

Monadic Second Order Logic as the Model Companion of Temporal Logic; a clearer version

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This is nothing more than another version of the 2016 paper by Ghilardi and van Gool. Since the proofs in [2] are mostly developed in the appendix, rewriting them completely in a more readable way is a quite useful exercise. Doing so, my goal is to understand deeply how the constructions work, in order to produce my own version for finite trees with counters by the end of my internship. If I explicit some point more than it was in the original paper, or if I change the order of a few definitions, it is because my understanding benefits from it. Conversely, if I move quickly on a topic, it is because it is already clear in my head.

1 Syntax and Semantics of fair CTL

Fair CTL, which is an extension of the well-known computational tree logic, is defined intuitively so it can simulate the termination condition for executions of tree automata. For example, the definition of EG is close to what we expect for a fair execution, ie. a branch on which a formula is true infinitely often (or a final state visited infinitely often).

We will define a *syntax* for the logic CTL^f , which will allow to interpret formulas as elements of a particular type of Boolean algebra, and to use results on duality for those algebras.

On the other hand, we will define *semantics*, which will allow to interpret such formulas on models, ie. trees whose nodes are labeled by sets of propositional variables supposed to hold on each node. Next section will then be dedicated to prove that these semantics are *complete* with respect to the syntax, in other words, that for every formula φ that holds in every model, we can syntactically deduce that φ is equivalent to the \top formula of the logic.

Syntax of fair CTL

Definition 1.1. Let $\bar{p} = \{p_1, \dots, p_n\}$ be a finite set of propositional variables. We define inductively a CTL^f -formula φ to be of the shape:

- \perp
- $p \in \bar{p}$
- $\neg\varphi$
- $\varphi \vee \psi$
- $\diamond\varphi$
- $EU(\varphi, \psi)$
- $EG(\varphi, \psi)$

For convenience, we then also define the De Morgan duals of our binary operations, respectively \wedge for \vee ; AR for EU; AF for EG. For the unary operator, we set $\Box\varphi = \neg\diamond(\neg\varphi)$; Finally, we set \top to be the negation of \perp , naturally.

Definition 1.2. We define the quasi-equational theory CTL^f by requiring the formulas to satisfy the following axioms:

1. Boolean algebra axioms for \vee, \neg, \perp ;
2. Unary operator \diamond preserves finite joins, including the empty joint \perp , making CTL^f into a modal logic;
3. $\diamond\top = \top$
4. Binary operators EU and EG satisfy the following fixpoint axioms for all a, b, c :

- $a \vee (b \wedge \diamond EU(a, b)) \leq EU(a, b)$
- $a \vee (b \wedge \diamond c) \leq c \implies EU(a, b) \leq c$
- $EG(a, b) \leq a \wedge \diamond EU(b \wedge EG(a, b), a)$
- $c \leq a \wedge \diamond EU(b \wedge c, a) \implies c \leq EG(a, b)$

In other words, $EU(a, b)$ is the least pre-fixpoint of the function $x \mapsto a \vee (b \wedge \diamond x)$. And $EG(a, b)$ is the greatest post-fixpoint of the function $x \mapsto a \wedge \diamond EU(b \wedge x, a)$.

Definition 1.3. Given what is said just above, we can deduce the definition of a CTL^f -algebra, which is a tuple $\mathbb{A} = (A, \perp, \neg, \vee, \diamond, EU, EG)$ which verifies every axiom of CTL^f quasi-equational theory.

Importantly, we remark that for all a, b , $AR(a, b)$ is the greatest post-fixpoint of $x \mapsto a \wedge (b \vee \Box x)$, while $AF(a, b)$ is the least pre-fixpoint of $x \mapsto a \vee \Box AR(b \vee c, a)$.

Also, the way EU and EG formulas are defined reminds the fixpoint operators of the μ -calculus. For example, for every a, b , $EU(a, b)$ could have been defined as $\mu x.(a \vee (b \wedge \Diamond x))$, the least fixpoint of this monotone function (it is enough to say the least *pre-fixpoint*). As for EG, it is a greatest (*post*-) fixpoint, so it could have been defined with a ν operator, where $\nu f = \neg \mu(\neg f)$. We can see that such a definition of EG makes apparent a nesting of a ν and a μ operator. This is what makes the logic hard to simplify, compared to LTL which only contains one such fixpoint operator.

