SOLUTIONS FOR THEORETICAL COMPETITION

Theoretical Question 1

1A

The ball is subjected to the pressure forces caused by the side wall $\vec{F}_1$ (which according to Newton's third law equals to the absolute value of the force exerted by the ball on the wall to be found) and by the force exerted by the liquid $\vec{F}_A$ (Archimedes's law). The center of the ball moves in the circle of the radius $R/2$ with the angular velocity ω. Newton's second law for the center of mass motion allows us to write the following equation

$$m\omega^2 \frac{R}{2} = F_1 + F_A, \tag{1.1}$$

where $m = \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 \rho$ is the mass of the ball. By the analogy to the derivation of Archimedes law using the liquid equilibrium condition, Archimedes's force can be written as follows

$$F_A = \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 \rho_0 \frac{\omega^2 R}{2}. \tag{1.2}$$

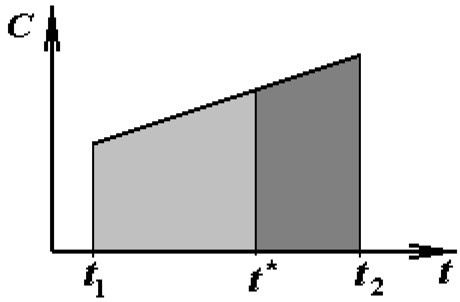
With the aid of (1.2) equation (1.1) gives rise to

$$F_1 = \frac{1}{6}\pi R^3 \rho \frac{\omega^2 R}{2} - \frac{1}{6}\pi R^3 \rho_0 \frac{\omega^2 R}{2} = \frac{1}{12}\pi R^4 \omega^2 (\rho - \rho_0). \tag{1.3}$$

Note that this formula is only applicable when the density of the ball is larger than that of the liquid, $\rho > \rho_0$. In the opposite case the ball does not touch the wall at all (floats to the vessel axis) which means that the pressure force equals zero at $\rho < \rho_0$.

1B

Let us draw the graphs showing the dependences of the heat capacities of the two bodies on temperature. The areas under these graphs are numerically equal to the amount of heat gained or given. Since the heat capacities of the bodies are equal, the heat balance condition corresponds to the equality of the areas of trapezoids from the temperature t_1 to the final temperature t^* and from t^* to t_2 (see the picture on the right). This allows us to write the following equation



$$\left(c(t_1) + c(t^*)\right)\cdot\left(t^* - t_1\right) = \left(c(t_2) + c(t^*)\right)\cdot\left(t_2 - t^*\right). \tag{1.4}$$

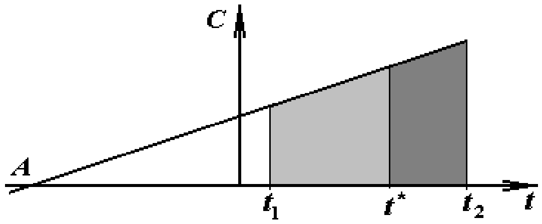
Substitution of the heat capacity expression gives rise to the equation

$$\left(1 + \alpha t_1 + 1 + \alpha t^*\right)\left(t^* - t_1\right) = \left(1 + \alpha t_2 + 1 + \alpha t^*\right)\left(t_2 - t^*\right), \tag{1.5}$$

which has the following positive root

$$t^* = \frac{1}{\alpha}\left(\sqrt{1 + \alpha(t_1 + t_2) + \frac{\alpha^2}{2}\left(t_1^2 + t_2^2\right)} - 1\right) = \frac{1}{\alpha}\left(\sqrt{\frac{(1+\alpha t_1)^2 + (1+\alpha t_2)^2}{2}} - 1\right). \tag{1.6}$$

Note.



