Forecast reconciliation: Geometry, optimization and beyond

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Outline

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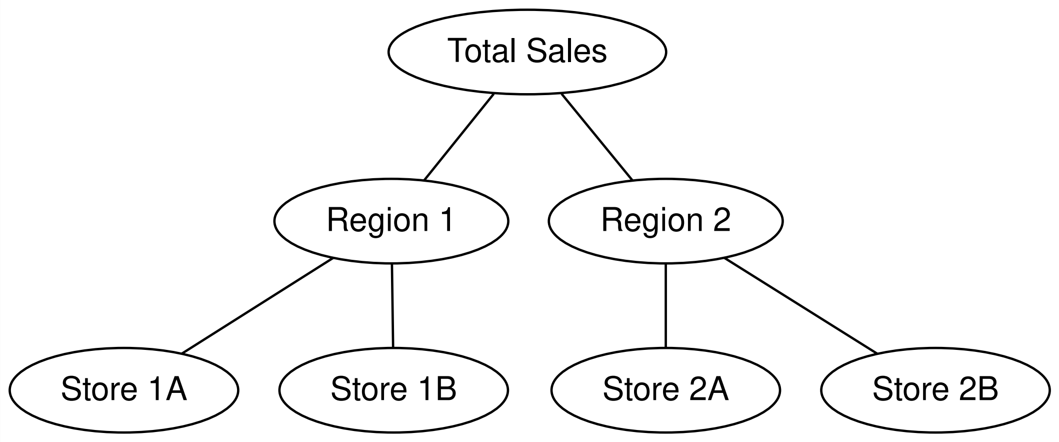
- Most commonly arise due to an **aggregation** structure, hence the name 'hierarchical'.
- Need not be hierarchical, alternative structures are grouped (or crossed) aggregation, or temporal aggregation.

Hierarchy


Total Sales
Region 1
Region 2
Store 1A
Store 1B
Store 2A
Store 2B

One representation

- For the simple hierarchy shown earlier:

$$\begin{pmatrix} y_{\text{Tot}} \\ y_1 \\ y_2 \\ y_{1A} \\ y_{1B} \\ y_{2A} \\ y_{2B} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} y_{1A} \\ y_{1B} \\ y_{2A} \\ y_{2B} \end{pmatrix}$$

$$\mathbf{y} = \mathbf{S} \times \mathbf{b}$$

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- This talk is about **two-stage** processes whereby incoherent **base** forecasts are adjusted to be coherent.
- Note there is also work on **end-to-end** forecasting (e.g. Rangapuram et al. 2021).

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$$\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{y}}$$

- This is called OLS reconciliation.

An optimization lens

- Ensure that $\hat{\mathbf{y}}$ and $\tilde{\mathbf{y}}$ are 'close'.

$$\tilde{\mathbf{y}} = \underset{\mathbf{y}}{argmin} ||\mathbf{y} - \hat{\mathbf{y}}||_2$$

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- Also has a game theoretic interpretation (see van Erven and Cugliari 2015)

Generalizations

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- Setting $\mathbf{W}^{-1}$ to the covariance matrix of $\mathbf{y} - \hat{\mathbf{y}}$ optimizes expected squared error loss.
- This is the well-known **MinT method** of Wickramasuriya, Athanasopoulos, and Hyndman (2019).

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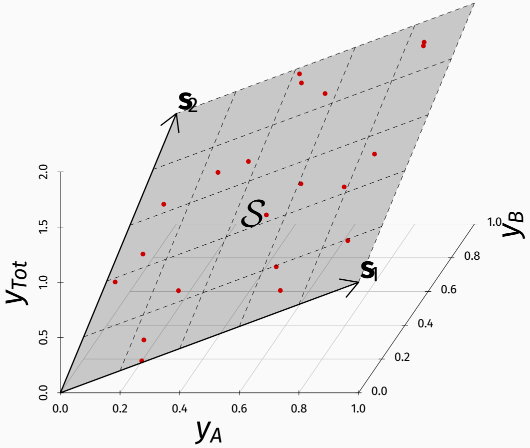
  - For discrete data $\mathcal{S}$ is a set of points (Zhang, Panagiotelis, and Kang 2023).
  - Non linear constraints will be covered later.
- The simplest three-variable hierarchy $y_{\mathrm{Tot}} = y_A + y_B$ for real-valued data is depicted on the next slide.