Definition 1.4. *For any finite set of propositional variables \bar{p} and any CTL^f -algebra \mathbb{A} , we can define a valuation $V : \bar{p} \rightarrow A$. Then any CTL^f -formula φ with variables in \bar{p} can be interpreted as a term in A (the interpretation of φ under V , $\varphi^A(V)$).*

Then, we can say the equality $\varphi(\bar{p}) = \psi(\bar{p})$ is valid if it interprets to \top under every valuation to every \mathbb{A} .

φ and ψ are said to be equivalent if the equation $\varphi = \psi$ is valid. φ is called a tautology if it is equivalent to \top , and is said to be consistent if it is not equivalent to \perp .

Finally, we say φ entails ψ if $\neg \varphi \vee \psi$ is a tautology, which we will note $\varphi \vdash \psi$ or $\varphi \leq \psi$.

Semantics of fair CTL

Definition 1.5. *We first define the notion of transition system, ie. a pair (S, R) , where S is a set, and R a binary relation on S . Then, a R -path is a (possibly infinite) sequence of nodes in S such that $s_i R s_{i+1}$ for all i . Finally, for a finite set \bar{p} of variables, define a \bar{p} -coloring $\sigma : S \rightarrow \mathcal{P}(\bar{p})$.*

We can define the forcing relation \models between nodes in S and formulas φ with variables in \bar{p} , by induction on φ :

- $s \not\models \perp$;
- $s \models p$ if $p \in \sigma(s)$;
- $s \models \neg \varphi$ if not $s \models \varphi$;
- $s \models \varphi \vee \psi$ if $s \models \varphi$ or if $s \models \psi$;
- $s \models \Diamond \varphi$ if there exists s' such that $s R s'$ and $s' \models \varphi$;
- $s \models EU(\varphi, \psi)$ if there exists $n \geq 0$ and a path $s_0 R s_1 R \dots R s_n$ ($s_0 = s$) such that $s_k \models \psi$ for every $k < n$ and $s_n \models \varphi$;

- $s \models EG(\varphi, \psi)$ if there exists an infinite path $s_0Rs_1R\ldots$ ($s_0 = s$) such that $s_k \models \varphi$ for every k and $s_j \models \psi$ infinitely often on the path.

Remark that as a consequence, $s \models \Box\varphi$ if and only if for every successor s' of s by R , $s' \models \varphi$ holds. Also, $s \models AR(\varphi, \psi)$ if and only if for every $n \geq 0$ and every path $s = s_0Rs_1R\ldots Rs_n$, either $s_k \models \psi$ for some $k < n$, or $s_n \models \varphi$.

Similarly, $s \models AF(\varphi, \psi)$ if and only if for all infinite paths $(s)_k$ starting from s_0 , if $s_j \not\models \psi$ infinitely often, then $s_k \models \varphi$ for some k .

Also, as a convention, we will consider the transition systems to be *serial*, ie. that every node has a successor. Syntactically, this is represented by the axiom $\Diamond\top = \top$.

Now, in order to prove completeness, we introduce the notion of *complex algebra* of a transition system. This can be seen as a canonical algebra for the semantics of our logic. Then, we will be able to say that a formula holds semantically if, and only if, it can be derived in the complex algebra.

Definition 1.6. Given a transition system (S, R) , its complex algebra is the tuple

$$\mathbb{P}_{(S,R)} = (\mathcal{P}(S), \emptyset, S \setminus (-), \cup, \Diamond_R, EU_R, EG_R)$$

which is the classical Boolean algebra over the powerset of S , along with operators

$$\Diamond_R(a) = R^{-1}[a] = \{s \in S \mid \exists t \in a, sRt\}$$

and EU_R and EG_R defined as fixpoints (see above) so that we finally have a CTL^f -algebra.

Lemma 1.7. For every a, b in $\mathbb{P}_{(S,R)}$ it holds that

$$EU_R(a, b) = \bigcup_{i=0}^{\infty} D_n(a, b)$$

where D_n is defined inductively as

$$\begin{cases} D_0(a, b) = \emptyset \\ D_{n+1}(a, b) = a \cup (b \cap \Diamond_R D_n(a, b)) \end{cases}$$

Proof. \supseteq By induction on n we show $D_n(a, b) \subseteq EU_R(a, b)$.

