Hilbert Spaces

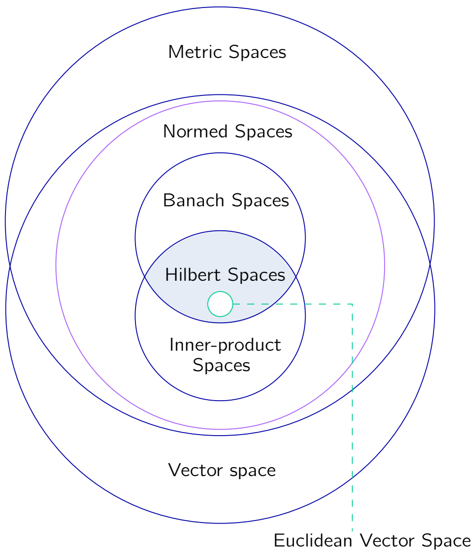

Figure 2.1: Relations among different spaces in functional analysis.

A Hilbert space $\mathcal{H}$ is a complete inner product space, and its theory can be considered as a generalization of the familiar Euclidean vector space (e.g. $\mathbb{R}^n$). It is a vector space where vectors (usually described using finite real or complex numbers), functions, and even more general objects can be represented. Here, the crucial idea to understand is that functions can be thought of as vectors with an infinite number of dimensions. Examples of relevant Hilbert Spaces are $\mathbb{C}^n$, the ℓ^2- and L^2-spaces. The ℓ^2 is the space of absolutely square-summable sequences and L^2-space of absolutely square-summable functions.

Theorem 2.1 (Complete normed spaces (Banach Spaces))**.** A normed space $\mathcal{H}$ is called complete if every Cauchy sequence of vectors in $\mathcal{H}$ converges to a vector in $\mathcal{H}$. A complete normed space is called a Banach space (Kennedy and Sadeghi, 2013).

As illustrated in Figure 2.1, all (finite-dimensional) Hilbert spaces are Banach spaces, which means that it is both normed and complete. However, not all Banach spaces are Hilbert spaces. In order for a Banach space to be considered a Hilbert space, the norm (or distance) must be induced by the inner product. The inner product of two signals is a scaled projection, i.e. a scalar which may contain complex values and can be defined as

$$\langle \alpha | \beta \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2 + ... + \alpha_N \beta_N. \tag{2.1}$$

This means that given an inner product $\langle \cdot \rangle$ in a vector space $\mathcal{H}$, the norm is defined as

$$||f|| = \sqrt{\langle f \rangle}. \tag{2.2}$$

It should be noted that the bases of a normed space are not necessarily orthonormal. A normed space is a type of vector space that has a norm, which is a mathematical function that gives each vector in the space a non-negative size or magnitude, and satisfies certain properties such as non-negativity, homogeneity, and the triangle inequality. A second important point to be aware of, is that the concept of completeness is closely linked to the norm. In a normed space, a sequence of vectors should converge, in terms of their norm, to a vector that is within the original space, in order for the space to be considered complete.