Coherent subspace



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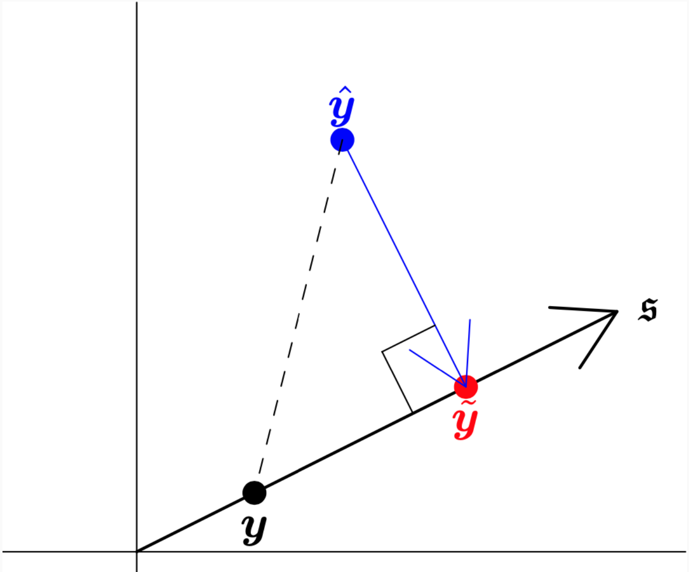
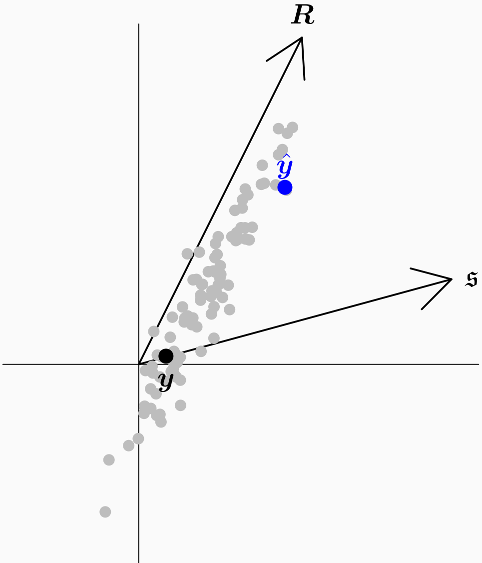
- These maps can be more general than projections.
- However projections have some nice properties
  - OLS always reduces forecast error.
  - MinT minimizes forecast error in expectation.

OLS Reconciliation



Why MinT?



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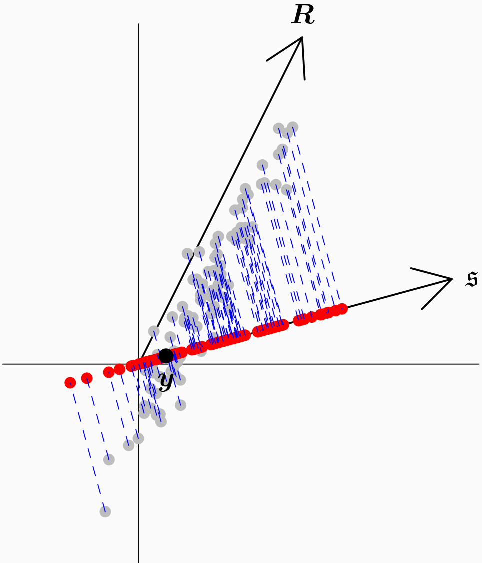

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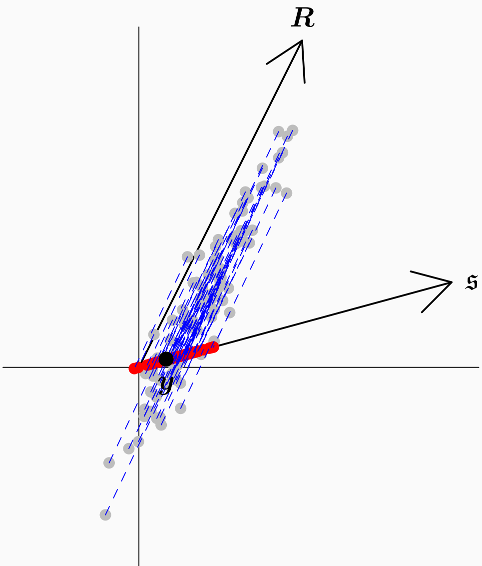

