6 Appendix (Proofs)

Proposition 1 (Relevance).
Given $x \in \Gamma$, if $(\Gamma; t) \Downarrow (\Gamma'; u)$ and $x \notin fv(t)$ then $(\Gamma \setminus x; t) \Downarrow (\Gamma' \setminus x; u)$.

Lemma 1.

1. Given $(S; t_1 t_2) \rightarrow_k (S''; v)$, then $\exists k_1, k_2 \in \mathbb{N}$, $k = k_1 + k_2$ such that
$$(S; t_1\ t_2) \rightarrow_{k_1} (S'; x\ t_2) \rightarrow_{k_2} (S''; v)$$
2. Given $(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow_k (S''; v)$, then $\exists k_1, k_2 \in \mathbb{N}, k = k_1 + k_2$ such that
$$(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow_{k_1} (S'; \text{split } x \text{ as } y, z \text{ in } t_2) \rightarrow_{k_2} (S''; v)$$

Proof.

1. By induction on k:
   - $k = 0$ holds vacuously because $(S; t_1 t_2)$ and $(S''; v)$ are different.
   - Assume the lemma holds for $k \leq k_0$ and we shall prove it for $k_0 + 1$.
     Assume that $(S; t_1 t_2) \rightarrow_{k_0+1} (S''; v)$.
     This means that the derivation sequence can be written as
     $$(S; t_1 t_2) \rightarrow \gamma \rightarrow_{k_0} (S''; v)$$
     for some configuration γ.
     If we apply the system rule $(S; E[t]) \rightarrow (S'; E[t'])$, we have $\gamma = (S_1; {t_1}' t_2)$ and
     $$(S; t_1 t_2) \rightarrow (S_1; {t_1}' t_2)$$
     because
     $$(S; t_1) \rightarrow_\beta (S_1; {t_1}')$$
     We therefore have
     $$(S_1; {t_1}' t_2) \rightarrow_{k_0} (S''; v)$$
     By induction hypothesis, there are natural numbers k_1 and k_2, such that
     $$(S_1; {t_1}' t_2) \rightarrow_{k_1} (S'; x t_2) \text{ and } (S'; x\ t_2) \rightarrow_{k_2} (S''; v)$$
     ,
     where $k_0 = k_1 + k_2$.
     Using the fact that $(S; t_1) \rightarrow_\beta (S_1; {t_1}')$ and $(S_1; {t_1}') \rightarrow_{k_1} (S'; x)$, we get
     $$(S; t_1 t_2) \rightarrow_{k_1+1} (S'; x t_2)$$
     We have already seen that $(S'; x\ t_2) \rightarrow_{k_2} (S''; v)$ and, since $(k_1+1)+k_2 = k_0 + 1$, we have proven the required result.

2. By induction on k:
   - $k = 0$ holds vacuously because $(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2)$ and $(S''; v)$ are different.
   - Assume the lemma holds for $k \leq k_0$ and we shall prove it for $k_0 + 1$.
     Assume that $(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow_{k_0+1} (S''; v)$.
     This means that the derivation sequence can be written as
     $$(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow \gamma \rightarrow_{k_0} (S''; v)$$
     for some configuration γ.
     If we apply the system rule $(S; E[t]) \rightarrow (S'; E[t'])$, we have
     $$\gamma = (S_1; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2)$$
     and
     $$(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow (S_1; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2)$$
     because
     $$(S; t_1) \rightarrow_\beta (S_1; {t_1}')$$
     We therefore have
     $$(S_1; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2) \rightarrow_{k_0} (S''; v)$$
     By induction hypothesis, there are natural numbers k_1 and k_2, such that
     $$(S_1; \text{split } {t_1}' \text{ as } y, z \text{ in} t_2) \rightarrow_{k_1} (S'; \text{split } x \text{ as } y, z \text{ in } t_2)$$
     and
     $$(S'; \text{split } x \text{ as } y, z \text{ in } t_2) \rightarrow_{k_2} (S''; v)$$
     where $k_0 = k_1 + k_2$. Using the fact that
     $$(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow (S_1; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2)$$
     and
     $$(S_1; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2) \rightarrow_{k_1} (S'; \text{split } x \text{ as } y, z \text{ in } t_2)$$
     we get
     $$(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \rightarrow_{k_1+1} (S'; \text{split } x \text{ as } y, z \text{ in } t_2)$$
     We have already seen that $(S'; \text{split } x \text{ as } y, z \text{ in } t_2) \rightarrow_{k_2} (S''; v)$ and, since $(k_1 + 1) + k_2 = k_0 + 1$, we have proven the required result.

□

Proposition 2 (Relevance). *If $a \notin fv(t)$ then $deref_{\Gamma, a:_\pi A = e}(t) = deref_\Gamma(t)$.*

Lemma 2 (*deref*$_\Gamma$ lemma).
Given $\Gamma, a :_\pi A = e$, *then* $\mathit{deref}_{\Gamma, a:_\pi A = e}(t) = \mathit{deref}_\Gamma(t[a \mapsto e])$.

Proof. By induction on term t, using Proposition 2. $\square$

Proposition 3 (Relevance). *If* $a \notin \mathit{fv}(t)$ *then* $\mathit{deref}_{S, a \mapsto_q e}(t) = \mathit{deref}_S(t)$.

Lemma 3 (*deref*$_S$ lemma).
Given $S, a \mapsto q\ e$, *then* $\mathit{deref}_{S, a \mapsto_q e}(t) = \mathit{deref}_S(t[a \mapsto e])$.

Proof. By induction on term t, using Proposition 3. $\square$

Proposition 4.
Given $(\Gamma, a :_\pi A = t_2; t_1[y \mapsto a])$, *where* $a \notin \mathit{fv}(t_1)$, *then*

$$\mathit{deref}_{\Gamma, a:_\pi A = t_2}(t_1[y \mapsto a]) = \mathit{deref}_\Gamma(t_1[y \mapsto t_2])$$

Lemma 4.
Given that $x \notin \mathit{fv}(t)$, *if* $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; v_1)$ *then*

$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; v_2)$ *and* $\mathit{deref}_{\Delta_2}(v_2) = \mathit{deref}_{\Delta_1}(v_1)$

Proof. By induction on $(\Gamma; t) \Downarrow (\Delta; v)$:

– Given $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \lambda_\pi x_1 : P.\ t)$, we have, by rule *s-abs*$_\mathcal{L}$

$$\begin{aligned} &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \lambda_\pi x_1 : P.\ t) \\ \Downarrow\ &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \lambda_\pi x_1 : P.\ t) \end{aligned}$$

And, by rule *s-abs*$_\mathcal{L}$

$$\begin{aligned} &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; \lambda_\pi x_1 : P.\ t) \\ \Downarrow\ &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; \lambda_\pi x_1 : P.\ t) \end{aligned}$$

We now have

$$\begin{aligned} &\mathit{deref}_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_{\pi_2} A_2 = e[z \mapsto x]}(\lambda_\pi x_1 : P.\ t) \\ = {}&\text{let } (t', \Gamma') = \mathit{deref}_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_{\pi_2} A_2 = e[z \mapsto x]}(t) \\ &\qquad = \mathit{deref}_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_{\pi_2} A_2 = e[z \mapsto x]}(t[y \mapsto e[z \mapsto x]]) \\ &\qquad \text{we know that } x \notin \mathit{fv}(\lambda x_1 : P.\ t)\text{, so by Proposition 1} \\ &\qquad = \mathit{deref}_{\Gamma, y:_{\pi_2} A_2 = e[z \mapsto x]}(t[y \mapsto e[z \mapsto t_2]]) \\ &\text{in } (\lambda_\pi x_1 : P.t', (\Gamma', y :_{\pi_2} A_2 = e[z \mapsto x])) \end{aligned}$$

We know that, after substituting all free occurrences of y in t, y does not occur in the resulting term. Therefore, by Lemma 2,

$$\mathit{deref}_{\Gamma, y:_{\pi_2} A_2 = e[z \mapsto x]}(t[y \mapsto e[z \mapsto t_2]]) = \mathit{deref}_\Gamma(t[y \mapsto e[z \mapsto t_2]])$$

And we can conclude that

$$deref_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_{\pi_2} A_2 = e[z \mapsto x]}(\lambda_\pi x_1 : P.\ t) = (\lambda_\pi x_1 : P.t', \Gamma')$$

Since we know that $x \notin fv(\lambda x_1 : P.\ t)$ and $(t', \Gamma') = deref_\Gamma(t[y \mapsto e[z \mapsto t_2]]$, then

$$deref_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_{\pi_2} A_2 = e[z \mapsto t_2]}(\lambda_\pi x_1 : P.\ t) = (\lambda_\pi x_1 : P.t', \Gamma')$$

And the property holds.

