Transient analysis

Objective: *Discuss time marching methods used in SPICE*

Outline:
1. Time marching methods
2. Explicit and implicit integration methods
3. Implicit methods used in circuit analysis
4. Implementation of discretization in SPICE

Supplemental reading: Vladimirescu, The SPICE Book,
Chapters: 6, 9.4 - 9.5, 10.4

1. Time marching methods

Two classes:

explicit - inexpensive per step, limited stability, not for circuit analysis

implicit – more expensive per step, better stability, suitable for circuit analysis.

Example of *RC* circuit

$$C\frac{dV}{dt} = -\frac{V-E}{R} \qquad \Rightarrow \qquad \frac{dV}{dt} = -\frac{1}{RC}V + \frac{1}{RC}E$$

$$\left.\begin{array}{l} R = 10[\Omega] \\ C = 1[pF] \end{array}\right\} \Rightarrow \quad RC = 10^{-11}[\sec]$$

Notation: $V(t)$ – **exact (theoretical) solution**

V_n – **approximate (numerical) solution at** $t = t_n$

We write the differential equation at time $t = t_n$ **in the form**

$$\left.\frac{dV(t)}{dt}\right|_{t=t_n} = -\frac{1}{RC} V(t_n) + \frac{1}{RC} E(t_n)$$

We can approximate the derivative using

a) $$\left.\frac{dV(t)}{dt}\right|_{t=t_n} \cong \frac{V(t_{n+1}) - V(t_n)}{h} \quad ; \quad h = t_{n+1} - t_n$$ **Forward Euler (F-E)**

formula or

b) $$\left.\frac{dV(t)}{dt}\right|_{t=t_n} \cong \frac{V(t_n) - V(t_{n-1})}{h} \quad ; \quad h = t_{n+1} - t_n$$ **Backward Euler (B-E)**

formula.

Using F-E formula we obtain

$$\frac{V(t_{n+1})-V(t_n)}{h} \cong -\frac{1}{RC}V(t_n)+\frac{1}{RC}E(t_n)$$

Computation of the solution on the basis of the above formula, from the initial condition $V(t_o)=V_o$, can be described by the difference equation

$$V_{n+1}=V_n-\frac{h}{RC}V_n+\frac{h}{RC}E(t_n)$$

For a constant excitation, $E(t)=A$, and zero initial condition the solution to the difference equation is

$$V_n = A\left[1-\left(1-\frac{h}{RC}\right)^n\right] \ .$$

The analytic solution is

$$V(t)=A\left(1-e^{-\frac{t}{RC}}\right)$$

and it approaches the level A as time increases.

The numerical solution, V_n, will tend to the same level iff

$$\left|1-\frac{h}{RC}\right|<1 \;\Rightarrow\; h<2RC.$$

2. Explicit and implicit integration methods

Discussed explicit F-E – method has stability problem
Step size is limited by stability. The explicit methods are not suitable for circuit analysis, where we want to be able to compute with large steps when the solution changes slowly (i.e. when the accuracy does not require small steps).

Backward-Euler (implicit method)

$$\left.\frac{dV}{dt}\right|_{t_{n+1}} \cong \frac{V(t_{n+1})-V(t_n)}{h}$$ this is the first order method.

This method will be applied to the equation of RC circuit

$$\left.\frac{dV(t)}{dt}\right|_{t_{n+1}} = -\frac{1}{RC}V(t_{n+1})+\frac{1}{RC}E(t_{n+1}).$$

Considering the finite difference approximation we have $V(t_n) \to V_n$ and the difference equation:

