Compressed sensing

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1 Introduction

Applications

2 Studying the l_0-minimization

We want to recover an s-sparse vector $\mathbf{x} \in \mathbb{K}^N$ knowing a vector of m measurements $\mathbf{y} \in \mathbb{K}^m$ and a measurement matrix $\mathbf{A} \in M_{m\times N}(\mathbb{K})$ with $m < N$, such that $\mathbf{Ax} = \mathbf{y}$. The system is underdetermined, so we have to look for alternative ways to solve it. The most straightforward approach is to solve the corresponding l_0-minimization problem.

Definition 2.1. *The **support** of a vector $\mathbf{x} \in \mathbb{K}^N$ is the set of indices of its nonzero entries:*

$$\operatorname{supp}(\mathbf{x}) = \{j \in [\![1, N]\!] \colon x_j \neq 0\}$$

Definition 2.2. *We define $\|\mathbf{x}\|_0$ as the cardinality of $\operatorname{supp}(\mathbf{x})$. We say that the vector $\mathbf{x}$ is s-**sparse** if $\|\mathbf{x}\|_0 \leq s$.*

Note that $\|\cdot\|_0$ is not an actual norm, nor is it a semi-norm. Now we can formalize the problem in the following form:

$$\text{minimize } \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{y}. \tag{P$_0$}$$

Proposition 2.3. *Let $\mathbf{A} \in M_{m\times N}(\mathbb{K})$, $\mathbf{x} \in \mathbb{K}^N$ s-sparse and $\mathbf{y} \in \mathbb{K}^m$. The following two statements are equivalent:*

(i) The vector $\mathbf{x}$ is the unique solution of the compressed sensing problem, i.e, it's the unique s-sparse vector such that $\mathbf{Ax} = \mathbf{y}$.

(ii) The vector $\mathbf{x}$ is the unique solution of P_0.

Proof. $(i) \Rightarrow (ii)$ If $\mathbf{x}$ is the only s-sparse vector that satisfies $\mathbf{Ax} = \mathbf{y}$, then there exists no such vector $\mathbf{z}$, that $\|\mathbf{z}\|_0 \leq \|\mathbf{x}\|_0 \leq s$, which makes $\mathbf{x}$ the unique minimizer of P_0.
$(ii) \Rightarrow (i)$ Immediate. □

Theorem 2.4. *Let $\mathbf{A} \in M_{m\times N}(\mathbb{K})$ and $\mathbf{x}, \mathbf{z} \in \mathbb{K}^N$. The following statements are equivalent:*

(i) If $\mathbf{Ax} = \mathbf{Az}$ and both $\mathbf{x}$ and $\mathbf{z}$ are s-sparse, then $\mathbf{x} = \mathbf{z}$.

(ii) $\operatorname{Ker} \mathbf{A} \cap \{\mathbf{z} \in \mathbb{K}^N \colon \|\mathbf{z}\|_0 \leq 2s\} = \{\mathbf{0}\}$, i.e., $\mathbf{0}$ is the only $2s$-sparse vector in the $\operatorname{Ker} \mathbf{A}$.

(iii) For every $S \subset [\![1, N]\!]$ with $\operatorname{card}(S) \leq 2s$, the submatrix $\mathbf{A}_S$ is injective as a map from $\mathbb{K}^{\operatorname{card}(S)}$ to $\mathbb{K}^m$.

(iv) Every set of $2s$ columns of $\mathbf{A}$ is linearly independent, i.e., $\operatorname{rang} \mathbf{A} \geq 2s$.

Proof. *(i)* $\Rightarrow$ *(ii)* Let $\mathbf{z} \in \mathbb{K}^N$ be $2s$-sparse and satisfy $\mathbf{Az} = \mathbf{0}$. On the other hand, we also have $\mathbf{A0} = \mathbf{0}$, so by the hypothesis it has to be that $\mathbf{z} = \mathbf{0}$.

(ii) $\Rightarrow$ *(i)* Now let $\mathbf{x}$ and $\mathbf{z}$ be two s-sparse vectors such that $\mathbf{Ax} = \mathbf{Az}$. Then $\mathbf{x} - \mathbf{z}$ is $2s$-sparse and $\mathbf{Ax} - \mathbf{Az} = \mathbf{A}(\mathbf{x} - \mathbf{z}) = \mathbf{0}$, which implies that $\mathbf{x} - \mathbf{z} \in \operatorname{Ker} \mathbf{A}$. By hypothesis, we conclude that $x = z$.

(ii) $\Rightarrow$ *(iii)* We recall that linear map $\mathbf{A}$ is injective iff $\operatorname{Ker} \mathbf{A} = \{\mathbf{0}\}$. Let $\in \mathbb{K}^N$ be a $2s$-sparse vector, such that $\mathbf{x}_S \in \operatorname{Ker} \mathbf{A}_S$, where $S = \operatorname{supp}(\mathbf{x})$. Then $\mathbf{Ax} = \mathbf{A}_S\mathbf{x}_S + \mathbf{A}_{\overline{S}}\mathbf{x}_{\overline{S}} = \mathbf{0} + \mathbf{A}_{\overline{S}}\mathbf{0} = \mathbf{0}$, where $\overline{S} = [\![1, N]\!] \setminus S$. Then by hypothesis, $\mathbf{x} = 0$ and $\mathbf{x}_S = 0$, and thus $\mathbf{A}_S$ is injective.

