USING PORTFOLIO OPTIMIZATION TECHNIQUES TO DETERMINE THE OPTIMAL PORTFOLIO

Critical Thinking Group 2

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Abstract

Building a portfolio investment is the most significant decision for an individual. This project will focus on the portfolio optimization of S&P 500 stocks with the support of the mathematical programming technique **Markowitz Mean-Variance Portfolio Analysis**, or simply the Mean-Variance Analysis, which is a normative theory. A normative theory describes a standard or norm of behavior that investors should pursue in constructing a portfolio rather than a prediction concerning actual behavior. The goal of modern portfolio theory is to reduce idiosyncratic risk which is a type of investment risk unique to an individual asset or class of assets. The eventual result of the model should give the best portfolio of the assets out of the set of all possible portfolios. This method combines multiple objective functions, minimizing the risk and maximizing the return, which can simply be achieved by maximizing the Sharpe Ratio. The higher a portfolio's Sharpe ratio, the better its risk-adjusted performance. This project will demonstrate how to optimize the portfolio investment for 50 stocks (sampling) using both historical data and investor expected returns.

Keywords

Keywords: Portfolio Optimization, Modern Portfolio Theory, Stochastic Simulation, Optimization Models, Markowitz Mean-Variance Model

Introduction

Portfolio optimization is the process of selecting the best portfolio or distribution of assets out of the set of all possible portfolios such that a given objective is satisfied. In portfolio optimization, the objective is typically to maximize the return on investment while minimizing the financial risk associated with the portfolio. In 1952, Harry Markovitz introduced the Markowitz Model, which has gone on to form the basis of modern portfolio theory. Modern portfolio theory is a mathematical framework that determines the minimum level of risk for an expected return. Specifically, modern portfolio theory provides an explanation for maximizing the ROI of a portfolio given a particular risk level. Modern portfolio theory is centered around the Markowitz Mean-Variance model which assumes that a portfolio with a lower risk will be favored over a portfolio with a higher risk. This assumption forms a central part of modern portfolio theory: that is how the risk and return of a given portfolio is affected by a certain asset (Chen, 2021). The motivation behind investigating portfolio optimization is that it is a classical optimization problem and even though it is one of the most common applications of optimization in the financial world, it happens to be one of the most complex problems. In response to the mathematical complexity surrounding portfolio optimization, a plethora of mathematical models and meta-heuristic algorithms have been developed (Zanjirdar, 2020). In general, portfolio optimization is a multi objective optimization problem since it involves minimizing risk and maximizing the return while adhering to various constraints. The constraints that are used in portfolio optimization problems vary widely depending on the portfolio that is being optimized but in general the constraints that are typically used include asset constraints, sector constraints and portfolio weight constraints which simply limit the weight of a given class of assets in a portfolio such as stocks, bonds, commodities, etc. These constraints thereby affect how diversified the portfolio is and in turn the return. The crux of any portfolio optimization problem is measuring as well as quantifying the risk of the portfolio so that it makes sense to a client.

One such benefit of using modern portfolio theory is that one can diversify the portfolio. Diversifying, or having a varied collection of assets helps minimize the variance of the portfolio. Another benefit of modern portfolio theory is that one can minimize the volatility of the portfolio. By incorporating assets that are inversely correlated such as treasuries and small cap stocks, one can expect the portfolio to generate stronger returns. Other advantages of modern portfolio theory are that it presents the ideas of risk and return in a straightforward way, thereby making the mean-variance model easy to use when solving portfolio optimization problems as well as the relative ease in formulating the mean variance model (Fernandez-Navarro, Martinez-Nieto, Carbonero-Ruz 2021). In the end, the biggest benefit of modern portfolio theory is that an optimal and efficient portfolio is created. The drawbacks of modern portfolio theory is that in the mean-variance model, the risk is defined by a dispersion parameter and it assumes that the returns are normally or elliptically distributed.

However, this often is not the case in the real world as the distributions of the returns are often asymmetric and have excess kurtosis. In addition, variance as a measure of risk has received widespread criticism by researchers due to its symmetrical measure in which desirable positive returns and undesirable negative returns end up being weighted equally. While Markowitz did recognize the limitations of the mean-variance model and proposed the semi-variance risk measure as an alternative in order to measure the variability of returns below the mean, researchers are now more concerned with underperforming portfolios rather than overperforming portfolios. As a result, researchers are now resorting to other measures of risk such as Value-at-Risk (VaR), and Conditional Value at Risk (CVaR), also known as expected shortfall (Lwin et al. 2017). Another limitation of the mean-variance model is that it has yet to be proven in out of sample validation which leads to further issues. For example, the issue of high concentration arises because assets that have either high expected returns or low expected variance will end up being overweighted, therefore making the portfolio less diversified and rendering the theory ineffective in minimizing the risk of the portfolio. In addition, the issue of instability arises because the assets that contain "good" features are prioritized and therefore the mean-variance model fails to take into account the estimation inaccuracy. This happens because when the asset undergoes even a minute change, the model ends up severely reallocating resources, regardless of the transaction cost or inaccuracies in the data. Lastly, the issue of sensitivity to input errors arises because assets that tend to have high expected returns are prioritized excess weight and therefore the model ends up being biased, thereby causing the model to be extremely sensitive to errors in the input data (Fernandez-Navarro, Martinez-Nieto, Carbonero-Ruz 2021).