1. It is possible to give a neat geometrical solution to the problem. Let us extend the heat capacity dependence to its intersection with the temperature axis (point A). Denote the areas of the triangles from the point A to the corresponding temperatures S_1, S^*, S_2. Since those triangles are similar, their areas are proportional to the squares of their heights, that is $c^2(t)$. The equality of the areas is then written as

$$S_2 - S^* = S^* - S_1.$$

Thus, we get

$$S^* = \frac{S_1 + S_2}{2},$$

or

$$\left(1+\alpha t^*\right)^2 = \frac{\left(1+\alpha t_1\right)^2 + \left(1+\alpha t_2\right)^2}{2}.$$

From this equation the final temperature can easily be found.

2. It is possible to obtain the expression for the internal heat by integrating the heat capacity and then making use of the conservation law.

1C

The kinetic energy of the oscillations is

$$E_k = n\frac{m\upsilon^2}{2} = n\frac{mx'^2}{2} = \frac{\alpha x'^2}{2}, \quad \text{where } \alpha = nm.$$

The potential energy of the same oscillations is

$$E_p = \frac{(k/n)(\Delta\ell)^2}{2} = \frac{(k/n)(2\pi x)^2}{2} = \frac{(4\pi^2 k/n)\cdot x^2}{2} = \frac{\beta x^2}{2},$$

where $\beta = 4\pi^2 k/n$. Thus, the cyclic frequency equals

$$\omega^2 = \beta/\alpha = 4\pi^2 k/n^2 m.$$

The period of oscillations is then simply found as

$$T = 2\pi/\omega = n\sqrt{m/k}.$$

1D

The light refraction causes the appearance of the pressure force acting on the biprism. This force is directed along the x axis and is parallel to the light propagation line. It equals to the change of momentum of the photons per unit of time. Passing through the biprizm each photon is refracted by the angle

$$\theta = (n-1)\gamma.$$

Momentum change of each photon equals

$$\Delta p_x = p(1-\cos\theta) \approx p\theta^2/2 = p(n-1)^2\gamma^2/2 = (h\nu/c)(n-1)^2\gamma^2/2$$

The number of photons refracted by the biprizm in the unit of time is written as

$$N = IS/h\nu.$$

The force is thus found as

$$F = N\Delta p_x = IS(n-1)^2 \gamma^2/2c = 4{,}2\ 10^{-7}\ N.$$

Since passing through the plate doesn't change the total momentum of photons, the second piece is subjected to the force with the same magnitude, but opposite direction. Thus, to separate two pieces, it is necessary to apply the force

$$F = 4{,}2\ 10^{-7}\ N.$$

Theoretical Question 2

Electromagnetic cannon

a) Forces acting on a mobile conductor are shown in Fig. 2. These are gravity force $m\vec{g}$, Ampere's force $\vec{F}_A$ and the force exerted by wires $\vec{N}$.

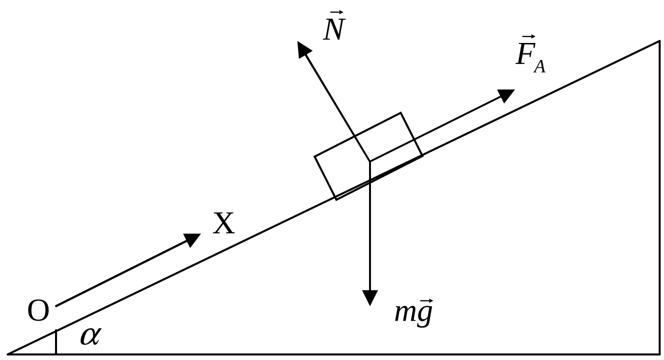


Fig 2. Forces acting on a mobile conductor.

