Problem 1

Absorption of radiation by a gas

A cylindrical vessel, with its axis vertical, contains a molecular gas at thermodynamic equilibrium. The upper base of the cylinder can be displaced freely and is made out of a glass plate; let's assume that there is no gas leakage and that the friction between glass plate and cylinder walls is just sufficient to damp oscillations but doesn't involve any significant loss of energy with respect to the other energies involved. Initially the gas temperature is equal to that of the surrounding environment. The gas can be considered as perfect within a good approximation. Let's assume that the cylinder walls (including the bases) have a very low thermal conductivity and capacity, and therefore the heat transfer between gas and environment is very slow, and can be neglected in the solution of this problem.

Through the glass plate we send into the cylinder the light emitted by a constant power laser; this radiation is easily transmitted by air and glass but is completely absorbed by the gas inside the vessel. By absorbing this radiation the molecules reach excited states, where they quickly emit infrared radiation returning in steps to the molecular ground state; this infrared radiation, however, is further absorbed by other molecules and is reflected by the vessel walls, including the glass plate. The energy absorbed from the laser is therefore transferred in a very short time into thermal movement (molecular chaos) and thereafter stays in the gas for a sufficiently long time.

We observe that the glass plate moves upwards; after a certain irradiation time we switch the laser off and we measure this displacement.

1. Using the data below and - if necessary - those on the sheet with physical constants, compute the temperature and the pressure of the gas after the irradiation. *[2 points]*
2. Compute the mechanical work carried out by the gas as a consequence of the radiation absorption. *[1 point]*
3. Compute the radiant energy absorbed during the irradiation. *[2 points]*
4. Compute the power emitted by the laser that is absorbed by the gas, and the corresponding number of photons (and thus of elementary absorption processes) per unit time. *[1.5 points]*
5. Compute the efficiency of the conversion process of optical energy into a change of mechanical potential energy of the glass plate. *[1 point]*

Thereafter the cylinder axis is slowly rotated by 90°, bringing it into a horizontal direction. The heat exchanges between gas and vessel can still be neglected.

6. State whether the pressure and/or the temperature of the gas change as a consequence of such a rotation, and - if that is the case – what is its/their new value. *[2.5 points]*

Data

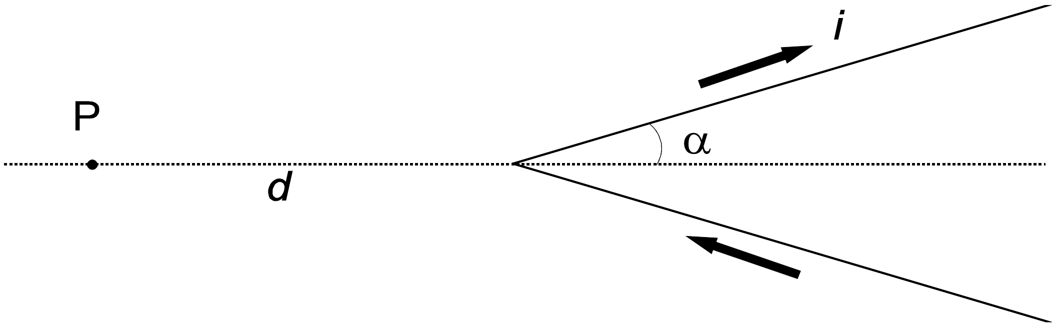
Room pressure: p_0 = 101.3 kPa
Room temperature: T_0 = 20.0°C
Inner diameter of the cylinder: 2r = 100 mm
Mass of the glass plate: m = 800 g
Quantity of gas within the vessel: n = 0.100 mol
Molar specific heat at constant volume of the gas: c_V = 20.8 J/(mol·K)
Emission wavelength of the laser: λ = 514 nm
Irradiation time: Δt = 10.0 s
Displacement of the movable plate after irradiation: Δs = 30.0 mm

Problem 2

Magnetic field with a V-shaped wire

Among the first successes of the interpretation by Ampère of magnetic phenomena, we have the computation of the magnetic field **B** *generated by wires carrying an electric current, as compared to early assumptions originally made by Biot and Savart.*

