Co-scattering: thermal average

ABSTRACT: In this document I describe the thermal averaging procedure for co-scattering cross section, taking into account the relativistic nature of bath particles.

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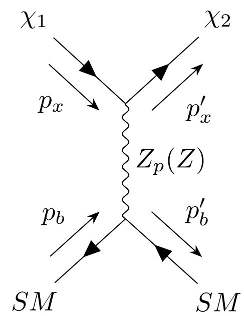
1 General expression

Let us consider the following scattering process



and assume that χ_1 is the dark matter candidate and that the bath particles are massless fermions: $m_b = 0$. The momentum distribution functions and number densities of the particles involved then read

$$\begin{aligned} f_\chi &= g_\chi e^{-E_\chi/T}, \qquad n_\chi = 4\pi g_\chi m_\chi^2 T K_2(m_\chi/T); \\ f_b &= \frac{g_b}{e^{E_b/T}+1}, \qquad n_b = 6\pi g_b \zeta(3) T^3, \end{aligned} \tag{1.1}$$

where $g_{\chi/b}$ are the internal degrees of freedom of the corresponding particles. The thermally averaged cross-section of the considered process can be expressed as[1, 2]

$$\langle \sigma v \rangle_{coscatt} = \int \frac{d^3p_\chi d^3p_b}{n_\chi n_b} \frac{p_\chi p_b}{E_\chi E_b} f_\chi f_b \sigma V_r \equiv \frac{1}{n_\chi n_b} \int d^3p_\chi d^3p_b \sigma v_{\text{møl}} f_\chi f_b, \tag{1.2}$$

where V_r is the relativistic relative velocity defined in [1] and $v_{\text{møl}}$ is the Møller velocity. The momentum integration in 1.2 boils down to the integration over the energies $E_{\chi/b}$ and the angle $\theta \equiv \theta_{\vec{p}_\chi \vec{p}_b}$. Similarly to [2, 3], we switch instead to the variables $E_\chi,\ E_b,\ s$[1] where $s = m_\chi^2 + 2E_b(E_\chi - p_\chi \cos\theta)$ is the Mandelstam variable,

$$\begin{aligned} d^3p_\chi d^3p_b = 4\pi p_\chi E_\chi dE_\chi 4\pi p_b E_b dE_b \frac{d\cos\theta}{2} &= 8\pi^2 p_\chi p_b E_\chi E_b dE_\chi dE_b ds \frac{1}{2p_\chi p_b} \\ &= 4\pi^2 E_\chi E_b dE_\chi dE_b ds. \end{aligned} \tag{1.3}$$

[1]Note however that we keep the integration over $E_{\chi/b}$, contrary to switcing to $E_\pm$ as in the references. The original reason of using $E_\pm$ is the sum of energies $E_\chi + E_b$ in the exponent that occurs if *both* particles follow the Boltzmann distribution.

The thermally-averaged cross section 1.2 then reads

$$\langle \sigma v \rangle_{coscatt} = g_\chi g_b \frac{4\pi^2}{n_\chi n_b} \int_{m_\chi}^{\infty} dE_\chi e^{-E_\chi/T} \int_0^\infty \frac{dE_b}{e^{E_b/T}+1} \int_{s(cos\theta=1)}^{s(cos\theta=-1)} ds \frac{s-m_\chi^2}{2}\sigma, \tag{1.4}$$

where we used the following kinematic relations:

$$v_{\text{møl}} \equiv \frac{\sqrt{(p_\chi \cdot p_b)^2 - m_\chi^2 m_b^2}}{E_\chi E_b} = \frac{s-m_\chi^2}{2E_\chi E_b} \tag{1.5}$$

$$E_b > 0, \quad E_\chi > m_\chi, \tag{1.6}$$

$$s(cos\theta = 1) \leq s \leq s(cos\theta = -1). \tag{1.7}$$

2 Cross section expansion

The cross section σ can be evaluated assuming that the following variables are small (see also Susanne's and Laura's notes):