The decision problem will consist of investigating a portfolio made up of a combination of various stocks, ideally from the S&P 500 and then using an optimization model to determine the optimal portfolio. The decision problem consists of maximizing the Sharpe ratio, which is simply a measure for how well an investment performs relative to its risk. Maximizing the Sharpe ratio is critical in the decision problem since it is a key metric in describing the overall performance of a portfolio. In addition, the Sharpe ratio combines maximizing the return and minimizing the risk as a single unit of measure, thereby making it the ideal objective to maximize. In the end, we hypothesize that if the portfolio is diversified and therefore has a higher return, then we would expect the Sharpe ratio to be maximized. Some of the research questions that we will address include what factors related to a portfolio affect the Sharpe ratio, what stocks affect the Sharpe ratio, and what other financial factors affect the Sharpe ratio, just to name a few. In our paper, we will be investigating some of the mathematical models and approaches that are used in portfolio optimization.

Literature review

The key characteristic of any portfolio optimization problem is determining the optimal portfolio such that the return is maximized and the risk is minimized. In addition, some of the main ideas of modern portfolio theory include the notion that risk and return are directly related, diversifying across a wide variety of security types will help minimize the overall risk of a portfolio, and that a rational person will end up choosing the portfolio that has the lower overall risk. Therefore, the relationship between risk, return, and diversification is essential to modern portfolio theory and hence a number of assumptions must be made in order to maintain. Some of these assumptions include the following: investors attempt to maximize returns given their unique situation, asset returns are normally distributed, investors are rational and avoid unnecessary risk, and unlimited amounts of capital can be borrowed at the risk-free rate (Anthony, 2020). Researchers have devised a multitude of mathematical models in response to the mathematical complexity that surrounds portfolio optimization. Portfolio optimization as a concept has been around since 1952 and over the years, many of the models that have been developed are centered around the principles that are central to modern portfolio theory, most importantly the mean-variance model. In addition, alternative approaches to the mean-variance model such as the expected shortfall and the nonparametric approaches to the mean-variance model as well as meta-heuristic algorithms have been proposed. In our paper, we will be addressing two such approaches that have been devised to address the problem of portfolio optimization and the mean-variance model.

The first optimization model builds on the limitations of the mean-variance model by formulating a nonparametric approach. In this model, the researchers achieved formulating an alternative approach to the mean-variance model in which the value at risk is being minimized rather than the variance. Formulating such an approach involved devising an optimal and efficient learning-guided hybrid multi-objective evolutionary algorithm (MODE-GL). The main reason for formulating such an algorithm is because of the immense mathematical complexity surrounding portfolio optimization. Once the MODE-GL algorithm was formulated, it was then benchmarked against two other similar algorithms, the Non-dominated Sorting Genetic Algorithm (NSGA-II) and the Strength Pareto Evolutionary Algorithm (SPEA2). The researchers opted to minimize the value at risk rather than the variance because VaR is an industry wide standardized risk measure and using the value at risk as the measure of risk allows researchers to better assess the risk of the portfolio that arises because of changes in financial and commodity asset prices. The value at risk, Ψ(w), at a given confidence level 1-α is simply the maximum expected loss that the portfolio cannot exceed with a given probability α. Measuring the value at risk can be achieved through one of three approaches: the parametric approach (variance-covariance), the non-parametric approach (historical simulation) which this specific model employs, and Monte-Carlo simulations. The parametric approach assumes that portfolio returns follow a normal or known distribution whereas the nonparametric approach doesn't assume that the returns will follow any known distribution and the Monte-Carlo approach simulates several random scenarios, therefore making it computationally challenging to determine the optimal portfolio. Minimizing the value at risk rather than the variance and constraints due to investor preferences and institutional regulations ends up rendering traditional approaches ineffective when solving portfolio optimization problems due to multiple local extrema and discontinuities. In the end, historical simulation ends up being the method of choice for calculating the value at risk and the value at risk has been widely accepted as the risk measure of choice among investment practitioners and financial regulators (Lwin et al. 2017). The formula for calculating the VaR and the mathematical formulation for this model are the following:

$$\psi(w) = \mathrm{VaR}_{\alpha}(w) = -\inf\{K_t(w) \mid \sum_{t=1}^{T} \rho_t \geq \alpha\}$$

where returns $K_t(w)$ are placed in an ascending order such that $K_1(w) \leq K_2(w) \leq \cdots \leq K_T(w)$.

minimize - $\psi(w)$ (Equation 1)
maximize - $\mu(w)$ (Equation 2)
s.t. $\sum_{i=1}^{N} w_i = 1,\ 0 \leq i \leq 1$ (Equation 3)

where N is the number of available assets and w_i ($0 \leq w_i \leq 1$) is the decision variable representing the proportion held of asset i. In addition, Equation 3 is the budget constraint for a feasible portfolio, that is all of the money should be invested in the portfolio and the non-negative property of w_i is the constraint that no short sales are allowed (Lwin et al. 2017).

