Conditional probability

Anastasiia Kim

January 29, 2020

Conditional probability

Conditional probability provides us with a way to reason about the outcome of an experiment, based on partial information

- In a word guessing game, the first letter of the word is a B. What is the likelihood that the second letter is an O?
- How likely is it that a person has a certain disease given that a medical test was negative?

How should we update our beliefs in light of the evidence we observe?

Definition

If A and B are events with $P(B) > 0$, then the conditional probability of A given B, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- A is the event whose uncertainty we want to update
- B is the evidence we observe (or want to treat as given)
- P(A) is the prior probability of A ('prior' means before updating based on the evidence)
- $P(A|B)$ the posterior probability of A ('posterior' means after updating based on the evidence)
- Note that there is no such event as $A|B$

Example

Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

- A $=$ {the first ball drawn is red} and B $=$ {the second ball drawn is red}
- Given the first ball selected is red, there are 7 remaining red balls and 4 white balls, so $P(B|A) = 7/11$
- $P(A) = 8/12 = 2/3$ is the probability to draw red ball out of 12 balls
- the desired probability is $P(A \cap B) = P(A)P(B|A) = (2/3)(7/11) = 14/33$

Of course, we could calculate this probability as $\binom{8}{2}/\binom{12}{2} = 14/33$

Consequences

- For any events A and B with positive probabilities

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

- *Multiplication rule*: for any events $A_1, ..., A_n$ with $P(A_1 \cap A_2 \cap ... \cap A_{n-1}) > 0$,

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_n|A_1 \cap ... \cap A_{n-1})$$

- For example,

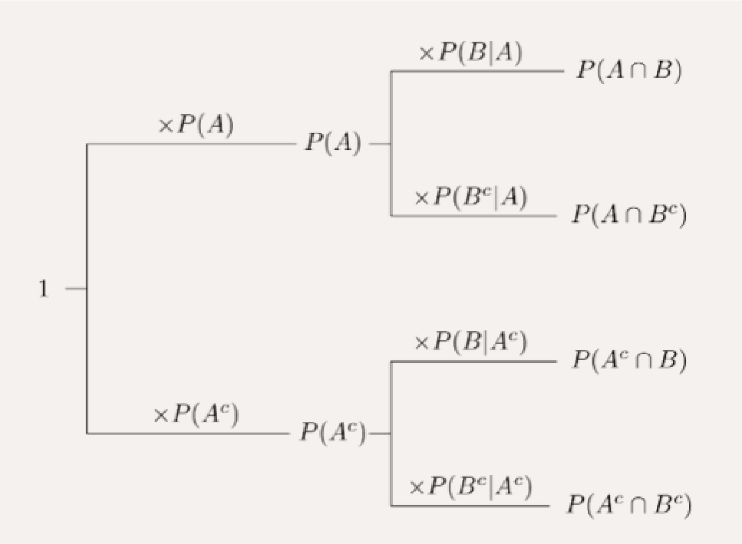
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) = P(A_2)P(A_3|A_2)P(A_1|A_2 \cap A_3)$$

Example

If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

- A $=$ $\{$an aircraft is present$\}$ and B $=$ $\{$the radar generates an alarm$\}$
- Define A^c and B^c
- Find P(not present, false alarm) and P(present, no detection)

Illustration



Bayes' theorem

- Bayes' rule
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$
- If the odds of an event A are $odds(A) = P(A)/P(A^c)$, then the odds form of Bayes' rule is
$$\frac{P(A|B)}{P(A^c|B)} = \frac{P(A)}{P(A^c)} \cdot \frac{P(B|A)}{P(B|A^c)}$$
  - In words, this says that the posterior odds are equal to the prior odds times the factor $P(B|A)/P(B|A^c)$, which is known in statistics as the likelihood ratio.

The law of total probability

- The law of total probability relates conditional probability to unconditional probability
- Conditional probability can be used to decompose complicated probability problems into simpler pieces, and it is often used in tandem with Bayes' rule

Let $A_1, ..., A_n$ are disjoint events of the sample space S, with $P(A_i) > 0$ for all i. Then for an event B

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

For example, for $i = 2$, the law of total probability is

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

Example

A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. We know that

- the probability that the test result is positive (suggesting the person has the disease), given that the person does not have the disease, is 2%
- the probability that the test result is negative (suggesting the person does not have the disease), given that the person has the disease, is only 1%.

A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?

Independence

Let A be the event that it rains tomorrow, and suppose that P(A)$=1/3$. Also suppose that I toss a fair coin; let B be the event that it lands heads up, P(B)$=1/2$. What is $P(A|B)$?

Independence

$P(A|B) = P(A) = 1/3$

- The result of my coin toss does not have anything to do with tomorrow's weather
- No matter if B happens or not, the probability of A should not change. This is an example of two independent events
- Two events are independent if one does not convey any information about the other.

Definition

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

- $P(A|B) = P(A)$ if $P(B) \neq 0$
- n events $A_1, ..., A_n$ are independent if

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2)...P(A_n)$$

$$P(A_1 \cup A_2 \cup ... \cup A_n) = 1 - (1 - P(A_1))(1 - P(A_2))...(1 - P(A_n))$$

- If A and B are independent then
  - A and B^c are independent
  - A^c and B are independent
  - A^c and B^c are independent

Example

Suppose that the probability of being killed in a single flight is $p_c = \frac{1}{4 \cdot 10^6}$ based on available statistics. Assume that different flights are independent. If a businessman takes 20 flights per year, what is the probability that he is killed in a plane crash within the next 20 years?

- The total number of flights that he will take during the next 20 years N $=$ (20)(20)=400
- if $p_s = 1 - p_c$ is the probability that he survives a single flight
- $P(\text{he survives N flights}) = p_s^N = (1 - p_c)^N$. The probability that the businessman will be killed in a plane crash within the next 20 years is

$$1 - P(\text{he survives 400 flights}) = 1 - (1 - p_c)^N = 9.9995 \cdot 10^{-5} \approx \frac{1}{10000}$$

Don't confuse *independence* and *being disjoint*

Concept	Meaning	Formulas
Disjoint	A and B cannot occur at the same time	$A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
Independent	A does not give any information about B	$P(A\|B) = P(A), P(B\|A) = P(B)$ $P(A \cap B) = P(A)P(B)$

Consider two events A and B, with $P(A) \neq 0$ and $P(B) \neq 0$. If A and B are disjoint, then they are not independent.

- Proof: since A and B are disjoint, we have

$$P(A \cap B) = 0 \neq P(A)P(B)$$

Thus, A and B are not independent.

Conditional Independence

Two events A and B are conditionally independent given an event C with $P(C) > 0$ if

$$P(A \cap B|C) = P(A|C)P(B|C)$$

A box contains two coins: a regular coin and one fake two-headed coin (P(H)=1). I choose a coin at random and toss it twice. Define the following events.

- A = {First coin toss results in an H}
- B = {Second coin toss results in an H}
- C = {Coin 1 (regular) has been selected}

Find $P(A|C), P(B|C), P(A \cap B|C), P(A), P(B), P(A \cap B)$. Note that A and B are not independent, but they are conditionally independent given C.