$$\delta = \frac{m_{\chi_2} - m_{\chi_1}}{m_{\chi_1}}, \tag{2.1}$$

$$\omega = v_{rel} - 4\delta - 2\delta^2, \tag{2.2}$$

where the latter results from the requirement that $s \geq \sum m_{fin}^2 = m_{\chi_2}^2 = m_{\chi_1}^2(1+\delta)^2$ and the relative velocity is defined via $s = m_{\chi_1}^2(1+v_{rel}/2)$. Expanding σ to the fist order in δ and second order in ω (which correspond to the lowest-order contributions), we obtain [4]

$$\sigma = -\frac{c_\theta^2\delta^2(\delta+2)^2(2\delta-1)e^2\epsilon^2 g_\chi^2 m_\chi^2 s_\theta^2}{8\pi m_{Z_p}^4} - \frac{c_\theta^2(2\delta-1)e^2\epsilon^2 g_\chi^2 m_\chi^2 s_\theta^2 v_{rel}^2}{32\pi m_{Z_p}^4} \tag{2.3}$$

$$+ \frac{c_\theta^2\delta(\delta+2)(2\delta-1)e^2\epsilon^2 g_\chi^2 m_\chi^2 s_\theta^2 v_{rel}}{8\pi m_{Z_p}^4}. \tag{2.4}$$

Expressing v_{rel} via s, we find that $\sigma = \sigma_0 + \sigma_1 s + \sigma_2 s^2$, were σ_i do not depend on the theory parameters only,

$$\sigma = -\frac{c_\theta^2(2\delta-1)(\delta+1)^4 e^2\epsilon^2 g_\chi^2 m_\chi^2 s_\theta^2}{8\pi m_{Z_p}^4} + \frac{c_\theta^2(2\delta-1)(\delta+1)^2 e^2\epsilon^2 g_\chi^2 s s_\theta^2}{4\pi m_{Z_p}^4} - \frac{c_\theta^2(2\delta-1)e^2\epsilon^2 g_\chi^2 s^2 s_\theta^2}{8\pi m_\chi^2 m_{Z_p}^4}. \tag{2.5}$$

3 Final result

Plugging in 2.5 to 1.4, we obtain the analytical expression for the averaged co-scattering cross section for relativistic medium [5],

$$\langle \sigma v \rangle_{coscatt}(x) =$$

$$-\frac{c_\theta^2(2\delta-1)e^2\epsilon^2 g_\chi^2 \mathrm{MD1}^2 s_\theta^2 \left(\delta(\delta+2)x^3\left(45\delta(\delta+2)x\zeta(3)K_2(x) - 7\pi^4 K_3(x)\right)\right)}{360\pi g_b g_\chi m_{Z_p}^4 x^4 \zeta(3) K_2(x)} \tag{3.1}$$

$$+\frac{2700\zeta(5)\left(\left(x^2+24\right)K_2(x) + 6xK_1(x)\right)}{360\pi g_b g_\chi m_{Z_p}^4 x^4\zeta(3)K_2(x)} \tag{3.2}$$

References

[1] M. Cannoni. Relativistic $< \sigma v_{\mathrm{rel}} >$ in the calculation of relics abundances: a closer look. *Phys. Rev. D*, 89(10):103533, 2014.

[2] Joakim Edsjo and Paolo Gondolo. Neutralino relic density including coannihilations. *Phys. Rev. D*, 56:1879–1894, 1997.

[3] Paolo Gondolo and Graciela Gelmini. Cosmic abundances of stable particles: Improved analysis. *Nucl. Phys. B*, 360:145–179, 1991.

[4] Laura & Nastya. Mathematica notebook, Nastya/Scatterings-fin_Nastya.nb.

[5] Nastya. Mathematica notebook, Nastya/Coscattering_cross-section_Nastya_minimal.nb.