$$\frac{V_{n+1} - V_n}{h} = -\frac{1}{RC} V_{n+1} + \frac{1}{RC} E_{n+1}$$

$$n = 0, 1, 2, \cdots$$

$$\Downarrow$$

$$V_{n+1} = aV_n + bE_{n+1}$$

Assume: $V_0 = 0, \quad E_n = E_0 = 5[V] = const$ **than the solution is**

$$V_{n+1} = bE_0(1 + a + a^2 + \cdots + a^n)$$

where

$$a = (1 + \frac{h}{RC})^{-1}, \qquad b = (1 + \frac{h}{RC})^{-1} \frac{h}{RC}$$

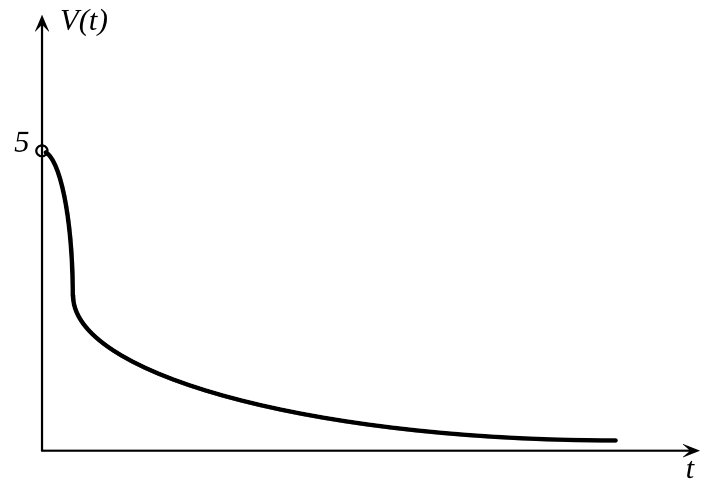
$V(t) \to 5$ **and also** $V_n \to 5$ **because** $|a| < 1$ **without any restriction on the integration step.**

Ass.: $V_0 = 5,\ E_n = 0$

Numerical approximation yields

$$V_{n+1} = aV_n$$

The solution is: $V_n = a^n V_0$

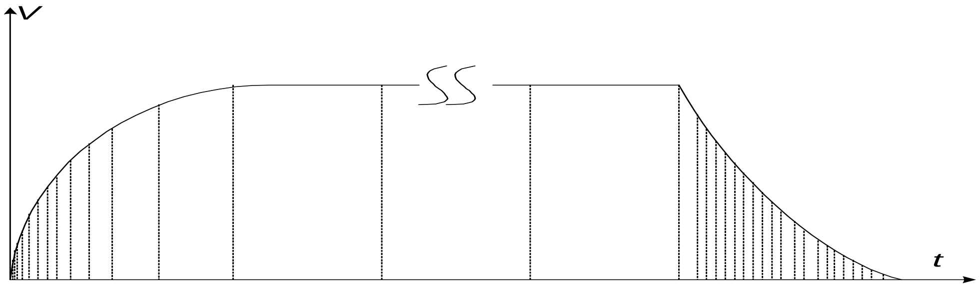


$$|a| < 1$$

STABILITY CONDITION

$$\frac{1}{1+\frac{h}{RC}} < 1$$

THE METHOD IS ABSOLUTELY STABLE



The importance of absolute stability

3. Implicit methods used in circuit analysis

B-E:

$$x_{n+1} = x_n + h_{n+1}\dot{x}_{n+1}$$ **1-rst order method**

$$\hat{x}_{n+1} = \left.\frac{dx}{dt}\right|_{t=t_{n+1}}$$

T-R **(default in SPICE)**

$$x_{n+1} = x_n + \frac{h_{n+1}}{2}(\dot{x}_n + \dot{x}_{n+1})$$ **2-nd order method**

THESE ARE ONE-STEP METHODS, ALSO CALLED SINGLE STEP METHODS.

EXERCISE-1

Apply T-R method to the RC circuit and obtain difference equation for the circuit.

<u>Hint</u>

$$\frac{dV}{dt} = -\frac{1}{RC}V + \frac{1}{RC}E$$

$$\frac{dV}{dt} = \dot{V}$$

For $t = t_n$ **we have**

$$\dot{V}(t_n) = -\frac{1}{RC}V(t_n) + \frac{1}{RC}E(t_n)$$

For $t = t_{n+1}$ **we have**

$$\dot{V}(t_{n+1}) = -\frac{1}{RC}V(t_{n+1}) + \frac{1}{RC}E(t_{n+1}) \quad .$$

The T-R formula is

$$V_{n+1} = V_n + \frac{h_{n+1}}{2}(\dot{V}_n + \dot{V}_{n+1})$$

Linear Multistep Methods (LMM)

General formula for LMM

$$x_{n+k} = \sum_{i=1}^{k} \alpha_i x_{n+k-i} + \sum_{i=0}^{k} h_{n+k-i} \beta_i \dot{x}_{n+k-i}$$

representing k-step methods.

When $\beta_0 = 0$, the method is explicit and it is not suitable for circuit simulation.
When $\beta_0 \neq 0$, the method is implicit and suitable for circuit simulation, the method is suitable for solving stiff problems.