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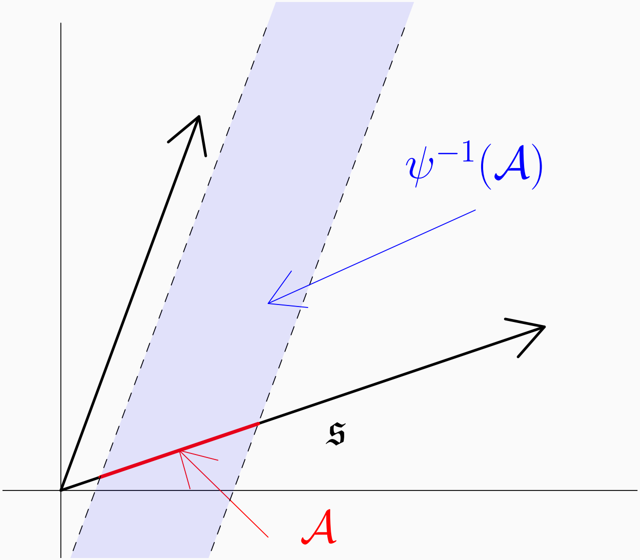
- Later formalized reconciliation as a **pushforward** (Panagiotelis et al. 2023).

Probabilistic Reconciliation



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$$\tilde{\mu}(\mathcal{A}) = \hat{\mu}(\psi^{-1}(\mathcal{A}))$$

- $\tilde{\mu}$ is the **pushforward** of $\hat{\mu}$ by ψ, denoted as $\psi\#\hat{\mu}$

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- What is the optimal ψ?
- In the point forecasting world the ψ given by MinT is optimal for squared loss.
- What does optimality even mean for distributional forecasts?

Scoring rules

- A scoring rule $S : \mathcal{P} \times \mathbb{R}^n \to \mathbb{R}$ takes a distributional forecast from a family $\mathcal{P}$ and an observation and assigns a *score* that measures forecast quality.

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- In the context of probabilistic forecasting, it has been proven that choosing ψ to be the same projection as MinT is optimal for log score when probabilistic forecasts are Gaussian (Wickramasuriya 2023)
- In other cases we can optimize using a data driven approach.

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$$\underset{\boldsymbol{\theta}}{argmin} \sum_t S(\psi_{\boldsymbol{\theta}} \# \hat{\mu}_t, \mathbf{y}_t)$$

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- Scoring rules that have been used include log score, energy score and variogram score.
- Paired forecasts and observations can be obtained using rolling or expanding window schemes.
- Different optimal values of $\boldsymbol{\theta}$ can be obtained for different forecast horizons.
- Often draw a sample from $\hat{\mu}$ rather than work with the distribution itself.
- Optimization by first order methods (e.g. SGD).

Energy Generation Example

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- Reconcile using projections (OLS, MinT) and also by optimising Energy score.

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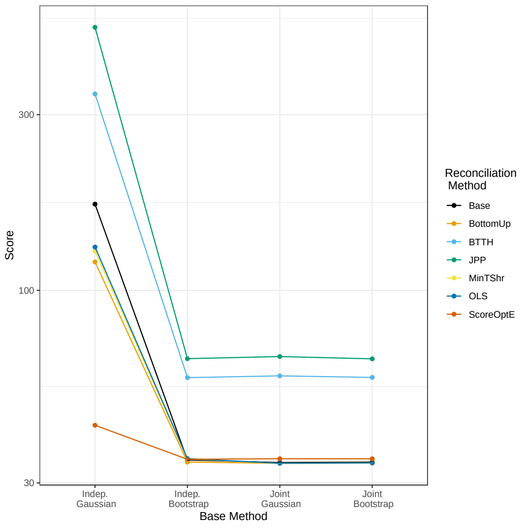

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- Interplay between reconciliation and model misspecification.
- Reconciliation via score optimization most effective when base models heavily misspecified.
- For reasonably well specified models:
  - Projections are robust.
  - Gains over base forecasts are not as big.
- Reconciliation gives forecaster a ‘second chance’.

Outline

Pinball loss

- Many forecasting problems involve optimizing pinball loss.

$$L_\alpha(y, q) = \alpha(y_i - q)I(y_i \geq q) + (1 - \alpha)(q - y_i)I(y_i < q)$$

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- Here, I(.) equals 1 when the statement in parentheses is true, 0 otherwise.
- Quantiles minimize expected pinball loss $E_Y\left[L_\alpha(y, q)\right]$

In reconciliation

- To target quantiles we optimize.

$$\underset{\theta}{argmin} \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_{\text{train}}} L_{\alpha}(y_{i,t}, \tilde{q}_{i,t})$$

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- Subject to the constraints

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Note $\tilde{y}_{i,t}$ depend on θ

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- It is also complicated by the need to approximate expectations with sample equivalents.

