Compressed sensing

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Outline:

1. Introducing the problem
2. Minimal number of measurements for 0-norm?
3. From 0-norm to 1-norm + questions to answer
4. When l_1-minimization solves the l_0-minimization problem (null space property)
5. Number of measurements with log + my plots
6. Phase transition + my plots
7. Conclusion

1 Introduction

We want to recover the sparse vector $\mathbf{x} \in \mathbb{K}^N$ knowing the vector of m measurements $\mathbf{y} \in \mathbb{K}^m$ and the measurement matrix $\mathbf{A} \in M_{m \times N}(\mathbb{K})$ with $m < N$.

Applications

2 Studying the l_0-minimization

We are looking for the sparsest solution of the underdetermined system of equations $\mathbf{Ax} = \mathbf{y}$. One way to approach this is to solve the corresponding l_0-minimization problem.

Definition 2.1 *The **support** of a vector $\mathbf{x} \in \mathbb{K}^N$ is the set of indices of its nonzero entries:*

$$\operatorname{supp}(\mathbf{x}) = \{j \in [\![1, N]\!] \mid x_j \neq 0\}.$$

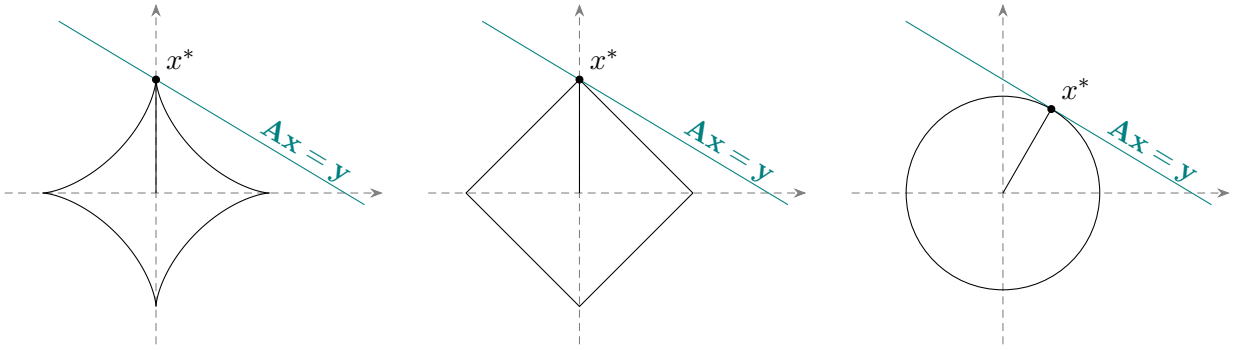
Definition 2.2 *We define $\|\mathbf{x}\|_0$ as the cardinality of $\operatorname{supp}(\mathbf{x})$. We say that the vector $\mathbf{x}$ is **s-sparse** if $\|\mathbf{x}\|_0 \leq s$.*

Note that $\|\cdot\|_0$ is not an actual norm, nor is it a semi-norm. Now we can formalize the problem in the following form:

$$\text{minimize } \|x\|_0 \quad \text{subject to } \mathbf{Ax} = \mathbf{y}. \tag{1}$$

Definition 2.3 *s-sparse*

Minimal number of measurements ($2s$)
NP-hardness



$p < 1$ $p = 1$ $p > 1$

3 Convex alternatives

$\|\cdot\|_p$: (preferably with pictures of unit balls)

- $0 < p < 1$: non-convex, NP, bad
- $p > 1$: convex, but doesn't solve the problem in general
- $p = 1$: convex, solves the problem, good

basis pursuit:

$$\text{minimize } \|x\|_1 \quad \text{subject to } \mathbf{Ax} = \mathbf{y} \tag{2}$$

Other algorithms from chapter 3?

4 Studying the l_1-minimization

When does it solve the problem 2? → chapter 4

Definition 4.1 *Null-space property*

Stability and robustness?

5 Number of measurements for l_1-minimization

Proposition 3.10 from The Convex Geometry of Linear Inverse Problems. My plots

6 Transition phase

Leaving on the Edge paper