THE GEAR'S METHODS

$$x_{n+k} = \sum_{i=1}^{k} \alpha_i x_{n+k-i} + h_{n+k} \beta_0 \dot{x}_{n+k}$$

IN **SPICE**

$$k = 2, 3, \cdots, 6$$

(for $k = 1$ we have $\alpha_1 = 1$ and $\beta_0 = 1$, which results in B-E)

<u>Example of 2-step method</u> $\quad k = 2 \quad \beta_0 = \frac{2}{3} \quad \alpha_1 = \frac{4}{3} \quad \alpha_2 = -\frac{1}{3}$

EXERCISE 2.

Derive difference formula for the RC circuit using Gear's method with k=2.

Check the method stability ($|\zeta| < 1$).

Hint:

Application of 2-step Gear method yields

$$x_{n+2}(1+\frac{2}{3}\gamma_{n+2})-\frac{4}{3}x_{n+1}+\frac{1}{3}x_n=\frac{2}{3}\gamma_{n+2}E_{n+2}$$

where $$\gamma_{n+2}=\frac{h_{n+2}}{\tau} \qquad \tau = RC$$

Checking the stability.

The characteristic polynomial is

$$\zeta^2(1+\frac{2}{3}\gamma_{n+2})-\frac{4}{3}\zeta+\frac{1}{3}=0$$

A general solution to the homogeneous difference equation is

$$x_n = C_A\zeta_1^n + C_B\zeta_2^n$$

Stability Condition is

$$|\zeta_1| < 1$$

$$|\zeta_2| < 1$$

Verify that the difference is always stable regardless of h_n.

An example of formula for 3-step Gear's method

$$k = 3 \qquad \beta_0 = \frac{6}{11} \qquad \alpha_1 = \frac{18}{11} \qquad \alpha_2 = -\frac{4}{11} \qquad \alpha_3 = \frac{2}{11}$$

$$x_{n+3} = \alpha_1 x_{n+2} + \alpha_2 x_{n+1} + \alpha_3 x_n + h_{n+3} \cdot \beta_0 \dot{x}_{n+3}$$

$$n = 0, 1, 2, \cdots$$

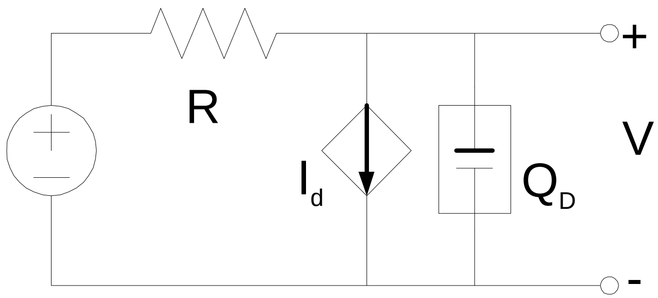
Note that

$$\sum_{i=1}^{k} \alpha_i = 1$$

which is an expression of the method consistency condition.

Exercise 3

Formulate B-E equation for the circuit below



Ass. : Q_D-junction cap. only

$$Q_D = Q(V)$$

$$I_d = I_s(e^{V/n_d V_t} - 1)$$

Result:

$$C(V_{n+1})\frac{V_{n+1} - V_n}{h_{n+1}} = -I_s(e^{V_{n+1}/\bar{V}_t} - 1) - \frac{1}{R}V_{n+1} + \frac{1}{R}E_{n+1}$$

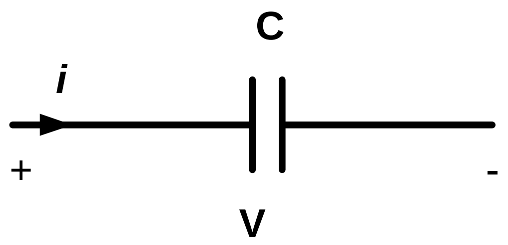
$$C(V) = \frac{dQ_D}{dV}, \ \bar{V}_t = n_d V_t$$

$$n = 0, 1, 2, \cdots$$

EXERCISE 3a.

Apply F-E and get the difference equations for the circuit in the Exercise 3.