For $n = 0$ this is obvious. For the induction step, let $s \in D_{n+1}(a, b)$. By the pre-fixpoint definition of EU we have $a \leq EU_R(a, b)$ and $(b \cap \Diamond EU_R(a, b)) \leq EU_R(a, b)$. It follows that if s is in a , it is directly in $EU_R(a, b)$; and in the other case, there exists s' in $D_n(a, b)$ such that sRs' . By hypothesis $s' \in EU_R(a, b)$. We have exactly $s \in b \cap \Diamond EU_R(a, b)$, from which we conclude $s \in EU_R(a, b)$.

\subseteq We will show that $D(a, b) := \bigcup_{i=1}^{\infty} D_i(a, b)$ is a pre-fixpoint of $x \mapsto a \cup (b \cap \diamond x)$. From the definition of EU the inclusion will directly follow.

Let $s \in a \cup (b \cap \diamond D(a, b))$. Note that $D_1(a, b) = a$ so if $s \in a$ it is in $D(a, b)$. And if $s \in (b \cap \diamond D_k(a, b))$ for some k (\diamond preserves countable unions), it is in $D_{k+1}(a, b)$ by definition. We can conclude here. \square

Note that $D_n(a, b)$ is the set:

$$\{s \in S \mid \exists s_1, \dots, s_n, s = s_0 R s_1 R \dots R s_n, s_k \in a \text{ for all } k < n \text{ and } s_n \in b\}$$

We can now state the corresponding lemma for EG :

Lemma 1.8. *For every a, b in $\mathbb{P}_{(S, R)}$ it holds that*

$$EG_R(a, b) = \bigcap_{i=0}^{\infty} D'_i(a, b)$$

where D'_n is defined inductively as

$$\begin{cases} D'_0(a, b) = \mathcal{P}(S) \\ D'_{n+1}(a, b) = a \cap \diamond_R EU_R(b \cap D'_n(a, b), a) \end{cases}$$

Note that $D'_n(a, b)$ is the set:

$$\{s \in S \mid \exists s_1, \dots, s_p, s = s_0 R s_1 R \dots R s_p, \forall k \leq p \ s_k \in a \text{ and } s_k \in b \text{ for } n \leq k\}$$

Proposition 1.9. *Consider a transition system (S, R) along with a \bar{p} -coloring $\sigma : S \rightarrow \mathcal{P}(\bar{p})$. This coloring corresponds exactly to a valuation $V_\sigma : \bar{p} \rightarrow \mathcal{P}(S)$, as we in fact have a bijection between \bar{p} -colorings and such valuations.*

Then, for every CTL^f -formula φ with variables in \bar{p} and for every $s \in S$, we have:

$$s \models \varphi \iff s \in \varphi^{\mathbb{P}_{(S, R)}}(V_\sigma(\bar{p}))$$

Proof. First, we show there is a bijection between \bar{p} -colorings $\sigma : S \rightarrow \mathcal{P}(\bar{p})$ and valuations $V : \bar{p} \rightarrow \mathcal{P}(S)$. Given σ , define $V_\sigma : \bar{p} \mapsto \{s \in S \mid p \in \sigma(s)\}$. And conversely, given V , we can define $\sigma_V : v \mapsto \{p \in \bar{p} \mid v \in V(p)\}$.

Second, we show the equivalence, by induction on the structure of φ . Looking at the definitions, it should be obvious for all cases up to diamond; as for EU and EG , the wanted equivalence is a consequence of lemmas 1.7 and 1.8. \square

2 Completeness proof

This section will be dedicated to showing completeness of the above axiomatization of CTL^f . Equivalently, we show that consistent formulas in an algebra (not equal to \perp) can be satisfied semantically.

Definition 2.1. *A tree is a transition system (S, R) with a root, such that for every s in S , there is a unique path from the root to s . This implies a partial order \preceq , the reflexive-transitive closure of R , which has the property that $v \preceq v'$ if and only if v lies on the unique path from the root to v' .*

Theorem 2.2. *For every consistent CTL^f -formula $\varphi_0(\bar{p})$, there exists a \bar{p} -colored tree with root s_0 , such that $s_0 \models \varphi_0$.*

The rest of this section consists of the proof of theorem 2.2. Let us fix a consistent formula $\varphi_0(\bar{p})$. We then have a CTL^f -algebra \mathbb{A} and a valuation $V : \bar{p} \rightarrow \mathbb{A}$ such that $\varphi_0^{\mathbb{A}}(V(\bar{p})) \neq \perp$. Now, since \mathbb{A} , \bar{p} and V are fixed in the rest of the section, we may leave them implicit and assimilate φ_0 to its interpretation.