– Given $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; < y_1, z_1 >_\pi)$, we have, by rule *s-pair*$_\mathcal{L}$,

$$\begin{aligned} &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; < y_1, z_1 >_\pi) \\ \Downarrow\ &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; < y_1, z_1 >_\pi) \end{aligned}$$

And, by rule *s-pair*$_\mathcal{L}$

$$\begin{aligned} &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; < y_1, z_1 >_\pi) \\ \Downarrow\ &(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; < y_1, z_1 >_\pi) \end{aligned}$$

Now, there are two cases:

- $\pi_2 = \omega$

  We have

$$\begin{aligned} & deref_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_\omega A_2 = e[z \mapsto x]}(< y_1, z_1 >_\pi) \\ = \text{let } & ({t_1}', \Gamma_1) = deref_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_\omega A_2 = e[z \mapsto x]}(y_1) \\ & \qquad = deref_{\Gamma, x:_{\pi_1} A_1 = t_2}(y_1[y \mapsto e[z \mapsto x]]) \text{ by Lemma 2} \\ & \qquad \text{we know that } x \notin fv(< y_1, z_1 >_\pi) \text{, so by Proposition 1,} \\ & \qquad = deref_\Gamma(y_1[y \mapsto e[z \mapsto t_2]]) \\ & ({t_2}', \Gamma_2) = deref_{\Gamma_1, x:_{\pi_1} A_1 = t_2, y:_\omega A_2 = e[z \mapsto x]}(z_1) \\ & \qquad = deref_{\Gamma_1, x:_{\pi_1} A_1 = t_2}(z_1[y \mapsto e[z \mapsto x]]) \text{ by Lemma 2} \\ & \qquad \text{we know that } x \notin fv(< y_1, z_1 >_\pi) \text{, so by Proposition 1,} \\ & \qquad = deref_{\Gamma_1}(z_1[y \mapsto e[z \mapsto t_2]]) \\ \text{in } & (< {t_1}', {t_2}' >_\pi, \Gamma_2) \end{aligned}$$

  and

$$deref_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_\omega A_2 = e[z \mapsto t_2]}(< y_1, z_1 >_\pi)$$

  Since we know that $x \notin fv(< y_1, z_1 >_\pi)$ and

  $({t_1}', \Gamma_1) = deref_\Gamma(y_1[y \mapsto e[z \mapsto t_2]])$ and $({t_2}', \Gamma_2) = deref_{\Gamma_1}(z_1[y \mapsto e[z \mapsto t_2]])$

  then

$$deref_{\Gamma, x:_{\pi_1} A_1 = t_2, y:_\omega A_2 = e[z \mapsto t_2]}(< y_1, z_1 >_\pi) = (< {t_1}', {t_2}' >_\pi, \Gamma_2)$$

  And the property holds.

- $\pi_2 = 1$
  If $y \in fv(y_1)$, then

$$\begin{aligned}
& deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto x]}(< y_1, z_1 >_\pi) \\
= \text{ let } & ({t_1}', \Gamma_1) = deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto x]}(y_1) \\
& \qquad\qquad = deref_{\Gamma,x:_{\pi_1}A_1=t_2}(y_1[y \mapsto e[z \mapsto x]]) \text{ by Lemma 2} \\
& \qquad\qquad \text{we know that } x \notin fv(< y_1, z_1 >_\pi)\text{, so by Proposition 1,} \\
& \qquad\qquad = deref_\Gamma(y_1[y \mapsto e[z \mapsto t_2]]) \\
& ({t_2}', \Gamma_2) = deref_{\Gamma_1}(z_1) \\
\text{in } & (< {t_1}', {t_2}' >_\pi, \Gamma_2)
\end{aligned}$$

and

$$deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto t_2]}(< y_1, z_1 >_\pi)$$

Since we know that $x \notin fv(< y_1, z_1 >_\pi)$ and

$$({t_1}', \Gamma_1) = deref_\Gamma(y_1[y \mapsto e[z \mapsto t_2]]) \text{ and } ({t_2}', \Gamma_2) = deref_{\Gamma_1}(z_1)$$

then

$$deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto t_2]}(< y_1, z_1 >_\pi) = (< {t_1}', {t_2}' >_\pi, \Gamma_2)$$

And the property holds.
If $y \in fv(z_1)$, then

$$\begin{aligned}
& deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto x]}(< y_1, z_1 >_\pi) \\
= \text{ let } & ({t_1}', \Gamma_1) = deref_{\Gamma,x:_{\pi_1}A_1=t_2}(y_1) \\
& \qquad\qquad \text{we know that } x \notin fv(< y_1, z_1 >_\pi)\text{, it can be removed} \\
& \qquad\qquad = deref_\Gamma(y_1) \\
& ({t_2}', \Gamma_2) = deref_{\Gamma_1,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto x]}(z_1) \\
& \qquad\qquad = deref_{\Gamma_1,x:_{\pi_1}A_1=t_2}(z_1[y \mapsto e[z \mapsto x]]) \text{ by Lemma 2} \\
& \qquad\qquad \text{we know that } x \notin fv(< y_1, z_1 >_\pi)\text{, so by Proposition 1,} \\
& \qquad\qquad = deref_\Gamma(z_1[y \mapsto e[z \mapsto t_2]]) \\
\text{in } & (< {t_1}', {t_2}' >_\pi, \Gamma_2)
\end{aligned}$$

and

$$deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto t_2]}(< y_1, z_1 >_\pi)$$

Since we know that $x \notin fv(< y_1, z_1 >_\pi)$ and $({t_1}', \Gamma_1) = deref_\Gamma(y_1)$ and $({t_2}', \Gamma_2) = deref_{\Gamma_1}(z_1[y \mapsto e[z \mapsto t_2]])$, then

$$deref_{\Gamma,x:_{\pi_1}A_1=t_2,y:_1A_2=e[z\mapsto t_2]}(< y_1, z_1 >_\pi) = (< {t_1}', {t_2}' >_\pi, \Gamma_2)$$

And the property holds.

- Given $(\Gamma, y_1 :_\omega B = t, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; y_1)$, we have

$$\frac{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; v_1)}{(\Gamma, y_1 :_\omega B = t, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; y_1) \Downarrow (\Delta_1, y_1 :_\omega B = v_1; v_1)} \textit{ s-unvar}_{\mathcal{L}}$$

and

$$\frac{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; v_2)}{(\Gamma, y_1 :_\omega B = t, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; y_1) \Downarrow (\Delta_2, y_1 :_\omega B = v_2; v_2)} \textit{ s-unvar}_{\mathcal{L}}$$

Since we know that $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; v_1)$ then, by induction hypothesis, there is

$$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; v_2)$$

and $\mathit{deref}_{\Delta_2}(v_2) = \mathit{deref}_{\Delta_1}(v_1)$.
We also know that $y_1 \notin \mathit{fv}(v_1)$ and $y_2 \notin \mathit{fv}(v_2)$, so we can conclude that

$$\mathit{deref}_{\Delta_2, y_1 :_\omega B = v_2}(v_2) = \mathit{deref}_{\Delta_1, y_1 :_\omega B = v_1}(v_1)$$

- Given $(\Gamma, y_1 :_1 B = t, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; y_1)$, we have

$$\frac{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; v_1)}{(\Gamma, y_1 :_1 B = t, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; y) \Downarrow (\Delta_1; v_1)} \textit{ s-linvar}_{\mathcal{L}}$$

and

$$\frac{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; v_2)}{(\Gamma, y_1 :_1 B = t, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; y) \Downarrow (\Delta_2; v_2)} \textit{ s-linvar}_{\mathcal{L}}$$

Since we know that $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; v_1)$ then, by induction hypothesis, there is

$$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; v_2)$$

and $\mathit{deref}_{\Delta_2}(v_2) = \mathit{deref}_{\Delta_1}(v_1)$. So the property holds.
- Given $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t\ y_1)$, we have

$$\frac{\begin{array}{c}(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; \lambda_\pi z_1 : P.t') \\ (\Delta_1; t'[z_1 \mapsto y_1]) \Downarrow (\Theta_1; v_1)\end{array}}{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t\ y_1) \Downarrow (\Theta_1; v_1)} \textit{ s-app}_{\mathcal{L}}$$

and

$$\frac{\begin{array}{c}(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; \lambda_\pi z_2 : P.t'') \\ (\Delta_2; t''[z_2 \mapsto y_1]) \Downarrow (\Theta_2; v_2)\end{array}}{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t\ y_1) \Downarrow (\Theta_2; v_2)} \textit{ s-app}_{\mathcal{L}}$$

Since we know that

$$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; t) \Downarrow (\Delta_1; \lambda_\pi z_1 : P.t')$$

then, by induction hypothesis, there is

$$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; t) \Downarrow (\Delta_2; \lambda_\pi z_2 : P.t'')$$

and $\mathit{deref}_{\Delta_1}(\lambda_\pi z_1 : P.t') = \mathit{deref}_{\Delta_2}(\lambda_\pi z_2 : P.t'')$. From this, we know that $\mathit{deref}_{\Delta_1}(t') = \mathit{deref}_{\Delta_2}(t'')$ then, by α-conversion, we can conclude that $\mathit{deref}_{\Theta_1}(v_1) = \mathit{deref}_{\Theta_2}(v_2)$.

- Given $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \text{split } e \text{ as } y_1, z_1 \text{ in } t)$, we have

$$\frac{\begin{array}{c}(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; e) \Downarrow (\Delta_1; < y_2, z_2 >_\pi) \\ (\Delta_1; t[y_1 \mapsto y_2][z_1 \mapsto z_2]) \Downarrow (\Theta_1; v_1)\end{array}}{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \text{split } e \text{ as } y_1, z_1 \text{ in } t) \Downarrow (\Theta_1; v_1)}\ \textit{s-split}_{\mathcal{L}}$$

and

$$\frac{\begin{array}{c}(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; e) \Downarrow (\Delta_2; < y_3, z_3 >_\pi) \\ (\Delta_2; t[y_1 \mapsto y_3][z_1 \mapsto z_3]) \Downarrow (\Theta_2; v_2)\end{array}}{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; \text{split } e \text{ as } y_1, z_1 \text{ in } t) \Downarrow (\Theta_2; v_2)}\ \textit{s-split}_{\mathcal{L}}$$

Since we know that

$$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; e) \Downarrow (\Delta_1; < y_2, z_2 >_\pi)$$

then, by induction hypothesis, there is

$$(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; e) \Downarrow (\Delta_2; < y_3, z_3 >_\pi)$$

and $\mathit{deref}_{\Delta_1}(< y_2, z_2 >_\pi) = \mathit{deref}_{\Delta_2}(< y_3, z_3 >_\pi)$.
Since we know that

$$\begin{aligned}\mathit{deref}_{\Delta_2}(< y_3, z_3 >_\pi) = \text{let } & ({t_1}', {\Delta_2}') = \mathit{deref}_{\Delta_2}(y_3) \\ & ({t_2}', {\Delta_2}'') = \mathit{deref}_{{\Delta_2}'}(z_3) \\ \text{in } & (< {t_1}', {t_2}' >_\pi, {\Delta_2}'')\end{aligned}$$

and

$$\begin{aligned}\mathit{deref}_{\Delta_1}(< y_2, z_2 >_\pi) = \text{let } & ({t_3}', {\Delta_1}') = \mathit{deref}_{\Delta_1}(y_2) \\ & ({t_4}', {\Delta_1}'') = \mathit{deref}_{{\Delta_1}'}(z_2) \\ \text{in } & (< {t_3}', {t_4}' >_\pi, {\Delta_1}'')\end{aligned}$$

then

$$\mathit{deref}_{\Delta_1}(y_2) = \mathit{deref}_{\Delta_2}(y_3) \text{ and } \mathit{deref}_{{\Delta_1}'}(z_2) = \mathit{deref}_{{\Delta_2}'}(z_3)$$

So we can conclude that $\mathit{deref}_{\Theta_1}(v_1) = \mathit{deref}_{\Theta_2}(v_2)$.