One advantage of MODE-GL is that it can solve mean-variance portfolio optimization problems using real world constraints such as cardinality (number of assets in the portfolio), quantity (floor and ceiling constraints), preassignment (investor subjective preferences), round-lot, and class constraints (asset and sector constraints). The assumptions of the basic mean-variance model are that there is a perfect market where securities are traded in any non-negative fractions, a portfolio can contain an unlimited number of assets, investors are not biased towards any particular asset, and investors are not worried about a portfolio being diversified. Unfortunately, these constraints are infeasible in the real world and by incorporating these real world constraints, the MODE-GL algorithm is able to better reflect practical, real world portfolio optimization. Another advantage is that meta-heuristic and hybrid algorithms such as MODE-GL provide viable alternatives for determining optimal or near optimal solutions in a much shorter time. The data for this model was derived from historical market data from the S&P 100 and S&P 500 and based on the experimental results, the MODE-GL algorithm was able to outperform two existing techniques in terms of solution quality and computational time (Lwin et al. 2017). Despite widespread adoption by investment and financial professionals, value at risk does have its limitations. One such limitation is that using value at risk as the objective leads to a non-convex and non-differential risk-return portfolio optimization problem where the number of local optima increases exponentially as the number of assets increases (Gaivoronski, Pflug 2004). The second limitation is that when the value at risk is the objective, the portfolio optimization problem is transformed into a non-convex NP-hard problem which is computationally difficult to solve (Benati, Rizzi 2011). Another limitation is that the non-convexity of value at risk prevents diversification of a portfolio which defeats the purpose of modern portfolio theory. In essence, the value at risk of a portfolio containing two securities may be greater than a combination of the value at risk of each of the securities in the portfolio. Lastly, a limitation of value at risk is that it fails to satisfy the subadditivity property for certain distributions of asset returns (Artzner, Delbaen, Eber, and Heath 2001).

The second optimization model involves modeling the mean-variance problem as a quadratic programming problem. Specifically, this model is proposed as the mean-squared variance model, an alternative to the mean-variance model. This model seems to be an appealing approach for portfolio optimization with regards to performance (Figure 9). The mean-squared variance model is formulated by linearly combining the conflicting objectives of maximizing returns while minimizing risk through a risk-aversion profile parameter. The resulting model is a non-convex QP problem, which has been reformulated as a mixed-integer linear programming problem. In the mean-squared variance model, for a portfolio consisting of N assets, the objective is to determine the optimal weights of the portfolio's value invested in each asset. The mathematical formulation of the mean-squared variance model is derived from the mean-variance model except that the mean of the portfolio is being squared instead. In the mean-squared variance model, performance measures such as the out of sample mean returns (MR) and the out of sample Sharpe ratio (SR) were computed, the hyperparameters were globally optimized via Bayesian optimization, and statistical tests such as hypothesis tests were conducted in order to provide statistical support when evaluating the results. In addition, the hyperparameters were globally optimized so that the model could be adapted to the behavior of the market in a given dataset. The advantage of the mean-squared variance model is that by reformulating the initial non-convex QP problem as an mixed-integer linear programming problem, future researchers and practitioners are able to obtain the global solution of the problem using current state of the art mixed-integer linear programming problem solvers. Another advantage is that by expressing the expected return in squared form, it overcomes the shortcomings of the mean-variance model. The potential problems that come with the mean-squared variance model is that both objectives are measured in different units. However, this issue can be accounted for by using the Sharpe ratio (Figure 7) which combines both objectives into a single unit measure, thereby making the Sharpe ratio the ideal objective to maximize. The results of the mean-squared variance model is that when it was benchmarked on 24 datasets, eight portfolio time series problems with three different estimation windows, the MSV model ended up performing well in most of the models with regards to the return and Sharpe ratio. When the Holm test was performed to compare the mean-squared variance model to other strategies, the mean-squared variance model generated a better expected return compared to other strategies for $\alpha = 0.10$ and statistically outperformed the global mean variance (GMV) strategy for $\alpha = 0.05$. While using the Sharpe ratio as the test metric, the mean-squared variance model ended up achieving much better results than other comparison methods for both $\alpha = 0.10$ and $\alpha = 0.05$ (Table II), therefore indicating that the mean-squared variance model achieved significantly better out of sample results compared to the mean-variance model (Fernandez-Navarro, Martinez-Nieto, Carbonero-Ruz 2021). The mathematical formulation for the mean-squared variance model is the following:

$$min_{w_1, \ldots, w_N} \; - \lambda \sum_{n.m=1}^{N} w_n w_m \sigma_{nm} \; - \; (1 - \lambda)\left(\sum_{n=1}^{N} w_n \mu_n\right)^2$$

s.t. $\sum_{n=1}^{N} w_n = 1$

$w_1, \ldots, w_N \geq 0$

where the term $\sum_{n=1}^{N} w_n \mu_n$ is the portfolio mean, $\sum_{n.m=1}^{N} w_n w_m \sigma_{nm}$, $(\sigma_{nn} = \sigma_n^2)$ is the portfolio variance and $\lambda \in [0, 1]$is a hyper-parameter of the problem that weights the relative importance of the risk with respect to the mean squared return

Methodology

The methodology for modeling the decision problem primarily involved formulating the appropriate LP model. The assumptions that explain the relationship between risk, return, and diversification are essential to modern portfolio theory and therefore formulating the LP model as such was important. The existence of a plethora of suitable optimization models made it quite difficult to find an appropriate LP model. In addition, many of the optimization models that currently exist go about determining the optimal portfolio by solving for different objectives. As a result, the key aspects involved in determining the optimal portfolio include determining the suitable objective that should be maximized/minimized, the amount of decision variables that should be solved for, and the constraints that need to be satisfied in the decision problem. In addition, we needed to be strategic about how to formulate the model for the decision problem in order to account for the potential limitations of modeling software such as Analytic Solver. If the model was formulated such that Analytic Solver was solving for too many decision variables, then Analytic Solver would not be able to solve the model and vice versa. The data that is used to formulate the model includes picking stocks from an equity index such as a stock exchange. For each of the stocks that were used in the model, we relied on historical data from 2015-2021, calculated the mean and standard deviation of each stock, and used the Excel PsiNormal function as a random number generator for the expected return.