Preliminaries

We begin by proving a few facts about EU , EG , etc. that will be useful in the rest of the section.

Proposition 2.3. *For every a, a', b, b', c in \mathbb{A} , the following hold:*

1. $EU(a \vee a', b) = EU(a, b) \vee EU(a', b)$;
2. $AR(a \wedge a', b) = AR(a, b) \wedge AR(a', b)$;
3. $a \leq a'$ and $b \leq b'$ implies $EG(a, b) \leq EG(a', b')$;
4. $a \leq a'$ and $b \leq b'$ implies $AF(a, b) \leq AF(a', b')$
5. $AR(a, b) \wedge \neg b \leq \Box AR(a, b)$
6. if $a \wedge c \leq \Box c$ and $b \wedge c \leq \Box c$, then $EU(a, b) \wedge c \leq EU(a \wedge \Box c, b \wedge \Box c)$.

Proof. 1. EU is defined as the fixpoint of a monotone function, therefore it is itself monotone. So we have $EU(a, b) \leq EU(a \vee a', b)$ and $EU(a', b) \leq EU(a \vee a', b)$, which entails $EU(a, b) \vee EU(a', b) \leq EU(a \vee a', b)$. For the other way, it suffices to prove that $EU(a, b) \vee EU(a', b)$

is a pre-fixpoint of the same function as $EU(a \vee a', b)$. We have, by monotonicity of \diamond and by definition of EU ,

$$\begin{aligned} & (a \vee a') \vee (b \wedge \diamond(EU(a, b) \vee EU(a', b))) \\ &= (a \vee (b \wedge \diamond(EU(a, b)))) \vee (a' \vee (b \wedge \diamond(EU(a', b)))) \\ &\leq EU(a, b) \vee EU(a', b) \end{aligned}$$

This proves our point.

2. this is clear from the fact that AR is the De Morgan dual of EU .
3. clear from the definition of EG as the fixpoint of a monotone function.
4. idem.
5. by definition of AR , we have $AR(a, b) \leq b \vee \Box(AR(a, b))$, which is equivalent to the statement we want to prove.
6. suppose that $a \wedge c \leq \Box c$ and $b \wedge c \leq \Box c$. Let us prove that $d := EU(a \wedge \Box c, b \wedge \Box c) \vee \neg c$ is a pre-fixpoint of the same function as $EU(a, b)$, which will entail our point by minimality of $EU(a, b)$.
 - $a \wedge c \leq a$, and by assumption $a \wedge c \leq \Box c$, so $a \wedge c \leq a \wedge \Box c \leq (a \wedge \Box c) \vee (b \wedge \Box c \wedge \diamond EU(a \wedge \Box c, b \wedge \Box c)) \leq EU(a \wedge \Box c, b \wedge \Box c)$;
 - note that $b \wedge c \wedge \diamond d = (b \wedge c \wedge \diamond(\neg c)) \vee ((b \wedge c) \wedge \diamond EU(a \wedge \Box c, b \wedge \Box c))$; but $b \wedge c \wedge \diamond(\neg c) \leq b \wedge \Box c \wedge \diamond \neg c = \perp$. Then we have $b \wedge c \wedge \diamond d \leq (b \wedge \Box c) \wedge \diamond EU(a \wedge \Box c, b \wedge \Box c) \leq EU(a \wedge \Box c, b \wedge \Box c)$.
 - From the two above points we conclude that $a \vee (b \wedge \diamond d) \leq d$.

□

Definition 2.4. We define ternary contextual operations EU_c and AF_c :

- $EU_c(p, q, r) = p \vee (q \wedge \diamond EU(p \wedge r, q \wedge r))$,
- $AF_c(p, q, r) = AF(p, q) \wedge (p \vee \Box AR(q \wedge r, p))$.

