Counting

Anastasiia Kim

January 24, 2020

General Information (updated)

Instructor (MWF 9.00-9.50 am): Anastasiia Kim
Email: anastasiiakim@unm.edu
Office Hours: W 2.30 - 4 pm, F 2 - 3.30 pm or by appointment, SMLC 319

Tutors: Jared DiDomenico, Md Rashidul Hasan
Emails: jdidomen@unm.edu, mdhasan@unm.edu
Recitation/Tutoring Hours: MTWR 4.30 pm - 5.30 pm at DSH 326

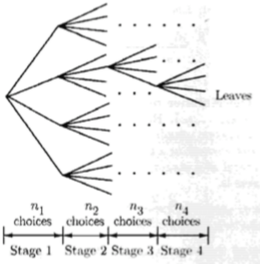
Multiplication rule (fundamental theorem of counting)

If an experiment can be described as a sequence of k stages, and

- the number of ways of completing stage 1 is n_1, and
- the number of ways of completing stage 2 is n_2 for each way of completing stage 1, and
- the number of ways completing stage 3 is n_3 for each way of completing stage 2, and so forth.

The total number of possible results of the k-stage experiment is

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$



Example 1

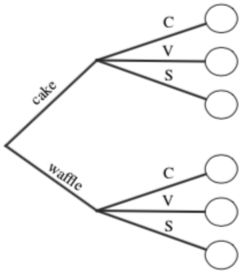


Figure 1: Tree diagram for choosing an ice cream cone. You can choose whether to have a cake cone or a waffle cone, and whether to have chocolate, vanilla, or strawberry as your flavor.

Example 2

A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How distinct telephone numbers are there?

- Visualize the choice of a subset as a sequential process: select one digit at a time

Example 2

A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How distinct telephone numbers are there?

- Visualize the choice of a subset as a sequential process: select one digit at a time
- There are 7 stages and we can choose one out of 10 elements at each stage, except for the first stage
- We have $10 - 2 = 8$ choices for the first stage

Using the multiplication rule, the answer is $8 \cdot 10 \cdot 10 \ldots \cdot 10 = 8 \cdot 10^6$.

Example 3

- How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers? 175,760,000
- How many license plates would be possible if repetition among letters or numbers were prohibited? 78,624,000

Example 4

Roll a die 3 times. What is the probability that you get *different* numbers?

- Identify the set of equally likely outcomes
- Compute the total number of outcomes and the number of good outcomes
- Compute the probability as #of good outcomes/total # of outcomes

Permutations

A *permutation* of the elements is an ordered sequence of the elements. The number of permutations of n different elements is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

- For example, there are $n!$ ways in which n people can line up for ice cream.
- How many different ordered arrangements of the letters a, b, and c are possible?
- 3! (3 letters). By direct enumeration we see that there are 6: abc, acb, bac, bca, cab, and cba. Each arrangement is a permutation. There are 6 possible permutations of a set of 3 objects.

Permutations of subsets

The number of ordered sequences (permutations) of subsets of r elements selected from a set of n different elements is

$$p_r^n = n \times (n-1) \times (n-2) \times ... \times (n-r+1) = \frac{n!}{(n-r)!}$$

If $r = n$, there are $n!$ permutations.

Exercise: count the the number of words that consist of four distinct letters. Answer: 358,800.

Permutations of similar objects. Multinomial coefficient

The number of permutations of $n = n_1 + n_2 + ... + n_r$ elements selected of which n_1 are of one type, n_2 are of a second type, ... , and n_r are of an rth type is

$$\frac{n!}{n_1!n_2!n_3!...n_r!}$$

- A hospital operating room needs to schedule 3 knee surgeries and 2 hip surgeries in a day. What is the number of possible sequences of these surgeries?
- How many different letter arrangements can be formed using the letters *STATISTICS*?