For the sake of simplicity, we decided to formulate the decision problem in which an optimal portfolio consisting of 50 S&P 500 stocks is being determined. In the decision problem, the objective will be to maximize the Sharpe ratio using different weights for each stock over a one year period, the decision variables will be the percentage of the portfolio that will be allocated for each stock, and the constraints for our model will be based on the assumptions that summarize the relationship between risk, return, and diversification. In particular, the hard constraint is that each stock will constitute no more than 20% of the portfolio and no less than 0.1% of the portfolio and the soft constraint is minimizing the overall risk of the portfolio. In addition, our model will involve using the covariance matrix to calculate the standard deviation of the portfolio that is used in the calculation of the Sharpe ratio. The Sharpe ratio was the objective function of choice primarily because the Sharpe ratio is a commonly used method that calculates the risk-adjusted return and is a key metric in describing the overall performance of the portfolio. That being said, the Sharpe ratio assumes that risk is equivalent to volatility and a higher Sharpe ratio is an indicator of the performance of the portfolio. The mathematical formulation of our model is in Figure 8 of the appendix. The formulas for calculating the expected return, variance and standard deviation of the portfolio are the following:

Expected Return: $E(R_p) = \sum_i w_i R_i$ (Equation 4)

where R_p represents the return of the portfolio, R_i represents the return for a given stock, and w_i represents the weight for each stock

Portfolio return variance: $\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j p_{ij}$ (Equation 5) where σ is the standard deviation of the periodic returns of the stocks and p_{ij} is the correlation coefficient between the returns of stocks i and j

Portfolio return volatility (standard deviation): $\sigma_p = \sqrt{\sigma_p^2}$ (Equation 6)

where σ_p^2 is the portfolio return variance

Sharpe ratio (Risk-Adjusted Return): $S_a = \frac{R_p - R_f}{\sigma_p}$ (Equation 7)

The Sharpe ratio is derived by rewriting Equation 4, Equation 5, and Equation 6 such that the return and the risk is expressed as a single unit of measure. This is important as the mean-variance model is a multi-objective LP model, thereby making it difficult to model. In the Sharpe ratio, R_p is the return of portfolio, R_f is the risk-free rate of return, and σ_p is the standard deviation of the portfolio's excess return. R_f is a hypothetical rate of return for a no-risk investment such as a government bond. Specifically, R_f is a representation of the expected interest that would be gained from a risk-free investment during a given period of time. R_f can be calculated by subtracting the current inflation rate from the yield of the treasury bond (Chen, 2021). Based on the data that we got we used the following decision variables: W_1 to W_{50} represents the percentage of the portfolio that will be allocated for each stock, R_f represents the risk-free rate as of 5/25/2021, R_p represent the return of the portfolio as calculated by Equation 1, and σ_p represents the standard deviation of the portfolio which is calculated by the covariance matrix. Therefore the setup of our model is as follows:

Objective:

Maximize - $\frac{R_p - R_f}{\sigma_p}$

Hard Constraints:
Constraint 1 - The weight of a particular stock can be no more than 20% and no less than 0.1%

Constraint 2 - The sum of the weights is 100%

Constraint 3 - Risk-free constraint as of May 25, 2021 is 0.06%

Soft Constraints
Constraint 1 - minimize risk of the portfolio

Computational Experiment and Results

In our model, we used the Simulation Optimization engine in Analytic Solver to perform Stochastic Simulation. Performing stochastic simulation allowed us to determine the weights that should be allocated for each stock that was picked. For each of the fifty stocks we looked at the historical data starting from the beginning of 2015 through May 24, 2021. The daily return for each stock was calculated. From the daily returns we calculated the average and standard deviation for each of the fifty stocks see figure 4. From the average and standard deviation we used the PsiNormal to formulate a random number generator from the normal distribution for returns for each of the fifty stocks see figure 5. Another critical component of the experiment was creating the covariance matrix for all all fifty stocks. Excel has a covariance function in the data analysis package to create a covariance matrix for the daily returns of all stocks see figure 3. The Sharpe ratio was calculated using the sumproduct of the weights annualized (multiply by 252 the number of trading days in a year) and randomly generated daily stock returns subtracted from the risk free rate (Figure 6). The risk free rate in this case was the return from the one year treasury bond since we are holding our portfolio for one year. The denominator of the Sharpe ratio was calculated using the formula provided in Figure 10 of the appendix. In Excel, the MMULT function was used (Figure 6). The stochastic optimization was performed using the Analytical Solver add-in for excel. The number of trials to be performed was set to 1000 and the number of simulations to be performed was set to 1 (Figure 2).