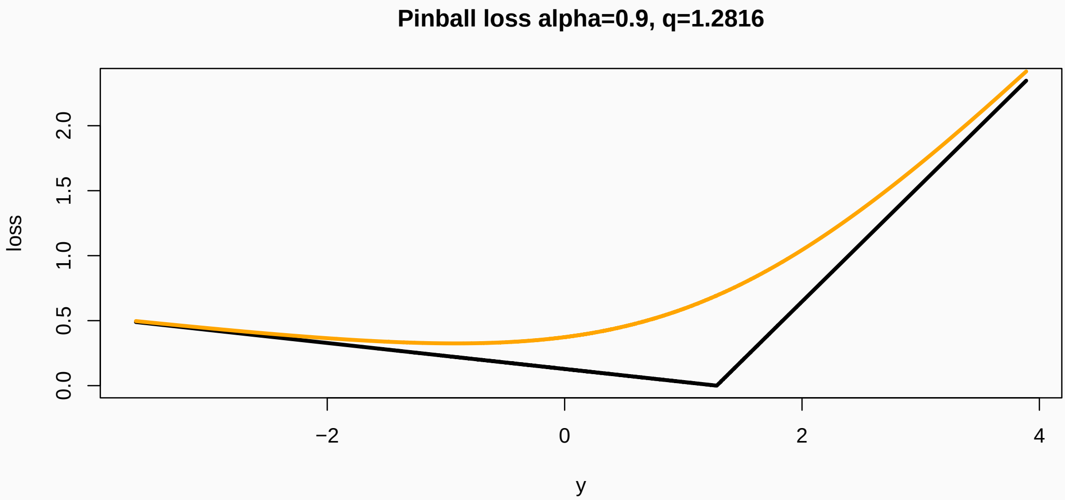
Smooth pinball loss

Use approximation converging to pinball loss as $\beta \to \infty$

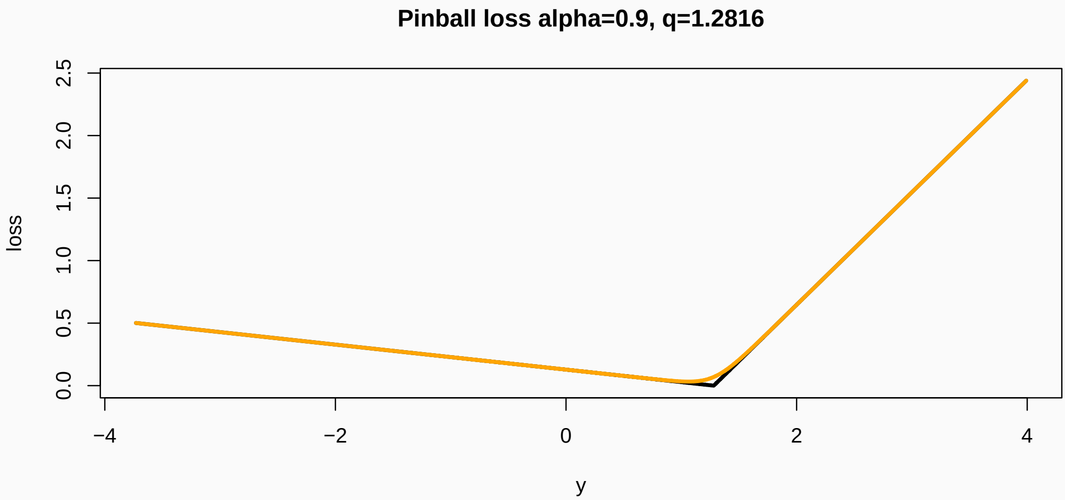
$$L_\alpha^\beta(y, q) = \frac{1}{\beta} \log \left(e^{\beta\alpha(y-q)} + e^{\beta(1-\alpha)(q-y)} \right)$$

Unlike the pinball function it is smooth, meaning we can use first order methods (like Stochastic Gradient Descent).