Proposition 2.5. For every p, q, r in \mathbb{A} , we have:

1. $EU_c(p, q, r)$ is the least pre-fixpoint of the monotone function $x \mapsto p \vee (q \wedge \diamond(r \wedge x))$;
2. $AF_c(p, q, r)$ is the greatest post-fixpoint of the monotone function $x \mapsto p \vee \Box AR(q \vee (r \wedge x), p)$.

Proof. 1. First, we have the equality

$$\begin{aligned} EU_c(p, q, r) \wedge r &= (p \wedge r) \vee (q \wedge r \wedge \diamond EU(p \wedge r, q \wedge r)) \\ &= EU(p \wedge r, q \wedge r) \end{aligned}$$

by definition of EU . So $EU_c(p, q, r)$ is a fixpoint of the required function, as by definition:

$$EU_c(p, q, r) = (\diamond(EU(p \wedge r, q \wedge r)) \wedge q) \vee p$$

Now, consider any fixpoint s of this function. Then, $(p \wedge r) \vee (q \wedge r \wedge \diamond(r \wedge s)) = r \wedge (p \vee (q \wedge \diamond(r \wedge s))) \leq r \wedge s$. By minimality, it entails that $EU(p \wedge r, q \wedge r) \leq r \wedge s$. Finally $EU_c(p, q, r) = p \vee (q \wedge \diamond EU(p \wedge r, q \wedge r)) \leq p \vee (q \wedge \diamond(r \wedge s)) \leq s$, which ends the proof.

2. Consider function $\sigma : x \mapsto p \vee \Box AR(q \vee (r \wedge x), p)$.

- $AF_c(p, q, r)$ is a pre-fixpoint of σ .

Note, by definition, that $AF_c(p, q, r) \leq AF(p, q)$. So

$$\begin{aligned} p \vee \Box AR(q \vee AF_c(p, q, r), p) \\ \leq p \vee \Box AR(q \vee AF(p, q), p) = AF(p, q) \end{aligned}$$

by definition of AF . By 2.3(2), we have

$$\begin{aligned} \sigma(AF_c(p, q, r)) &= (p \vee \Box AR(q \vee r, p)) \wedge (p \vee \Box AR(q \vee AF_c(p, q, r), p)) \\ &\leq (p \vee \Box AR(q \vee r, p)) \wedge AF(p, q) \\ &\leq AF_c(p, q, r). \end{aligned}$$

- $AF_c(p, q, r)$ is the least such pre-fixpoint. Let $s \in A$ be such that $\sigma(s) \leq s$. Write $\alpha = AR(a \vee r, p)$. Then the following claim is true:

$$AF(p \vee \diamond \neg \alpha, q \vee s) \wedge \Box \alpha \leq s$$

Indeed, by hypothesis $p \leq s$ and $\Box AR(q \vee (r \wedge s), p) \leq s$. From there, $\neg q \wedge \neg s \wedge \alpha \leq \neg p \wedge \alpha \leq \Box \alpha$ by 2.3(5). Applying 2.3(6), we get that

$$\begin{aligned} \neg AR(q \vee s, p) \wedge \alpha &= EU(\neg q \wedge \neg s, \neg p) \wedge \alpha \\ &\leq EU(\neg q \wedge \neg s \wedge \Box \alpha, \neg p \wedge \Box \alpha). \end{aligned}$$

Now, negation inverts the inequation:

$$AR(q \vee s \vee \Diamond \neg \alpha, p \vee \Diamond \neg \alpha) \leq AR(q \vee s, p) \vee \neg \alpha,$$

to which we can apply \Box on both sides. Now, $\neg \alpha = \neg AR(q \vee r, p)$ by definition; and 2.3(2) tells us that

$$AR(q \vee (r \wedge s), p) = AR(q \vee r, p) \wedge AR(q \vee s, p).$$

The two combined give that

$$\begin{aligned} \Box(AR(q \vee s, p) \wedge \neg \alpha) &= \Box(AR(q \vee (r \wedge s), p) \vee \neg \alpha) \\ &\leq \Box AR(q \vee (r \wedge s), p) \vee \Diamond \neg \alpha, \end{aligned}$$

by a property of modal algebras. We can combine all of the above and get that

$$\begin{aligned} \Box AR(q \vee s \vee \Diamond \neg \alpha, p \vee \Diamond \neg \alpha) &\leq \Box AR(q \vee (r \wedge s), p) \vee \Diamond \neg \alpha \\ &\leq s \vee \Diamond \neg \alpha. \end{aligned}$$