4. Implementation of discretization in SPICE
(Example of T-R method)
The capacitor



$$V_{n+1} = V_n + \frac{h_{n+1}}{2}(\dot{V}_n + \dot{V}_{n+1})$$ **T-R formula**

The circuit relations

$$i = C\frac{dV}{dt} \quad \Rightarrow \quad \dot{V} = \frac{1}{C} i$$

Using the following notation for the numerical approximation to the current

$$I_n \cong i(t_n)$$

$$I_{n+1} \cong i(t_{n+1})$$

and the circuit relations we obtain

$$V_{n+1} = V_n + \frac{h_{n+1}}{2C}(I_n + I_{n+1}) \quad .$$

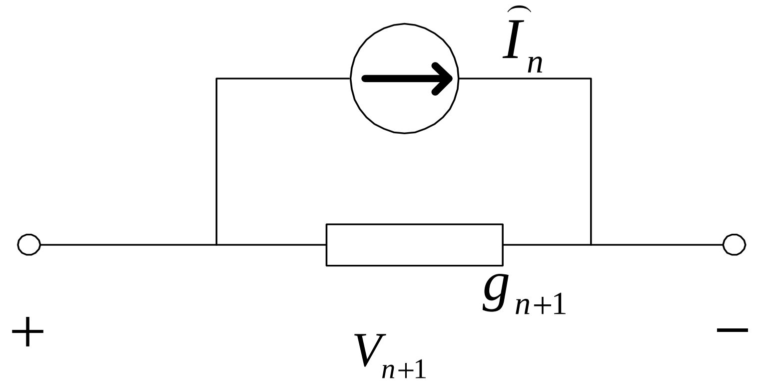
The explicit formula for the current is

$$I_{n+1} = \underbrace{\frac{2C}{h_{n+1}}}_{g_{n+1}} V_{n+1} \underbrace{- \frac{2C}{h_{n+1}} V_n - I_n}_{\widehat{I}_n}$$

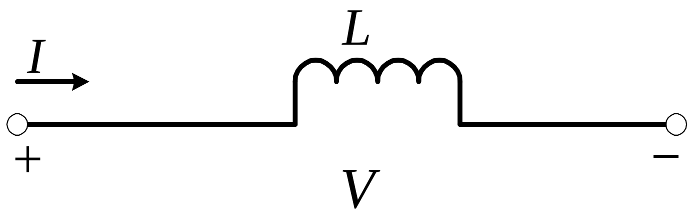
or using the notation for the coefficients

$$I_{n+1} = g_{n+1} V_{n+1} + \widehat{I}_n \qquad \text{where} \quad g_{n+1} = \frac{2C}{h_{n+1}} \text{ and } \widehat{I}_n = -I_n - \frac{2C}{h_{n+1}} V_n .$$

The circuit interpretation yields the following equivalent circuit for the capacitance



The Inductor



The circuit relations

$$V = L\frac{dI}{dt}$$

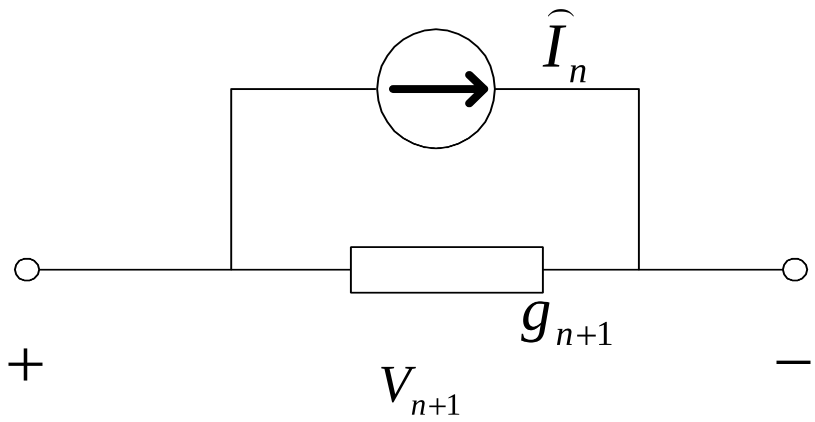
$$\frac{dI}{dt} = \frac{1}{L}V$$

The T-R formula: $I_{n+1} = I_n + \frac{h_{n+1}}{2}(\dot{I}_n + \dot{I}_{n+1})$ **where** $\dot{I} = \frac{dI}{dt}$.

Using the circuit relations we obtain

$$I_{n+1} = \underbrace{\frac{h_{n+1}}{2L}}_{g_{n+1}} V_{n+1} + \underbrace{I_n + \frac{h_{n+1}}{2L}V_n}_{\hat{I}_n}$$

The circuit interpretation of the formula yields the equivalent circuit



Replacing the capacitors and inductors by their equivalent circuits we obtain
$\Rightarrow$ COMPANION NETWORK

Exercise: Use B-E formula to get equivalent circuits for the capacitor and inductor.