Newton's second law for a mobile conductor projected on axis *OX* can be written as

$$ma = F_A - mg\sin\alpha\,. \tag{2.1}$$

Ampere's force is given by

$$F_A = BIh\,. \tag{2.2}$$

For the mobile conductor to start moving upwards one needs $a \geq 0$, and Ohm's law can be written as

$$I = \frac{E_{\min}}{R}, \tag{2.3}$$

so we get

$$E_{\min} = \frac{mgR\sin\alpha}{Bh}. \tag{2.4}$$

b) The current reaches a constant value when the acceleration becomes zero. Then from (2.1) and (2.2) we get

$$I_0 = \frac{mg\sin\alpha}{Bh}. \tag{2.5}$$

c) The current I in the mobile conductor is given by Ohm's law

$$I = \frac{E + E_{ind}}{R}. \tag{2.6}$$

Here E_{ind} is an emf of induction due to change of the magnetic flux through the contour. It equals

$$E_{ind} = -\frac{d\Phi}{dt} = -B\frac{dS}{dt} = -Bh\frac{dx}{dt} = -Bhu\,, \tag{2.7}$$

where u is the velocity of the mobile conductor. From (2.6) and (2.7) we get

$$I = \frac{E - Bhu}{R}\,. \tag{2.8}$$

When the velocity reaches the constant value, current equals $I = I_0$, so from (2.5) and (2.8) we obtain

$$u_0 = \frac{E}{Bh} - \frac{mgR\sin\alpha}{B^2h^2}\,. \tag{2.9}$$

d) From (2.1) and (2.2)

$$m\Delta u = BIh\Delta t - mg\Delta t\sin\alpha\,. \tag{2.10}$$

Using $\Delta q = I\Delta t$, from (2.10) we get

$$mu_0 = Bhq - mg\tau\sin\alpha\,, \tag{2.11}$$

where τ is the total time for the mobile conductor to reach the end of the wires.
Multiplying (2.8) by Δt and using $u = \Delta x / \Delta t$, we get

$$\Delta q = \frac{E\Delta t - Bh\Delta x}{R}\,. \tag{2.12}$$

Integrating (2.12) we get

$$q = \frac{E\tau - BhL}{R}\,. \tag{2.13}$$

From (2.11) and (2.13) we obtain

$$q = \frac{m(E^2Bh - mgER\sin\alpha + B^3h^3Lg\sin\alpha)}{B^2h^2(BEh - mgR\sin\alpha)}\,. \tag{2.14}$$

The final answer is

$$C_1 = \frac{mBhg\sin\alpha}{(BEh - mgR\sin\alpha)} \quad C_2 = \frac{mE}{B^2h^2}\,.$$

e) The work done by the source is given by

$$A = Eq\,. \tag{2.15}$$

The mobile conductor acquires kinetic energy

$$E_{kin} = \frac{mu_0^2}{2} \tag{2.16}$$

and potential energy

$$E_{pot} = mgL\sin\alpha\,. \tag{2.17}$$

The energy conservation law gives

$$A = E_{kin} + E_{pot} + Q\,, \tag{2.18}$$

so

$$Q = \frac{mE(E^2Bh - mgER\sin\alpha + B^3h^3Lg\sin\alpha)}{B^2h^2(BEh - mgR\sin\alpha)} - mgL\sin\alpha - \frac{m}{2}\left(\frac{E}{Bh} - \frac{mgR\sin\alpha}{B^2h^2}\right)^2. \tag{2.19}$$

The final answer is

$$C_3 = \frac{m^2g^2R\sin^2\alpha}{(BEh - mgR\sin\alpha)} \quad C_4 = \frac{mE^2}{B^2h^2} - \frac{m}{2}\left(\frac{E}{Bh} - \frac{mgR\sin\alpha}{B^2h^2}\right)^2$$

Theoretical Question 3
Helium atom

a) According to Bohr's postulate angular momentum of each electron takes the following discrete values

$$m_e \upsilon \cdot r = n\hbar \text{, где } n = 1, 2, 3, \ldots$$

For ground state $n = 1$. Then

$$p \cdot r = \hbar$$

b) Potential energy of the system consists of the potential energies of attraction between each electron and nucleus, and the potential energy of repulsion between electrons. Classically system can only perform a circular motion if electrons are always located on opposite sides of the nucleus. The potential energy of the system for such configuration is

$$E_p = -\frac{1}{\pi\varepsilon_0}\frac{e^2}{r} + \frac{1}{8\pi\varepsilon_0}\frac{e^2}{r} = -\frac{7}{8\pi\varepsilon_0}\frac{e^2}{r}$$

c) Newton's second law for each electron is written as

$$m_e \frac{\upsilon^2}{r} = \frac{1}{4\pi\varepsilon_0}\left(\frac{2e^2}{r^2} - \frac{e^2}{4r^2}\right)$$

