Gamma and other continuous distributions

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March 4/6, 2020

Gamma function

The gamma function, shown by $\Gamma(x)$ is an extension of the factorial function to real (and complex) numbers. Specifically, if $n \in \{1, 2, 3, ...\}$, then

$$\Gamma(n) = (n-1)!$$

More generally, for any positive real number α, $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \alpha > 0$$

Properties of the gamma function:

- $$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$
- $$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$
- $$\Gamma(1/2) = \sqrt{\pi}$$

Gamma function

- Find
$$\Gamma(3/2)$$
- Find
$$\int_0^\infty x^6 e^{-5x}\, dx$$

Gamma distribution

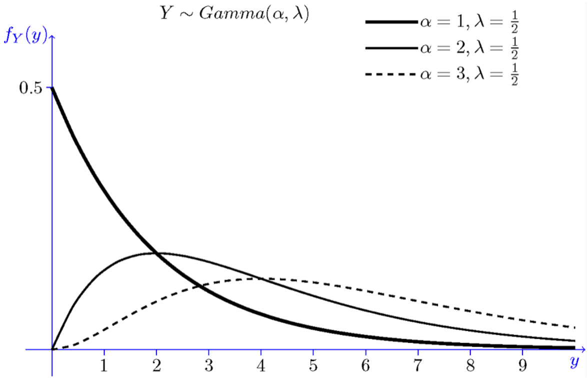
A continuous random variable X is said to have a gamma distribution with parameters (α, λ), $\alpha > 0, \lambda > 0$, if its density function is given by

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad 0 < x < \infty$$

$$E(x) = \frac{\alpha}{\lambda}, Var(x) = \frac{\alpha}{\lambda^2}$$

If we let $\alpha = 1$, $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, so $Gamma(1, \lambda) = Exponential(\lambda)$
The sum of n independent $Exponential(\lambda)$ random variables is a $Gamma(n, \lambda)$ random variable.

Probability density function of gamma random variables



The parameter α is known as the shape parameter, since it most influences the peakedness of the distribution. The parameter λ is called the scale parameter, since most of its influence is on spread of the distribution.

Gamma distribution. Example

The total service time of a multistep manufacturing operation has a gamma distribution with mean 18 minutes and standard deviation 6.

- Determine λ and α.
- Assume that each step has the same distribution for service time. What distribution for each step and how many steps produce this gamma distribution of total service time?

Erlang and Chi-squared distributions

- An exponential random variable describes the time (length) until the first count is obtained in a Poisson process. A generalization of the exponential distribution is the time until α events occur in a Poisson process. if X is the time until the αth event in a Poisson process, then

$$P(X > t) = \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

  A continuous random variable X is said to have an Erlang distribution with parameters (α, λ), $\alpha > 0, \lambda > 0$, if its density function is given by

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad 0 < x < \infty$$

  Important: α is an integer! Otherwise, the distribution is gamma.
- the Chi-squared distribution is a special case of the gamma distribution in which $\lambda = 1/2$ and $\alpha = 1/2, 1, 3/2, 2, \ldots$

Erlang distribution. Example

Patients arrive at a hospital emergency department according to a Poisson process with a mean of 6.5 per hour.

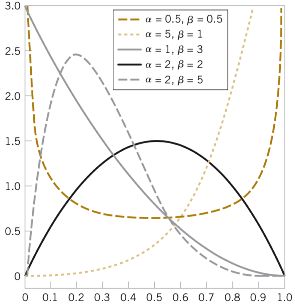
- What is the mean time until the 10th arrival?

Beta distribution

A continuous random variable X is said to have a beta distribution with parameters (α, β), $\alpha > 0, \beta > 0$, if its density function is given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1$$

$$E(x) = \frac{\alpha}{\alpha+\beta}, Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$



Beta distribution. Example

Consider the completion time of a large commercial development. The proportion of the maximum allowed time to complete a task is modeled as a beta random variable with $\alpha = 2.5$ and $\beta = 1$. What is the probability that the proportion of the maximum time exceeds 0.7? Suppose that X denotes the proportion of the maximum time required to complete the task. The probability is

$$P(X > 0.7) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(3.5)}{\Gamma(2.5)\Gamma(1)} \int_0^1 x^{1.5}dx = 0.59$$

Weibull distribution

A continuous random variable X is said to have a Weibull distribution with parameters (α, β), $\alpha > 0, \beta > 0$, if its density function is given by

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]}, \quad x > 0$$

cdf:

$$F(x) = 1 - e^{-\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]}, \quad E(X) = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)$$

Consider an object consisting of many parts, and suppose that the object experiences death (failure) when any of its parts fail. It has been shown (both theoretically and empirically) that under these conditions a Weibull distribution provides a close approximation to the distribution of the lifetime of the item.

Weibull distribution. Example

The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily modeled as a Weibull random variable with $\alpha = 1/2$ and $\beta = 5000$ hours. Determine the mean time E(X) until failure. Determine the probability that a bearing lasts at least 6000 hours: $P(X > 6000)$.

Lognormal distribution

The lifetime of a product that degrades over time is often modeled by a lognormal random variable. For example, this is a common distribution for the lifetime of a semiconductor laser.

Let X have a normal distribution with mean μ and variance σ^2; then $Y = e^X$ is a lognormal random variable.

Example: The lifetime (in hours) of a semiconductor laser has a lognormal distribution with mean 10 and standard deviation 1.5. What is the probability that the lifetime exceeds 10,000 hours?

$$P(Y > 10,000) = 1 - P(e^X \leq 10,000) = 1 - P(X \leq ln(10,000)) =$$

$$= 1 - \Phi\left(\frac{ln(10,000) - 10}{1.5}\right) = 1 - \Phi(-0.52) = 0.7$$