- Given

  $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \text{let}_\pi\ a_1 : B_1 = e_1, \ldots, a_n : B_n = e_n \text{ in } t)$

  we have

  $$\frac{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x], a_1 :_\pi B_1 = e_1, \ldots, a_n :_\pi B_n = e_n; t) \Downarrow (\Delta_1; v_1)}{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x]; \text{let}_\pi\ a_1 : B_1 = e_1, \ldots, a_n : B_n = e_n \text{ in } t) \Downarrow (\Delta_1; v_1)}\ \textit{s-let}_{\mathcal{L}}$$

  and

  $$\frac{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2], a_1 :_\pi B_1 = e_1, \ldots, a_n :_\pi B_n = e_n; t) \Downarrow (\Delta_2; v_2)}{(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2]; \text{let}_\pi\ a_1 : B_1 = e_1, \ldots, a_n : B_n = e_n \text{ in } t) \Downarrow (\Delta_2; v_2)}\ \textit{s-let}_{\mathcal{L}}$$

  Since we know that

  $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto x], a_1 :_\pi B_1 = e_1, \ldots, a_n :_\pi B_n = e_n; t) \Downarrow (\Delta_1; v_1)$

  then, by induction hypothesis, there is

  $(\Gamma, x :_{\pi_1} A_1 = t_2, y :_{\pi_2} A_2 = e[z \mapsto t_2], a_1 :_\pi B_1 = e_1, \ldots, a_n :_\pi B_n = e_n; t) \Downarrow (\Delta_2; v_2)$

  and $\textit{deref}_{\Delta_2}(v_2) = \textit{deref}_{\Delta_1}(v_1)$.

$\square$

Lemma 5 (Γ-Substitution Lemma).
If $(\Gamma, a :_\pi A = t_2; t_1[y \mapsto a]) \Downarrow (\Gamma_1; t_1')$ and $a \notin fv(t_1)$, then

$$(\Gamma; t_1[y \mapsto t_2]) \Downarrow (\Gamma_2; t_2') \text{ and } \textit{deref}_{\Gamma_1}(t_1') = \textit{deref}_{\Gamma_2}(t_2')$$

Proof. By induction on $(\Gamma; t) \Downarrow (\Gamma'; t')$:

– Given $(\Gamma, a :_\omega A = t_2; x[y \mapsto a])$, there are two cases:
  - $x = y$
    Since $x[y \mapsto a] = a$, we have
    $$\frac{(\Gamma; t_2) \Downarrow (\Delta; v_1)}{(\Gamma, a :_\omega A = t_2; a) \Downarrow (\Delta, a :_\omega A = v_1; v_1)}\ \textit{s-unvar}_{\mathcal{L}}$$
    and $x[y \mapsto t_2] = t_2$, We already know that $(\Gamma; t_2) \Downarrow (\Delta; v_1)$, therefore, since $a \notin \Gamma$, and we can conclude that $a \notin fv(v_1)$ so, by Proposition 2,
    $$\textit{deref}_{\Delta, a :_\omega A = v_1}(v_1) = \textit{deref}_{\Delta}(v_1)$$
  - $x \neq y$
    Since $x[y \mapsto a] = x$, we have $(\Gamma, a :_\omega A = t_2; x) \Downarrow (\Delta; v)$ and, since x is a variable and $x \neq a$, then $a \notin fv(x)$ so, by Proposition 1, we can conclude that, since $x[y \mapsto t_2] = x$,
    $$(\Gamma; x) \Downarrow (\Delta; v)$$
    Therefore, the property holds.

– Given $(\Gamma, a :_1 A = t_2; x[y \mapsto a])$, there are two cases:
  - $x = y$
    Since $x[y \mapsto a] = a$, we have
    $$\frac{(\Gamma; t_2) \Downarrow (\Delta; v_2)}{(\Gamma, a :_1 A = t_2; a) \Downarrow (\Delta; v_2)}\ \textit{s-linvar}_{\mathcal{L}}$$
    and, since $x[y \mapsto t_2] = t_2$ we already know that $(\Gamma; t_2) \Downarrow (\Delta; v_2)$. So, the property holds.
  - $x \neq y$
    Since $x[y \mapsto a] = x$, we have $(\Gamma, a :_1 A = t_2; x) \Downarrow (\Delta; v)$ and, since x is a variable and $x \neq a$, then $a \notin \mathit{fv}(x)$ so, by Proposition 1, we can conclude that, since $x[y \mapsto t_2] = x$,
    $$(\Gamma; x) \Downarrow (\Delta; v)$$

– Given $(\Gamma, a :_{\pi_2} A = t_2; \lambda_{\pi_1} x : A.t)$, we assume by α-equivalence that $x \neq y$. Since $(\lambda_{\pi_1} x : A.\ t)[y \mapsto a] = \lambda_{\pi_1} x : A.\ t[y \mapsto a]$, we have, by rule $\textit{s-abs}_{\mathcal{L}}$
$$\begin{array}{c}(\Gamma, a :_{\pi_2} B = t_2; \lambda_{\pi_1} x : A.\ t[y \mapsto a]) \\ \Downarrow (\Gamma, a :_{\pi_2} B = t_2; \lambda_{\pi_1} x : A.\ t[y \mapsto a])\end{array}$$
and, since $(\lambda_{\pi_1} x : A.\ t)[y \mapsto t_2] = \lambda_{\pi_1} x : A.\ t[y \mapsto t_2]$, we have
$$\frac{}{(\Gamma; \lambda_{\pi_1} x : A.\ t[y \mapsto t_2]) \Downarrow (\Gamma; \lambda_{\pi_1} x : A.\ t[y \mapsto t_2])}\ \textit{s-abs}_{\mathcal{L}}$$
Now, we have
$$\begin{array}{l}\mathit{deref}_{\Gamma, a:_{\pi_2} B = t_2}(\lambda_{\pi_1} x : A.\ t[y \mapsto a]) \\ = \text{let } (t', \Gamma') = \mathit{deref}_{\Gamma, a:_{\pi_2} B = t_2}(t[y \mapsto a]) \\ \qquad\qquad \text{by Proposition 4} \\ \qquad\qquad = \mathit{deref}_{\Gamma}(t[y \mapsto t_2]) \\ \quad \text{in } (\lambda_{\pi_1} x : A.t', \Gamma')\end{array}$$
Since we already know that $(t', \Gamma') = \mathit{deref}_{\Gamma}(t[y \mapsto t_2])$ then, we can conclude that
$$\mathit{deref}_{\Gamma}(\lambda_{\pi_1} x : A.t[y \mapsto t_2]) = (\lambda_{\pi_1} x : A.t', \Gamma')$$
and the property holds.

– Given $(\Gamma, a :_{\pi_1} A = t_2; < y_1, y_2 >_{\pi})$,
Since $(< y_1, y_2 >_{\pi})[y \mapsto a] =< y_1[y \mapsto a], y_2[y \mapsto a] >_{\pi}$, we have, by rule $\textit{s-pair}_{\mathcal{L}}$
$$\begin{array}{c}(\Gamma, a :_{\pi_1} B = t_2; < y_1[y \mapsto a], y_2[y \mapsto a] >_{\pi}) \\ \Downarrow (\Gamma, a :_{\pi_1} B = t_2; < y_1[y \mapsto a], y_2[y \mapsto a] >_{\pi})\end{array}$$

and, since $(< y_1, y_2 >_\pi)[y \mapsto t_2] =< y_1[y \mapsto t_2], y_2[y \mapsto t_2] >_\pi$, we have, by rule *s-pair*$_\mathcal{L}$

$$\begin{array}{c}(\Gamma; < y_1[y \mapsto t_2], y_2[y \mapsto t_2] >_\pi)\\ \Downarrow (\Gamma; < y_1[y \mapsto t_2], y_2[y \mapsto t_2] >_\pi)\end{array}$$

Now, we have

$$\begin{array}{l} \quad deref_{\Gamma, a:_{\pi_1} B=t_2}(< y_1[y \mapsto a], y_2[y \mapsto a] >_\pi)\\ = \text{let } ({t_1}', \Gamma_1) = deref_{\Gamma, a:_{\pi_1} B=t_2}(y_1[y \mapsto a])\\ \qquad\qquad \text{by Proposition 5.4,}\\ \qquad\qquad = deref_\Gamma(y_1[y \mapsto t_2])\\ \qquad ({t_2}', \Gamma_2) = deref_{\Gamma_1, a:_{\pi_1} B=t_2}(y_2[y \mapsto a])\\ \qquad\qquad \text{by Proposition 5.4,}\\ \qquad\qquad = deref_{\Gamma_1}(y_2[y \mapsto t_2])\\ \quad \text{in } (< {t_1}', {t_2}' >_\pi, \Gamma_2)\end{array}$$

and

$$deref_\Gamma(< y_1[y \mapsto t_2], y_2[y \mapsto t_2] >_\pi)$$

Since we know that

$$({t_1}', \Gamma_1) = deref_\Gamma(y_1[y \mapsto t_2]) \text{ and } ({t_2}', \Gamma_2) = deref_{\Gamma_1}(y_2[y \mapsto t_2])$$

Then the property holds.

