1. In terms of triple products the expressions may be rewritten as

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\left(\left(\vec{a}\times\vec{b}\right),\vec{c},\vec{d}\right)=\left(\vec{d},\left(\vec{a}\times\vec{b}\right),\vec{c}\right)=\vec{d}\cdot\left(\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right)=-\vec{d}\cdot\left(\vec{c}\times\left(\vec{a}\times\vec{b}\right)\right)=$$
$$=-\vec{d}\cdot\left(\vec{a}\left(\vec{b}\cdot\vec{c}\right)-\vec{b}\left(\vec{a}\cdot\vec{c}\right)\right)=\left(\vec{b}\cdot\vec{d}\right)\left(\vec{a}\cdot\vec{c}\right)-\left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right),$$

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\vec{c}\left(\left(\vec{a}\times\vec{b}\right)\cdot\vec{d}\right)-\vec{d}\left(\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right)=\vec{c}\left(\vec{d},\vec{a},\vec{b}\right)-\vec{d}\left(\vec{c},\vec{a},\vec{b}\right)=\vec{c}\left(\vec{a},\vec{b},\vec{d}\right)-\vec{d}\left(\vec{a},\vec{b},\vec{c}\right)=$$
$$=\vec{c}\left(\vec{a}\cdot\left(\vec{b}\times\vec{d}\right)\right)-\vec{d}\left(\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right),$$

q.e.d.

2. Let's denote $\vec{A}=(\vec{a}\cdot\vec{r})\,\vec{b}$, $\vec{B}=(\vec{a}\cdot\vec{r})\,\vec{r}$, $\vec{C}=\vec{a}\times\vec{r}$, $\vec{D}=(\vec{a}\times\vec{r})\,\phi(\vec{r})$, $\vec{E}=\vec{r}\times(\vec{a}\times\vec{r})$, where $\vec{r}$ is the radius-vector. Then

$$\nabla\cdot\vec{A}=\left(\vec{b}\cdot\nabla\right)(\vec{a}\cdot\vec{r})+(\vec{a}\cdot\vec{r})\left(\nabla\cdot\vec{b}\right)=\left(\vec{b}\cdot\nabla\right)(\vec{a}\cdot\vec{r})=b_xa_x+b_ya_y+b_za_z=\vec{a}\cdot\vec{b},$$
$$\nabla\times\vec{A}=(\vec{a}\cdot\vec{r})\left(\nabla\times\vec{b}\right)-\left(\vec{b}\times\nabla\right)(\vec{a}\cdot\vec{r})=-\left(\vec{b}\times\nabla\right)(\vec{a}\cdot\vec{r})=-\vec{b}\times\vec{a}=\vec{a}\times\vec{b},$$
$$\nabla\cdot\vec{B}=(\vec{r}\cdot\nabla)(\vec{a}\cdot\vec{r})+(\vec{a}\cdot\vec{r})(\nabla\cdot\vec{r})=(xa_x+ya_y+za_z)+3(\vec{a}\cdot\vec{r})=4(\vec{a}\cdot\vec{r}),$$
$$\nabla\times\vec{B}=(\vec{a}\cdot\vec{r})(\nabla\times\vec{r})-(\vec{r}\times\nabla)(\vec{a}\cdot\vec{r})=-(\vec{r}\times\nabla)(\vec{a}\cdot\vec{r})=-\vec{r}\times\vec{a}=\vec{a}\times\vec{r},$$
$$\nabla\cdot\vec{C}=(\nabla,\vec{a},\vec{r})=(\vec{r},\nabla,\vec{a})=\vec{r}\cdot(\nabla\times\vec{a})=0,$$
$$\nabla\times\vec{C}=\vec{a}(\nabla\cdot\vec{r})-\vec{r}(\nabla\cdot\vec{a})=3\vec{a},$$
$$\nabla\cdot\vec{D}=((\vec{a}\times\vec{r})\cdot\nabla)\,\phi(\vec{r})+\phi(\vec{r})\,(\nabla\cdot(\vec{a}\times\vec{r}))=((\vec{a}\times\vec{r})\cdot\nabla)\,\phi(\vec{r}),$$
$$\nabla\times\vec{D}=\phi(\vec{r})\left(\nabla\times\vec{C}\right)-((\vec{a}\times\vec{r})\times\nabla)\,\phi(\vec{r})=3\phi(\vec{r})\,\vec{a}+\vec{a}\,(\vec{r}\cdot\nabla)\,\phi(\vec{r})-\vec{r}\,(\vec{a}\cdot\nabla)\,\phi(\vec{r}),$$

where it's denoted

$$(\vec{a}\cdot\nabla)=a_x\frac{\partial}{\partial x}+a_y\frac{\partial}{\partial y}+a_z\frac{\partial}{\partial z}$$

and

$$(\vec{a}\times\nabla)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ a_x&a_y&a_z\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\end{vmatrix},$$

both of these are *operators*. The last ones are easier to calculate with a simplification $\vec{E}=r^2\vec{a}-\vec{r}\,(\vec{a}\cdot\vec{r})$. We obtain:

$$\nabla\cdot\vec{E}=2(a_xx+a_yy+a_zz)-\nabla\cdot\vec{B}=-2\,(\vec{a}\cdot\vec{r}),$$
$$\nabla\times\vec{E}=r^2\,(\nabla\times\vec{a})-(\vec{a}\times\nabla)\,r^2-\nabla\times\vec{B}=-2\,(\vec{a}\times\vec{r})-(\vec{a}\times\vec{r})=-3\,(\vec{a}\times\vec{r}).$$

3. The plane $z=0$ will obviously not be equipotential. The potential at an arbitrary point $(x_0,\,y_0,\,z_0)$ can be found from

$$\varphi(x_0,y_0,z_0)=\frac{\sigma_0}{4\pi\varepsilon_0}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{\sin(\alpha x)\sin(\beta y)dxdy}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}}.$$

In the plane $z = 0$ this simplifies to

$$\varphi(x_0, y_0, 0) = \frac{\sigma_0}{4\pi\varepsilon_0} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{\sin(\alpha(x + x_0)) \sin(\beta(y + y_0)) dx dy}{\sqrt{x^2 + y^2}} = \frac{\sigma_0}{2\varepsilon_0 \sqrt{\alpha^2 + \beta^2}} \sin(\alpha x_0) \sin(\beta y_0).$$