A particular case of Stone duality: $(\Diamond, \mathbf{X}, I)$-*algebras*

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The proof of theorem 3.5 in [2] uses the Jónsson-Tarski representation theorem for a particular kind of "S4-algebras". However, the duality can be understood using only the extended Stone duality explained in [1]. Indeed, $(\Diamond, \mathbf{X}, I)$-*algebras* are Boolean algebras equipped with additional items, along with several axioms that the structure satisfies. Then, we can extend the extended Stone duality for Boolean algebras to $(\Diamond, \mathbf{X}, I)$-*algebras*, by highlighting to which subclass of Boolean spaces they correspond.

Definition 1. *As defined in [2], a* $(\Diamond, \mathbf{X}, I)$-algebra *is a Boolean algebra* $(A, \vee, \neg, \perp)$ *equipped with a join-preserving, unary modal operator* $\Diamond$*, a Boolean endomorphism* $\mathbf{X}$*, and an element* I *of* A *distinct from* $\perp$*. In addition, for any* $a \in A$*, the following axioms are true:*

(i) $a \vee \mathbf{X}\Diamond a \leq \Diamond a$

(ii) if $\mathbf{X}\Diamond a \leq a$ *then* $\Diamond a \leq a$

(iii) if $a \neq \perp$ *then* $I \leq a$

(iv) $\mathbf{X}I = \perp$

Note that our first item is fact is an equality for every a*:* $\mathbf{X}(\mathbf{X}\Diamond a \vee a) = \mathbf{X}a \vee \mathbf{X}\mathbf{X}\Diamond a \leq \mathbf{X}\Diamond a \vee \mathbf{X}\Diamond a$*, by monotonicity of* $\mathbf{X}$ *and by* (i)*. This is itself below* $\mathbf{X}\Diamond a \vee a$*. Now by* (ii) *it holds that* $\Diamond a \leq \Diamond(\mathbf{X}\Diamond a \vee a) \leq \mathbf{X}\Diamond a \vee a$*, which means we have exactly an equality in* (i)*.*

Now, given a $(\Diamond, \mathbf{X}, I)$-*algebra*, we have an underlying Boolean algebra, with dual Y (its set of ultrafilters). The Boolean endomorphism $\mathbf{X}$ is transposed to a continuous function $f\colon Y \to Y$, while $\Diamond$ is only a join-preserving unary operator, so it corresponds to a compatible relation $R \subseteq Y \times Y$.

Finally, we can prove from our axioms that I is an atom. For any $a \in A$ such that $\perp < a \leq I$, by (iii) we have $I \leq \Diamond a$. On the other side, by (iv), $\mathbf{X}I = \perp \leq I$, so (ii) implies $\Diamond I \leq I$. Finally $\Diamond I \leq \Diamond a$, and monotonicity of $\Diamond$ makes us conclude that $I = a$. As a consequence, $\uparrow I$ is an ultrafilter x_0 in Y.
It remains to deduce what properties the so-called $(\Diamond, \mathbf{X}, I)$-*space* dual to A must have.

Theorem 2. *Let Y be the space dual to A, equipped with f, R, and x_0 as defined above. Then the following hold:*

- *x_0 is a predecessor for every x,* ie. $R[x_0] = Y$*;*
- *x_0 has no predecessor,* ie. $f^{-1}(x_0) = \emptyset$*;*
- *R is a preorder,* ie. *it is reflexive and transitive;*
- *for every clopen $K \subseteq Y$, we have $f^{-1}(K) \subseteq K \implies R^{-1}[K] \subseteq K$;*
- *$f(x)$ is the "least successor of" x for every x in Y, meaning $xRf(x)$ and $\forall y \neq x$, $xRy \implies f(x)Ry$.*

Conversely, if all of the above hold in the space Y, then the original Boolean algebra verifies all the axioms in definition 1.

Proof. First, let us show the first part of this theorem. We suppose that axioms from (i) from (iv) hold.

- (iii) is equivalent to saying $x_0 \in R^{-1}[K]$ for every non-empty clopen K of Y. This is exactly saying that $R[x_0] = Y$: the left-to-right implication comes from the fact that $\{x\}$ is a clopen for every $x \in Y$; for right-to-left, note that any clopen $K \neq \emptyset$ is included in $Y = R[x_0]$, so that we can pick any $k \in K$ and get x_0Rk.
- quite obviously, (iv) expresses that $f^{-1}(x_0) = \emptyset$.

Now, we can observe that (i) is equivalent to three points, say $(i1) : a \vee \mathbf{X}\Diamond a \geq \Diamond a$, $(i2) : a \leq \Diamond a$, $(i3) : \mathbf{X}\Diamond a \leq \Diamond a$.

- $(i2)$ expresses exactly the fact that relation R is reflexive.
- Transitivity comes from the fact that $(i3)$ holds, which we can apply to (ii) to get that $\Diamond\Diamond a \leq \Diamond a$. This is equivalent to transitivity in space Y.

- $(i3)$ can be rephrased in terms of space by saying $f^{-1}(R^{-1}[K]) \subseteq R^{-1}[K]$ for every clopen K. This implies $xRf(x)$ for every x. By contraposition, if some x belonged to $f^{-1}R^{-1}[K]$ but not to $R^{-1}[K]$, we would get a k in K with $f(x)Rk$, while for all $y \in K$,$x \not R y$. Having $xRf(x)$ would entail an immediate contradiction, by transitivity.

- $(i1)$, in terms of space, means that for any clopen K, $R^{-1}[K] \subseteq K \cup f^{-1}(R^{-1}[K])$. In particular, since every $\{y\}$ is clopen in Y, if we have xRy for some $y \neq x$, then $f(x)Ry$.

- Finally, (ii) translates to : $f^{-1}(K) \subseteq K \implies R^{-1}[K] \subseteq K$ for every clopen $K \subseteq Y$.

To prove the reciprocal implication, if we suppose that all of the properties hold on Y, we already have (ii), $(i2)$, (iii) and (iv), as we had reasoned with equivalences. It remains to show only the following points:

- If $xRf(x)$ for every x, then if $x \in f^{-1}R^{-1}[K]$, K clopen, *ie.* $f(x)Ry$ for some y in K, it holds by transitivity that xRy *ie.* $x \in R^{-1}[K]$. This fact proves $(i3)$.

- Let xRk for some k in clopen K. If $x = k$ then obviously $x \in K$. If not, then by supposition $f(x)Rk$ *ie.* $x \in f^{-1}(R^{-1}[K])$. We conclude that $(i1)$ holds.

This ends the proof of theorem 2. $\square$

References

[1] Mai Gehrke and Sam van Gool. *Topological duality for distributive lattices, and applications.* ., 2022.

[2] Silvio Ghilardi and Sam van Gool. A model-theoretic characterization of monadic second order logic on infinite words. *The Journal of Symbolic Logic*, 82(1), 2017.