Physics,Theoretical competition.

Problem 1(10 points)

One of the ends of the uniform massive rod of length L is connected using a hinge to a vertical rotating axis (fig.1.1). The hinge is such that in the reference frame of the axis the rod can freely move in a certain vertical plane, without any friction in the hinge. The axis is rotating with an angular velocity ω, acceleration of gravity equals **g**.

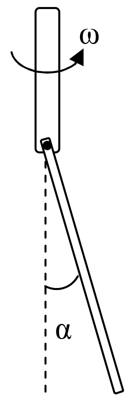


Fig.1.1

1) Calculate the stationary values of the angle α.
2) Analyze the stability of the system for each stationary state.

Suppose that the rod is given a small kick with respect to its stable state.

3) Calculate the period of oscillations.

Solution

It is easier to solve the problem in noninertial frame of reference connected with the rotating axis, where the rod is in equilibrium under the action of forces including centrifugal inertial forces. Let as a result of the rotation the rod is deflected from the vertical by an angle. Consider an element of the rod at the distance ℓ from the point O with a length $d\ell$ and a mass dm. Given that the mass of the rod with a length L is m, we have $dm = dl \cdot m/L$. The centrifugal force and the acting on that element is equal to (see fig.1.2)

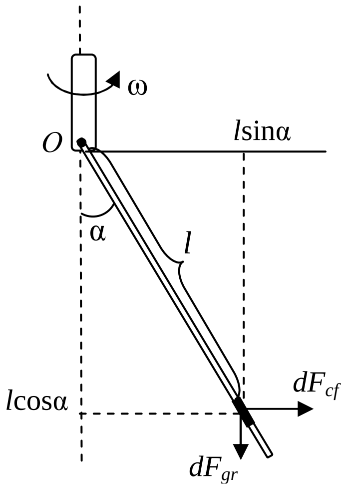


Рис.1.2

$$dF_{cf.} = \omega^2 \ell \sin\alpha \cdot dm = (m/L)\omega^2 \sin\alpha \cdot \ell \cdot d\ell \quad (1.1)$$ ***(0,5 points)***

$$dF_{gr} = g \cdot dm = (mg/L)d\ell \quad (1.2)$$ ***(0,5 points)***

The torques of these forces about the point O are

$dM_{cf} = \ell\cos\alpha \cdot dF_{cf}$ and $dM_{gr} = \ell\sin\alpha \cdot dF_{gr}$

Thus the torque of the force of gravity is

$$M_{gr} = -(mgL/2)\sin\alpha \quad (1.3)$$ ***(0,5 points)***

and the torque of the centrifugal force is equal to

$$M_{cf} = (m\omega^2 L^2/3)\cos\alpha \cdot \sin\alpha \quad (1.4)$$ ***(0,5 points)***

At equilibrium we have $\sum_i \vec{M}_i = 0$, i.e. $\left(\frac{m\omega^2 L^2}{3}\cos\alpha - \frac{mgL}{2}\right)\sin\alpha = 0$, consequently

a1) $\sin\alpha = 0$, hence $\alpha = 0$ and $\alpha = \pi$; ***(1.0 point)***

a2) $\cos\alpha = \omega_0^2/\omega^2$, where $\omega_0^2 \equiv 3g/(2L)$. Thus $\alpha = \arccos(\omega_0^2/\omega^2)$ with $\omega^2 > \omega_0^2$. ***(1.0 point)***

The variation of the total momentum is defined by

$$\frac{\partial M}{\partial\alpha} = -\frac{mgL}{2}\cos\alpha + \frac{m\omega^2 L^2}{3}(\cos^2\alpha - \sin^2\alpha). \quad (1.5)$$

The equilibrium state is stable provided $(\partial M/\partial\alpha) < 0$. Then

b1) For $\alpha = 0$ the derivative of the total torque is equal to

$$\partial M / \partial \alpha = (mL^2/3)\cdot(\omega^2 - \omega_0^2)\,, \qquad (1.6)$$

consequently that state is stable if $\omega^2 < \omega_0^2$; ***(1.0 point)***

For $\alpha = \pi$ we have $\partial M / \partial \alpha = (mL^2/3)\cdot(\omega^2 + \omega_0^2) \geq 0$ hence that state is always unstable. ***(1.0 point)***

b2) At $\cos\alpha = \omega_0^2/\omega^2$ the derivative of the torque is

$$\partial M / \partial \alpha = (mL^2/3)\cdot(\omega_0^2 - \omega^2)(\omega_0^2 + \omega^2)/\omega^2\,, \qquad (1.7)$$ ***(1.0 point)***

hence the state is stable for all $\omega^2 > \omega_0^2$.