Our results were an average Sharpe ratio of 1.71 (Figure 1). As per industry standards, a Sharpe ratio between 1 and 2 is considered good, a Sharpe ratio above 2 is great, and a Sharpe ratio above 3 excellent. Our model produced a Sharpe ratio of 1.71 which was above our goal of 1 and is therefore considered good by industry standards. Beta is a measurement of how the stock moves with the overall market. Beta is not something that should be maximized or minimized but rather beta should be optimal given the portfolio in question. Portfolios containing assets that primarily have high betas will be overexposed to market risk and portfolios containing assets that primarily have low betas will not have the upside potential. To check if our model has a good mix of stocks, we evaluated the top 6 stocks in our portfolio based on their weight and respective beta. As we can see from the top 6 stocks, there is a good mix of high and low beta stocks. This mix of high and low betas means our portfolio has stuck a good balance in volatility. The full list of stocks with the respective weights is in Table I of the appendix.

Stock	Weight	Beta
AMZN	20.00%	1.15
NVDA	20.00%	1.37
LLY	12.59%	0.25
COST	8.80%	0.64
DHR	8.10%	0.64
TSLA	8.00%	1.98

Discussion and Conclusions

If we compare the Sharpe ratio results that we got (1.71) with the Sharpe value S&P 500 which was 2.7, we can reason the difference as such:

1. Amount of variables - S&P 500 use 500 stocks vs our model 50 stocks. More stocks can help and increase the Sharpe ratio.
2. Adding some constraints to the weights on the class, by experimenting with the model that we had we could increase the minimum amount of shares that will have a weight of more than 0.1%. This will increase the Sharpe ratio value.
3. Our portfolio had a good mix of betas, adding some constraints for the beta can also help improve the sharpe ratio.

It is important to note that overall we did reach our goal as we got Sharpe Ratio above 1.0 and the stocks that the model picked were from different industries. Few limitations that should be notes:

1. Since we are using standard deviation of returns in the denominator we assume the returns are normally distributed, this is not always the case. Covid-19 impact macny stocks are very skewed.
2. High spikes can impact the results of the model and give different weights some years.

In order to overcome some of those challenges we could have picked more balanced stocks. In addition, distributing the weights of the investment across more stocks could have helped increase the ration but also decrease the return. Overall the model performed well and it will be interesting to test a similar model in other fields like agriculture in order to determine the best mix of crops that a farming company should invest in to reduce the risk. Cryptocurrency is another interesting topic that could use this model or even involve it in our current model.

