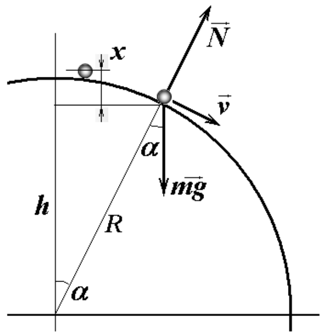
SOLUTION OF THE THEORETICAL COMPETITION

Problem 1

1A (2 points)



The Newtonsecond law for the body in the radial projection takes the from

$$m\frac{v^2}{R} = mg\cos\alpha - N \qquad (1)$$

and the energy conservation law is written as

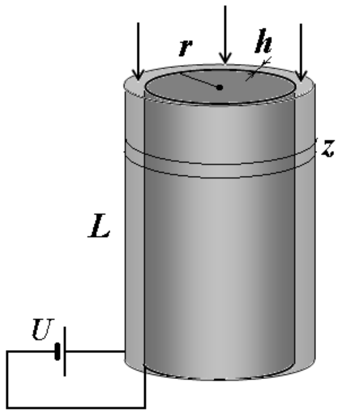
$$\frac{mv^2}{2} = mgx \qquad (2)$$

Using $N = 0$ as a condition for the body to lose contact with the sphere surface we get from Eqs. (1) and (2)

$$h = 2x,$$

where $h = R\cos\alpha$ is the altitude of the body above the table. Thus, the body loses contact with the surface when its attitude above the table becomes twice that from its initial position. Thus, $h = \frac{2}{3}h_0$.

1B (3 points)



Consider a thing layer of water of width z which is perpendicular to the axis of the system. This layer passes through the heater at a time $\tau = L/v$ where v is the water velocity to be found. The water heats up due to passing of the electric current. The amount of this heat is determined by the Joule law

$$Q = \frac{U^2}{R}\tau . \qquad (1)$$

Here R is the electrical resistance of the layer.Taking into account that the electric current pases perpendicular to the surfaces of cylinders, the resistance of the layer is

$$R = \rho\frac{h}{2\pi rz} . \qquad (2)$$

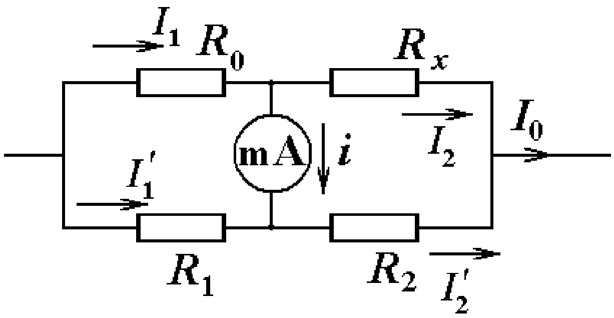
The total amount of heat released due to the electric current is exposed into the heating of the water whose amount is determined as $Q = cm\Delta t$. The mass of the layer is found as $m = V\gamma = 2\pi rzh\gamma$. Thus, the heat balance equation is read as

$$\frac{U^2}{\rho\frac{h}{2\pi rz}}\frac{L}{v} = c\cdot 2\pi rzh\gamma\cdot\Delta t . \qquad (3)$$

From this equation we find

$$v = \frac{U^2 L}{\rho h^2 c\gamma\Delta t} . \qquad (4)$$

1C (2 points)



In the figure on the right you can see the notation and directions of electric currents in the circuit. The electric currents passing through the resistors are found from the followinf obvious relations

$$\begin{cases} I_1 + I_1' = I_0 \\ I_1 R_0 = I' R_1 \end{cases} \Rightarrow \quad I_1 = I_0 \frac{R_1}{R_1 + R_0}, \tag{1}$$

$$\begin{cases} I_2 + I_2' = I_0 \\ I_2 R_0 = I' R_2 \end{cases} \Rightarrow \quad I_2 = I_0 \frac{R_2}{R_2 + R_x}, \tag{2}$$

From the figure one can see that the lectric current through the milliammeter equals

$$i = I_1 - I_2 = I_0 \left(\frac{R_1}{R_1 + R_0} - \frac{R_2}{R_2 + R_x} \right). \tag{3}$$