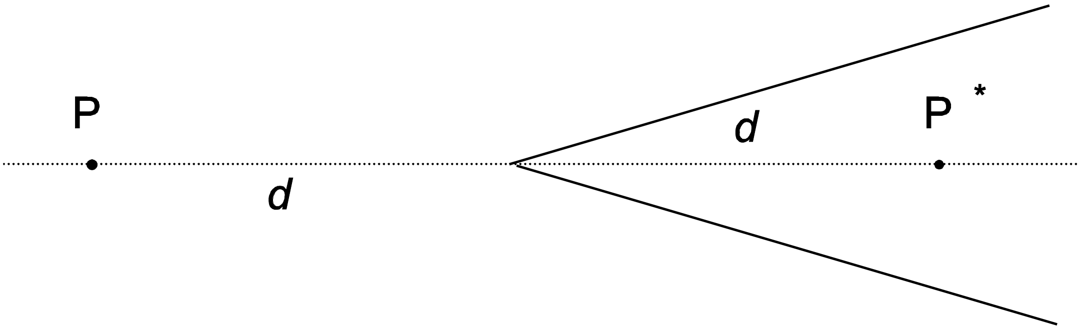
A particularly interesting case is that of a very long thin wire, carrying a constant current i, made out of two rectilinear sections and bent in the form of a "V", with angular half-span[1] α (see figure). According to Ampère's computations, the magnitude B of the magnetic field in a given point P lying on the axis of the "V", outside of it and at a distance d from its vertex, is proportional to $\tan\left(\frac{\alpha}{2}\right)$. Ampère's work was later embodied in Maxwell's electromagnetic theory, and is universally accepted.



Using our contemporary knowledge of electromagnetism,

1. Find the direction of the field **B** in P. *[1 point]*
2. Knowing that the field is proportional to $\tan\left(\frac{\alpha}{2}\right)$, find the proportionality factor k in $\left|\mathbf{B}(\mathrm{P})\right| = k\tan\left(\frac{\alpha}{2}\right)$. *[1.5 points]*
3. Compute the field **B** in a point P^* symmetric to P with respect to the vertex, *i.e.* along the axis and at the same distance d, but inside the "V" (see figure). *[2 points]*

[1] Throughout this problem α is expressed in radians



4. In order to measure the magnetic field, we place in P a small magnetic needle with moment of inertia I and magnetic dipole moment $\boldsymbol{\mu}$; it oscillates around a fixed point in a plane containing the direction of **B**. Compute the period of small oscillations of this needle as a function of B. *[2.5 points]*

In the same conditions Biot and Savart had instead assumed that the magnetic field in P might have been (we use here the modern notation) $B(\mathrm{P}) = \frac{i \mu_0 \alpha}{\pi^2 d}$, where μ_0 is the magnetic permeability of vacuum. In fact they attempted to decide with an experiment between the two interpretations (Ampère's and Biot and Savart's) by measuring the oscillation period of the magnetic needle as a function of the "V" span. For some α values, however, the differences are too small to be easily measurable.

5. If, in order to distinguish experimentally between the two predictions for the magnetic needle oscillation period T in P, we need a difference by at least 10%, namely $T_1 > 1.10\, T_2$ (T_1 being the Ampere prediction and T_2 the Biot-Savart prediction) state in which range, approximately, we must choose the "V" half-span α for being able to decide between the two interpretations. *[3 points]*

Hint

Depending on which path you follow in your solution, the following trigonometric equation might be useful: $\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha}$

Problem 3

A space probe to Jupiter

We consider in this problem a method frequently used to accelerate space probes in the desired direction. The space probe flies by a planet, and can significantly increase its speed and modify considerably its flight direction, by taking away a very small amount of energy from the planet's orbital motion. We analyze here this effect for a space probe passing near Jupiter.

