Exponential distribution and Poisson process

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Exponential Distribution

In practice, the exponential distribution often arises as the distribution of the amount of time until some specific event occurs.

- the amount of time (starting from now) until an earthquake occurs
- until a new war breaks out
- waiting time between phone calls

Exponential Distribution

A continuous random variable X whose probability density function is given, for some $\lambda > 0$

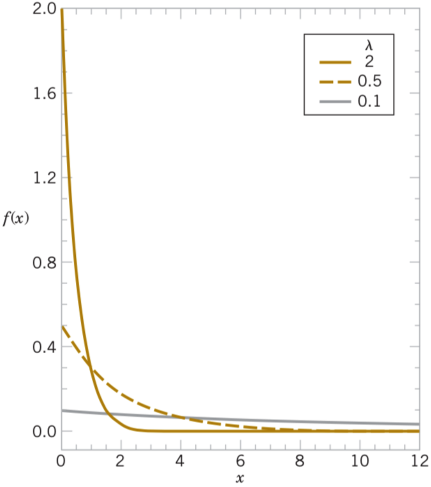
$$f(x) = \lambda e^{-\lambda x}, \quad 0 \leq x < \infty$$

and $f(x) = 0$ otherwise, is said to be an exponential random variable with rate λ.
For $X \sim Exp(\lambda)$: $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.
The cumulative distribution function of an exponential random variable is obtained by integration. If $0 \leq x < \infty$,

$$F(x) = \int_0^\infty \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x}$$

and $F(x) = 0, x < 0$.

Probability density function of exponential random variables



Example

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public 10 telephone booth, find the probability that you will have to wait more than 10 minutes Let X denote the length of the call made by the person in the booth.

- wait more than 10 minutes

$$P(X > 10) = 1 - P(X < 10) = 1 - F(10) = 1 - (1 - e^{-1/10 \cdot 10}) = .368$$

- the probability that you will wait between 10 and 20 minutes

$$P(10 < X < 20) = F(20) - F(10) = e^{-1} - e^{-2} = .233$$

The lack of memory property

We say that a nonnegative random variable X is memoryless if for all $s, t \geq 0$

$$P(X > s + t | X > t) = P(X > s)$$

The equation is equivalent to

$$\frac{P(X > s + t \cap X > t)}{P(X > t)} = P(X > s)$$

or

$$P(X > s + t) = P(X > s)P(X > t)$$

Think of X as being the lifetime of some instrument. The memoryless property says that the probability that the instrument survives for at least $s + t$ hours, given that it has survived t hours, is the same as the initial probability that it survives for at least s hours. The instrument does not 'remember' that it has already been in use for a time t.

Example

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery? What can be said when the distribution is not exponential?

- if the lifetime distribution is Exponential with $\lambda = 1/10$:

$$P(\text{remaining time} > 5) = 1 - F(5) = e^{-1/2} = .604$$

- If the lifetime distribution not Exponential, let's denote by t the number of miles that the battery had been in use prior to the start of the trip, then

$$P(\text{lifetime} > t + 5 | \text{lifetime} > t) = \frac{1 - F(t+5)}{1 - F(t)}$$

where $F(t)$ is the cdf for the corresponding distribution.

Example

The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.

- What is the probability that you do not receive a message during a two-hour period?
- If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?
- What is the expected time between your fifth and sixth messages?

Poisson process

A Poisson process is a sequence of arrivals occurring at different points on a timeline, such that the number of arrivals in a particular interval of time has a Poisson distribution.

A process of arrivals in continuous time is called a Poisson process with rate λ if the following two conditions hold:

- The number of arrivals that occur in an interval of length t is a $\textit{Poisson}(\lambda t)$ random variable.
- The numbers of arrivals that occur in disjoint intervals are independent of each other. For example, the numbers of arrivals in the intervals $(0, 10)$ and $[10, 12)$ are independent.

Poisson process

Suppose we can model the number of calls arriving during an t-minute time window with a Poisson distribution. Assume that the calls arrive completely at random in time during the t-minutes

- Let the expected number of calls during a 1-minute interval be $\lambda = 2$ (a rate)
- For example, the expected number of calls in 2-minutes is 4 calls
- Then the expected number of calls in t-minutes is $\lambda t = 2t$ calls

Let N (r.v.) denote the number of calls in an t-minute time interval in a Poisson process with a rate parameter of λ events per minute. Then,

$$N \sim Poisson(\lambda t)$$

Exponential distribution and Poisson process

How long you have to wait for an event depends on how often events occur. N is the number of calls in an t-minute time interval

$$N \sim Poisson(\lambda t)$$

Let X be the wait time (continuous) until the first call.

$$P(X > t) = P(\text{you wait at least t minutes for first call}) =$$

$$= P(\text{there were no calls in the first t minutes}) = P(N = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-\lambda t}$$

is the cumulative distribution function of X.

Wait time in a Poisson process is modeled with the Exponential distribution.

Example

The time between arrivals of small aircraft at a county airport is exponentially distributed with a mean of one hour.

- What is the probability that more than two aircraft arrive within an hour?
- If 5 separate one-hour intervals are chosen, what is the probability that no interval contains more than two arrivals?
- Determine the length of an interval of time (in hours) such that the probability that no arrivals occur during the interval is 0.10.