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Appendix

I. Spreadsheet Model Setup

Figure 1

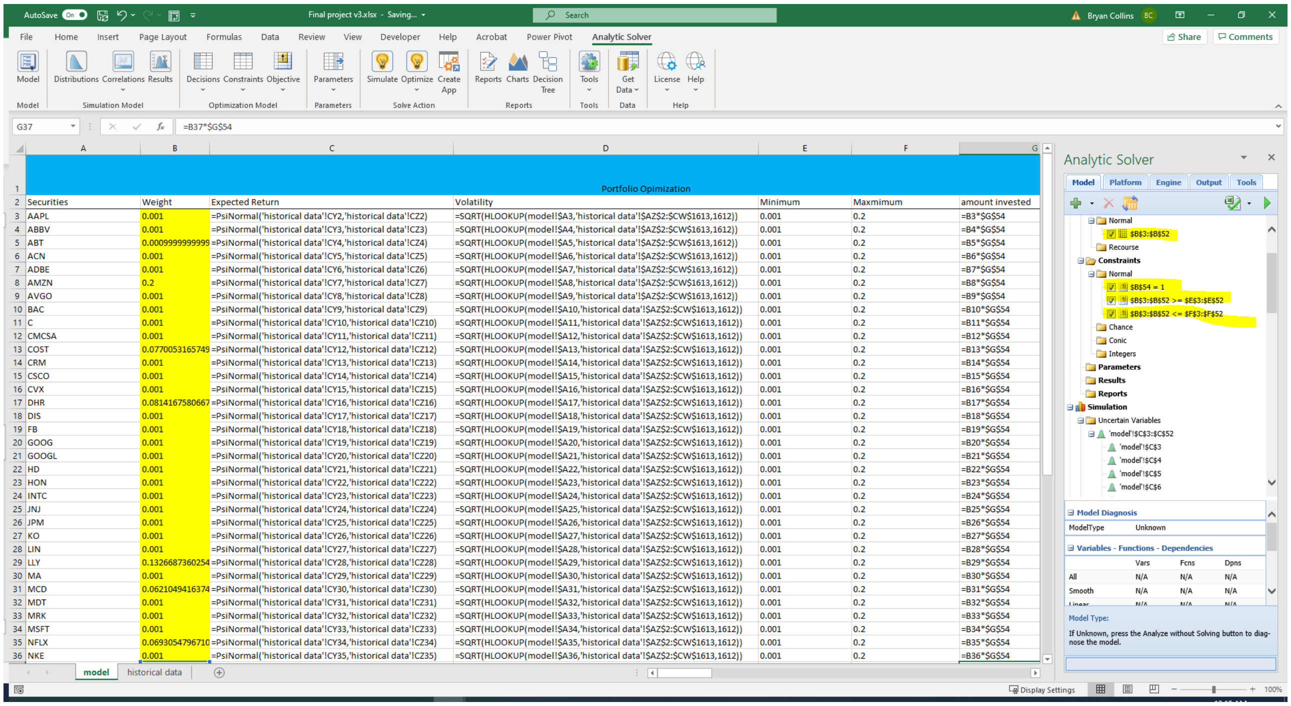

Figure 2

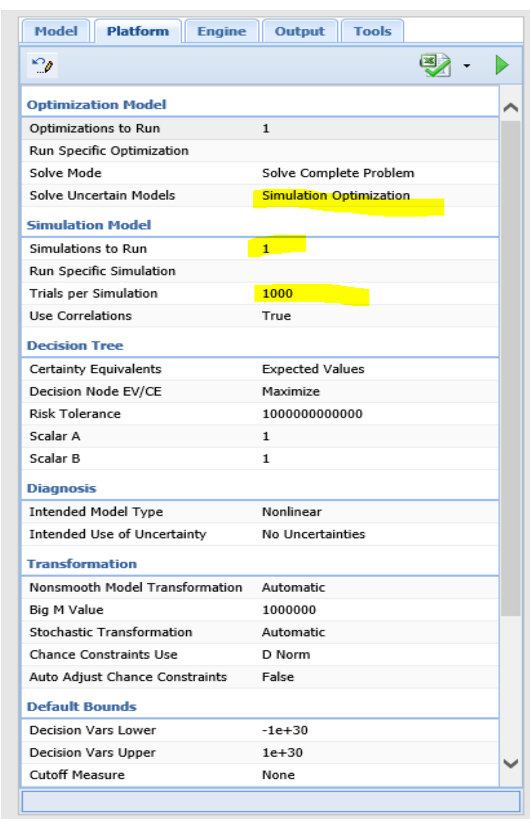

Figure 3

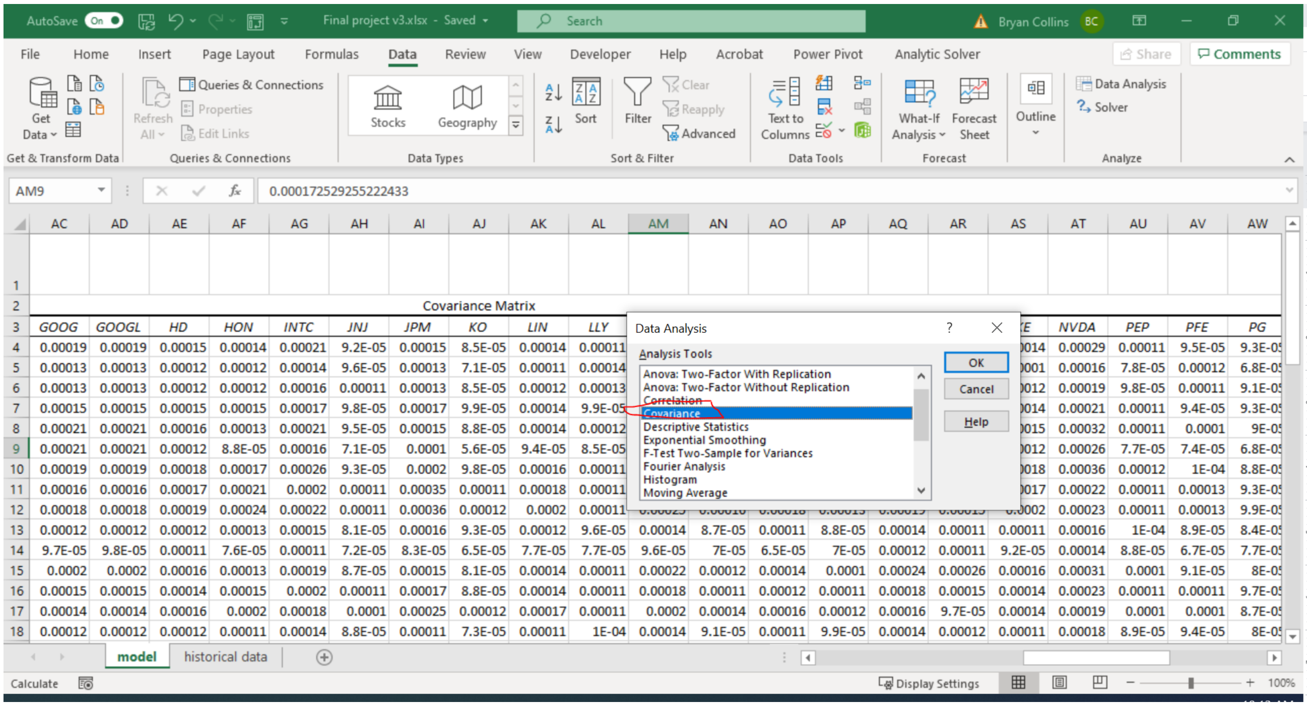

Figure 4

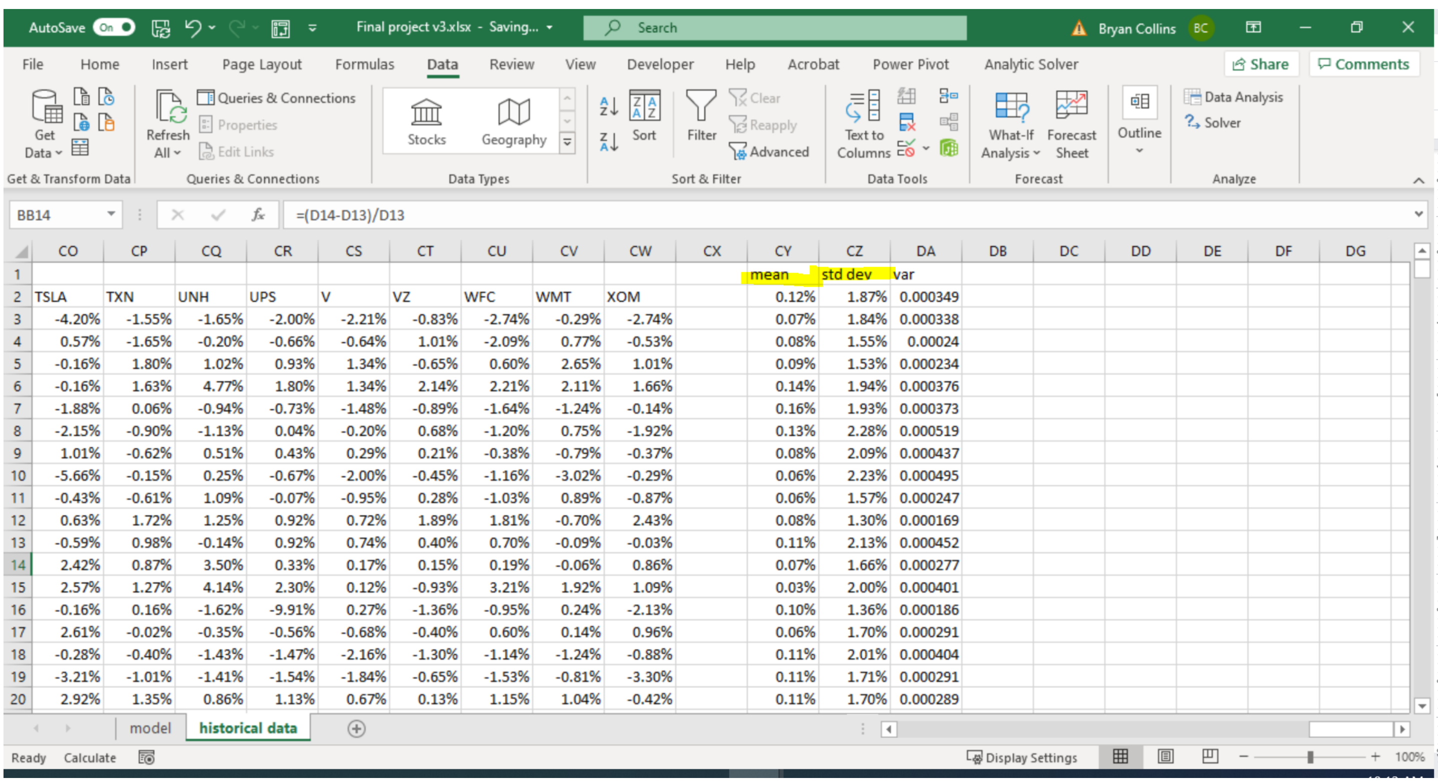

Figure 5

50	WFC	0.001	=PsiNormal('historical data'!CY49,'historical data'!CZ49)	=SQRT(HLOOKUP(model!$A50,'historical data'!$AZ$2:$CW$1613,1612))	0.001
51	WMT	0.000999999999999	=PsiNormal('historical data'!CY50,'historical data'!CZ50)	=SQRT(HLOOKUP(model!$A51,'historical data'!$AZ$2:$CW$1613,1612))	0.001
52	XOM	0.001	=PsiNormal('historical data'!CY51,'historical data'!CZ51)	=SQRT(HLOOKUP(model!$A52,'historical data'!$AZ$2:$CW$1613,1612))	0.001
53			Expected Return	Sd (Risk)	
54	Portfolio	=+SUM(B3:B52)	=SUMPRODUCT(C3:C52,B3:B52)*252	=SQRT(MMULT(MMULT(TRANSPOSE(B3:B52),L4:BI53),B3:B52*252))	
55	Sharpe Ratio	=(C54-B59)/D54	>=	1	good
56		=IF(B57<=1,"not ac	>=	2	great
57	mean	=PsiMean(B55)	>=	3	excellent
58					
