Probability

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Naive definition in terms of relative frequency

An experiment, whose sample space is S, is repeatedly performed under exactly the same conditions.

- The probability of the event A is defined as the (limiting) proportion of time that A occurs:
$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$$
- where $n(A)$ is the number of times in the first n repetitions of the experiment that the event A occurs
- this naive definition assumes that all outcomes are equally likely

Naive definition is applicable

- when there is *symmetry* in the problem that makes outcomes equally likely. It is common to assume that a coin has a 50% chance of landing Heads when tossed, due to the physical symmetry of the coin
- when the outcomes are equally likely by *design*. For example, consider conducting a survey of n people in a population of N people
- when the naive definition serves as a useful *null model*. We assume that the naive definition applies just to see what predictions it would yield, and then we can compare observed data with predicted values to assess whether the hypothesis of equally likely outcomes is tenable.

History of probability

In 1654 the Flemish aristocrat Chevalier de Méré sent a letter to the mathematician Blaise Pascal:

- I used to bet even money that I would get at least one 6 in four rolls of a fair die. The probability of this is 4 times the probability of getting a 6 in a single die, i.e., $4/6 = 2/3$; clearly I had an advantage and indeed I was making money. Now I bet even money that within 24 rolls of two dice I get at least one double 6. This has the same advantage ($24/36 = 2/3$), but now I am losing money. Why?

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- de Méré's reasoning was faulty: if the number of rolls were 7 in the first game, the logic would give the nonsensical probability $7/6$.

History of probability

How to compute probabilities for de Méré's games?

- Game 1: there are 4 rolls and he wins with at least one 6
- Game 2: there are 24 rolls of two dice and he wins by at least one pair of 6's rolled.

The birthday problem

What is the probability that, in a group of k people, at least two of them will have been born on the same day of the year?

Assumptions

- Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29)
- Assume that people's birthdays are independent: knowing some people's birthdays gives us no information about other people's birthdays. This would not hold if, e.g., we knew that two of the people were twins

The birthday problem

Insight: a group of only 23 people is large enough to have about a $50-50$ chance of at least one coincidental birthday!

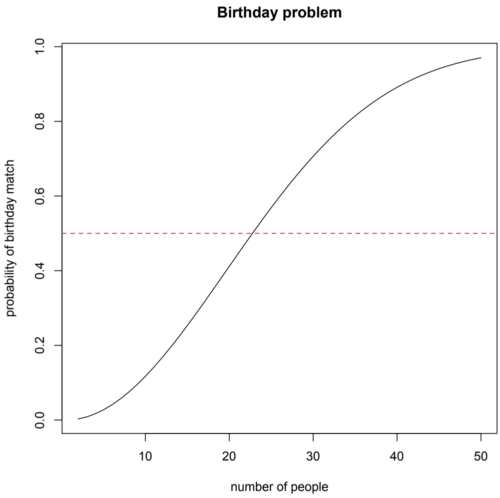
Solution

- There are 365^k ways to assign birthdays to the people in the room, since we can imagine the 365 days of the year being sampled k times, with replacement.
- The number of ways to assign birthdays to k people such that no two people share a birthday is $356 \cdot 364 \cdot 363 ...(365-k+1)$

$$P(\text{no birthday match}) = \frac{365 \cdot 364 \cdot 363 ...(365-k+1)}{365^k}$$

$$P(\text{at least 1 birthday match}) = 1 - \frac{365 \cdot 364 \cdot 363 ...(365-k+1)}{365^k}$$

The birthday problem



Axioms of probability (Kolmogorov's axioms)

For any event A and sample space S

- $0 \leq P(A) \leq 1$
- $P(S) = 1$
- For any sequence of mutually exclusive (disjoint) events A_1 , A_2 , ... (that is, events for which $A_i \cap A_j = \emptyset$ when $i \neq j$),

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i)$$

Propositions

Many useful properties can be derived from these axioms. For example,

- $P(A^c) = 1 - P(A)$
- $P(\emptyset) = 0$
- $P(A \backslash B) = P(A) - P(A \cap B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If $A \subset B$, then $P(A) \leq P(B)$
- For any events $A_1, \ldots, A_n$,

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} P(A_1 \cap \ldots \cap A_n)$$

Examples

In a presidential election, there are three candidates. Call them A, B, and C. Based on our polling analysis, we estimate that A has a 20 percent chance of winning the election, while B has a 40 percent chance of winning. What is the probability that A or B win the election?

- P(A wins or B wins) = P(A wins) + P(B wins) = 0.6

A retail establishment accepts either the American Express or the VISA credit card. A total of 24% of its customers carry an American Express card, 61% carry a VISA card, and 11% carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

- P(Amex or Visa) = P(Amex) + P(Visa) - P(Amex and Visa) = 0.24 + 0.61 - 0.11 = 0.74

de Montmort's (1713), matching problem

Consider a well-shuffled deck of n cards, labeled 1 through n. You flip over the cards one by one, saying the numbers 1 through n as you do so. You win the game if, at some point, the number you say aloud is the same as the number on the card being flipped over (for example, if the 10th card in the deck has the label 10). What is the probability of winning?

- Let A_i be the event that the ith card in the deck has the number i written on it
- Find $P(A_1 \cup A_2 ... \cup A_n)$: as long as at least one of the cards has a number matching its position in the deck, you will win the game
- How to solve? Use inclusion-exclusion formula and the fact that the Taylor series for $1/e$ is

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...$$

Example

Suppose we have the following information:

- There is a 60% chance that it will rain today.
- There is a 50% chance that it will rain tomorrow.
- There is a 30% chance that it does not rain either day.

Find the following probabilities:

- a. The probability that it will rain today or tomorrow.
- b. The probability that it will rain today and tomorrow.
- c. The probability that it will rain today but not tomorrow.
- d. The probability that it either will rain today or tomorrow, but not both.