6.1.1 Type Theory

At the early 20th century, Bertrand Russell introduced type theory to cope with a paradox in naive set theory, which was expressed as follows:

$$H = \{x|x \notin x\} \Rightarrow H \in H \Leftrightarrow H \notin H. \tag{6.1}$$

The paradox arises when one considers whether H contains itself or not. If H contains itself, then by definition it is a set that does not contain itself, which is a contradiction. On the other hand, if H does not contain itself, then by definition it is a set that should contain itself, which is also a contradiction. This problem arises when impredicative universal quantification is allowed, which means that the definition of the object involves quantifying over all objects, including the object being defined itself. Russel's type theory resolved this problem by defining objects as part of a specific group. Consider a type n, then we can redefine H as

$$H^n = \{x^{n-1}|x^{n-1} \notin x^{n-1}\} \Rightarrow H^n \in H^n \Leftrightarrow H^n \notin H^n. \tag{6.2}$$

In this case, the paradox is false since the sets are defined by the types, and type $n-1$ excludes type n (Eades, 2012).

Type theory became the formal presentation that models objects and relations, such as a variable, function or substitution, with types. For example, variable 10 has the type of natural numbers ($\mathbb{N}$), which is in the built-in notation written as $10 : \mathrm{nat}$. From this term, other typed terms can be constructed. As illustrated in (Hoang, 2014), terms of the type $\mathbb{N}$ can be constructed just by defining the variable as we defined before, or as a successor function $succ(n) : \mathbb{N}$:

$$0 \xrightarrow{succ} 1 :\equiv \mathrm{succ}(0) \xrightarrow{succ} 2 :\equiv \mathrm{succ}(1)$$

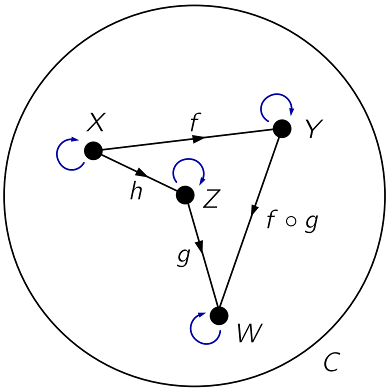

Figure 6.2: Category C with a set of objects $\{X, Y, Z, W\}$, morphisms $\{f, g, h\}$ and composition of morphisms $\{f \circ g\}$. Each object also has an identity morphism: an arrow points to the object itself.

6.1.2 Category Theory

Tom Leinster described category theory as a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level (Leinster, 2016). Formally, category theory is the study of mathematical structures using abstractions of functions called morphisms, as well as a mathematical workspace and theory (Barr and Wells, 2012; Eades, 2012). It provides the concepts to meaningfully compare and combine unrelated systems by understanding their patterns (Harley, 2020). A category C is an abstract object consisting of a set of objects, morphisms and compositions of morphisms. The objects and relations can visually be represented as directed graphs, as illustrated in figure 6.2.

Categories are connected through structure-preserving maps named functors. They can be considered as morphisms in a category of subcategories. This theoretical framework of formalization is especially useful for combining different levels of abstraction (e.g. the unification of CHARM with other modules as will be explained further), and for formal descriptions of systems.

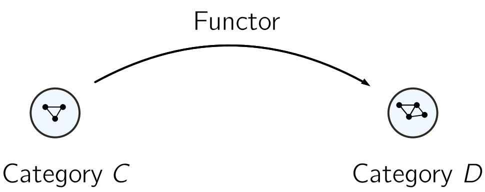

Figure 6.3: A functor represented as a directed edge from category C to category D.