Methods of Point Estimation. Method of Moments

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Motivation

- Practical situation: we know that random data is drawn from a parametric model (distribution), whose parameters we do not know.

  For example, in an election between two candidates, data will be drawn from a *Bernoulli*(p) distribution with unknown parameter p. Use the data to estimate the value of the parameter p, as p predicts the result of the election.
- *Before...* If the distribution (model) and its parameters are known then we can calculate the probability of data.
- *Now with Stat. Inference...* Estimate the probability of parameters given a parametric model and observed data drawn from it.

  I know that data follows the Normal distributions but don't know the values of the parameters μ and σ^2. However, data from a random sample is available to draw inference about μ and σ^2.

Methods of Point Estimation

- How to estimate a parameter?
- Estimating a parameter with its sample analogue is usually reasonable
- Still need a more methodical way of estimating parameters
- Method of Moments (MOM) is the oldest method of finding point estimators
- MOM is simple and often doesn't give best estimates
- Method of maximum likelihood (ML or MLE)
- MLEs have better efficiency properties than MOM estimates. But moment estimators are sometimes easier to compute
- Both ML and MOM can produce unbiased point estimators
- Bayesian Estimation of Parameters: prior information + sample results

Method of Moments

Idea: equate the first k population moments, which are defined in terms of expected values, to the corresponding k sample moments. Solve the system of equations.

Let $X_1, X_2, ..., X_n$ be a random sample from the probability distribution (discrete or continuous). The *kth* population moment (or distribution moment) is $E(X^k), k = 1, 2, ...$ The corresponding *kth* sample moment is

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k, \quad k = 1, 2, ...$$

Example: the first population moment is $E(X) = \mu$, and the first sample moment $(k = 1)$ is $\bar{X}$. Thus, by equating the population and sample moments, we find that $\hat{\mu} = \bar{X}$. The sample mean is the moment estimator of the population mean.

Exponential Distribution Moment Estimator

Let $X_1, X_2, ..., X_n$ be a random sample from the *Exponential*(λ) distribution. The question: which exponential distribution?!

- need to estimate one parameter λ, so $k = 1$
- MOM: equate $E(X) = \bar{X}$ (population mean = sample mean)

$$E(X) = 1/\lambda = \bar{X}$$

$$\bar{X} = \frac{1}{\lambda}$$

$$\hat{\lambda} = \frac{1}{\bar{X}}$$

is the moment estimator λ.

Suppose that the time to failure of an electronic module is exponentially distributed. Eight units are randomly selected and tested, resulting in the following failure time (in hours): $11.96, 5.03, 67.40, 16.07, 31.50, 7.73, 11.10, 22.38$. The moment estimate of λ is

$$\hat{\lambda} = \frac{1}{\bar{x}} = \frac{1}{21.65} = 0.0462$$

Normal Distribution Moment Estimators

Let $X_1, X_2, ..., X_n$ be a random sample from the $Normal(\mu, \sigma^2)$ distribution. For the normal distribution, $E(X) = \mu$ and $E(X^2) = \mu^2 + \sigma^2$.

- need to estimate two parameters, so $k = 2$
- MOM: equate

$$E(X) = \bar{X}, \qquad E(X^2) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\mu = \bar{X}, \qquad \mu^2 + \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Solve

$$\hat{\mu} = \bar{X}$$

and

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} X_i^2 - n(\sum_{i=1}^{n} X_i)^2}{n} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$$

Notice that the moment estimator of σ^2 is not an unbiased estimator.