However, $\mathbf{v}^* - \mathbf{v}^n \sim \Delta t \frac{\partial \mathbf{v}}{\partial t}$ so the last term is proportional to $(\Delta t)^3$ for small Δt and can be neglected. Equation (8.65) reduces the solution process to the sequential inversion of tridiagonal matrices in alternating directions. Thus, using the component notation for the velocities (cf. Eqs. (5.49) and (5.50)):

$$\left(1 - \frac{\mu \Delta t}{2\rho} \frac{\delta^2}{\delta x^2}\right) \overline{v_i}^x = \left(1 + \frac{\mu \Delta t}{2\rho} \frac{\delta^2}{\delta x^2}\right) v_i^n + \frac{\Delta t}{\rho} [AP] , \qquad (8.66)$$

$$\left(1 - \frac{\mu \Delta t}{2\rho} \frac{\delta^2}{\delta y^2}\right) \overline{v_i}^y = \left(1 + \frac{\mu \Delta t}{2\rho} \frac{\delta^2}{\delta y^2}\right) \overline{v_i}^x . \qquad (8.67)$$

$$\left(1 - \frac{\mu \Delta t}{2\rho} \frac{\delta^2}{\delta z^2}\right) v_i^* = \left(1 + \frac{\mu \Delta t}{2\rho} \frac{\delta^2}{\delta z^2}\right) \overline{v_i}^y . \qquad (8.68)$$

Each of these systems of equations has a tridiagonal coefficient matrix and thus can be solved with the efficient TDMA method; this requires no iteration. However, this solution method is only applicable to structured grids (both Cartesian and non-orthogonal ones); it cannot be applied with such simplicity and efficiency to unstructured grids.

8.3.3 *The Poisson Equation for Pressure*

The Poisson equation for pressure can be solved by any elliptic equation solver, e.g., multigrid, ADI, conjugate gradient, GMRES, etc. (see Chap. 5). Here, we address some issues related to the Poisson equation discretization.

1. For non-iterative fractional-step methods, Hirt and Harlow (1967) argued that, given that a Poisson equation is solved iteratively at every step to enforce continuity, one should take care to avoid accumulation of incompressibility errors. Their insight was that one can (1) carry the iterative solution of the equation to a high level of accuracy or (2) use some self-correcting procedure. Interestingly, for Eq. (7.44) it was noted that formulation of the pressure Poisson equation begins with inclusion of the divergence of the velocity at both the current (n) and future ($n+1$) time steps; we can show that the formulation laid out in Sect. 7.2.1 actually includes the ability to account for a divergent v^n. We begin by writing Eq. (7.53) without the viscous terms (they play no role in this illustration) as

$$(\rho \mathbf{v})^* = (\rho \mathbf{v})^n - \Delta t \left[\frac{3}{2} C(\mathbf{v}^n) - \frac{1}{2} C(\mathbf{v}^{n-1})\right] - \Delta t G(p^{n-1/2}) , \qquad (8.69)$$

and introducing it into the pressure Poisson equation (7.57) which was derived to force the divergence of $\mathbf{v}^{n+1}$ to be zero. The result is

$$\begin{aligned} D(G(p')) &= \frac{D(\rho \mathbf{v})^*}{\Delta t} = \\ \frac{D(\rho \mathbf{v})^n}{\Delta t} - D\left(\left[\frac{3}{2} C(\mathbf{v}^n) - \frac{1}{2} C(\mathbf{v}^{n-1})\right]\right) - D(G(p^{n-1/2})) &= \\ \frac{D(\rho \mathbf{v})^n}{\Delta t} - \dots \text{. QED} \end{aligned} \qquad (8.70)$$

Accordingly, any error in the divergence in the previous step is fed back as a correction in the next step. Hirt and Harlow (1967) show that this cuts down the accumulation of incompressibility errors in cases where, e.g., computing time is saved by terminating iteration with a smaller number of iterations and so less accuracy.