Thus for different values of the parameters of the systems oscillations are possible about the angles $\alpha = 0$ and $\alpha = \arccos(\omega_0^2/\omega^2)$. The equation for oscillatory motion has the form

$$I\cdot(\Delta\alpha)'' = \Delta M\,, \qquad (1.7)$$ ***(0,5 points)***

where I is the moment of inertia of the rod about the point *O*:

$$I = mL^2/3\,.$$ ***(0,5 points)***

For small $\Delta\alpha$ the torque can be written as $\Delta M = (\partial M / \partial \alpha)\cdot\Delta\alpha$. Then the period of oscillations is equal to

$$T = 2\pi\sqrt{\frac{I}{-\partial M / \partial \alpha}} \qquad (1.8)$$ ***(1.0 point)***

From 1.6 and 1.8 we get for $\alpha = 0$ ($\omega^2 < \omega_0^2$)

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \omega^2}}\,. \qquad (1.9)$$ ***(0,5 points)***

For $\alpha = \arccos\left(\omega_0^2/\omega^2\right)$ ($\omega^2 > \omega_0^2$) from 1.7 and 1.8 the period of oscillations is equal to

$$T = \frac{2\pi\omega}{\sqrt{\omega^4 - \omega_0^4}} \qquad (1.10)$$ ***(0,5 points)***

Problem 2 (8 points)

The stove is used to heat the room, and the temperature in the room is t_1= 17°C, while the temperature outside is t_o= 7°C. It is suggested to use the ideal thermal pump instead, using the reversed Carnot cycle. Degree of efficiency of the engine, used in reverse Carnot cycle is η= 60%. Assuming that the heat exchange between the room and the outside is proportional to the temperature difference, and that the engine uses the same amount of fuel per unit time as the stove, calculate the temperature in the room, if

a) engine is located outside of the room
b) engine is located inside the room.

Solution

Let the heat transmission factor be *k*, then the energy losses of the stove are given by

$$A = k(T_1 - T_0),$$ ***(1.0 point)***

where *A* – is the energy consumed by the stove in a unit time, $T = t^0$C+273. The thermal pump operates using the reversed Carnot cycle, with maximal temperature being equal to the room temperature, and the minimal – to the temperature outside.

Denote the romm temperature by *T*, then the efficiency of the Carnot cycle would be $(T - T_0)/T$, and in the reversed Carnot cycle the following equality will hold

$$\frac{A_{Carnot}}{Q_2} = \frac{T - T_0}{T}\,,$$ ***(2 points)***

c) where Q_2 is the heat transferred to the room in a unit time,

$$A_{Carnot} = \eta A,$$ **(*1.0 point)***

where A is the work performed by the engine in a unit time. The energy balance in the case of engine located outside of the room is

$$k(T-T_0) = Q_2 = \frac{TA_{Carnot}}{T-T_0} = \frac{\eta T}{T-T_0}k(T_1-T_0) \Rightarrow T-T_0 = \frac{T}{T-T_0}\eta(T_1-T_0),$$ ***(1,5 points)***

whence we find

$$(T-T_0)^2 - \eta(T_1-T_0)(T-T_0) - \eta T_0(T_1-T_0) = 0.$$

Solving this equation we find $T-T_0 = 44.1^0$, hence the temperature in the room would be

$$t_2 = 54{,}1^0\text{C}.$$ **(*1.0 point)***

When the engine is inside the room an additional energy equal to $(1-\eta)A$ is dissipated in the room, thus the energy balance equation

$$k(T-T_0) = \frac{T\eta}{T-T_0}k(T_1-T_0) + (1-\eta)(T_1-T_0)k,$$ **(*1.0 point)***

hence

$$(T-T_0)^2 - \eta(T_1-T_0)(T-T_0) - T_0(T_1-T_0) = 0.$$

The positive solution of that equation is $T-T_0 = 56^0$ and thus the temperature established in the room will be

$$t_3 = 66^0\text{C}$$ ***(0,5 points)***

Problem 3 (12 points)

Given a ring of radius R with a current I flowing in it.