- Given $(\Gamma, a :_\pi A = t_2; e\ x)$, there are two cases:
  - $x = y$
    Since $(e\ x)[y \mapsto a] = e[y \mapsto a]\ x[y \mapsto a] = e[y \mapsto a]\ a$, we have
    $$\frac{\begin{array}{c}(\Gamma, a :_{\pi_1} A = t_2; e[y \mapsto a]) \Downarrow (\Gamma'; \lambda_{\pi_2} z_1 : B.\ t)\\ (\Gamma'; t[z_1 \mapsto a]) \Downarrow (\Delta_1; v_1)\end{array}}{(\Gamma, a :_{\pi_1} A = t_2; e[y \mapsto a]\ a) \Downarrow (\Delta_1; v_1)}\ \textit{s-app}_\mathcal{L}$$
    We know that $(\Gamma, a :_{\pi_1} A = t_2; e[y \mapsto a]) \Downarrow (\Gamma'; \lambda_{\pi_2} z_1 : B.\ t)$ then, by induction hypothesis, $(\Gamma; e[y \mapsto t_2]) \Downarrow (\Gamma''; \lambda_{\pi_2} z_2 : B.\ t')$ and
    $$deref_{\Gamma'}(\lambda_{\pi_2} z_1 : B.\ t) = deref_{\Gamma''}(\lambda_{\pi_2} z_2 : B.\ t')$$
    We also know that
    $$\begin{aligned}(e\ x)[y \mapsto t_2] &= e[y \mapsto t_2]\ x[y \mapsto t_2]\\ &= e[y \mapsto t_2]\ t_2\end{aligned}$$
    so, by rule *app*$_\mathcal{L}$, we have
    $$\frac{\begin{array}{c}(\Gamma; e[y \mapsto t_2]) \Downarrow (\Gamma''; \lambda_{\pi_2} z_2 : B.\ t')\\ (\Gamma''; t'[z_2 \mapsto t_2]) \Downarrow (\Delta_2; v_2)\end{array}}{(\Gamma; e[y \mapsto t_2]\ t_2) \Downarrow (\Delta_2; v_2)}$$

Since we know that $deref_{\Gamma'}(\lambda_{\pi_2} z_1 : B.\ t) = deref_{\Gamma''}(\lambda_{\pi_2} z_2 : B.\ t')$ then if

$$(\Gamma'; t[z_1 \mapsto a]) \Downarrow (\Delta_1; v_1)$$

then, by induction hypothesis,

$$(\Gamma''; t'[z_2 \mapsto t_2]) \Downarrow (\Delta_2; v_2)$$

which is true by α-conversion, and $deref_{\Delta_1}(v_1) = deref_{\Delta_2}(v_2)$. So the property holds.

- $x \neq y$
  Since $(e\ x)[y \mapsto a] = e[y \mapsto a]\ x$, we have

$$\frac{\begin{array}{c}(\Gamma, a :_{\pi_1} A = t_2; e[y \mapsto a]) \Downarrow (\Gamma'; \lambda_{\pi_2} z_1 : B.\ t) \\ (\Gamma'; t[z_1 \mapsto x]) \Downarrow (\Delta_1; v_1)\end{array}}{(\Gamma, a :_{\pi_1} A = t_2; e[y \mapsto a]\ x) \Downarrow (\Delta_1; v_1)}\ \textit{s-app}_{\mathcal{L}}$$

  We know that $(\Gamma, a :_{\pi_1} A = t_2; e[y \mapsto a]) \Downarrow (\Gamma'; \lambda_{\pi_2} z_1 : B.\ t)$ then, by induction hypothesis, $(\Gamma; e[y \mapsto t_2]) \Downarrow (\Gamma''; \lambda_{\pi_2} z_2 : B.\ t')$ and

$$deref_{\Gamma'}(\lambda_{\pi_2} z_1 : B.\ t) = deref_{\Gamma''}(\lambda_{\pi_2} z_2 : B.\ t')$$

  We also know that

$$\begin{aligned}(e\ x)[y \mapsto t_2] &= e[y \mapsto t_2]\ x[y \mapsto t_2] \\ &= e[y \mapsto t_2]\ x\end{aligned}$$

  so, by rule *s-app*$_{\mathcal{L}}$,

$$\frac{\begin{array}{c}(\Gamma; e[y \mapsto t_2]) \Downarrow (\Gamma''; \lambda_{\pi_2} z_2 : B.\ t') \\ (\Gamma''; t'[z_2 \mapsto x]) \Downarrow (\Delta_2; v_2)\end{array}}{(\Gamma; e[y \mapsto t_2]\ x) \Downarrow (\Delta_2; v_2)}$$

  Since we know that $deref_{\Gamma'}(\lambda_{\pi_2} z_1 : B.\ t) = deref_{\Gamma''}(\lambda_{\pi_2} z_2 : B.\ t')$ then if

$$(\Gamma'; t[z_1 \mapsto x]) \Downarrow (\Delta_1; v_1)$$

  then, by induction hypothesis,

$$(\Gamma''; t'[z_2 \mapsto x]) \Downarrow (\Delta_2; v_2)$$

  which is true by α-conversion, and $deref_{\Delta_2}(v_2) = deref_{\Delta_1}(v_1)$, so the property holds.

– Given $(\Gamma, a :_{\pi_1} A = t_2; \text{split } e_1 \text{ as } y_1, z_1 \text{ in } e_2)$,
since

$$(\text{split } e_1 \text{ as } y_1, z_1 \text{ in } e_2)[y \mapsto a] = \text{split } e_1[y \mapsto a] \text{ as } y_1, z_1 \text{ in } e_2[y \mapsto a]$$

we have

$$\frac{\begin{array}{c}(\Gamma, a :_{\pi_1} A = t_2; e_1[y \mapsto a]) \Downarrow (\Gamma'; < y_2, z_2 >_{\pi_2}) \\ (\Gamma'; e_2[y \mapsto a][y_1 \mapsto y_2][z_1 \mapsto z_2]) \Downarrow (\Delta_1; v_1)\end{array}}{(\Gamma, a :_{\pi_1} A = t_2; \text{split } e_1[y \mapsto a] \text{ as } y_1, z_1 \text{ in } e_2[y \mapsto a]) \Downarrow (\Delta_1; v_1)}\ \textit{s-split}_{\mathcal{L}}$$

We know that $(\Gamma, a :_{\pi_1} A = t_2; e_1[y \mapsto a]) \Downarrow (\Gamma'; < y_2, z_2 >_{\pi_2})$ then, by induction hypothesis, $(\Gamma; e_1[y \mapsto t_2]) \Downarrow (\Gamma''; < y_3, z_3 >_{\pi_2})$ and

$$\textit{deref}_{\Gamma'}(< y_2, z_2 >_{\pi_2}) = \textit{deref}_{\Gamma''}(< y_3, z_3 >_{\pi_2})$$

We also know that

$$(\text{split } e_1 \text{ as } y_1, z_1 \text{ in } e_2)[y \mapsto t_2] = \text{split } e_1[y \mapsto t_2] \text{ as } y_1, z_1 \text{ in } e_2[y \mapsto t_2]$$

so, by rule *s-split*$_{\mathcal{L}}$,

$$\frac{\begin{array}{c}(\Gamma; e_1[y \mapsto t_2]) \Downarrow (\Gamma''; < y_3, z_3 >_{\pi_2}) \\ (\Gamma''; e_2[y \mapsto t_2][y_1 \mapsto y_3][z_1 \mapsto z_3]) \Downarrow (\Delta_2; v_2)\end{array}}{(\Gamma; \text{split } e_1[y \mapsto t_2] \text{ as } y_1, z_1 \text{ in } e_2[y \mapsto t_2]) \Downarrow (\Delta_2; v_2)}$$

Since we know that $\textit{deref}_{\Gamma'}(< y_2, z_2 >_{\pi_2}) = \textit{deref}_{\Gamma''}(< y_3, z_3 >_{\pi_2})$, if $(\Gamma'; e_2[y \mapsto a][y_1 \mapsto y_2][z_1 \mapsto z_2]) \Downarrow (\Delta_1; v_1)$ then, by induction hypothesis,

$$(\Gamma''; e_2[y \mapsto t_2][y_1 \mapsto y_3][z_1 \mapsto z_3]) \Downarrow (\Delta_2; v_2)$$

and $\textit{deref}_{\Delta_2}(v_2) = \textit{deref}_{\Delta_1}(v_1)$. So the property holds.

– Given $(\Gamma, a :_{\pi_2} A = t_2; \text{let}_{\pi_1}\ a_1 : B_1, \ldots,\ a_n : B_n \text{ in } e$, since

$$\begin{aligned}&(\text{let}_{\pi_1}\ a_1 : B_1 = e_1, \ldots,\ a_n : B_n = e_n \text{ in } t)[y \mapsto a] \\ &= \text{let}_{\pi_1}\ a_1 : B_1 = e_1[y \mapsto a], \ldots,\ a_n : B_n = e_n[y \mapsto a] \text{ in } t[y \mapsto a]\end{aligned}$$

we have

$$\frac{(\Gamma, a :_{\pi_2} A = t_2, a_1 :_{\pi_1} B_1 = e_1[y \mapsto a], \ldots,\ a_n :_{\pi_1} B_n = e_n[y \mapsto a]; t[y \mapsto a]) \Downarrow (\Delta_1; v_1)}{(\Gamma, a :_{\pi_2} A = t_2; \text{let}_{\pi_1}\ a_1 : B_1 = e_1[y \mapsto a], \ldots,\ a_n : B_n = e_n[y \mapsto a] \text{ in } t[y \mapsto a]) \Downarrow (\Delta_1; v_1)}\ \textit{s-let}_{\mathcal{L}}$$

by Lemma 4, if

$$\begin{aligned}&(\Gamma, a :_{\pi_2} A = t_2, a_1 : B_1 = e_1[y \mapsto a], \ldots,\ a_n : B_n = e_n[y \mapsto a]; t[y \mapsto a]) \\ &\Downarrow (\Delta_1; v_1)\end{aligned}$$

then

$$\begin{aligned}&(\Gamma, a :_{\pi_2} A = t_2, a_1 : B_1 = e_1[y \mapsto t_2], \ldots,\ a_n : B_n = e_n[y \mapsto t_2]; t[y \mapsto a]) \\ &\Downarrow (\Delta_1'; v_1')\end{aligned}$$

