Continuous Distributions

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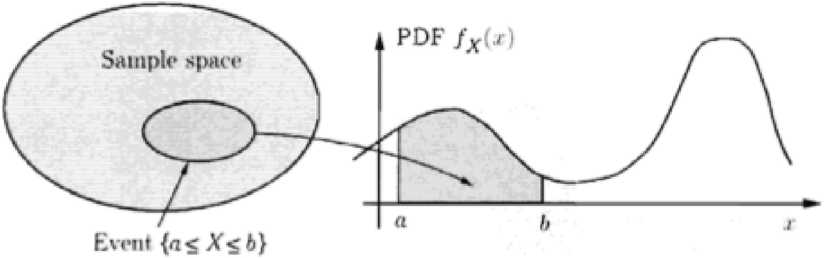
Continuous Random Variables

Continuous Random Variables are random variables whose set of possible values is uncountable (interval of real numbers).

- the time that a train arrives at a specified stop
- the lifetime of a transistor

Probability density function (pdf)



- The probability that X takes a value in an interval $[a, b]$ is $\int_a^b f(x)dx$, which is the shaded area in the figure.
- This integral is the area under the density function over this interval [a,b], and it can be loosely interpreted as the sum of all the values over this interval.
- A continuous probability model assigns probabilities as areas under a density curve. The area under the curve and above any range of values is the probability of an outcome in that range.

Probability density function (pdf)

For a continuous random variable X , a probability density function is a function such that

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = \int_a^b f(x)dx =$ area under f(x) from a to b for any a and b
- also for any value a $P(X = a) = \int_a^a f(x)dx = 0$ (only for continuous r.v.), instead represent x as $a \leq x \leq b$
- if X is a continuous random variable, for any a and b

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Example

The probability density function of the length of a cutting blade is $f(x) = 1.25$ for $74.6 < x < 75.4$ millimeters. Determine the following:

- $P(X < 74.8) = \int_{-\infty}^{\infty} f(x)dx = \int_{74.6}^{74.8} 1.25dx = (1.25x)\Big|_{74.6}^{74.8} =$ $1.25(74.8) - 1.25(74.6) = 0.25$
- If the specifications for this process are from 74.8 to 75.4 millimeters, what proportion of blades meets specifications?

$$P(74.8 < X < 75.4) = \int_{74.8}^{75.4} 1.25dx = (1.25x)\Big|_{74.8}^{75.4} = (1.25)(75.4) - (1.25)(74.8) = 0.75$$

$$P(74.8 < X < 75.4) = 1 - P(X < 74.8) = 1 - 0.25 = 0.75$$

Example

Suppose that X is a continuous random variable with probability density function $f(x) = C(4x - 2x^2)$ for $0 < x < 2$ and $f(x) = 0$ otherwise.

- What is the value of C?
- Find $P(X > 1)$?

Cumulative Distribution Function (cdf)

The cdf of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

The cumulative distribution function is defined for all real numbers. The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating. The fundamental theorem of calculus states that

$$\frac{d}{dx}\int_{-\infty}^{x} f(y)dy = f(x)$$

Then, given F(x),

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

Example

The cumulative density function F(x) of the length of a cutting blade (recall $f(x) = 1.25$ for $74.6 < x < 75.4$) consists of three expressions. If $x < 74.6$, f(x)=0, therefore

$$F(x) = 0, \quad for \quad x < 74.6$$

and by definition of cdf

$$F(x) = \int_{-\infty}^{x} f(y)dy) = \int_{74.6}^{x} f(y)dy = (1.25x)\Big|_{74.6}^{x} = 1.25x{-}93.25 \quad for \quad 74.6 < x < 75.4$$

Finally,

$$F(x) = \int_{-\infty}^{x} f(y)dy) = \int_{74.6}^{x} f(y)dy = 1 \quad for \quad 75.4 \leq x$$

Why need cdf?
Now we can calculate $P(X < 74.8) = F(74.8) = 1.25(74.8) - 93.25 = 0.25$.

Expected Value

Suppose that X is a continuous random variable with probability density function f(x). The mean or expected value of X, denoted as E(X), is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

for any real-valued function g,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

for any constants a and b

$$E(aX + b) = aE(X) + b$$

Variance

The variance of a continuous random variable X with probability density function f(x), denoted as Var(X) is

$$Var(X) = E([X - E(X)]^2) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx =$$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - (E(X))^2 = E(X^2) - (E(X))^2$$

for any constants a and b

$$Var(aX + b) = a^2 Var(X)$$

Example

The expected value of the length of a cutting blade (recall $f(x) = 1.25$ for $74.6 < x < 75.4$) is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{74.6}^{75.4} 1.25xdx = \frac{1.25x^2}{2}\Bigg|_{74.6}^{75.4} =$$

$$= \frac{1.25(75.4)^2}{2} - \frac{1.25(74.6)^2}{2} = 75$$

To find the variance of the length of a cutting blade we need to find $E(X^2)$ first

$$E(X^2) = \int_{74.6}^{75.4} 1.25x^2dx = \frac{1.25x^3}{3}\Bigg|_{74.6}^{75.4} = \frac{1.25(75.4)^3}{3} - \frac{1.25(74.6)^3}{3} = 5625.053$$

and the variance is

$$Var(X) = E(X^2) - (E(X))^2 = 5625.053 - (75)^2 = 0.053$$

Example

X is a continuous random variable with the density function $f(x) = 2x$ if $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Find the following

- cdf $F(x)$
- expected values $E(X)$, $E(X^2)$
- expected value of the function $g(X) = 3X + 5, E(g(X))$
- variance $Var(X)$
- variance of the function $g(X) = 3X + 5, Var(g(X))$