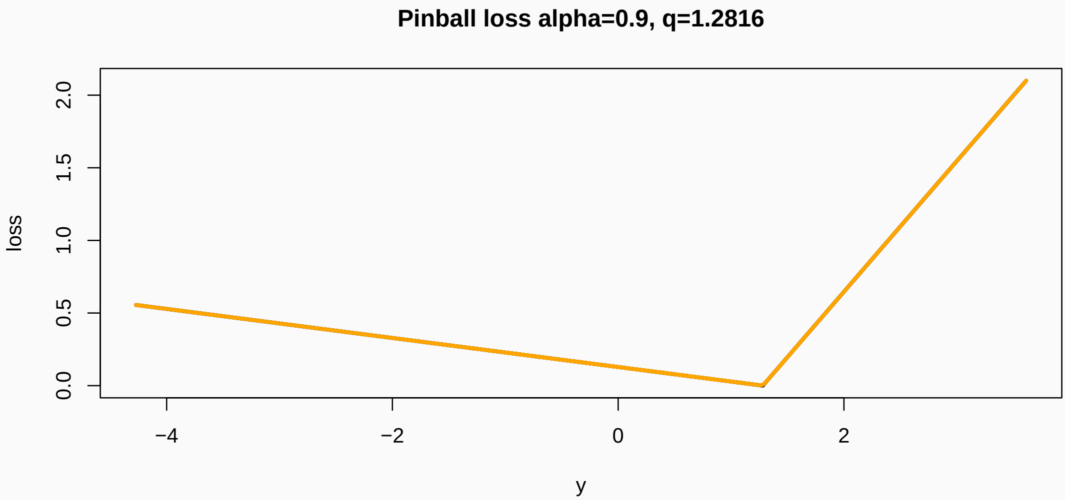
Smoothed pinball loss (β = 1)



Smoothed pinball loss (β = 10)



Smoothed pinball loss (β = 100)



What we want to solve

$$\underset{\theta}{argmin} \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_{\text{train}}} L_\alpha(y_{i,t}, \tilde{q}_{i,t})$$

- Subject to the constraints

$$\tilde{q}_{i,t} = \underset{q}{argmin} E_{\tilde{Y}_{i,t}} \left[L_\alpha(\tilde{y}_{i,t}, q) \right]$$

What we can solve

$$\underset{\theta}{argmin} \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_{\text{train}}} L_{\alpha}^{\beta}(y_{i,t}, \tilde{q}_{i,t})$$

- Subject to the constraints

$$\tilde{q}_{i,t|t-1} = \underset{q}{argmin} \sum_{j} L_{\alpha}^{\beta}(\tilde{y}_{i,t}^{(j)}, q)$$

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$$\sup_{\theta \in \boldsymbol{\Theta}} \left| f^{\beta}(\theta) - f(\theta) \right| \rightarrow 0 \text{ as } \beta \rightarrow \infty$$

$$\sup_{\theta \in \boldsymbol{\Theta}} \left| f^{(J)}(\theta) - f^{\beta}(\theta) \right| \rightarrow 0 \text{ as } J \rightarrow \infty.$$

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  - Bounded second moment of the stochastic gradient.

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- Train on 10 years (120 observations), evaluation on 7 years (84 observations).

Pinball Loss - Out of Sample (Normal errors)

	Quantile Level			
Method	0.05	0.2	0.8	0.95
Base	32*	85*	101	46
OLS	32*	84*	104	51
WLS	**31***	**82***	112	65
MinT	**31***	**82***	111	65
QOpt	35*	85*	100*	**41***

Bold denotes best performing method, asterisk(*) denotes inclusion in model confidence set (Hansen et. al., 2011).

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$$\tilde{\mathbf{y}} = \underset{\mathbf{y}}{argmin}(\mathbf{y} - \hat{\mathbf{y}})'\mathbf{W}(\mathbf{y} - \hat{\mathbf{y}})$$

- Subject to $\mathbf{y} \in \mathcal{S}$.