(iii) $\Rightarrow$ *(ii)* Let $\mathbf{x}$ be $2s$-sparse, $S = \operatorname{supp}(\mathbf{x})$ and $\mathbf{A}_S$ an injective map. Suppose that $\mathbf{x} \in \operatorname{Ker} \mathbf{A}$. Then by extension, $\mathbf{x}_S \in \operatorname{Ker} \mathbf{A}_S$ and $\mathbf{x}_S = \mathbf{0}$. Thus, $\mathbf{x} = 0$.

For the last two implications we have to assume that $2s \leq m$.

(iii) $\Rightarrow$ *(iv)* Let $S \subset [\![1, N]\!]$, $\operatorname{card}(S) = 2s$. Then $\operatorname{rang}(A_S) = 2s - \dim(\operatorname{Ker} \mathbf{A}_S)) = 2s - 0 = 2s$.

(iv) $\Rightarrow$ *(iii)* Let $S \subset [\![1, N]\!]$, $\operatorname{card}(S) \leq 2s$. Then $\operatorname{rang}(A_S) = \operatorname{card}(S)$ and $\dim(\operatorname{Ker} \mathbf{A}_S)) = \operatorname{card}(S) - \operatorname{rang}(A_S) = 0$. Thus, $\operatorname{Ker} \mathbf{A}_S = \{\mathbf{0}\}$. $\square$

The importance of this theorem is that it gives us the necessary condition for a successful recovery of any s-sparse vector $\mathbf{x}$ from P_0: the number of measurements m has to be at least $2s$. Indeed, if it is possible to reconstruct the vector by solving l_0-problem, then statement (i) holds, and then according to the theorem, $\operatorname{rang} \mathbf{A} \geq 2s$. On the other hand, the rank of a matrix cannot be greater than its smallest dimension, which in this case is m. This gives us the necessary condition $m \geq 2s$.

However, this lower bound is true for the most general case, where matrix A is arbitrary. In reality, there are many cases where the lower threshold is much softer. For example, in the case of random matrices, we can find a lower value of m that is enough for the problem to be solved with a reasonably high probability). This fact will be further studied in the next sections. For now, we will only state a theorem from [math intro to cs], that looks at this problem from another angle: here, we assume that the vector $\mathbf{x}$ is known and that we can choose a measurement matrix by ourselves.

Theorem 2.5. *For any $N \geq s+1$, given an s-sparse vector $\mathbf{x} \in \mathbb{K}^N$, there exists a measurement matrix $\mathbf{A}$ $inM_{m\times N}(\mathbb{K})$ with $m = s + 1$ rows such that the vector $\mathbf{x}$ can be recovered from its measurement vector $\mathbf{y} = \mathbf{Ax}$ by solving P_0.*

Unfortunately, as good as it seems in theory, the l_0-minimization is not effective in practice: problem P_0 happens to be NP-hard [source]. In fact, any non-convex optimization problem is NP-hard, which means that we have to look for convex alternatives instead. And one such alternative approach is l_1-minimization.

3 From l_0 to l_1

$\|\cdot\|_p$: (preferably with pictures of unit balls)

- $0 < p < 1$: non-convex, NP, bad
- $p > 1$: convex, but doesn't solve the problem in general
- $p = 1$: convex, solves the problem, good

basis pursuit:

$$\text{minimize } \|x\|_1 \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{y} \tag{1}$$

Other algorithms from chapter 3?

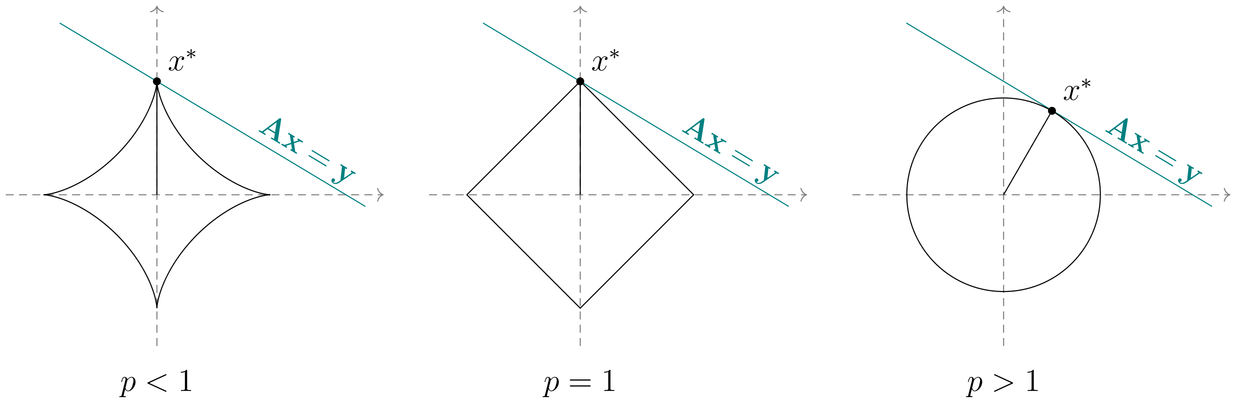

Figure 1: Unit balls

4 Studying the l_1-minimization

When does it solve the problem P_0? → chapter 4

Definition 4.1. *Null-space property*

Stability and robustness?

5 Number of measurements for l_1-minimization

Proposition 3.10 from The Convex Geometry of Linear Inverse Problems. My plots

6 Transition phase

Leaving on the Edge paper