1. Calculate the magnetic field at point O_1 located on the axis of the ring. The total angular span of the ring at point O_1 equals 2α. (See Fig.3.1)

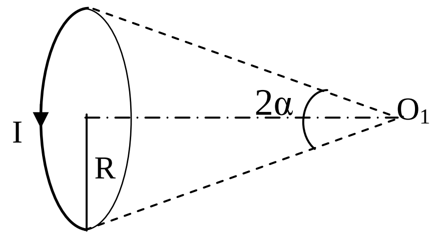


Fig.3.1

The solenoid of radius R and length l consists of N coils, distributed uniformly. The current through the solenoid equals I.

2. Find the magnetic field on the axis of the solenoid at the point, from which the angular span of the ends of solenoid equals 2α and 2β. (See Fig.3.2)

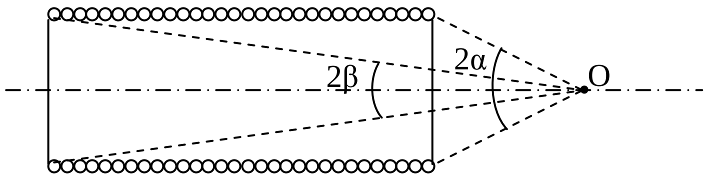


Fig.3.2

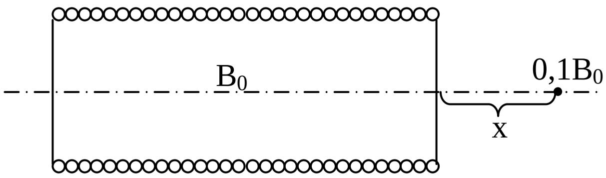


Fig.3.3

3. In the following we assume that $\ell \gg R$.
   a) Calculate the magnetic field B_0 inside the solenoid far away from its ends.
   в) Find the distance x, for which $B=0.1\cdot B_0$ (See Fig.3.3)
   c) Calculate the inductance of the coil L, assuming that magnetic field inside the solenoid far away from its ends is uniform.

Magnetic bullet is moving along the axis of the solenoid. Solenoid is connected to a capacitor C. Magnetic moment of the bullet M is parallel to the axis of the solenoid. We will neglect the change of the velocity of the bullet.

4. a) Write the condition, that the time it takes the bullet to pass the region of non uniform magnetic field is much smaller than the period of oscillations in LC circuit. Assume in the following, that this condition is always satisfied.

b) Find the values of the bullet velocities υ, such that the amplitude of oscillations in LC circuit after the bullet passes the solenoid is maximal.

c) What is the value of the current oscillation amplitude $I_{\text{мах}}$ for this case? Draw the graph of $I(t)$.

d) Prove that the magnetic force acting on a bullet equals $M\dfrac{\partial B}{\partial x}$ and is parallel to the axis of the solenoid.

Comment:

You may consider the bullet to be a small ring of area S_0,with a current I_0, and $M=S_0I_0$.

In the theory of the magnetism the following theorem is proven: *If one denotes the flux of the magnetic field created by the first contour through the second contour as* $L_{12}I_1$, and *the flux of the magnetic field created by the second contour through the first contour as* $L_{21}I_2$ *,then* $L_{12}=L_{21}$*. It is assumed that the positive direction of the flux is consistent with the positive direction of the current in each contour .*

Solution

a) From the symmetry consideration the resulting magnetic field on the axis is directed along the axis. According to the Bio-Savaro, the field of a small element of a current I with a length $R\cdot d\varphi$ is directed at an angle $\pi/2-\alpha$ to the axis and is equal to

$$\frac{\mu_0 I d\varphi}{4\pi R}\sin^2\alpha \qquad (2.1)$$

By projecting it on the axis and integrating over the φ from 0 to 2π we get

$$B=\frac{\mu_0 I}{2R}\sin^3\alpha \quad (2.2) \qquad \textbf{(1.0 point)}$$

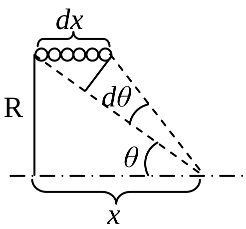


Fig.3.4

b) Since $x=Rctg\theta$ (see fig..3.4), the length of a strip defined by angles θ and $\theta+d\theta$ is

$$dx=\frac{R}{\sin^2\theta}d\theta. \quad (2.3)$$

Using this and the result of the previous item we find

$$B=\int\frac{\mu_0 NI}{2lR}\sin^3\theta dx=\frac{\mu_0 NI}{2l}\int_\alpha^\beta\sin\theta d\theta=\frac{\mu_0 NI}{2l}(\cos\beta-\cos\alpha). \quad (2.4) \qquad \textbf{(2 points)}$$