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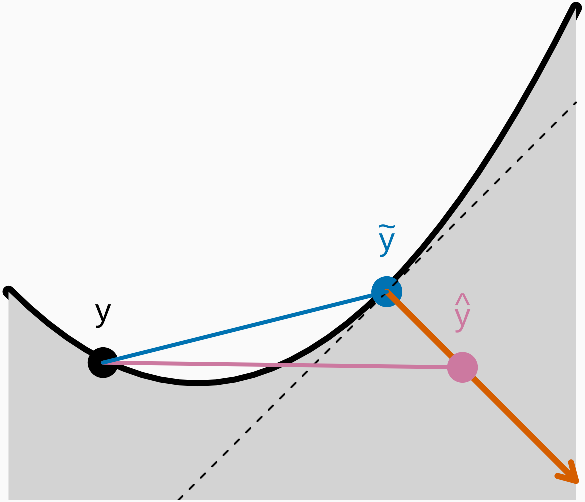
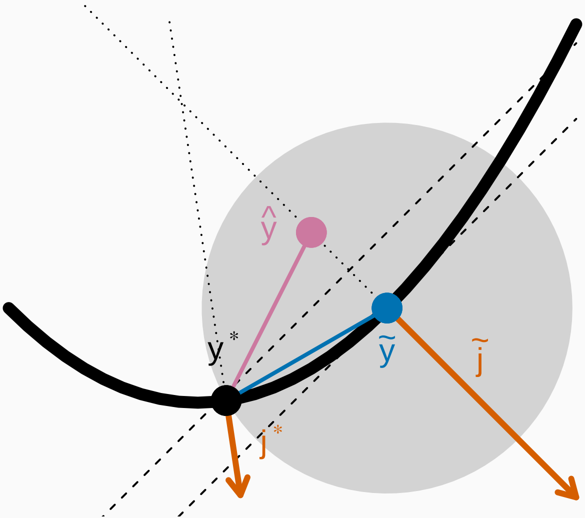
  - Find closest point on the coherent manifold equidistant from the base and reconciled forecast.
  - This defines a ball in which reconciliation always outperforms base forecasts.

Convex Function: Hypograph



Any function



Radius of the Ball

- The radius of the ball on the previous slide is given by

$$r = \sqrt{\boldsymbol{\kappa}'\mathbf{J}^{*\prime}\mathbf{J}^{*}\boldsymbol{\kappa} + \mu\boldsymbol{\kappa}'\mathbf{J}^{*\prime}\tilde{\mathbf{J}}\boldsymbol{\lambda} + \frac{\mu^2}{4}\boldsymbol{\lambda}\tilde{\mathbf{J}}'\tilde{\mathbf{J}}\boldsymbol{\lambda}}$$

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- λ and $\boldsymbol{\kappa}$ are Lagrange multipliers associated with certain optimization problems.

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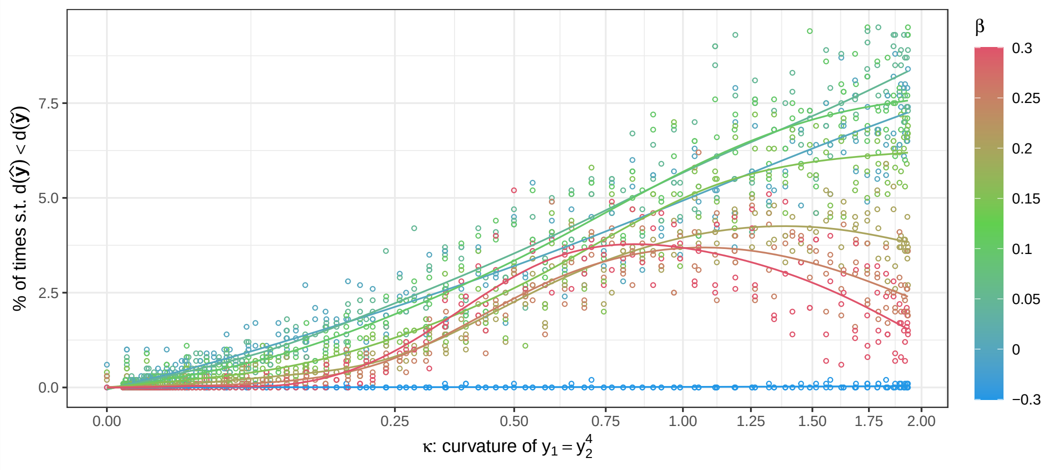
  - When reconciled forecast is in a high probability region of the true DGP.
  - When some constraints are convex and the bast forecast is more likely to lie in the hypographs of these constraints.

Simulation results



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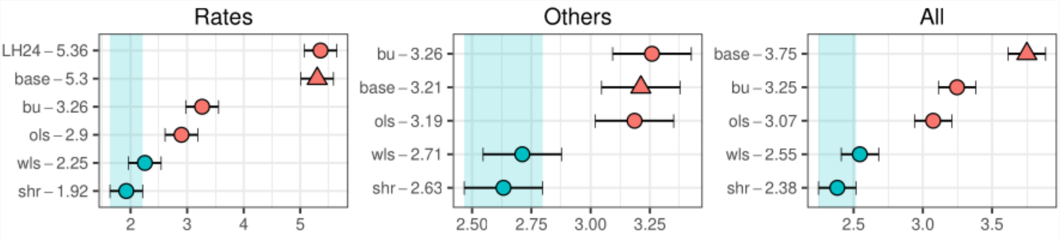
- However $M = D/E$ for each region.

