Hw-1: Fourier derivation

Prove that we can derive from one representation into another.

Let’s derive the sine-cosine form from exponential:

Fourier series, exponential form

$$s_n(x) = \sum_{n=-N}^{N} C_n e^{i2\pi \frac{n}{P}x}$$

Fourier series, sine-cosine form

$$s_n(x) = A_0 + \sum_{n=1}^{N}(A_n \cos(2\pi\frac{n}{P}x) + B_n \sin(2\pi\frac{n}{P}x))$$

we now that

$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$
$\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$

$$\begin{aligned} s_n(x) = \sum_{n=-N}^{N} C_n \frac{2i}{2i}(e^{i2\pi\frac{n}{P}x} - \frac{1}{2}e^{-i2\pi\frac{n}{P}x} + \frac{1}{2}e^{-i2\pi\frac{n}{P}x}) = \\ = \sum_{n=-N}^{N} C_n i(\frac{e^{i2\pi\frac{n}{P}x} - e^{-i2\pi\frac{n}{P}x}}{2i} + \frac{e^{i2\pi\frac{n}{P}x} + e^{-i2\pi\frac{n}{P}x}}{2i}) = \\ = \sum_{n=-N}^{N} C_n i(\sin 2\pi\frac{n}{P}x + \frac{\cos 2\pi\frac{n}{P}x}{i}) = \\ = \sum_{n=-N}^{N} C_n(i\sin 2\pi\frac{n}{P}x + \cos 2\pi\frac{n}{P}x) \end{aligned}$$

- if n=0:

  $C_n(i\sin 2\pi\frac{0}{P}x + \cos 2\pi\frac{0}{P}x) = C_0$

- if n from -N to -1:
  $\sum_{n=-N}^{-1} C_n(i\sin 2\pi\frac{n}{P}x+\cos 2\pi\frac{n}{P}x) =$
  $\sum_{n=1}^{N} C_{-n}(i\sin 2\pi\frac{-n}{P}x+\cos 2\pi\frac{-n}{P}x) =$
  $\sum_{n=1}^{N} C_{-n}(-i\sin 2\pi\frac{n}{P}x+\cos 2\pi\frac{n}{P}x)$

Then, we can devive the sum into three components: when n from -N to -1, when n=0 and when n from 1 to N

$$s_n(x) = \sum_{n=-N}^{-1}(C_n(i\sin 2\pi\frac{n}{P}x+\cos 2\pi\frac{n}{P}x)) + C_0 + \sum_{n=1}^{N}(C_n(i\sin 2\pi\frac{n}{P}x+\cos 2\pi\frac{n}{P}x)) =$$
$$= \sum_{n=1}^{N} C_{-n}(-i\sin 2\pi\frac{n}{P}x+\cos 2\pi\frac{n}{P}x) + C_n + \sum_{n=1}^{N}(C_n(i\sin 2\pi\frac{n}{P}x+\cos 2\pi\frac{n}{P}x)) =$$
$$= C_0 + \sum_{n=1}^{N}((C_n - C_{-n})i\sin 2\pi\frac{n}{P}x + (C_n + C_{-n})\cos 2\pi\frac{n}{P}x)$$

Let's assume $\begin{cases} C_0 = A_0 \\ A_n = (C_n - C_{-n})i \\ B_n = C_n + C_{-n} \end{cases}$

then,

$$s_n(x) = A_0 + \sum_{n=1}^{N}(A_n\cos(2\pi\frac{n}{P}x) + B_n\sin(2\pi\frac{n}{P}x))$$