59	risk free rate	0.0006			
60					
61					
62					

Figure 6

7	UPS	0.10%	-0.30%	1.58%	0.1%	20.0%	$	100.00	
8	V	0.10%	-0.35%	1.60%	0.1%	20.0%	$	100.00	
9	VZ	0.10%	3.62%	1.16%	0.1%	20.0%	$	100.00	
0	WFC	0.10%	0.59%	2.01%	0.1%	20.0%	$	100.00	
1	WMT	0.10%	-1.32%	1.38%	0.1%	20.0%	$	100.00	
2	XOM	0.10%	-0.93%	1.77%	0.1%	20.0%	$	100.00	
3									
4	Portfolio	100%	64.35%	22.80%			$	100,000.00	
5	Sharpe Ratio	2.82	>=	1	good				
6		good	>=	2	great				
7	mean	1.71	>=	3	excellent				
8									
9	risk free rate	0.06%							
0									
1									

II. Data collection

Table I - Final weights of 50 stocks chosen from the S&P 500

Securities	Weight	Securities	Weight	Securities	Weight	Securities	Weight	Securities	Weight
AAPL	0.10%	AVGO	0.10%	CSCO	0.10%	GOOGL	0.10%	KO	0.10%
ABBV	0.10%	BAC	0.10%	CVX	0.10%	HD	0.10%	LIN	0.10%
ABT	0.10%	C	0.10%	DHR	8.14%	HON	0.10%	LLY	13.27%
ACN	0.10%	CMCSA	0.10%	DIS	0.10%	INTC	0.10%	MA	0.10%
ADBE	0.10%	COST	7.70%	FB	0.10%	JNJ	0.10%	MCD	6.21%
AMZN	20.00%	CRM	0.10%	GOOG	0.10%	JPM	0.10%	MDT	0.10%

