Homework 6

Stat 345 - Spring 2020

Name: ______________________________

Problem 1

We load on a plane 100 packages whose weights are independent random variables that are uniformly distributed between 5 and 50 pounds. What is the probability that the total weight will exceed 3000 pounds? It is not easy to calculate the cdf of the total weight and the desired probability, but an approximate answer can be quickly obtained using the central limit theorem.

Problem 2

Let $X_1, X_2, ..., X_n$ be i.i.d. Gamma random variables with parameters α and λ. The likelihood function is difficult to differentiate because of the gamma function. Rather than finding the maximum likelihood estimators, what are the method of moments estimators of both parameters α and λ?

Problem 3

Let $X_1, X_2, ..., X_n$ be i.i.d. Geometric(θ), $\theta = 1, 2, 3, ...$ random variables.

a) Find the maximum likelihood estimator of θ.

The nice property of maximum likelihood estimators is the *invariance property*. If $\hat{\theta}$ is the MLE of θ, then $g(\hat{\theta})$ is the MLE of $g(\theta)$. For example, if I have a random sample from a Bernoulli distribution $X_i \sim Bernoulli(\theta), i = 1, ..., n$ and I need to find the MLE of the function of θ, say $g(\theta) = (1 - \theta^2)$, then I can first find the MLE of θ which is $\hat{\theta} = \bar{X}$, and by the invariance property the MLE of $g(\theta)$ is $g(\hat{\theta}) = 1 - \hat{\theta}^2 = 1 - \bar{X}^2$.

b) In a certain hard video game a player is confronted with a series of AI opponents and has an θ probability of defeating each one. Success with any opponent is independent of previous encounters. Until first win, the player continues to AI contest opponents. Let X denote the number of opponents contested until the player's first win. Suppose that data of 10 players was collected:

$$7, 4, 3, 1, 12, 10, 2, 1, 4, 6$$

What is the MLE of the probability that a player contests five or more AI opponents in a game until the first win?

Problem 4

Let $X_1, X_2, ..., X_n$ be a random sample from a uniform distribution on the interval $(0, a)$. Recall that the maximum likelihood estimator (MLE) of a is $\hat{a} = max(X_i)$.

a) Let $Y = max(X_i)$. Use the fact that $Y \leq y$ if and only if each $X_i \leq y$ to derive the cumulative distribution function of Y.

b) Find the probability density function of Y from cdf.

c) Use the obtained pdf to show that MLE for a ($\hat{a} = max(X_i)$) is biased.

d) Say I would like to consider another estimator for a, I will call it $\hat{b} = 2\bar{X}$. Is it unbiased estimator of a (show)? How you can explain someone without calculations why $\hat{b} = 2\bar{X}$ is a reasonable estimator of a?

e) Based on the result in (c), I will propose to use unbiased estimator for a instead of $\hat{a} = max(X_i)$, say $\hat{c} = \frac{n}{n+1} max(X_i)$. Given that the relative efficiency of any two unbiased estimators $\hat{b}, \hat{c}$ is the ratio of their variances

$$\frac{Var(\hat{b})}{Var(\hat{c})},$$

explain which of these two unbiased estimators is more efficient. You can obtain the $Var(\hat{c}) = Var(\frac{n}{n+1} max(X_i))$ from $Var(\hat{a}) = Var(Y)$. The variance of the $Y = max(X_i)$ is

$$Var(Y) = \frac{n}{(n+1)^2(n+2)} a^2$$