2. The set-up and strategy for staggered grids are essentially the same as those described in Sect. 8.1 so we shall not focus on staggered grids here; rather we spend our time on colocated grids. Issues and solutions related to the calculation of pressure on a colocated grid were discussed for SIMPLE and related schemes in Sect. 8.2.1. The checkerboard pressure pattern for the sparse Laplacian form was demonstrated and an alternative compact Laplacian was derived. However, there and in Sects. 7.1.3 and 7.1.5.1, the loss of energy conservation when the compact form is used was noted. We describe here two alternative formulations that do not experience an uncoupling of the pressure field; one uses the compact Laplacian with its accompanying continuity error and one uses a modified method with the sparse Laplacian.
3. On a colocated grid using the equations described in Sect. 7.2.1, the pressure Poisson equation (7.57) can be discretized in two ways.[1] We first discretize the equation in two dimensions by using CDS on the Laplacian directly; this yields (cf. Eq. (8.52)):

$$\left(\frac{p'_E - 2p'_P + p'_W}{(\Delta x)^2}\right) + \left(\frac{p'_N - 2p'_P + p'_S}{(\Delta y)^2}\right) = \frac{1}{\Delta t}\left(\frac{(\rho u)_e - (\rho u)_w}{\Delta x} + \frac{(\rho v)_n - (\rho v)_s}{\Delta y}\right)^* . \qquad (8.71)$$

On the other hand, constructing the equation by using the discrete forms of the divergence and gradient operators as indicated in Sect. 7.1.5.1 yields (see Fig. 3.5 for $2\Delta x$, etc. grid points, namely, EE, WW, etc.)

$$\left(\frac{p'_{EE} - 2p'_P + p'_{WW}}{4(\Delta x)^2}\right) + \left(\frac{p'_{NN} - 2p'_P + p'_{SS}}{4(\Delta y)^2}\right) = \frac{1}{\Delta t}\left(\frac{(\rho u)_e - (\rho u)_w}{\Delta x} + \frac{(\rho v)_n - (\rho v)_s}{\Delta y}\right)^* . \qquad (8.72)$$

Using the compact Laplacian of Eq. (8.71) actually produces an error in continuity equal to

$$-\Delta t\left[(\Delta x)^2\frac{\partial^4 p'}{\partial x^4}+(\Delta y)^2\frac{\partial^4 p'}{\partial y^4}\right]=$$
$$-(\Delta t)^2\left[(\Delta x)^2\frac{\partial^5 p}{\partial t\partial x^4}+(\Delta y)^2\frac{\partial^5 p}{\partial t\partial y^4}\right] \quad (8.73)$$

because $p' \sim \Delta t(\partial p/\partial t)$. The equivalent result was obtained in Sect. 8.2.1. In contrast, for staggered grids, the FV formulation using the discrete divergence and gradient forms gives both a compact Laplacian and no continuity error. Equation (8.71) performed well in the *non-iterative* simulations of Armfield (2000); Armfield and Street (2005) and Armfield et al. (2010). There, at each time step, using the current p' values, the CV velocities are corrected such that, for example,

$$u_P^{n+1}=u_P^*-\frac{\Delta t}{2\rho\Delta x}(p'_E-p'_W) \quad (8.74)$$

and each of the CV-face velocities is also corrected according to, for example,

$$(\rho u)_e^{n+1}=(\rho u)_e^*-\frac{\Delta t}{\Delta x}(p'_E-p'_P) \quad (8.75)$$

which yields a divergence-free velocity field for these convecting velocities. It was observed that higher-order errors in the pressure appear to limit the growth of grid-scale error in the pressure field. Also, the feedback of continuity error into the next time step (Eq. (8.70)) reduces the accumulation of continuity error.

4. Use of the sparse form (8.72) is inefficient and leads to pressure oscillations in iterative colocated schemes (see Sect. 8.2.1). Armfield et al. (2010) made an improvement in the sparse scheme for the non-iterative case (cf. the staggered-grid fractional-step method of Choi and Moin 1994). The modified scheme is as follows (the original fractional-step procedure is described in Sect. 7.2.1): Solve Eqs. (7.50) and (7.51) by

   a. finding an estimate of the new velocity $\mathbf{v}^*$ using the old value of the pressure $p^{n-1/2}$:

$$\frac{(\rho\mathbf{v})^*-(\rho\mathbf{v})^n}{\Delta t}+\left[\frac{3}{2}C(\mathbf{v}^n)-\frac{1}{2}C(\mathbf{v}^{n-1})\right]=$$
$$-G(p^{n-1/2})+\frac{L(\mathbf{v}^*)+L(\mathbf{v}^n)}{2}, \quad (8.76)$$

   b. adding the old pressure gradient to the estimated velocity field

$$(\rho\hat{\mathbf{v}})^*=(\rho\mathbf{v})^*+\Delta t\,G(p^{n-1/2}), \quad (8.77)$$

and so approximately canceling the pressure gradient in the previous equation. The cancellation would be exact in the absence of the implicit viscous term.