and $\textit{deref}_{\Delta_1}(v_1) = \textit{deref}_{\Delta_1'}(v_1')$. So, by induction hypothesis,

$$(\Gamma, a_1 : B_1 = e_1[y \mapsto t_2], \ldots,\ a_n : B_n = e_n[y \mapsto t_2]; t[y \mapsto t_2]) \Downarrow (\Delta_2; v_2)$$

and $\mathit{deref}_{\Delta_2}(v_2) = \mathit{deref}_{\Delta_1}(v_1)$. We also know that

$$\begin{aligned} &(\text{let}_{\pi_1}\ a_1 : B_1 = e_1, \ldots,\ a_n : B_n = e_n \text{ in } t)[y \mapsto t_2] \\ =\ &\text{let}_{\pi_1}\ a_1 : B_1 = e_1[y \mapsto t_2], \ldots,\ a_n : B_n = e_n[y \mapsto t_2] \text{ in } t[y \mapsto t_2] \end{aligned}$$

so, by rule $\textit{s-let}_{\mathcal{L}}$,

$$\frac{(\Gamma, a_1 :_{\pi_1} B_1 = e_1[y \mapsto t_2], \ldots,\ a_n :_{\pi_1} B_n = e_n[y \mapsto t_2]; t[y \mapsto t_2]) \Downarrow (\Delta_2; v_2)}{(\Gamma; \text{let}_{\pi_1}\ a_1 : B_1 = e_1[y \mapsto t_2], \ldots,\ a_n : B_n = e_n[y \mapsto t_2] \text{ in } t[y \mapsto t_2]) \Downarrow (\Delta_2; v_2)}$$

Since we already know that $\mathit{deref}_{\Delta_2}(v_2) = \mathit{deref}_{\Delta_1}(v_1)$, the property holds.

□

Lemma 6 (S-Substitution Lemma).
If $(S, a \mapsto q\ t; M) \twoheadrightarrow (S_1; N_1)$ *then*

$$(S; M[a \mapsto t]) \twoheadrightarrow (S_2; N_2) \textit{ and } \mathit{deref}_{S_2}(N_2) = \mathit{deref}_{S_1}(N_1)$$

Proof. By induction on $(S; t) \twoheadrightarrow (S'; v)$:

– Given $(S, a \mapsto q\ e; q_1\ \lambda y : q_2\ P.\ t)$, we have, by rule $\textit{s-abs}_{\mathcal{W}}$,

$$\begin{aligned} &(S, a \mapsto q\ e; q_1\ \lambda y : q_2\ P.\ t) \\ \to\ &(S, a \mapsto q\ e, a_1 \mapsto q_1\ \lambda y : q_2\ P.\ t; a_1) \end{aligned}$$

We know that, since $q_1\ \lambda y : q_2\ P.$ t is not evaluated, $a \notin \mathit{fv}(q_1\ \lambda y : q_2\ P.\ t)$, so we have $(q_1\ \lambda y : q_2\ P.\ t)[a \mapsto e] = q_1\ \lambda y : q_2\ P.$ t, for which, by rule s-abs$_{\mathcal{W}}$,

$$\begin{aligned} &(S; q_1\ \lambda y : q_2\ P.\ t) \\ \to\ &(S, a_2 \mapsto q_1\ \lambda y : q_2\ P.\ t; a_2) \end{aligned}$$

We know that

$$\begin{aligned} &\mathit{deref}_{S, a \mapsto q\ e, a_1 \mapsto q_1\ \lambda y : q_2\ P.\ t}(a_1) \\ =\ &\mathit{deref}_{S, a \mapsto q\ e}(q_1\ \lambda y : q_2\ P.\ t) \\ &\text{since } a_1 \text{ is a fresh variable, it can be removed from the heap} \\ =\ &\text{let } (t', S') = \mathit{deref}_{S, a \mapsto q\ e}(t) \\ &\qquad\qquad \text{since we know that } a \notin \mathit{fv}(t) \\ &\qquad\qquad = \mathit{deref}_S(t), \text{ by Lemma 5.3} \\ &\text{in } (q_1\ \lambda y : q_2\ P.t', S') \end{aligned}$$

and

$$\begin{aligned} &\mathit{deref}_{S, a_2 \mapsto q_1\ \lambda y : q_2\ P.\ t}(a_2) \\ =\ &\mathit{deref}_S(q_1\ \lambda y : q_2\ P.\ t) \\ &\text{since } a_2 \text{ is a fresh variable, it can be removed from the heap} \end{aligned}$$

We already know that $(t', S') = \mathit{deref}_S(t)$, therefore

$$\mathit{deref}_S(q_1\ \lambda y : q_2\ P.t) = (q_1\ \lambda y : q_2\ P.t', S')$$

So the property holds.

– Given $(S, a \mapsto q\ e; q_1 < t_1, t_2 >)$, we have

$$\frac{}{(S, a \mapsto q\ e; q_1 < t_1, t_2 >) \rightarrow (S, a \mapsto q\ e, a_1 \mapsto q_1 < t_1, t_2 >; a_1)}\ \textit{s-pair}_{\mathcal{W}}$$

We know that, since $q_1 < t_1, t_2 >$ is not evaluated, $a \notin \mathit{fv}(q_1 < t_1, t_2 >)$, we have

$$\begin{aligned}(q_1 < t_1, t_2 >)[a \mapsto e] &= q_1 < t_1[a \mapsto e], t_2[a \mapsto e] > \\ &= q_1 < t_1, t_2 >\end{aligned}$$

and

$$\frac{}{(S; q_1 < t_1, t_2 >) \rightarrow (S, a_2 \mapsto q_1 < t_1, t_2 >; a_2)}\ \textit{s-pair}_{\mathcal{W}}$$

Since we know that $a \notin q_1 < t_1, t_2 >$, we do not need to prove this for each possibility of q, so we will then assume $q = \mathrm{un}$. We know that,

$$\begin{aligned}
&\mathit{deref}_{S, a \mapsto q\ e, a_1 \mapsto q_1 < t_1, t_2 >}(a_1) \\
= &\ \mathit{deref}_{S, a \mapsto q\ e}(q_1 < t_1, t_2 >) \\
&\ \text{since } a_1 \text{ is a fresh variable, it can be removed from the heap} \\
= &\ \text{let } ({t_1}', S_1) = \mathit{deref}_{S, a \mapsto q\ e}(t_1) \\
&\qquad\qquad \text{since we know that } a \notin \mathit{fv}(t_1) \\
&\qquad\qquad = \mathit{deref}_S(t_1), \text{ by Lemma 5.3} \\
&\qquad ({t_2}', S_2) = \mathit{deref}_{S_1, a \mapsto q\ e}(t_2) \\
&\qquad\qquad \text{since we know that } a \notin \mathit{fv}(t_2) \\
&\qquad\qquad = \mathit{deref}_{S_1}(t_2), \text{ by Lemma 5.3} \\
&\ \text{in } (q_1 < {t_1}', {t_2}' >, S_2)
\end{aligned}$$

and

$$\begin{aligned}
&\mathit{deref}_{S, a_2 \mapsto q_1 q_1 < t_1, t_2 >}(a_2) \\
= &\ \mathit{deref}_S(q_1 < t_1, t_2 >) \\
&\ \text{since } a_2 \text{ is a fresh variable, it can be removed from the heap}
\end{aligned}$$

We know that $({t_1}', S_1) = \mathit{deref}_S(t_1)$ and $({t_2}', S_2) = \mathit{deref}_{S_1}(t_2)$, therefore

$$\mathit{deref}_S(q_1 < t_1, t_2 >) = (q_1 < {t_1}', {t_2}' >, S_2)$$

So the property holds.

– Given $(S, a \mapsto q\ e; x\ t)$, we have

$$\frac{S(x) = q_1\ \lambda y : q_2\ P.\ t'}{(S, a \mapsto q\ e; x\ t) \to (S \overset{q}{\sim} x, a \mapsto q\ e; t'[y \mapsto t])}\ \textit{s-app}_{\mathcal{W}}$$

We know that if x occurs unrestricted then it will stay in S but, if x occurs linearly then it will not be used again, thus we can use $S \overset{q}{\sim} x = S$ instead of distinguishing both cases.
So there are two cases,

- $x = a$
  Since we know that $(x\ t)[a \mapsto e] = x[a \mapsto e]\ t[a \mapsto e] = e\ t[a \mapsto e]$, then $e \equiv q_1\ \lambda y : q_2\ P.\ t'$ because its value must be equal to $S(x)$, so we have

  $$(S; (q_1\ \lambda y : q_2\ P.\ t')\ t[a \mapsto e]) \to (S, a_1 \mapsto q_1\ \lambda y : q_2\ P.\ t'; a_1\ t[a \mapsto e])$$

  and, by rule *s-app*$_{\mathcal{W}}$

  $$(S, a_1 \mapsto q_1\ \lambda y : q_2\ P.\ t'; a_1\ t[a \mapsto e]) \to (S \overset{q_1}{\sim} a_1; t'[y \mapsto t[a \mapsto e]])$$

  We know that a_1 is a fresh variable, so, since it will not be used again, we can use $S \overset{q_1}{\sim} a_1 = S$.
  Now we have

  $$\begin{aligned} &\mathit{deref}_{S,a\mapsto q\ e}(t'[y \mapsto t]) \\ =\ &\mathit{deref}_S(t'[y \mapsto t][a \mapsto e]) \text{ by Lemma 5.3} \end{aligned}$$

  We know that, since t' is not evaluated and a is an address variable, then $a \notin \mathit{fv}(t')$, therefore we can conclude that

  $$\mathit{deref}_S(t'[y \mapsto t][a \mapsto e]) = \mathit{deref}_S(t'[y \mapsto t[a \mapsto e]])$$