Mortality Data



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- New work on Forecast Linear Augmented Projects (FLAP)

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- Reconcile using MinT.

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- It has been proven that the same is true when only looking at $\mathbf{y}$.
- It is this result that allows the benefits of reconciliation to be applied to problems where there are no constraints at all!
- We also prove that the forecast variance in non-increasing as more synthetic components are added.

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- The quality of covariance matrix estimates deteriorate with higher dimension.
- However for finite dimension, the benefit of FLAP outweighs errors in estimating covariance matrix.

Geometry of FLAP

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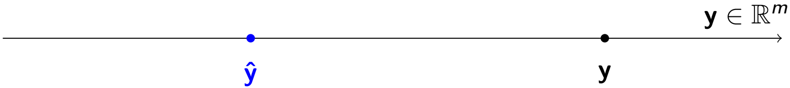

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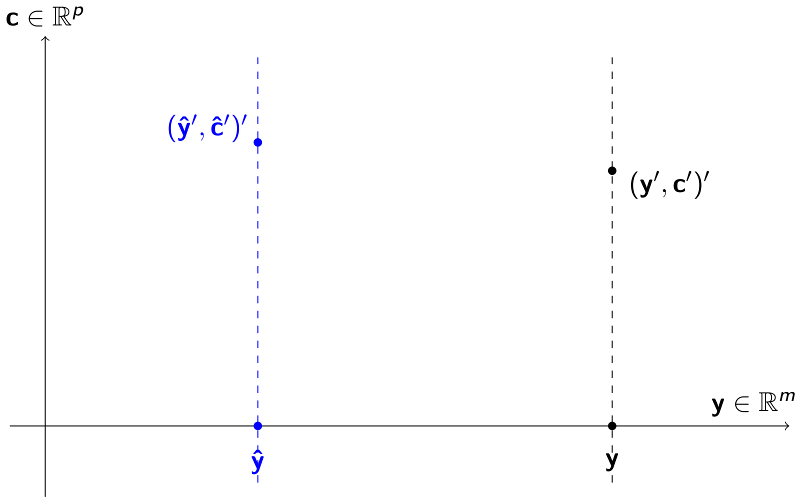

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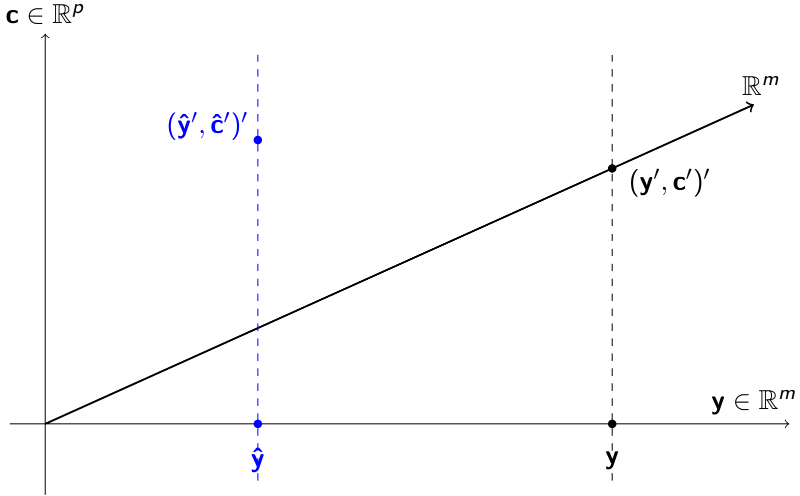

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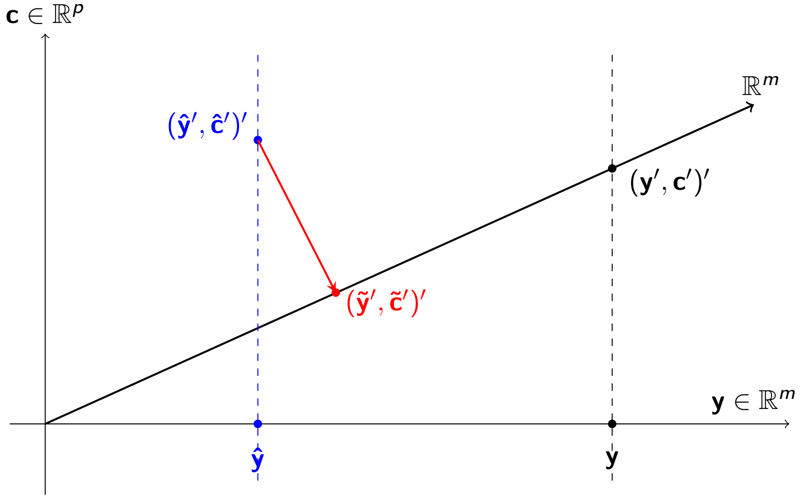