For the current through the milliasmmeter to be equal zero it is necessary for the bridge to be balanced and the balancecondition is written as

$$\frac{R_1}{R_1 + R_0} = \frac{R_2}{R_2 + R_x},$$

or

$$\frac{R_1}{R_0} = \frac{R_2}{R_x} \tag{4}$$

from which we determine the unknown resistance

$$R_x = R_0 \frac{R_2}{R_1}. \tag{5}$$

To determine the relative error of this formula it is necessary to solve Eq.(3) using the condition that the electric current i is relatively small. Making use of notation $\frac{i}{I_0} = \eta$ and taking into account that $\eta << 1$ we get

$$\frac{R_1}{R_1 + R_0} - \frac{R_2}{R_2 + R_x} = \eta \quad \Rightarrow \quad \frac{R_2}{R_2 + R_x} = \frac{R_1}{R_1 + R_0} - \eta$$

$$\frac{R_2 + R_x}{R_2} = \left(\frac{R_1}{R_1 + R_0} - \eta \right)^{-1} = \frac{R_1 + R_0}{R_1} \left(1 - \eta \frac{R_1 + R_0}{R_1} \right)^{-1} \approx \frac{R_1 + R_0}{R_1} \left(1 + \eta \frac{R_1 + R_0}{R_1} \right).$$

$$1 + \frac{R_x}{R_2} = \left(1 + \frac{R_0}{R_1} \right) \left(1 + \eta \frac{R_1 + R_0}{R_1} \right) = 1 + \frac{R_0}{R_1} + \eta \left(\frac{R_1 + R_0}{R_1} \right)^2 \quad \Rightarrow \quad R_x = R_2 \frac{R_0}{R_1} + \eta \left(\frac{R_1 + R_0}{R_1} \right)^2$$

Rewriting the last expression in the form

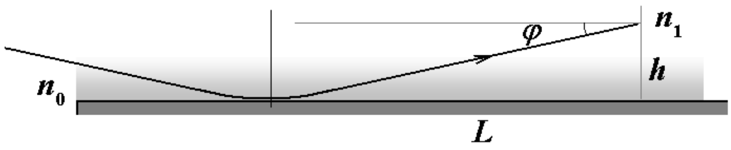
$$R_x = R_2 \frac{R_0}{R_1} + \eta \left(\frac{R_1 + R_0}{R_1} \right)^2 = R_2 \frac{R_0}{R_1} \left(1 + \eta \frac{(R_1 + R_0)^2}{R_1 R_2} \right), \tag{6}$$

w obtain the relative error sought

$$\varepsilon = \eta \frac{(R_1 + R_0)^2}{R_1 R_2}. \tag{7}$$

1D (3 points

In reality, the mirage-“puddles” appear due to the reflection of rays coming from the sky. The reflection itself is caused by heated layer of air right near the concrete. In the figure on the right you can see one of such rays.



The condition of absolute reflection takes the form

$$n_0 = n_1 \cos\varphi, \tag{1}$$

where n_0, n_1 are the refraction indexes right above the concrete and at a certain attitude from it, respectively.

The refraction index depends on the molecule number density and, consequently, on the air temperature, too. Using the ideal gas equation of state

$$P = \gamma kT \tag{2}$$

we express the number density and substitute it into the formula for the refraction index to get

$$n_1 = 1 + \frac{\alpha P}{2kT}, \quad n_0 = 1 + \frac{\alpha P}{2k(T+\Delta T)}. \tag{3}$$

The ratio of the refraction indexes at zero attitude from the concrete and at the attitude of the driver is simply found as

$$\frac{n_0}{n_1} = \frac{1 + \dfrac{\alpha P}{2k(T+\Delta T)}}{1 + \dfrac{\alpha P}{2kT}} \approx \frac{1 + \dfrac{\alpha P}{2kT}\left(1 - \dfrac{\Delta T}{T}\right)}{1 + \dfrac{\alpha P}{2kT}} \approx 1 - \frac{\alpha P \Delta T}{2kT^2}. \tag{4}$$

Sincethe angle φ is small, we use the approximate formula $\cos\varphi \approx 1 - \dfrac{\varphi^2}{2}$. In this case from (1) and (4) we get

$$\varphi = \sqrt{\frac{\alpha P \Delta T}{kT^2}}. \tag{5}$$

The distance at which the mirage-“puddles” appear is then obtained as

$$L = \frac{h}{\varphi} = h\sqrt{\frac{kT^2}{\alpha P \Delta T}} = 1{,}2\sqrt{\frac{1{,}38\cdot 10^{-23}\cdot (293)^2}{2{,}3\cdot 10^{-29}\cdot 1{,}0\cdot 10^5\cdot 2{,}0}} \approx 6{,}1\cdot 10^2\,м. \tag{6}$$

Marking scheme

№	Content	Points
	1A	
1	The Newton second law (1)	1
2	Cnservation law (2)	0,5
3	Correct final result	0,5
	1B	
1	Joule law (1)	0,5
2	Water resistance (2)	1
3	The amount of energy to heat the water	0,5

4	The heat balance	0,5
5	Correct formula (4)	0,5
	1C	
1	Expressions for electric currents (1)-(2)	0,5
2	Correct formula (4) for resistance	0,5
3	Electric current (3) through the milliammeter	0,5
4	Correct final result(7)	0,5
	1D	
1	Condition (1) of absolute reflection	1
2	Correct dependence (3) of the refraction index	1
3	Correct formula (5) for the angle	1
4	Correct expression (6) for the distance	0,5
5	Correct numerical value for the distance	0,5

Problem 2
Electromagnetic swing (10 points)

1.[1 point] The forces acting on the mobile conductor are depicted in Figure 1. The condition for the mobile conductor to start moving is written as

$$mg\sin\alpha > F_{friction} = \mu N = \mu mg\cos\alpha\,. \quad (1)$$

Thus, the inequality sought is $\tan\alpha > \mu$.

