```
np.fft.fftshift()
```

to put these into "logical" order.

In [6]:
```
# . . Your answer goes here
c2= [-0.5, 0, 0.5]
c4= [1/12, -2/3, 0, 2/3, -1/12]
c6= [-1/60, 3/20, -3/4, 0, 3/4, -3/20, 1/60]
c8=[1/280, -4/105, 1/5, -4/5, 0 ,4/5, -1/5, 4/105, -1/280]
N=64

pc2= np.zeros(64); pc2[0:3]=c2;
pc4= np.zeros(64); pc4[0:5]=c4;
pc6= np.zeros(64); pc6[0:7]=c6;
pc8= np.zeros(64); pc8[0:9]=c8;

FFTpc2= np.fft.fftshift(np.fft.fft(pc2))
FFTpc4= np.fft.fftshift(np.fft.fft(pc4))
FFTpc6= np.fft.fftshift(np.fft.fft(pc6))
FFTpc8= np.fft.fftshift(np.fft.fft(pc8))
```

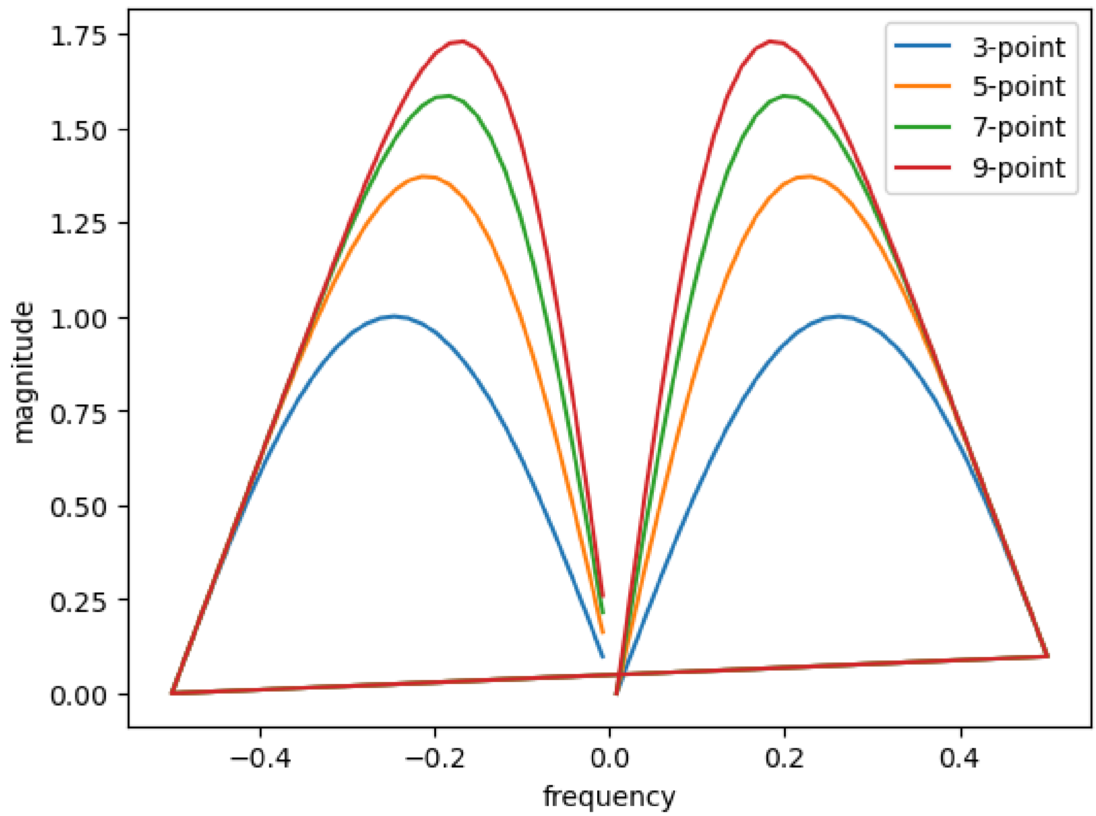
(b) Compute the frequency axis (also in "logical order") using

In [7]:
```
# . . Your answer goes here
freq= np.fft.fftshift(np.linspace(-0.5, 0.5, 64))
```

(c) Plot the Fourier spectra from your DFT calculation for these four approximations on the same plot.

