# A particular case of Stone duality: $(\Diamond, \mathbf{X}, I)$-*algebras*

Anatole Leterrier

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The proof of theorem 3.5 in [2] uses the Jónsson-Tarski representation theorem for a particular kind of "S4-algebras". However, the duality can be understood using only the extended Stone duality explained in [1]. Indeed, $(\Diamond, \mathbf{X}, I)$-*algebras* are Boolean algebras equipped with additional items, along with several axioms that the structure satisfies. Then, we can extend the extended Stone duality for Boolean algebras to $(\Diamond, \mathbf{X}, I)$-*algebras*, by highlighting to which subclass of Boolean spaces they correspond.

**Definition 1.** *As defined in [2], a* $(\Diamond, \mathbf{X}, I)$-algebra *is a Boolean algebra* $(A, \vee, \neg, \perp)$ *equipped with a join-preserving, unary modal operator* $\Diamond$*, a Boolean endomorphism* $\mathbf{X}$*, and an element* $I$ *of* $A$ *distinct from* $\perp$*. In addition, for any* $a \in A$*, the following axioms are true:*

*(i)* $a \vee \mathbf{X}\Diamond a \leq \Diamond a$

*(ii) if* $\mathbf{X}\Diamond a \leq a$ *then* $\Diamond a \leq a$

*(iii) if* $a \neq \perp$ *then* $I \leq a$

*(iv)* $\mathbf{X}I = \perp$

*Note that our first item is fact is an equality for every* $a$*:* $\mathbf{X}(\mathbf{X}\Diamond a \vee a) = \mathbf{X}a \vee \mathbf{X}\mathbf{X}\Diamond a \leq \mathbf{X}\Diamond a \vee \mathbf{X}\Diamond a$*, by monotonicity of* $\mathbf{X}$ *and by* $(i)$*. This is itself below* $\mathbf{X}\Diamond a \vee a$*. Now by* $(ii)$ *it holds that* $\Diamond a \leq \Diamond(\mathbf{X}\Diamond a \vee a) \leq \mathbf{X}\Diamond a \vee a$*, which means we have exactly an equality in* $(i)$*.*

Now, given a $(\Diamond, \mathbf{X}, I)$-*algebra*, we have an underlying Boolean algebra, with dual $Y$ (its set of ultrafilters). The Boolean endomorphism $\mathbf{X}$ is transposed to a continuous function $f\colon Y \to Y$, while $\Diamond$ is only a join-preserving unary operator, so it corresponds to a compatible relation $R \subseteq Y \times Y$.

Finally, we can prove from our axioms that I is an atom. For any $a \in A$ such that $\perp < a \leq I$, by $(iii)$ we have $I \leq \Diamond a$. On the other side, by $(iv)$, $\mathbf{X}I = \perp \leq I$, so $(ii)$ implies $\Diamond I \leq I$. Finally $\Diamond I \leq \Diamond a$, and monotonicity of $\Diamond$ makes us conclude that $I = a$. As a consequence, $\uparrow I$ is an ultrafilter $x_0$ in $Y$.
It remains to deduce what properties the so-called $(\Diamond, \mathbf{X}, I)$-*space* dual to A must have.

**Theorem 2.** *Let $Y$ be the space dual to $A$, equipped with $f$, $R$, and $x_0$ as defined above. Then the following hold:*

- *$x_0$ is a predecessor for every $x$,* ie. $R[x_0] = Y$*;*
- *$x_0$ has no predecessor,* ie. $f^{-1}(x_0) = \emptyset$*;*
- *$R$ is a preorder,* ie. *it is reflexive and transitive;*
- *for every clopen $K \subseteq Y$, we have $f^{-1}(K) \subseteq K \implies R^{-1}[K] \subseteq K$;*
- *$f(x)$ is the "least successor of" $x$ for every $x$ in $Y$, meaning $xRf(x)$ and $\forall y \neq x$, $xRy \implies f(x)Ry$.*

*Conversely, if all of the above hold in the space $Y$, then the original Boolean algebra verifies all the axioms in definition 1.*

*Proof.* First, let us show the first part of this theorem. We suppose that axioms from $(i)$ from $(iv)$ hold.

- $(iii)$ is equivalent to saying $x_0 \in R^{-1}[K]$ for every non-empty clopen $K$ of $Y$. This is exactly saying that $R[x_0] = Y$: the left-to-right implication comes from the fact that $\{x\}$ is a clopen for every $x \in Y$; for right-to-left, note that any clopen $K \neq \emptyset$ is included in $Y = R[x_0]$, so that we can pick any $k \in K$ and get $x_0Rk$.
- quite obviously, $(iv)$ expresses that $f^{-1}(x_0) = \emptyset$.

Now, we can observe that $(i)$ is equivalent to three points, say $(i1) : a \vee \mathbf{X}\Diamond a \geq \Diamond a$, $(i2) : a \leq \Diamond a$, $(i3) : \mathbf{X}\Diamond a \leq \Diamond a$.

- $(i2)$ expresses exactly the fact that relation $R$ is reflexive.
- Transitivity comes from the fact that $(i3)$ holds, which we can apply to $(ii)$ to get that $\Diamond\Diamond a \leq \Diamond a$. This is equivalent to transitivity in space $Y$.

- $(i3)$ can be rephrased in terms of space by saying $f^{-1}(R^{-1}[K]) \subseteq R^{-1}[K]$ for every clopen $K$. This implies $xRf(x)$ for every $x$. By contraposition, if some $x$ belonged to $f^{-1}R^{-1}[K]$ but not to $R^{-1}[K]$, we would get a $k$ in $K$ with $f(x)Rk$, while for all $y \in K$,$x \not R y$. Having $xRf(x)$ would entail an immediate contradiction, by transitivity.

- $(i1)$, in terms of space, means that for any clopen $K$, $R^{-1}[K] \subseteq K \cup f^{-1}(R^{-1}[K])$. In particular, since every $\{y\}$ is clopen in $Y$, if we have $xRy$ for some $y \neq x$, then $f(x)Ry$.

- Finally, $(ii)$ translates to : $f^{-1}(K) \subseteq K \implies R^{-1}[K] \subseteq K$ for every clopen $K \subseteq Y$.

To prove the reciprocal implication, if we suppose that all of the properties hold on $Y$, we already have $(ii)$, $(i2)$, $(iii)$ and $(iv)$, as we had reasoned with equivalences. It remains to show only the following points:

- If $xRf(x)$ for every $x$, then if $x \in f^{-1}R^{-1}[K]$, $K$ clopen, *ie.* $f(x)Ry$ for some $y$ in $K$, it holds by transitivity that $xRy$ *ie.* $x \in R^{-1}[K]$. This fact proves $(i3)$.

- Let $xRk$ for some $k$ in clopen $K$. If $x = k$ then obviously $x \in K$. If not, then by supposition $f(x)Rk$ *ie.* $x \in f^{-1}(R^{-1}[K])$. We conclude that $(i1)$ holds.

This ends the proof of theorem 2. $\square$

## References


[1] Mai Gehrke and Sam van Gool. *Topological duality for distributive lattices, and applications.* ., 2022.

[2] Silvio Ghilardi and Sam van Gool. A model-theoretic characterization of monadic second order logic on infinite words. *The Journal of Symbolic Logic*, 82(1), 2017.