Remember also that $p \leq s$. This gives that

$$(p \vee \Diamond \neg \alpha) \vee \Box AR(q \vee s \vee \Diamond \neg \alpha, p \vee \Diamond \neg \alpha) \leq s \vee \Diamond \neg \alpha.$$

Apply definition of AF and get:

$$AF(p \vee \Diamond \neg \alpha, q \vee s) \leq s \vee \Diamond \neg \alpha,$$

which is equivalent to our very first claim. In the end, by definition we have

$$\begin{aligned} AF_c(p, q, r) &= AF(p, q) \wedge (p \vee \Box \alpha) \\ &= (AF(p, q) \wedge p) \vee (AF(p, q) \wedge \Box \alpha) \\ &\leq p \vee (AF(p, q) \wedge \Box \alpha) \\ &\leq p \vee (AF(p \vee \Diamond \neg \alpha, q \vee s) \wedge \Box \alpha) \text{ by monotonicity of } AF \\ &\leq s \text{ by everything above.} \end{aligned}$$

This is what we wanted to prove.

□

Note that replacing EU by EU_c and AF by AF_c does not change the expressivity of the syntax, as $EU(a, b) = EU_c(a, b, \top)$ and $AF(a, b) = AF_c(a, b, \top)$. We then use notations EU and AF for either, as this presents no ambiguity.

Proposition 2.6. *For every p, q, r, γ in \mathbb{A} , we have:*

1. *if $\gamma \wedge EU(p, q, r) \neq \perp$, then $\gamma \wedge EU(p, q, r \wedge \neg\gamma) \neq \perp$;*
2. *if $\gamma \wedge AF(p, q, r) \neq \perp$, then $\gamma \wedge AF(p, q, r \wedge \neg\gamma) \neq \perp$.*

Proof. Let p, q, r and γ be elements of A . We can write $\delta = \neg\gamma$ and prove the contraposition, namely

$$EU(p, q, r \wedge \delta) \leq \delta \Rightarrow EU(p, q, r) \leq \delta.$$

Suppose that $EU(p, q, r \wedge \delta) \leq \delta$, ie. $EU(p, q, r \wedge \delta) = EU(p, q, r \wedge \delta) \wedge \delta$. By 2.5, writing $\sigma : (p, q, x) \mapsto p \vee (q \wedge \diamond x)$ we have:

$$\begin{aligned} \sigma(p, q, r \wedge EU(p, q, r \wedge \delta)) &= \sigma(p, q, r \wedge EU(p, q, r \wedge \delta) \wedge \delta) \\ &= EU(p, q, r \wedge \delta). \end{aligned}$$

But then, by minimality, we get that

$$EU(p, q, r) \leq EU(p, q, r \wedge \delta) \leq \delta \text{ by assumption.}$$

The proof for AF is strictly analogous. \square

Definition 2.7. *A CTL^f -formula is in negative normal form (NNF) if it is constructed by applying operators $(\wedge, \vee, \Box, \Diamond, EU_c, AF_c, AR, EG)$ to literals, ie. formulas of the form $\perp \mid \top \mid p \mid \neg p$.*

Proposition 2.8. *Every CTL^f -formula is equivalent to one that is in NNF.*

Proof. Negations can be pushed inwards, thanks to dualities, eg. $\neg\Box \equiv \Diamond\neg$, etc. \square

This is now that we make use of the results of (Jónsson-Tarski) duality on modal algebras, shown for example in [1] (chapter 4).

Definition 2.9. *Given a modal algebra \mathbb{A} , we define its dual frame as the pair $\mathbb{A}_* = (A_*, R_*)$, where*

- A_* is the set of ultrafilters of \mathbb{A} (given that \mathbb{A} is a Boolean algebra, its filters are all incomparable for the inclusion order; they correspond to upward-closures of the atoms of \mathbb{A});

- R_* is the relation on A_* induced by \diamond , so that xR_*y if and only if $\diamond[y] \subseteq x$.

Still from [1], we can state the following theorem:

Theorem 2.10. *Any modal algebra \mathbb{A} is isomorphic to the complex algebra of its dual frame.*

The difficulty here is that this is not true for every CTL^f -algebra.