The planet Jupiter orbits around the Sun along an elliptical trajectory, that can be approximated by a circumference of average radius R; in order to proceed with the analysis of the physical situation we must first:

1. Find the speed V of the planet along its orbit around the Sun. *[1.5 points]*
2. When the probe is between the Sun and Jupiter (on the segment Sun-Jupiter), find the distance from Jupiter where the Sun's gravitational attraction balances that by Jupiter. *[1 point]*

A space probe of mass m = 825 kg flies by Jupiter. For simplicity assume that the trajectory of the space probe is entirely in the plane of Jupiter's orbit; in this way we neglect the important case in which the space probe is expelled from Jupiter's orbital plane.

We only consider what happens in the region where Jupiter's attraction overwhelms all other gravitational forces.

In the reference frame of the Sun's center of mass the initial speed of the space probe is v_0 $=1.00\cdot10^4$ m/s (along the positive y direction) while Jupiter's speed is along the negative x direction (see figure 1); by "initial speed" we mean the space probe speed when it's in the interplanetary space, still far from Jupiter but already in the region where the Sun's attraction is negligible with respect to Jupiter's. We assume that the encounter occurs in a sufficiently short time to allow neglecting the change of direction of Jupiter along its orbit around the Sun. We also assume that the probe passes behind Jupiter, i.e. the x coordinate is greater for the probe than for Jupiter when the y coordinate is the same.

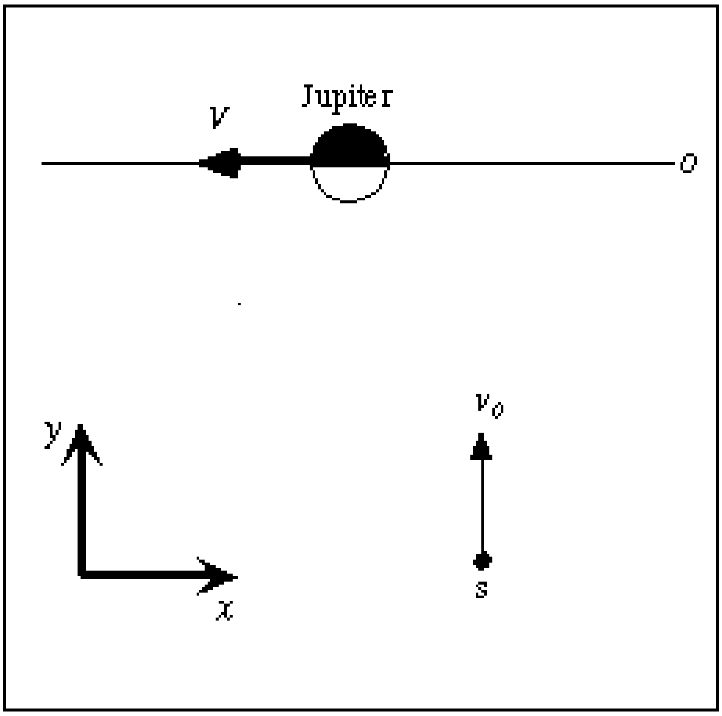


Figure 1: View in the Sun center of mass system. O denotes Jupiter's orbit, s is the space probe.

3. Find the space probe's direction of motion (as the angle φ between its direction and the x axis) and its speed v' in Jupiter's reference frame, when it's still far away from Jupiter. *[2 points]*
4. Find the value of the space probe's total mechanical energy E in Jupiter's reference frame, putting – as usual – equal to zero the value of its potential energy at a very large distance, in this case when it is far enough to move with almost constant velocity owing to the smallness of all gravitational interactions. *[1 point]*

The space probe's trajectory in the reference frame of Jupiter is a hyperbola and its equation in polar coordinates in this reference frame is

$$\frac{1}{r} = \frac{GM}{v'^2 b^2}\left(1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}\cos\theta\right) \qquad (1)$$

where b is the distance between one of the asymptotes and Jupiter (the so called *impact parameter*), E is the probe's total mechanical energy in Jupiter's reference frame, G is the gravitational constant, M is the mass of Jupiter, r and θ are the polar coordinates (the radial distance and the polar angle).