In [8]:
```
# . . Your answer goes here
plt.plot(freq, np.abs(FFTpc2), label="3-point")
plt.plot(freq, np.abs(FFTpc4), label="5-point")
plt.plot(freq, np.abs(FFTpc6), label="7-point")
plt.plot(freq, np.abs(FFTpc8), label="9-point")
plt.xlabel("frequency")
plt.ylabel("magnitude")
plt.legend()
```

Out[8]:
```
<matplotlib.legend.Legend at 0x2154d148210>
```

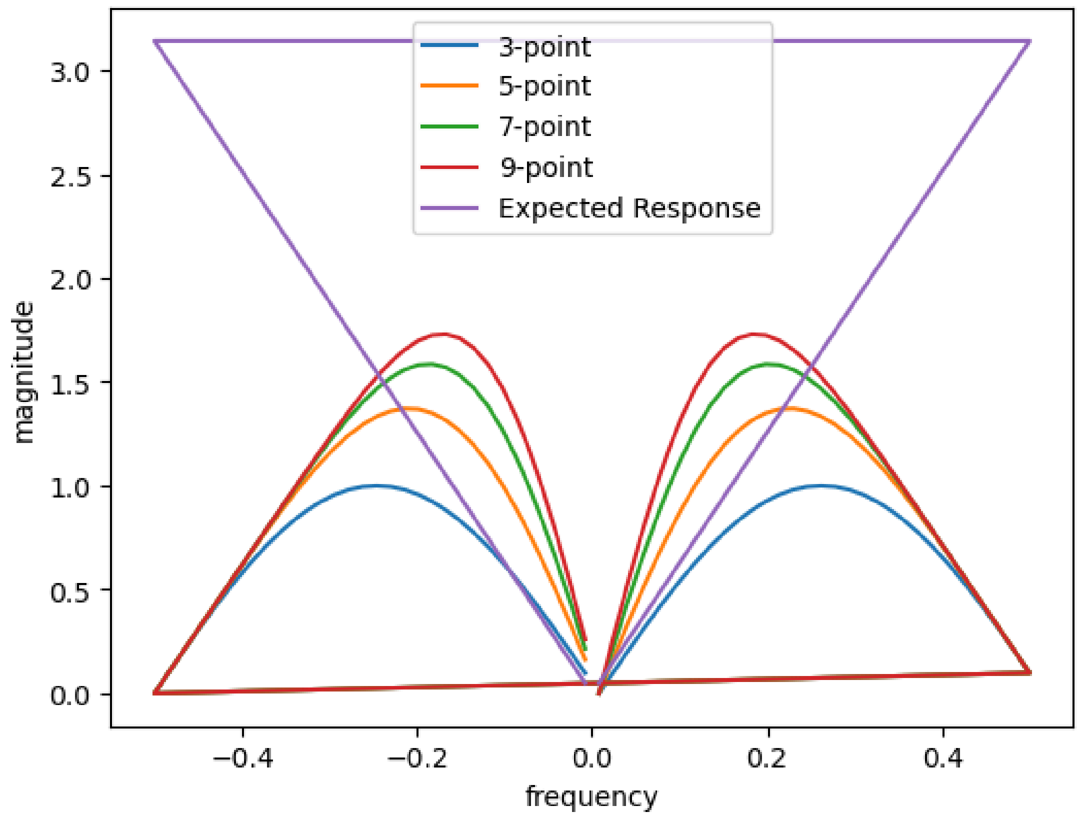


(d) To examine the accuracy of these four approximations, include a plot on the same graph of the expected **analytic spectral response** (i.e., $\left|\frac{\partial}{\partial t}\right| \iff |2\pi i f| = 2\pi|f|$). Don't forget to include a legend!

In [9]:
```
# . . Your answer goes here
Er= 2 * np.pi * np.abs(freq)

plt.plot(freq, np.abs(FFTpc2), label="3-point")
plt.plot(freq, np.abs(FFTpc4), label="5-point")
plt.plot(freq, np.abs(FFTpc6), label="7-point")
plt.plot(freq, np.abs(FFTpc8), label="9-point")
plt.plot(freq, Er, label="Expected Response")
plt.xlabel("frequency")
plt.ylabel("magnitude")
plt.legend()
```

Out[9]:
```
<matplotlib.legend.Legend at 0x2154d247510>
```



3-point
5-point
7-point
9-point
Expected Response
magnitude
frequency
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.4
−0.2
0.0
0.2
0.4