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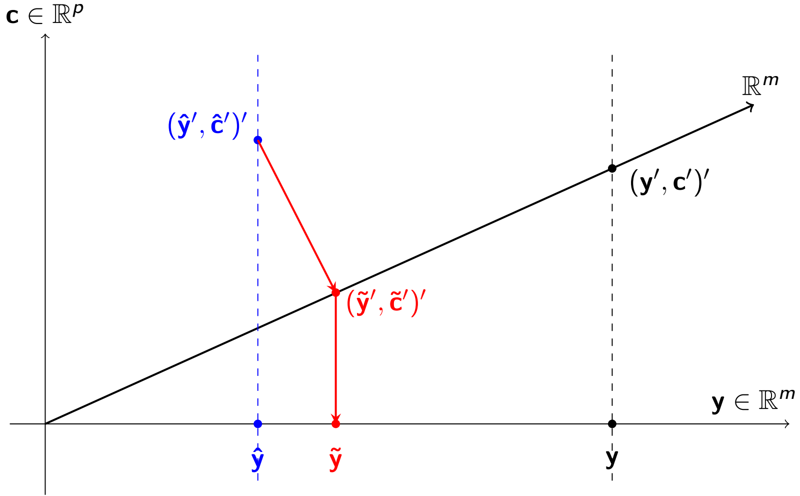

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- Expanding time series cross-validation with initial size of 25 years and forecast horizon 12 months.

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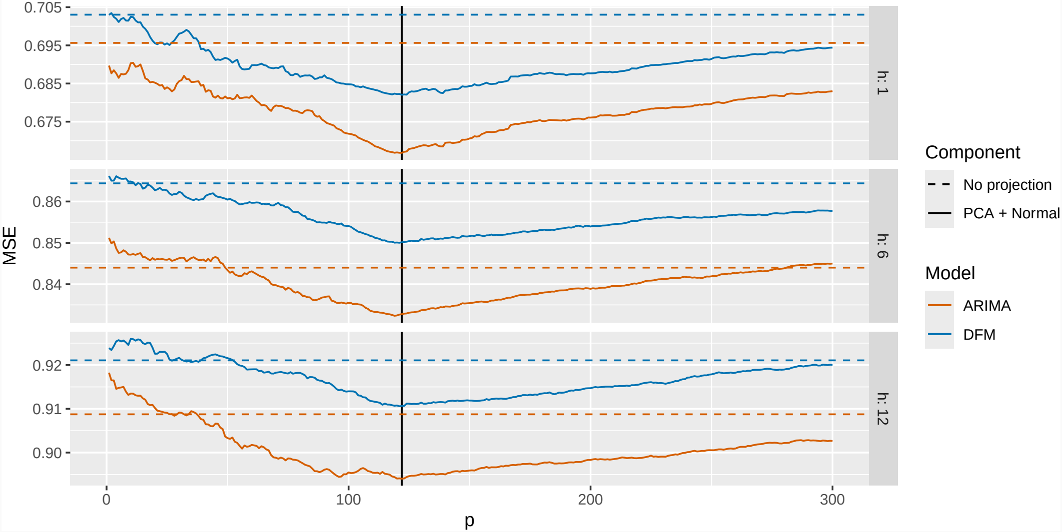
0.705
0.695
0.685
0.675
h: 1
0.86
0.85
0.84
h: 6
0.92
0.91
0.90
h: 12
MSE
0
100
200
300
p
Component
No projection
PCA + Normal
Model
ARIMA
DFM

Working Paper and R Package

YF Yang, G Athanasopoulos, RJ Hyndman, and A Panagiotelis (2024). “Forecast Linear Augmented Projection (FLAP): A free lunch to reduce forecast error variance”.

Department of Econometrics and Business Statistics, Monash University, Working Paper Series 13/24.

You can install the stable version from CRAN

```r
## CRAN.R-project.org/package=flap
install.packages("flap")
```

or the development version from Github

```r
## github.com/FinYang/flap
# install.packages("remotes")
remotes::install_github("FinYang/flap")
```

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The right people









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  - Other loss functions?
- Jump on the bandwagon!

Links

Postdoc opportunity

Link to slides