  We know that $(S, a \mapsto q\ e; t'[y \mapsto t]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis, since we already know that $a \notin \mathit{fv}(t')$,

  $$(S; t'[y \mapsto t[a \mapsto e]]) \twoheadrightarrow (S_2; v_2)$$

  and $\mathit{deref}_{S_2}(v_2) = \mathit{deref}_{S_1}(v_1)$. So the property holds.
- $x \neq a$
  Since we know that $(x\ t)[a \mapsto e] = x[a \mapsto e]\ t[a \mapsto e] = x\ t[a \mapsto e]$, so we have

  $$\frac{S(x) = q_1\ \lambda y : q_2\ P.\ t'}{(S; x\ t[a \mapsto e]) \to (S \overset{q_1}{\sim} x; t'[y \mapsto t[a \mapsto e]])}\ \textit{s-app}_{\mathcal{W}}$$

  We know that if x occurs unrestricted then it will stay in S but, if it occurs linearly then it will not be used again, thus we can use $S \overset{q_1}{\sim} x = S$ instead of distinguishing both cases. Then we have

  $$\begin{aligned} &\mathit{deref}_{S,a\mapsto q\ e}(t'[y \mapsto t]) \\ =\ &\mathit{deref}_S(t'[y \mapsto t][a \mapsto e]) \text{ by Lemma 5.3} \end{aligned}$$

We know that, since t' is not evaluated, and both a and x are address variables, $a, x \notin fv(t')$, therefore we can conclude that

$$deref_S(t'[y \mapsto t][a \mapsto e]) = deref_S(t'[y \mapsto t[a \mapsto e]])$$

We know that $(S, a \mapsto q\ e; t'[y \mapsto t]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis,

$$(S; t'[y \mapsto t[a \mapsto e]]) \twoheadrightarrow (S_2; v_2)$$

and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$. So the property holds.

– Given $(S, a \mapsto q\ e; t_1 t_2)$, we have

$$(S, a \mapsto q\ e; t_1 t_2) \twoheadrightarrow (S', a \mapsto q\ e; {t_1}' t_2)$$

because

$$(S, a \mapsto q\ e; t_1) \twoheadrightarrow (S', a \mapsto q\ e; {t_1}')$$

where, necessarily, ${t_1}' \equiv q_1\ \lambda y : q_2\ P.t$. We then have

$$(S', a \mapsto q\ e; {t_1}' t_2) \rightarrow (S', a \mapsto q\ e, a_1 \mapsto {t_1}'; a_1 t_2)$$

because, by rule $s\text{-}abs_{\mathcal{W}}$,

$$(S', a \mapsto q\ e; {t_1}') \rightarrow (S', a \mapsto q\ e, a_1 \mapsto {t_1}'; a_1)$$

and, by rule $s\text{-}app_{\mathcal{W}}$,

$$(S', a \mapsto q\ e, a_1 \mapsto {t_1}'; a_1 t_2) \rightarrow (S' \overset{q_1}{\sim} a_1, a \mapsto q\ e; t[y \mapsto t_2])$$

We know that $q_1\ \lambda y : q_2\ P.t$ is not evaluated and S' saves address variables contained in t_2. Therefore we know that $a \notin fv(t)$.
We then have $(t_1 t_2)[a \mapsto e] = t_1[a \mapsto e] t_2[a \mapsto e]$.
Since we know that $(S, a \mapsto q\ e; t_1) \twoheadrightarrow (S', a \mapsto q\ e; {t_1}')$ then, by induction hypothesis, $(S; t_1[a \mapsto e]) \twoheadrightarrow (S''; {t_1}'')$ and $deref_{S', a \mapsto q\ e}({t_1}') = deref_{S''}({t_1}'')$.
Therefore, we have

$$(S; t_1[a \mapsto e] t_2[a \mapsto e]) \twoheadrightarrow (S''; {t_1}'' t_2[a \mapsto e])$$

where, necessarily, ${t_1}'' \equiv {q_1}'\ \lambda z : q_2,'\ P'.t'$.
We then have

$$(S''; {t_1}'' t_2[a \mapsto e]) \rightarrow (S'', a_2 \mapsto {t_1}''; a_2 t_2[a \mapsto e])$$

because, by rule $s\text{-}abs_{\mathcal{W}}$,

$$(S''; {t_1}'') \rightarrow (S'', a_2 \mapsto {t_1}''; a_2)$$

and, by rule $s\text{-}app_{\mathcal{W}}$,

$$(S'', a_2 \mapsto {t_1}''; a_2 t_2[a \mapsto e]) \rightarrow (S'' \overset{q_1}{\sim} a_2; t'[z \mapsto t_2[a \mapsto e]])$$

Since both a_1 and a_2 are fresh variables, they will not be used again regardless of q_1 and ${q_1}'$, so we can use $S' \overset{q_1}{\sim} a_1 = S'$ and $S'' \overset{{q_1}'}{\sim} a_2 = S''$. Thus

$$\begin{aligned} deref_{S',a\mapsto q\ e}(t[y \mapsto t_2]) &= deref_{S'}(t[y \mapsto t_2][a \mapsto e]) \text{ by Lemma 5.3} \\ &\text{since } a \notin fv(t), \text{ we have} \\ &= deref_{S'}(t[y \mapsto t_2[a \mapsto e]]) \end{aligned}$$

We know that, since $a \notin fv(t)$, $deref_{S',a\mapsto q\ e}(q_1\ \lambda y : q_2\ P.t) = deref_{S'}(q_1\ \lambda y : q_2\ P.t)$ by Proposition 1. We also know that $deref_{S'}(q_1\ \lambda y : q_2\ P.t) = deref_{S''}({q_1}'\ \lambda z : {q_2}'\ P'.t')$.
Therefore, we can conclude that

$$deref_{S',a\mapsto q\ e}(t[y \mapsto t_2[a \mapsto e]]) = deref_{S''}(t'[z \mapsto t_2[a \mapsto e]])$$

Now, we know that if $(S', a \mapsto q\ e; t[y \mapsto t_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis,

$$(S''; t'[z \mapsto t_2[a \mapsto e]]) \twoheadrightarrow (S_2; v_2)$$

and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$. So the property holds.

– Given $(S, a \mapsto q\ e; \text{split } x \text{ as } y, z \text{ in } t_2)$, we have, by rule *s-split*$_{\mathcal{W}}$

$$\begin{aligned} &(S, a \mapsto q\ e; \text{split } x \text{ as } y, z \text{ in } t) \\ \to\ &(S \overset{q_1}{\sim} x, a \mapsto q\ e; t[y \mapsto t_1][z \mapsto t_2]) \end{aligned}$$

where $S(x) = q_1 < t_1, t_2 >$
We know that if x occurs unrestricted then it will stay in S but, if it occurs linearly then it will not be used again, thus we can use $S \overset{q_1}{\sim} x = S$.
So there are two cases,

- $x = a$
  Since we know that

$$\begin{aligned} (\text{split } x \text{ as } y, z \text{ in } t)[a \mapsto e] &= \text{split } x[a \mapsto e] \text{ as } y, z \text{ in } t[a \mapsto e] \\ &= \text{split } e \text{ as } y, z \text{ in } t[a \mapsto e] \end{aligned}$$

  then $e \equiv q_1 < t_1, t_2 >$ because its value must be equal to $S(x)$, so we have

$$\begin{aligned} &(S; \text{split } (q_1 < t_1, t_2 >) \text{ as } y, z \text{ in } t[a \mapsto e]) \\ \to\ &(S, a_1 \mapsto q_1 < t_1, t_2 >; \text{split } a_1 \text{ as } y, z \text{ in } t[a \mapsto e]) \end{aligned}$$

  and, by rule *s-split*$_{\mathcal{W}}$

$$\begin{aligned} &(S, a_1 \mapsto q_1 < t_1, t_2 >; \text{split } a_1 \text{ as } y, z \text{ in } t[a \mapsto e]) \\ \to\ &(S \overset{q_1}{\sim} a_1; t[a \mapsto e][y \mapsto t_1][z \mapsto t_2]) \end{aligned}$$

We know that a_1 is a fresh variable, so it will not be used anymore and we can use $S \overset{q_1}{\sim} a_1 = S$.
Now we have

$$\begin{aligned} & \mathit{deref}_{S,a\mapsto q\ e}(t[y \mapsto t_1][z \mapsto t_2]) \\ = \ & \mathit{deref}_S(t[y \mapsto t_1][z \mapsto t_2][a \mapsto e]) \text{ by Lemma 5.3} \end{aligned}$$

We know that, since t_1 and t_2 are not evaluated and a is an address variable, then $a \notin \mathit{fv}(t_1)$ and $a \notin \mathit{fv}(t_2)$, therefore we can conclude that

$$\mathit{deref}_S(t[y \mapsto t_1][z \mapsto t_2][a \mapsto e]) = \mathit{deref}_S(t[a \mapsto e][y \mapsto t_1][z \mapsto t_2])$$

We know $(S, a \mapsto q\ e; t[y \mapsto t_1][z \mapsto t_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis, $(S; t[a \mapsto e][y \mapsto t_1][z \mapsto t_2]) \twoheadrightarrow (S_2; v_2)$ and $\mathit{deref}_{S_2}(v_2) = \mathit{deref}_{S_1}(v_1)$. So the property holds.

- $x \neq a$
  Since we know that

$$\begin{aligned} & (\text{split } x \text{ as } y, z \text{ in } t)[a \mapsto e] \\ = \ & \text{split } x[a \mapsto e] \text{ as } y, z \text{ in } t[a \mapsto e] \\ = \ & \text{split } x \text{ as } y, z \text{ in } t[a \mapsto e] \end{aligned}$$

we have, by rule *s-split*$_{\mathcal{W}}$

$$\begin{aligned} & (S; \text{split } x \text{ as } y, z \text{ in } t[a \mapsto e]) \\ \to \ & (S \overset{q_1}{\sim} x; t[a \mapsto e][y \mapsto t_1][z \mapsto t_2]) \end{aligned}$$

where $S(x) = q_1 < t_1, t_2 >$.
We know that if x occurs unrestricted then it will stay in S but, if it occurs linearly then it will not be used again, thus we can use $S \overset{q_1}{\sim} x = S$ Then we have

$$\mathit{deref}_{S,a\mapsto q\ e}(t[y \mapsto t_1][z \mapsto t_2]) = \mathit{deref}_S(t[y \mapsto t_1][z \mapsto t_2][a \mapsto e])$$

We know that, since t_1 and t_2 are not evaluated, and both a and x are address variables, then $a, x \notin \mathit{fv}(t_1)$ and $a, x \notin \mathit{fv}(t_2)$, therefore we can conclude that

$$\mathit{deref}_S(t[y \mapsto t_1][z \mapsto t_2][a \mapsto e]) = \mathit{deref}_S(t[a \mapsto e][y \mapsto t_1][z \mapsto t_2])$$

Since $(S, a \mapsto q\ e; t[y \mapsto t_1][z \mapsto t_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis, $(S; t[a \mapsto e][y \mapsto t_1][z \mapsto t_2]) \twoheadrightarrow (S_2; v_2)$ and $\mathit{deref}_{S_2}(v_2) = \mathit{deref}_{S_1}(v_1)$. So the property holds.