Lemma 2.11. *Let \mathbb{A} and \mathbb{A}_* be a modal algebra and its dual frame. Let $x \in A_*$. Consider $a \in \mathbb{A}$ such that $\diamond a \in x$. Then there exists $y \in A_*$ such that $a \in y$ and xR_*y .*

Proof. Consider the set $\{b \in A \mid \diamond b \notin x\}$. It is an ideal in \mathbb{A} not containing a . Indeed, it is downwards-closed, because if $c \leq b$, and $\diamond b \notin x$, then $\diamond c \notin x$ by monotonicity of \diamond . And it is stable by join, as \diamond preserves join, so if it contained $\diamond(b \vee c)$ it would contain $\diamond b$. Therefore its complementary is a filter $:= y$ containing a , and xR_*y quite obviously from definition. \square

Definition 2.12. *Let ρ be a finite a set of formulas. For any $x, x' \in \mathbb{A}_*$, define*

$$x \sim_\rho x' \Leftrightarrow x \cap \rho = x' \cap \rho$$

We call the equivalence class of a point x under \sim_ρ the ρ -type of x . For any $x \in \mathbb{A}_$, define the characteristic formula*

$$\kappa(x, \rho) := \bigwedge \{\gamma \mid \gamma \in \rho \cap x\} \wedge \bigwedge \{\neg\gamma \mid \gamma \in \rho \setminus x\}$$

Lemma 2.13. *For any set of formulas ρ and points $x, x' \in \mathbb{A}_*$, we have*

$$x \sim_\rho x' \Leftrightarrow \kappa(x, \rho) \in x'.$$

Proof. x' being an ultrafilter, we have $\kappa \in x'$ if and only if for all $\gamma \in x \cap \rho$, $\gamma \in x'$, and for all $\gamma \in \rho \setminus x$, $\gamma \notin x'$. This is another way of saying $x \cap \rho = x' \cap \rho$. \square

We combine this lemma with proposition 2.6 to obtain the following facts, which will allow to make ‘jumps’ in the ultrafilter frame of \mathbb{A} .

Lemma 2.14. *Let ρ be a finite set of formulas, let $\heartsuit \in \{EU, AF\}$, and let φ, ψ , and χ be formulas. For any $x \in \mathbb{A}_*$ such that $\heartsuit(\varphi, \psi, \chi) \in x$, there exists $x' \in \mathbb{A}_*$ such that $x \sim_\rho x'$ and $\heartsuit(\varphi, \psi, \chi \wedge \neg\kappa(x, \rho)) \in x'$.*

Proof. We have $\kappa(x, \rho) \wedge \heartsuit(\varphi, \psi, \chi) \in x$, it is $\neq \perp$. Then, by 2.6, $\kappa(x, \rho) \wedge \heartsuit(\varphi, \psi, \chi \wedge \neg\kappa(x, \rho)) \neq \perp$. Then it must belong to some ultrafilter x' in \mathbb{A}_* , which is upwards-closed, so $\kappa(x, \rho) \in x'$ and $\heartsuit(\varphi, \psi, \chi \wedge \neg\kappa(x, \rho)) \in x'$. Moreover, by 2.13, we get our equivalence. \square

Definition 2.15. *A set of CTL^f -formulas Γ is called (Fischer-Ladner) closed if the following hold:*

- $EU(\top, \top, \top) \in \Gamma$;
- if φ is in Γ , then any subformula of φ is in Γ too;
- if $EG(\varphi, \psi) \in \Gamma$, then $\diamond EU(\psi \wedge EG(\varphi, \psi), \varphi) \in \Gamma$;
- if $AR(\varphi, \psi) \in \Gamma$, then $\Box AR(\varphi, \psi) \in \Gamma$;
- if $EU(\varphi, \psi, \chi) \in \Gamma$, then $\diamond(\chi \wedge EU(\varphi, \psi, \chi)) \in \Gamma$;
- if $AF(\varphi, \psi, \chi) \in \Gamma$, then $\Box AR(\psi \vee \chi, \varphi) \in \Gamma$;

The closure of a set of CTL^f -formulas is the smallest closed set containing it.

