Task 5

1. (a) Let's look for a solution of form

$$u(x,y) = f(ax+by).$$

Substituting this into the initial equation yields

$$2a^2 - 5ab + 3b^2 = 0.$$

So, such a solution fits the equation if $a = b$ or $2a = 3b$. Then the full form is

$$u(x,t) = Af(x+y) + Bg(3x+2y).$$

(b) Let's look for a solution of form

$$u(x,y) = e^{ax+by} f(cx+dy).$$

Substitution yields a system of equations, stating that each term collected by f, f' and f'' is equal to zero:

$$\begin{cases} 2a^2 + 6ab + 4b^2 + a + b = 0, \\ 4ac + 6ad + 6bc + 8bd + c + d = 0, \\ 2c^2 + 6cd + 4d^2 = 0. \end{cases}$$

This system has 2 solutions: $a = -b$ with $c = -d$, and $2a = -1 - 4b$ with $c = -2d$. So, the general solution looks like

$$u(x,t) = Af(x-y) + Be^{-x/2} g(2x-y).$$

(c) In this equation the solution of the inhomogeneous equation is obtained from the solution of the homogeneous equation by adding $2e^x$. The substitution from the previous equation yields

$$\begin{cases} b^2 - 2ab + 2a - b = 0, \\ 2bd - 2ad - 2bc + 2c - d = 0, \\ d^2 - 2cd = 0. \end{cases}$$

This system also has 2 solutions: $a = 0$ with $d = 0$, and $b = 2a$ with $d = 2c$. So, the general solution is

$$u(x,y) = 2e^x + Ae^y f(x) + Bg(x+2y).$$

2. By differentiation we obtain:

$$v_x = -\frac{x^2+t^2}{\left(x^2-t^2\right)^2} u^{(1,0)} - \frac{2xt}{\left(x^2-t^2\right)^2} u^{(0,1)},$$

$$v_t = \frac{2xt}{\left(x^2-t^2\right)^2} u^{(1,0)} + \frac{x^2+t^2}{\left(x^2-t^2\right)^2} u^{(0,1)},$$

$$v_{xx} = \frac{2x\left(x^2+3t^2\right)}{\left(x^2-t^2\right)^3} u^{(1,0)} + \frac{\left(x^2+t^2\right)^2}{\left(x^2-t^2\right)^4} u^{(2,0)} + \frac{2t\left(3x^2+t^2\right)}{\left(x^2-t^2\right)^3} u^{(0,1)} + \frac{4x^2t^2}{\left(x^2-t^2\right)^4} u^{(0,2)},$$

$$v_{tt} = \frac{2x\left(x^2+3t^2\right)}{\left(x^2-t^2\right)^3} u^{(1,0)} + \frac{\left(x^2+t^2\right)^2}{\left(x^2-t^2\right)^4} u^{(2,0)} + \frac{2t\left(3x^2+t^2\right)}{\left(x^2-t^2\right)^3} u^{(0,1)} + \frac{4x^2t^2}{\left(x^2-t^2\right)^4} u^{(0,2)} = v_{xx}.$$

So, the function v satisfies the equation.

3. The coordinates and differentials are parametrized as follows:

$$\begin{cases} x = 0, \\ y = 2(1+\cos t), \\ z = 2(1+\sin t), \\ dx = 0, \\ dy = -2\sin t\, dt, \\ dz = 2\cos t\, dt. \end{cases}$$

The integral is taken from $t = 0$ to $t = 2\pi$, as the curve is periodic with a period of 2π. By substitution of the coordinates we get:

$$I = \int\limits_0^{2\pi} 12(1+\cos t)\cos t\, dt = \int\limits_0^{2\pi} (12\cos t + 6 + 6\cos 2t)\, dt = 12\pi.$$