– Given $(S, a \mapsto q\ e; \text{split } t_1 \text{ as } y, z \text{ in } t)$, we have

$$(S, a \mapsto q\ e; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \twoheadrightarrow (S', a \mapsto q\ e; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2)$$

because

$$(S, a \mapsto q\ e; t_1) \twoheadrightarrow (S', a \mapsto q\ e; {t_1}')$$

where, necessarily, ${t_1}' \equiv q_1 < e_1, e_2 >$. We then have

$$(S', a \mapsto q\, e; \text{split}\, {t_1}' \,\text{as}\, y, z \,\text{in}\, t_2) \rightarrow (S', a \mapsto q\, e, a_1 \mapsto {t_1}'; \text{split}\, a_1 \,\text{as}\, y, z \,\text{in}\, t_2)$$

because, by rule *s-pair*$_{\mathcal{W}}$,

$$(S', a \mapsto q\ e; {t_1}') \rightarrow (S', a \mapsto q\ e, a_1 \mapsto {t_1}'; a_1)$$

and, by rule *s-split*$_{\mathcal{W}}$,

$$(S', a \mapsto q\ e, a_1 \mapsto {t_1}'; \text{split } a_1 \text{ as } y, z \text{ in } t_2) \rightarrow (S' \overset{q_1}{\sim} a_1; t_2[y \mapsto e_1][z \mapsto e_2])$$

We know that $q_1 < e_1, e_2 >$ is not evaluated and S' saves address variables contained in t_2. Therefore we know that $a \notin \mathit{fv}(e_1)$ and $a \notin \mathit{fv}(e_2)$.
We then have $(\text{split } t_1 \text{ as } y, z \text{ in } t_2)[a \mapsto e] = \text{split } t_1[a \mapsto e] \text{ as } y, z \text{ in } t_2[a \mapsto e]$.
Since we know that $(S, a \mapsto q\ e; t_1) \twoheadrightarrow (S', a \mapsto q\ e; {t_1}')$ then, by induction hypothesis, $(S; t_1[a \mapsto e]) \twoheadrightarrow (S''; {t_1}'')$ and $\mathit{deref}_{S''}({t_1}'') = \mathit{deref}_{S', a \mapsto q_1\ e}({t_1}')$.
Therefore,

$$(S; \text{split } t_1[a \mapsto e] \text{ as } y, z \text{ in } t_2[a \mapsto e]) \twoheadrightarrow (S''; \text{split } {t_1}'' \text{ as } y, z \text{ in } t_2[a \mapsto e])$$

where, necessarily, ${t_1}'' \equiv {q_1}' < {e_1}', {e_2}' >$.
We then have

$$(S''; \text{split}\, {t_1}'' \,\text{as}\, y, z \,\text{in}\, t_2[a \mapsto e]) \rightarrow (S'', a_2 \mapsto {t_1}''; \text{split}\, a_2 \,\text{as}\, y, z \,\text{in}\, t_2[a \mapsto e])$$

because, by rule *s-pair*$_{\mathcal{W}}$,

$$(S''; {t_1}'') \rightarrow (S'', a_2 \mapsto {t_1}''; a_2)$$

and, by rule *s-split*$_{\mathcal{W}}$,

$$(S'', a_2 \mapsto {t_1}''; \text{split}\, a_2 \,\text{as}\, y, z \,\text{in}\, t_2[a \mapsto e]) \rightarrow (S'' \overset{{q_1}'}{\sim} a_2; t_2[a \mapsto e][y \mapsto {e_1}'][z \mapsto {e_2}'])$$

Since both a_1 and a_2 are fresh variables, they will not be used again regardless of q_1 and ${q_1}'$, so we can use $S' \overset{q_1}{\sim} a_1 = S'$ and $S'' \overset{{q_1}'}{\sim} a_2 = S''$. Thus

$$\begin{aligned}
&\mathit{deref}_{S', a \mapsto q\ e}(t[y \mapsto e_1][z \mapsto e_2]) \\
= \;&\mathit{deref}_{S'}(t[y \mapsto e_1][z \mapsto e_2][a \mapsto e]) \text{ by Lemma 5.3} \\
&\text{ since } a \notin \mathit{fv}(e_1) \text{ and } a \notin \mathit{fv}(e_2), \text{ we have} \\
= \;&\mathit{deref}_{S'}(t[a \mapsto e][y \mapsto e_1][z \mapsto e_2])
\end{aligned}$$

We know that, since $a \notin fv(e_1)$ and $a \notin fv(e_2)$,

$$deref_{S',a \mapsto q\ e}(q_1 < e_1, e_2 >) = deref_{S'}(q_1 < e_1, e_2 >)$$

by Proposition 1. We also know that $deref_{S'}(q_1 < e_1, e_2 >) = deref_{S''}({q_1}' < {e_1}', {e_2}' >)$.
Therefore, we can conclude that

$$deref_{S'}(t_2[a \mapsto e][y \mapsto e_1][z \mapsto e_2]) = deref_{S''}(t_2[a \mapsto e][y \mapsto {e_1}'][z \mapsto {e_2}'])$$

Now, we have that if $(S, a \mapsto q\ e; t_2[y \mapsto e_1][z \mapsto e_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis,

$$(S; t_2[a \mapsto e][y \mapsto {e_1}'][z \mapsto {e_2}']) \twoheadrightarrow (S_2; v_2)$$

and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$. So the property holds.

$\square$

Lemma 7 (Term Substitution Lemma).
Given $M_{\mathcal{W}}[y \mapsto t]$*, then* $(M_{\mathcal{W}}[y \mapsto t])^*_{\mathcal{L}} = (M_{\mathcal{W}})^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$.

Proof. By induction on $M_{\mathcal{W}}$:

- $M_{\mathcal{W}} \equiv x$
  There are two cases:
  - $x = y$
    We have $(x[y \mapsto t])^*_{\mathcal{L}} = t^*_{\mathcal{L}}$ and $x^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] = x[y \mapsto t^*_{\mathcal{L}}] = t^*_{\mathcal{L}}$. So the property holds.
  - $x \neq y$
    We have $(x[y \mapsto t])^*_{\mathcal{L}} = x^*_{\mathcal{L}} = x$ and $x^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] = x[y \mapsto t^*_{\mathcal{L}}] = x$. So the property holds.
- $M_{\mathcal{W}} = q\ \lambda x : q'\ P.\ t_1$
  By α-equivalence, we know that $x \neq y$. So we have

  $$((q\ \lambda x : q'\ P.\ t_1)[y \mapsto t])^*_{\mathcal{L}} = (q\ \lambda x : q'\ P.\ t_1[y \mapsto t])^*_{\mathcal{L}} = \lambda_\pi x : P_{\mathcal{L}}.(t_1[y \mapsto t])^*_{\mathcal{L}}$$

  where $\pi = \mathrm{q}'_{\mathcal{L}}$, and

  $$(q\ \lambda x : q'\ P.\ t_1)^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] = (\lambda_\pi x : P_{\mathcal{L}}.{t_1}^*_{\mathcal{L}})[y \mapsto t^*_{\mathcal{L}}] = \lambda_\pi x : P_{\mathcal{L}}.{t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$$

  By induction hypothesis, we know that $(t_1[y \mapsto t])^*_{\mathcal{L}} = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

  $$\lambda_\pi x : P_{\mathcal{L}}.(t_1[y \mapsto t])^*_{\mathcal{L}} = \lambda_\pi x : P_{\mathcal{L}}.{t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$$
- $M_{\mathcal{W}} = q < t_1, t_2 >$

We have

$$
\begin{aligned}
&((q\ < t_1, t_2 >)[y \mapsto t])^*_{\mathcal{L}} \\
&= (q\ < t_1[y \mapsto t], t_2[y \mapsto t] >)^*_{\mathcal{L}} \\
&= \text{let}_\pi\ a_1 : A_1 = (t_1[y \mapsto t])^*_{\mathcal{L}}, a_2 : A_2 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } < a_1, a_2 >_\pi \\
&\quad \text{where } \pi = q_{\mathcal{L}}
\end{aligned}
$$

and

$$
\begin{aligned}
&(q\ < t_1, t_2 >)^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \\
&= (\text{let}_\pi\ b_1 : B_1 = {t_1}^*_{\mathcal{L}}, b_2 : B_2 = {t_2}^*_{\mathcal{L}} \text{ in } < b_1, b_2 >_\pi)[y \mapsto t^*_{\mathcal{L}}] \\
&= \text{let}_\pi\ b_1 : B_1 = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}], b_2 : B_2 = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } (< b_1, b_2 >_\pi)[y \mapsto t^*_{\mathcal{L}}] \\
&= \text{let}_\pi\ b_1 : B_1 = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}], b_2 : B_2 = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } < b_1, b_2 >_\pi
\end{aligned}
$$

By induction hypothesis, we know that $(t_1[y \mapsto t])^*_{\mathcal{L}} = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$ and $(t_2[y \mapsto t])^*_{\mathcal{L}} = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$
\begin{aligned}
&\text{let}_\pi\ a_1 : A_1 = (t_1[y \mapsto t])^*_{\mathcal{L}}, a_2 : A_2 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } < a_1, a_2 >_\pi \\
&= \text{let}_\pi\ b_1 : B_1 = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}], b_2 : B_2 = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } < b_1, b_2 >_\pi
\end{aligned}
$$