Securities	Weight	Securities	Weight	Securities	Weight	Securities	Weight
MRK	0.10%	PFE	0.10%	TXN	0.10%	WMT	0.10%
MSFT	0.10%	PG	0.10%	UNH	5.29%	XOM	0.10%
NFLX	6.93%	QCOM	0.10%	UPS	0.10%		
NKE	0.10%	T	0.10%	V	0.10%		
NVDA	20.00%	TMO	0.10%	VZ	0.10%		
PEP	0.10%	TSLA	8.36%	WFC	0.10%		

III. Formulas that were used to formulate the decision problem

Figure 7 - Formula for calculating the Sharpe ratio

Formula and Calculation of Sharpe Ratio

$$Sharpe\ Ratio = \frac{R_p - R_f}{\sigma_p}$$

where:

R_p = return of portfolio

R_f = risk-free rate

σ_p = standard deviation of the portfolio's excess return

Figure 8 - Mathematical Formulation for MPT

In general:

- Expected return:

$$\mathrm{E}(R_p) = \sum_i w_i \, \mathrm{E}(R_i)$$

where R_p is the return on the portfolio, R_i is the return on asset *i* and w_i is the weighting of component asset i (that is, the proportion of asset "i" in the portfolio).

- Portfolio return variance:

$$\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \rho_{ij},$$

where σ is the (sample) standard deviation of the periodic returns on an asset, and ρ_{ij} is the correlation coefficient between the returns on assets *i* and *j*. Alternatively the expression can be written as:

$$\sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij},$$

where $\rho_{ij} = 1$ for $i = j$, or

$$\sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_{ij},$$

where $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$ is the (sample) covariance of the periodic returns on the two assets, or alternatively denoted as $\sigma(i, j)$, cov_{ij} or $\mathrm{cov}(i, j)$.

- Portfolio return volatility (standard deviation):

$$\sigma_p = \sqrt{\sigma_p^2}$$

IV. Additional Figures and Tables

Figure 9 - Cumulative monthly returns plot of the four portfolios implemented from 1 September 2017–1 August 2020.

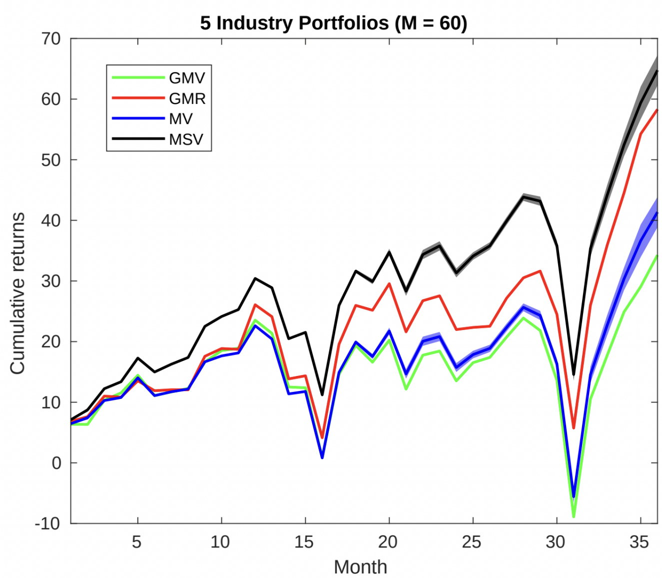

Figure 10 - Formula for calculating the expected variance of the portfolio

Expected portfolio variance= SQRT (W^T * (Covariance Matrix) * W)

Table II - Statistical results for the Holm test for α = 0.10 and α = 0.05 using the mean squared variance (MSV) strategy as the control method: mean MR and SR ranking of the strategies implemented (R_{MR} and R_{SR}), z-statistics and p-values of the Holm tests for the MR and SR analysis and adjusted α Holm values ($\alpha_{0.10}$ and $\alpha_{0.05}$). The best results are in boldface and second-best results in italics

MR Analysis					
Method	$\overline{R}_{MR}$	z-statistic	p-value	$\alpha_{0.10}$	$\alpha_{0.05}$
GMV$_{\bullet,\circ}$	3.2500	4.0249	1×10^{-4}	0.0333	0.0167
GMR$_{\bullet}$	2.5208	2.0683	0.0386	0.0500	0.0250
MV$_{\bullet}$	*2.4792*	1.9566	0.0504	0.1000	0.0500
MSV	**1.7500**	-	-	-	-
SR Analysis					
Method	$\overline{R}_{SR}$	z-statistic	p-value	$\alpha_{0.10}$	$\alpha_{0.05}$
GMV$_{\bullet,\circ}$	3.0208	3.5218	4×10^{-4}	0.0333	0.0167
GMR$_{\bullet,\circ}$	2.7292	2.7394	0.0062	0.0500	0.0250
MV$_{\bullet,\circ}$	*2.5417*	2.2362	0.0253	0.1000	0.0500
MSV	**1.7083**	-	-	-	-

•: Statistical difference with α = 0.10; ○: Statistical difference with α = 0.05.