c) By formula (2.4) with $\cos\beta\approx 1$ и $\cos\alpha\approx -1$ we have

$$B_0=\frac{\mu_0 NI}{l} \quad (2.5) \qquad \textbf{(0,5 points)}$$

d) Inserting $\cos\beta\approx 1$ into (2.4) we get an equation for $\cos\alpha$

$$1-\cos\alpha=0{,}2, \quad (2.6)$$

whence

$$x=4R/3 \quad (2.7) \qquad \textbf{(1.0 point)}$$

e) The total flux through the solenoid is equal to

$$\Phi=B_0SN=\mu_0\frac{\pi R^2N^2}{l}I, \quad (2.9)$$

hence

$$L=\mu_0\frac{\pi R^2N^2}{l} \quad (2.10) \qquad \textbf{(0,5 points)}$$

f)

$$\frac{\upsilon\cdot\sqrt{LC}}{R}>>1 \qquad (2.11) \qquad \textbf{(1.0 point)}$$

g) Consider the variation of the flux of the magnetic field through the bullet during its flight. Using the theorem of the comment we have

$$\Phi = L_{12} I_0 = \frac{B(x,I)S_0}{I} I_0 = \frac{B(x,I)}{I} M \quad (2.12)$$ **(1.0 point)**

where $B(x,I)$ is the induction of the field of the solenoid, I – its current. Since $B(x,I)/I$ doesn't depend on I, the flux through the bullet changes only when it flies through the nonuniform region of the field. According to (2.11) the total flux through the solenoid doesn't change during that time of flight, and hence from the flux conservation we find that the current jump I_1 induced in the solenoid would be defined by equation

$$LI_1 = M \frac{\mu_0 N}{l}. \quad (2.13)$$ **(1.0 point)**

Inserting L from (2.10) we get

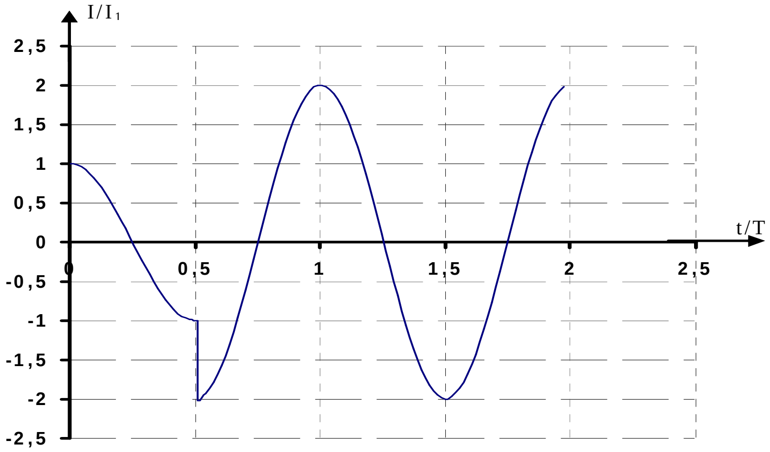
$$I_1 = \frac{M}{\pi R^2 N}. \quad (2.14)$$

After the bullet's emergence from the nonuniform region the magnetic field flux through it doesn't vary, hence in the current in the solenoid oscillates harmonically. When the bullet arrives to the exit from the solenoid, it again enters a nonunifom region of the magnetic field and another sudden change of the current, with a sign opposite to that at the entrance into the solenoid, occurs. To have a maximal residual current in the oscillator it shoud have the same sign as the current in the circuit, hence the number of oscillations of the contour must be equal odd number of half oscillations. Hence

$$\upsilon = \frac{l}{(2n-1)\pi\sqrt{LC}}, \text{ где } n=1, 2, 3,\ldots \quad (2.15)$$ **(1.0 point)**

h)

$$I_{max} = 2I_1 = \frac{2M}{\pi N R^2} \quad (2.16)$$ **(0,5 points)**



(1.0 point)

Fig.3.5

i) The Ampere force acting on the bullet is $2\pi B_\perp I_0 R$. Using the fact that the flux of the magnetic field through the closed surface is zero we find that

$$-(B_x - B_{x+dx})\pi R^2 = B_\perp 2\pi R dx,$$ **(0,5 points)**

$$B_\perp 2\pi R I_0 = -\frac{B_x - B_{x+dx}}{dx}\pi R^2 I_0,$$ **(0,5 points)**

$$F = M\frac{\partial B}{\partial x}$$ **(0,5 points)**