Ordered sampling with replacement

- Select an item at random r times from a collection of n distinct items, replacing the selected item each time before the next selection.
- The total number of ways of choosing r items from a set of n items when ordering matters and repetition is allowed

$$n \times n \times \ldots \times n = n^r$$

- Imagine a jar with 3 balls, labeled from 1 to 3. Sample balls one at a time with replacement, meaning that each time a ball is chosen, it is returned to the jar.
- Each sampled ball is a sub-experiment (stage) with 3 possible outcomes, and there are 2 sub-experiments. Thus, by the multiplication rule there are $3^2 = 9$ ways to obtain a sample of size 2.

Combinations: unordered sampling without replacement

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

- This is the same as the problem of counting the number of k-element subsets of a given n-element set.
- But forming a combination is different than forming a k-permutation in a combination there is *no ordering* of the selected elements.

For example, whereas the 2-permutations of the letters A, B, and C are
AB, BA, AC, CA, BC, CB
the combinations (no duplicates!) of two out of these three letters are
AB, AC, BC

Combinations

- $n \times (n-1) \times (n-2) \times ... \times (n-r+1) = \frac{n!}{(n-r)!}$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant
- each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is $\binom{n}{r}$

$$\binom{n}{r} = \frac{n(n-1)...(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

We call $C_r^n = \binom{n}{r}$ *n choose r* or *binomial coefficient*
Exercise: A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Example

A bin of 30 parts contains 3 defective parts and 27 nondefective parts. What is the probability of getting exactly 2 defective parts in a sample of size 5 if the sampling is done without replacement (repetition not allowed)?

- Calculate the number of ways we can choose 2 defective parts from the 3 defective parts
- Calculate the number of ways to select the remaining $5 - 2 = 3$ nondefective parts
- Calculate the total number of different subsets of size 5
- Using the multiplication rule, calculate the probability

History of probability

In 1654 the Flemish aristocrat Chevalier de Méré sent a letter to the mathematician Blaise Pascal:

- I used to bet even money that I would get at least one 6 in four rolls of a fair die. The probability of this is 4 times the probability of getting a 6 in a single die, i.e., $4/6 = 2/3$; clearly I had an advantage and indeed I was making money. Now I bet even money that within 24 rolls of two dice I get at least one double 6. This has the same advantage $(24/62 = 2/3)$, but now I am losing money. Why?

History of probability

In 1654 the Flemish aristocrat Chevalier de Méré sent a letter to the mathematician Blaise Pascal:

- I used to bet even money that I would get at least one 6 in four rolls of a fair die. The probability of this is 4 times the probability of getting a 6 in a single die, i.e., $4/6 = 2/3$; clearly I had an advantage and indeed I was making money. Now I bet even money that within 24 rolls of two dice I get at least one double 6. This has the same advantage $(24/62 = 2/3)$, but now I am losing money. Why?
- de Méré's reasoning was faulty: if the number of rolls were 7 in the first game, the logic would give the nonsensical probability $7/6$.

History of probability

How to compute probabilities for de Méré's games?

- Game 1: there are 4 rolls and he wins with at least one 6
- Game 2: there are 24 rolls of two dice and he wins by at least one pair of 6's rolled.

The birthday problem

What is the probability that, in a group of k people, at least two of them will have been born on the same day of the year?

Assumptions

- Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29)
- Assume that people's birthdays are independent: knowing some people's birthdays gives us no information about other people's birthdays. This would not hold if, e.g., we knew that two of the people were twins

The birthday problem

Insight: a group of only 23 people is large enough to have about a $50-50$ chance of at least one coincidental birthday!

Solution

- There are 365^k ways to assign birthdays to the people in the room, since we can imagine the 365 days of the year being sampled k times, with replacement.
- The number of ways to assign birthdays to k people such that no two people share a birthday is $356 \cdot 364 \cdot 363 ... (365-k+1)$

$$P(\text{no birthday match}) = \frac{365 \cdot 364 \cdot 363 ... (365-k+1)}{365^k}$$

$$P(\text{at least 1 birthday match}) = 1 - \frac{365 \cdot 364 \cdot 363 ... (365-k+1)}{365^k}$$

The birthday problem