Figure 2 shows the two branches of a hyperbola as described by equation (1); the asymptotes and the polar co-ordinates are also shown. Note that equation (1) has its origin in the "attractive focus" of the hyperbola. The space probe's trajectory is the attractive trajectory (the emphasized branch).

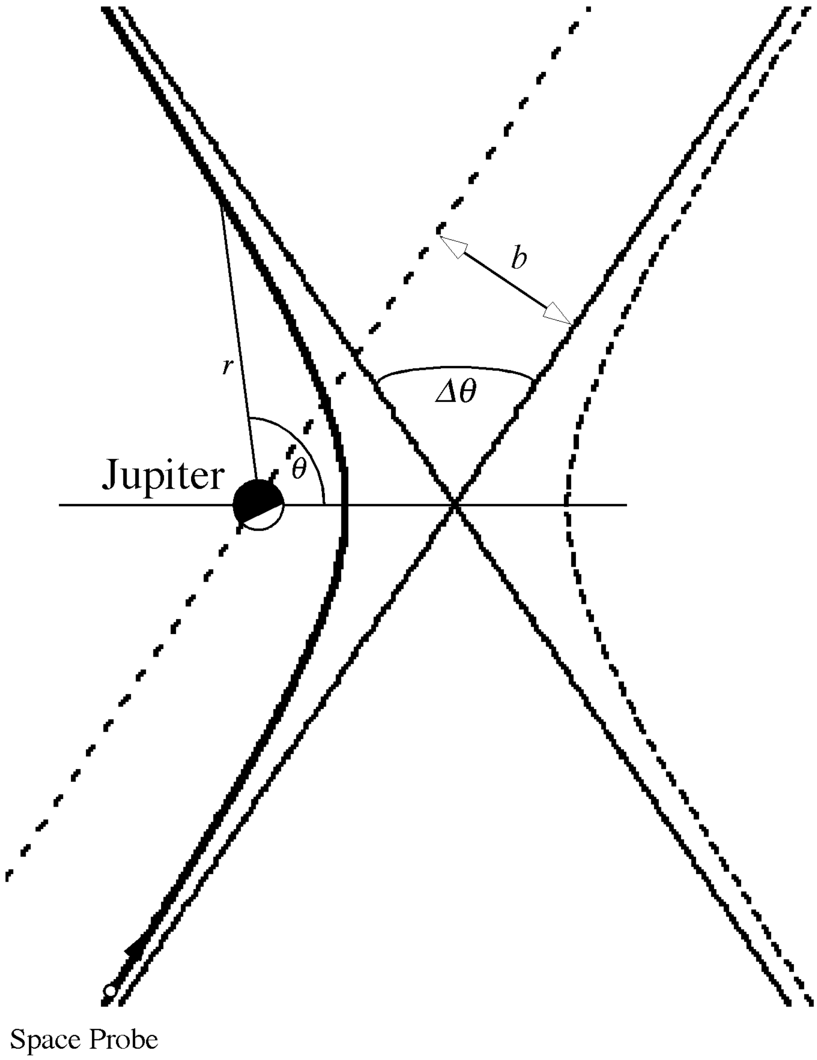


Figure 2

5. Using equation (1) describing the space probe's trajectory, find the total angular deviation $\Delta\theta$ in Jupiter's reference frame (as shown in figure 2) and express it as a function of initial speed v' and impact parameter b. *[2 points]*
6. Assume that the probe cannot pass Jupiter at a distance less than three Jupiter radii from the center of the planet; find the minimum possible impact parameter and the maximum possible angular deviation. *[1 point]*
7. Find an equation for the final speed v'' of the probe in the Sun's reference frame as a function only of Jupiter's speed V, the probe's initial speed v_0 and the deviation angle $\Delta\theta$. *[1 point]*
8. Use the previous result to find the numerical value of the final speed v'' in the Sun's reference frame when the angular deviation has its maximum possible value. *[0.5 points]*

Hint

Depending on which path you follow in your solution, the following trigonometric formulas might be useful:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$