2. [2 points] The electromagnetic law of Faraday immediately provides the expression for the emf

$$E_{induction} = -\frac{d\Phi}{dt} = -B\frac{dS}{dt} = Bhu\,. \quad (2)$$

For the circuit created by the coil, the rods and the mobile conductor we have

$$E_{induction} + E_{selfinduction} = 0\,, \quad (3)$$

where the emf of self-induction in the coil is written as

$$E_{selfinduction} = -L\frac{dI}{dt}\,. \quad (4)$$

Here I is the electric current in the coil.
Solving eqns. (2)-(4) altogether gives rise

$$I = \frac{Bh}{L}x\,. \quad (5)$$

3. [1 point] The forces acting on the mobile conductor are depicted in Figure 1. The equation of motion along the x axis (directed downward along the inclined plane) takes the form

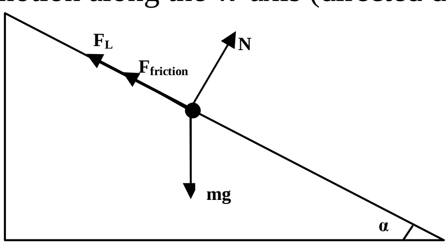


Рисунок 1.

$$m\ddot{x} = mg\sin\alpha - \mu mg\cos\alpha - F_L\,, \quad (6)$$

where $F_L = BIh = \frac{B^2h^2}{L}x$ is the Lorenz force.

Thus, eq. (6) is rewritten as follows

$$\ddot{x} = -\frac{B^2h^2}{mL}x + g\sin\alpha - \mu g\cos\alpha\,. \quad (7)$$

Eq. (7) is the simplest equation for harmonic oscillations with friction and its solution is found as

$$x = x_0 + A\cos\omega t\,. \quad (8)$$

Here $\omega^2 = \dfrac{B^2h^2}{mL}$ is the natural-vibration frequency. Substituting (8) into (7) at the initial conditions $x(0)=0$ and $u(0)=\dot{x}(0)=0$ gives rise

$$x(t) = \frac{g(\sin\alpha - \mu\cos\alpha)}{\omega^2}(1-\cos\omega t)\,. \tag{9}$$

Consequently, the velocity of the mobile conductor is

$$u(t) = \dot{x}(t) = \frac{g(\sin\alpha - \mu\cos\alpha)}{\omega}\sin\omega t\,. \tag{10}$$

Hence

$$u_{\max} = \frac{g(\sin\alpha - \mu\cos\alpha)}{Bh}\sqrt{mL}\,. \tag{11}$$

4. [1 point] Substituting (9) into (5) gives the electric current strength

$$I(t) = \frac{Bh}{L}x(t) = \frac{mg(\sin\alpha - \mu\cos\alpha)}{Bh}(1-\cos\omega t)\,. \tag{12}$$

Thus, we find

$$I_{\max} = \frac{2mg(\sin\alpha - \mu\cos\alpha)}{Bh}\,. \tag{13}$$

5. [3 points] Figure 2 shows the forces acting on the mobile conductor when it moves upward. The corresponding equation of motion along the x axis is now written as

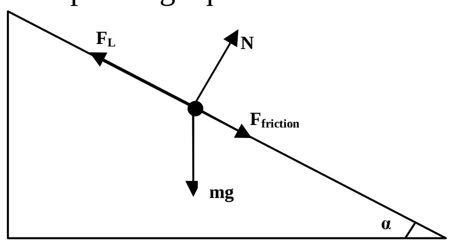


Рисунок 2.

$$m\ddot{x} = mg\sin\alpha + \mu mg\cos\alpha - F_L \tag{14}$$

or taking into account eq. (5)

$$\ddot{x} = -\frac{B^2h^2}{mL}x + g\sin\alpha + \mu g\cos\alpha\,. \tag{15}$$

Again solving this equation, but for the initial conditions $x(\pi/\omega) = \dfrac{2g(\sin\alpha - \mu\cos\alpha)}{\omega^2}$ and $u(\pi/\omega) = \dot{x}(\pi/\omega) = 0$, we get

$$x(t) = \frac{g(\sin\alpha - 3\mu\cos\alpha)}{\omega^2}(1-\cos\omega t) + \frac{4\mu g\cos\alpha}{\omega^2}\,. \tag{16}$$