Combining this equation with angular momentum quantization rule we get

$$r = \frac{16\pi\varepsilon_0}{7}\frac{\hbar^2}{m_e e^2}$$

Numerical value is

$$r = 3{,}02 \cdot 10^{-11} \text{ m}$$

d) The full ground state energy equals

$$E = E_k + E_p = 2\frac{p^2}{2m_e} - \frac{7}{8\pi\varepsilon_0}\frac{e^2}{r}$$

Expressing *p* as a function of *r* using quantization rule and using the result of previous section we obtain

$$E = -\left(\frac{7}{16\pi\varepsilon_0}\right)^2 \frac{me^4}{\hbar^2}$$

Numerical value is

$$E = -133{,}32 \cdot 10^{-19} \text{ J} = -83{,}3 \text{ eV}$$

e) After single ionization helium atom looks similar to hydrogen atom with nucleus charge $+2e$. Energy of such configuration is given by

$$E_1 = -\frac{1}{8\pi^2\varepsilon_0^2}\frac{me^4}{\hbar^2} = -87{,}06 \cdot 10^{-19} \text{ J} = -54{,}4 \text{ eV}$$

Then ionization energy is written as

$$E_{ion} = E_1 - E = 28{,}9 \text{ eV}$$

f) Atoms will be ionized by external pressure when the work which needs to be done to considerably change the volume of one helium atom is of the order of ionization energy. As an estimate this condition can be written as

$$E_{ion} \sim \frac{4\pi r^3}{3} p_{ion}$$

Numerical value is of the order of

$$p_{ion} \sim 10^{14}\,\text{Pa}$$

Marking scheme for solution to theoretical problems

Task 1

A

Marked results	Marks
Equation (1):	0,5 point
Expression for Archimedes's force	0,5 point
Formula (3) for the total force	0,5 point
$F_1 = 0$ at $\rho < \rho_0$	0,5 point

B

Marked results	Marks
Heat balance equation	1.0 point
Expression for the heat	1.0 point
Formula (3) for the final temperature	1.0 point

C

Marked results	Marks
Potential energy of oscillations	1,0 point
Kinetic energy for oscillations	0,5 point
Correct formula for the period	0,5 point

D

Marked results	Marks
Right idea	0,5 point
Change in light momentum after passing through the biprizm	1,0 point
Formula for the pressure exerted on the biprizm	0,5 point
Pressure on the second biprizm	0,5 point
Correct numerical result	0,5 point

Task 2

Marked results	Marks
Correct expression for $E_{\min}$	1 point
Correct expression for I_0	1 point
Correct expression for u_0	2 points
Correct expression for Ohm's law	0,5 point
Correct equation of motion (Newton's second law)	0,5 point
Correct expression for C_1	1 point

Correct expression for C_2	1 point
Correct expression for the energy conservation law	1 point
Correct expression for C_3	1 point
Correct expression for C_4	1 point

Task 3

Marked results	Marks
Quantization rule	1 point
Correct relative position of electrons	1 point
Correct expression for potential energy	1 point
Correct expression for radius	1,5 points
Correct numerical value for radius	0,5 points
Correct expression and numerical value of total energy	1 point
Correct expression and numerical value of single ionization energy	2 points
Reasonable expression for estimation of pressure	1.5 points
Numerical value of the order $10^{13} - 10^{15}$ Pa	0.5 point

Note: solution which completely neglects interaction between electrons gets 1 point for parts a)-e)