   c. defining a correction to $\hat{\mathbf{v}}^*$ of the form

$$(\rho\mathbf{v})^{n+1}=(\rho\hat{\mathbf{v}})^*-\Delta t\,G(p^{n+1/2}), \quad (8.78)$$

and finally

   d. finding $p^{n+1/2}$ by substituting Eq. (8.78) into the continuity equation (7.51) to obtain

$$D(G(p^{n+1/2}))=\frac{D(\hat{\mathbf{v}}^*)}{\Delta t}. \quad (8.79)$$

Using the sparse Laplacian (8.72) produces

$$\left(\frac{p_{EE}-2p_P+p_{WW}}{4\Delta x^2}\right)^{n+1/2}+\left(\frac{p_{NN}-2p_P+p_{SS}}{4\Delta y^2}\right)^{n+1/2}=$$
$$\frac{1}{\Delta t}\left(\frac{\hat{u}_e^*-\hat{u}_w^*}{\Delta x}+\frac{\hat{v}_n^*-\hat{v}_s^*}{\Delta y}\right). \quad (8.80)$$

The initial pressure used at each time step in the Poisson equation solver should be the pressure from the previous time step.

In Sect. 7.2.1, we used the results of application of the fractional-step method to show in Eq. (7.59) that an $O(\Delta t)^2$ error $(1/2)\Delta t\,L(G(p'))$ was made, but it was consistent with the basic discretization. Following the same procedure here shows precisely the same error, i.e.,

$$\frac{1}{2}\Delta t\,L(G(p^{n+1/2}-p^{n-1/2}))=\frac{1}{2}\Delta t\,L(G(p')). \quad (8.81)$$

This means that the extra pressure cancellation step in this procedure reduces the fractional-step error by an order of magnitude compared to the P1 method; again recall that $L=D(G(\,))$. Results presented in Armfield et al. (2010) show that this sparse scheme has essentially no divergence error. However, their compact pressure-correction scheme (which was derived by directly differencing the Poison equation; see warnings in Sects. 7.1.3 or 7.1.5.1) has a divergence error that is approximately the same irrespective of the degree to which the pressure Poisson equation is converged.

Consistent with the result obtained for the methods' errors, both schemes have approximately the same accuracy. No grid scale oscillations were observed with either scheme, but solving for a new full pressure in the modified scheme prevents grid scale oscillations from accumulating in any case.

8.3.4 *Initial and Boundary Conditions*

The initial condition on the velocity should be divergence-free. Also, most of the fractional-step schemes employ the Adams–Bashforth scheme for convection. Because Adams–Bashforth schemes are multilevel methods, solutions cannot be started using only data at the initial time point. One has to use other methods to get the calculation started, e.g., the Crank–Nicolson method with iteration for the pressure; see Sect. 6.2.2. Often, the pressure-correction field computed at the first step can include a first-order-in-time error because the initial pressure is everywhere zero and prescribed at the wrong time, i.e., not at a 1/2 time level; Fringer et al. (2003) give an example. However, this issue will not arise if a multilevel start is used because the full pressure is calculated at the right time, i.e., halfway through the time step, until enough temporal data is collected to use the regular method.

Boundary conditions are a more complex issue. For the original P1-schemes, special intermediate boundary conditions were needed after the first step of estimating the new velocity (Kim and Moin 1985; Zang et al. 1994). In general, however, the physical boundary conditions can be used for velocities and scalars, while for the pressure the conditions needed depend on the method as follows:

1. For staggered grids, no pressure boundary condition is required, but setting the gradient of the pressure correction (or pseudo-pressure in P1-schemes) normal to surfaces equal to zero is appropriate.[2]
2. For colocated schemes using auxiliary nodes outside boundaries, the normal momentum equation at the immediate interior node requires the pressure at the immediate exterior node, so a high-order extrapolation from the interior is used. Again, setting the gradient of the pressure correction (or pseudo-pressure in P1 schemes) normal to surfaces equal to zero is appropriate.[3]
3. For the tangential velocity at a wall, it is worth noting that, while a boundary condition can be set for the estimated velocity in the velocity-estimation step of the method, the projection step, which is essentially an irrotational step, cannot constrain the tangential velocity correction; thus, a small error is made (Armfield and Street 2002). It is less for schemes with a small fractional-step error (see Eq. (7.59) and (8.81)), i.e., when the correction by the divergence constraint is small because the original velocity estimate is better.

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[2]This is true where there are Dirichlet *velocity* boundary conditions; at outflows where the velocity has a Neumann boundary condition, there are a number of options for pressure boundary conditions, depending on the flow. Professional code documentation usually describes these options. See also