Lemma 2.16. *The closure of a finite set of CTL^f -formulas is finite.*

Proof. Obvious from definition 2.15. \square

Model construction

Remember that we have fixed a consistent formula φ_0 at the beginning of this section (along with valuation $V : \bar{p} \rightarrow \mathbb{A}$), and that we need to construct a model (a tree) in which it is satisfied. By 2.16, define Γ_0 to be the (finite) closure of $\{\varphi_0\}$.

Definition 2.17. *We will call an eventuality formula any formula of the form $\heartsuit(\varphi, \psi, \chi)$, with \heartsuit being AF or EU .*

Moreover, we note the set of eventuality formulas in propositional variables \bar{p} as $Ev(\bar{p})$.

The tree S we will construct will be a union of finite trees, which use a notion of *partial tableau for Γ_0 in \mathbb{A}* . Informally, a partial tableau will consist of a finite tree T , along with two labellings α and β . Labelling α assigns to each node of T an ultrafilter of \mathbb{A} , which corresponds to the set of formulas we wish to force in the node. On the other end, β records the ‘current status’ of eventuality formulas in Γ_0 , via a data structure consisting of a finite list of tuples of the form $(\theta, \sigma, \rho, \tilde{\chi})$ for each node v . Each tuple is

made of an eventuality formula θ of Γ_0 which lies in $\alpha(v')$ for some $v' \preceq v$; a status σ (a for active, f for frozen or e for extinguished); a finite set of formulas ρ , called the *relevance set*; and a *context formula* $\tilde{\chi}$ which will be a strenghtening of χ .

Definition 2.18. Let Γ_0 be a finite, closed set of CTL^f -formulas with variables in \bar{p} . Define

$$\Sigma := (\Gamma_0 \cap Ev(\bar{p})) \times \{a, f, e\} \times \mathcal{P}_f(CTL^f(\bar{p})) \times CTL^f(\bar{p}).$$

A partial tableau for Γ_0 in \mathbb{A} is a tuple (T, α, β) , where

- T is a finite tree;
- α is a function from T to \mathbb{A}_* , the set of ultrafilters of \mathbb{A} ;
- β is a function from T to Σ_* , the set of finite words over Σ .

For each $v \in T$ we write $\ell(v)$ for the length of $\beta(v)$.

For each $1 \leq k \leq \ell(v)$, we write $\beta(v)_k = (\theta(v)_k, \sigma(v)_k, \rho(v)_k, \tilde{\chi}_k)$, with $\theta(v)_k = \heartsuit(v)_k(\varphi(v)_k, \psi(v)_k, \chi(v)_k)$.

We also need to impose a few conditions on partial tableaux so that they are ‘well-formed’, namely:

- (a) elements of the list $\beta(v)$ persist in $\beta(v')$ for tree successors v' of v ;
- (b) if the first coordinate φ of an eventuality formula lies in $\alpha(v)$, then it is extinguished;
- (c) ρ always contains Γ_0 ;
- (d) EU -formulas are never frozen;
- (e) $\tilde{\chi}$ is a strenghtening of χ ;
- (f) eventuality formulas that occur earlier in β always lie in the relevance set;
- (g) α contains all non-extinguished eventuality formulas.

Formally:

Definition 2.19. The partial tableau (T, α, β) for Γ_0 in \mathbb{A} is said to be well-formed if the following conditions hold for all $v \in T$ and for all $1 \leq k \leq \ell(v)$:

- (a) $\forall v' \succeq v, \ell(v') \geq \ell(v)$ and $\theta(v')_k = \theta(v)_k$;
- (b) $\varphi(v)_k \in \alpha(v) \Rightarrow \sigma(v)_k = e$;

- (c) $\Gamma_0 \subseteq \rho(v)_k$
- (d) $\heartsuit(v)_k = EU \Rightarrow \sigma(v)_k \neq f;$
- (e) $\tilde{\chi}(v)_k \vdash \chi(v)_k;$
- (f) $k' < k \Rightarrow \heartsuit_{k'}(\varphi(v)_{k'}, \psi(v)_{k'}, \tilde{\chi}(v)_{k'}) \in \rho(v)_k;$
- (g) $\sigma(v)_k \neq e \Rightarrow \heartsuit_k(\varphi(v)_k, \psi(v)_k, \tilde{\chi}(v)_k) \in \alpha(v)_k.$

References

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