From expressions (9) and (16) we conclude that after each turn in the direction of the motion the amplitude decreases by the value $\delta A = 2\mu g\cos\alpha/\omega^2$. The final stop of the moving conductor occurs if the turning point is less than $\delta A/2$ in distance from the equilibrium position $x_0 = g\sin\alpha/\omega^2$ in the absence of friction. Denote the position of the final stop by x_f, the amount of heat released due to the friction is determined by the energy conservation law as

$$Q(x_f) = mgx_f\sin\alpha - LI_f^2/2 = -\frac{B^2h^2(x_f - x_0)^2}{2L} + \frac{m^2g^2L\sin^2\alpha}{2B^2h^2}. \tag{17}$$

In case of $\mu << tg\alpha$ we can neglect the first term in equation (17) and we get

$$Q = \frac{m^2g^2L\sin^2\alpha}{2B^2h^2}. \tag{18}$$

6. [2 points] In case of final μ it is necessary to know the value of x_f. For $\mu = \dfrac{\mathrm{tg}\alpha}{2009}$ from the previous solution we get

$$x_f - x_0 = \delta A/2. \tag{19}$$

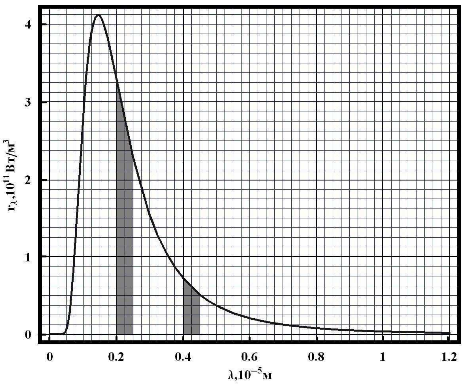
In this case the relative error equals

$$\frac{\delta Q}{Q} = -\left(\frac{\delta A/2}{x_0}\right)^2 = -\frac{1}{2009^2}. \quad (20)$$

Marking scheme

№	**Content**	**Points**
1	The correct inequality $\tan\alpha > \mu$	1
2	Expression (2) or irs analog Correct formula (5)	1 1
3	Formula (11)	1
4	Formula (13)	1
5	Description of the stop point Formula (18)	1 2
6	The coordinate of the stop point (19) Formula (20)	1 1

Problem 3
Thermal radiation (10 points)



1. The ratio of energies radiated in the wavelength intervals $(\lambda_1, \lambda_1 + \Delta\lambda)$ and $(\lambda_2, \lambda_2 + \Delta\lambda)$ is equal to the ratio of squares depicted in the figure to the right

$$\frac{E_1}{E_2} = \frac{S_1}{S_2} = 12.8$$

Claculation is made bycounting a number of squares.

2. For each figure the wavelrngths of the maximum is simnplyfound

$\lambda_{\max} = 1{,}45 \circ 10^{-6}$ m at $T_1 = 2000K$

$\lambda_{\max} = 2{,}23 \circ 10^{-6}$ m at $T_1 = 1300K$.

From the relation $\lambda = bT^n$, we find $n = -1$, $b = 2{,}9 \cdot 10^{-3}$ m·K.

3. The total squares in figures provided are equal $R_1 = 0{,}91$ MW/m^2 at $T_1 = 2000$ K and $R_2 = 0{,}16$ MW/m^2 at $T_2 = 1300$ K.

From the relation $R = \sigma T^m$ we obtain: $m = 4$, $\sigma = 5{,}7 \cdot 10^{-8}$ W/(m^2·K^4).

4. In accordance with Wien law, the sun radiation correspondce to the black body temperature

$$T_0 = \frac{b}{\lambda_m}.$$

The power of solar radiation is determined by the Stefan-Boltzmann law

$$W = \sigma T^4 4\pi R^2$$

where R is the sun radius.

The sun loses the amount of energy

$$U = 0.01Mc^2,$$

where M is the sun mass, c denotes the speed of light.

Thus, the time sought is

$$t = \frac{U}{W} = \frac{0.01Mc^2}{\sigma(b/\lambda_m)^4 4\pi R^2} = 3.8 \cdot 10^{18} s$$

Marking scheme

№	**Content**	**Points**
1	For searching the ratio od appropriate squares Correct numerical value for the ratio with relative error 5%	0,5 0,5
2	Correct numerical values for the wavelengths of maxima with relative error 5% Correct value of n Correct value of b with relative error 5%	1 0,5 0,5
3	Correct numerical value of R_1 with relative error 5% Correct numerical value of R_2 with relative error 5% Correct value of m Correct value of σ with relative error 5%	1 1 0,5 0,5
4	Wien law to determine the sun temperature T_0 The power of solar radiation W The energy for the sun to lose U Correct time t with relative error 5%	1 1 1 1