1. Due to the Planck distribution the spectral energy in a wavelength range $(\lambda,\ \lambda+d\lambda)$ is given by

$$f(\lambda)d\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\dfrac{hc}{kT}\right)-1},$$

which peaks at

$$\frac{hc}{\lambda kT} = 4.965.$$

That gives the peak wavelength

$$\lambda_0 = \frac{hc}{4.965\,kT}.$$

Then the surface temperature of Betelgeuse is about $T_1 = 3000$ K and the surface temperature of Rigel is about $T_2 = 20000$ K.

2. As $f(x)$ is differentiable, then in the limit of $n \to \infty$

$$\left(\frac{f\left(x+\dfrac{1}{n}\right)}{f(x)}\right)^n \approx \left(\frac{f(x)+\dfrac{1}{n}f'(x)}{f(x)}\right)^n = \left(1+\frac{1}{n}\frac{f'(x)}{f(x)}\right)^n = \exp\left(\frac{f'(x)}{f(x)}\right).$$

3. Guess the van der Waals equation looks like

$$p = \frac{NkT}{V-bN} - \frac{aN^2}{V^2} = \frac{nkT}{1-bn} - an^2,$$

where n is the volumetric concenration of the gas. As there's considered to be no substance exchange with the surroundings, then $\partial N/\partial V = 0$, and

$$\frac{\partial C_V}{\partial V} = \frac{3}{2}k\frac{\partial N}{\partial V} = 0.$$

As the entropy is given by

$$dS = \frac{dQ}{T} = \frac{C_V dT}{T},$$

then (within an additive constant)

$$S = C_V \ln T = \frac{3}{2}Nk\ln T.$$

The energy (comparing with a $T = const$ and $V \to \infty$ state)

$$E = \int pdV = \nu RT\ln(1-bn) + an^2V.$$

The work to be done is

$$A = E_2 - E_1 = \nu RT_2\ln\left(1-\frac{bN}{V_2}\right) + \frac{aN^2}{V_2} - \nu RT_2\ln\left(1-\frac{bN}{V_2}\right) - \frac{aN^2}{V_1},$$

where T_2 can be found from the van der Waals equation using the initial (or final) pressure and the amount of substance of the gas, namely

$$T_2 = \frac{V_2 - bN}{Nk}\left(p_2 + \frac{aN^2}{V_2^2}\right).$$

4. As the dust sticks to the spacecraft, its mass increases with time. Its rate of increase is $dm/dt = \rho A v$. Also, due to momentum conservation $mv = m_0 v_0$, so

$$\frac{d}{dt}(mv) = v\frac{dm}{dt} + m\frac{dv}{dt} = 0.$$

Using $m = m_0 v_0 / v$, we get

$$\rho A v^2 = -\frac{m_0 v_0}{v}\frac{dv}{dt},$$

which implies

$$\frac{dv}{dt} = -\frac{\rho A v^3}{m_0 v_0}.$$

The solution of this equation is

$$\frac{1}{v^2} = \frac{1}{v_0^2} + \frac{2\rho A t}{m_0 v_0},$$

or

$$v(t) = \frac{v_0}{\sqrt{1 + \frac{\rho v_0 A t}{m_0}}}.$$