– $M_{\mathcal{W}} \equiv x\ t_1$
  There are two cases:
  - $x = y$
    We have

$$
\begin{aligned}
((x\ t_2)[y \mapsto t])^*_{\mathcal{L}} &= (x[y \mapsto t]\ t_2[y \mapsto t])^*_{\mathcal{L}} \\
&= (t\ t_2[y \mapsto t])^*_{\mathcal{L}} \\
&= \text{let}_\pi\ a_1 : A_1 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } t^*_{\mathcal{L}}\ a_1
\end{aligned}
$$

    and

$$
\begin{aligned}
&(x\ t_2)^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \\
&= (\text{let}_\pi\ b_1 : B_1 = {t_2}^*_{\mathcal{L}} \text{ in } x^*_{\mathcal{L}}\ b_1)[y \mapsto t^*_{\mathcal{L}}] \\
&= \text{let}_\pi\ b_1 : B_1 = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } x[y \mapsto t^*_{\mathcal{L}}]\ b_1[y \mapsto t^*_{\mathcal{L}}] \\
&= \text{let}_\pi\ b_1 : B_1 = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } t^*_{\mathcal{L}}\ b_1
\end{aligned}
$$

    By induction hypothesis, we know that $(t_2[y \mapsto t])^*_{\mathcal{L}} = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$
\text{let}_\pi\ a_1 : A_1 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } t^*_{\mathcal{L}}\ a_1 = \text{let}_\pi\ b_1 : B_1 = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } t^*_{\mathcal{L}}\ b_1
$$

  - $x \neq y$
    We have

$$
\begin{aligned}
((x\ t_2)[y \mapsto t])^*_{\mathcal{L}} &= (x[y \mapsto t]\ t_2[y \mapsto t])^*_{\mathcal{L}} \\
&= (x\ t_2[y \mapsto t])^*_{\mathcal{L}} \\
&= \text{let}_\pi\ a_1 : A_1 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } x\ a_1
\end{aligned}
$$

and

$$
\begin{aligned}
&(x\ t_2)^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]\\
&= (\text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}} \text{ in } x^*_{\mathcal{L}}\ b_1)[y \mapsto t^*_{\mathcal{L}}]\\
&= \text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } x[y \mapsto t^*_{\mathcal{L}}]\ b_1[y \mapsto t^*_{\mathcal{L}}]\\
&= \text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } x\ b_1
\end{aligned}
$$

By induction hypothesis, we know that $(t_2[y \mapsto t])^*_{\mathcal{L}} = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$\text{let}_\pi\ a_1 : A_1 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } x\ a_1 = \text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } x\ b_1$$

- $M_{\mathcal{W}} = t_1\ t_2$
  We have

$$
\begin{aligned}
((t_1\ t_2)[y \mapsto t])^*_{\mathcal{L}} &= (t_1[y \mapsto t] t_2[y \mapsto t])^*_{\mathcal{L}}\\
&= \text{let}_\pi\ a_1 : A_1 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } (t_1[y \mapsto t])^*_{\mathcal{L}}\ a_1
\end{aligned}
$$

and

$$
\begin{aligned}
&(t_1\ t_2)^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]\\
&= (\text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}} \text{ in } t_1{}^*_{\mathcal{L}}\ b_1)[y \mapsto t^*_{\mathcal{L}}]\\
&= \text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } t_1{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]\ b_1[y \mapsto t^*_{\mathcal{L}}]\\
&= \text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } t_1{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]\ b_1
\end{aligned}
$$

By induction hypothesis, we know that $(t_1[y \mapsto t])^*_{\mathcal{L}} = t_1{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$ and $(t_2[y \mapsto t])^*_{\mathcal{L}} = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$
\begin{aligned}
&\text{let}_\pi\ a_1 : A_1 = (t_2[y \mapsto t])^*_{\mathcal{L}} \text{ in } (t_1[y \mapsto t])^*_{\mathcal{L}}\ a_1\\
&= \text{let}_\pi\ b_1 : B_1 = t_2{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ in } t_1{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]\ b_1
\end{aligned}
$$

- $M_{\mathcal{W}} = \text{split } x \text{ as } y, z \text{ in } t_2$
  There are two cases:
  - $x = x_1$
    We have

$$
\begin{aligned}
&((\text{split } x \text{ as } y, z \text{ in } t_2)[x_1 \mapsto t])^*_{\mathcal{L}}\\
&= (\text{split } x[x_1 \mapsto t] \text{ as } y, z \text{ in } t_2[x_1 \mapsto t])^*_{\mathcal{L}}\\
&= (\text{split } t \text{ as } y, z \text{ in } t_2[x_1 \mapsto t])^*_{\mathcal{L}}\\
&= \text{split } t^*_{\mathcal{L}} \text{ as } y, z \text{ in } (t_2[x_1 \mapsto t])^*_{\mathcal{L}}
\end{aligned}
$$

    and

$$
\begin{aligned}
&(\text{split } x \text{ as } y, z \text{ in } t_2)^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}]\\
&= (\text{split } x^*_{\mathcal{L}} \text{ as } y, z \text{ in } t_2{}^*_{\mathcal{L}})[x_1 \mapsto t^*_{\mathcal{L}}]\\
&= \text{split } x[x_1 \mapsto t^*_{\mathcal{L}}] \text{ as } y, z \text{ in } t_2{}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}]\\
&= \text{split } t^*_{\mathcal{L}} \text{ as } y, z \text{ in } t_2{}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}]
\end{aligned}
$$

By induction hypothesis, we know that $(t_2[x_1 \mapsto t])^*_{\mathcal{L}} = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$\text{split } t^*_{\mathcal{L}} \text{ as } y, z \text{ in } (t_2[x_1 \mapsto t])^*_{\mathcal{L}} = \text{split } t^*_{\mathcal{L}} \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}]$$

- $x \neq x_1$
  We have

$$\begin{aligned} & ((\text{split } x \text{ as } y, z \text{ in } t_2)[x_1 \mapsto t])^*_{\mathcal{L}} \\ = {} & (\text{split } x[x_1 \mapsto t] \text{ as } y, z \text{ in } t_2[x_1 \mapsto t])^*_{\mathcal{L}} \\ = {} & (\text{split } x \text{ as } y, z \text{ in } t_2[x_1 \mapsto t])^*_{\mathcal{L}} \\ = {} & \text{split } x^*_{\mathcal{L}} \text{ as } y, z \text{ in } (t_2[x_1 \mapsto t])^*_{\mathcal{L}} \\ = {} & \text{split } x \text{ as } y, z \text{ in } (t_2[x_1 \mapsto t])^*_{\mathcal{L}} \end{aligned}$$

  and

$$\begin{aligned} & (\text{split } x \text{ as } y, z \text{ in } t_2)^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}] \\ = {} & (\text{split } x^*_{\mathcal{L}} \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}})[x_1 \mapsto t^*_{\mathcal{L}}] \\ = {} & \text{split } x[x_1 \mapsto t^*_{\mathcal{L}}] \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}] \\ = {} & \text{split } x \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}] \end{aligned}$$

  By induction hypothesis, we know that $(t_2[x_1 \mapsto t])^*_{\mathcal{L}} = {t_2}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$\text{split } x \text{ as } y, z \text{ in } (t_2[x_1 \mapsto t])^*_{\mathcal{L}} = \text{split } x \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}[x_1 \mapsto t^*_{\mathcal{L}}]$$

– $M_{\mathcal{L}} = \text{split } t_1 \text{ as } y, z \text{ in } t_2$
  We have

$$\begin{aligned} ((\text{split } t_1 \text{ as } y, z \text{ in } t_2)[y \mapsto t])^*_{\mathcal{L}} &= (\text{split } t_1[y \mapsto t] \text{ as } y, z \text{ in } t_2[y \mapsto t])^*_{\mathcal{L}} \\ &= \text{split } (t_1[y \mapsto t])^*_{\mathcal{L}} \text{ as } y, z \text{ in } (t_2[y \mapsto t])^*_{\mathcal{L}} \end{aligned}$$

  and

$$\begin{aligned} (\text{split } t_1 \text{ as } y, z \text{ in } t_2)^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] &= (\text{split } {t_1}^*_{\mathcal{L}} \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}})[y \mapsto t^*_{\mathcal{L}}] \\ &= \text{split } {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \end{aligned}$$

  By induction hypothesis, we know that $(t_1[y \mapsto t])^*_{\mathcal{L}} = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$ and $(t_2[y \mapsto t])^*_{\mathcal{L}} = {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$, so we can conclude that

$$\begin{aligned} & \text{split } (t_1[y \mapsto t])^*_{\mathcal{L}} \text{ as } y, z \text{ in } (t_2[y \mapsto t])^*_{\mathcal{L}} \\ = {} & \text{split } {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}] \end{aligned}$$

□

Lemma 8. *Given $(\Gamma; M_{\mathcal{L}})$, then $deref_{\Gamma_{\mathcal{W}}}(({M_{\mathcal{L}}}^*)_{\mathcal{W}}) = deref_{\Gamma_{\mathcal{W}}}((M_{\mathcal{L}})_{\mathcal{W}})$.*

Proof. By induction on $(\Gamma; M_{\mathcal{L}})$:

- Given $(\Gamma, x :_1 A = t; x)$, we have

$$(\Gamma_{\mathcal{W}}, x \mapsto \text{lin } t_{\mathcal{W}}; x_{\mathcal{W}}) = (\Gamma_{\mathcal{W}}, x \mapsto \text{lin } t_{\mathcal{W}}; x)$$

and

$$\begin{aligned}(\Gamma_{\mathcal{W}}, x \mapsto \text{lin } t_{\mathcal{W}}; (x^*)_{\mathcal{W}}) &= (\Gamma_{\mathcal{W}}, x \mapsto \text{lin } t_{\mathcal{W}}; x_{\mathcal{W}}) \\ &= (\Gamma_{\mathcal{W}}, x \mapsto \text{lin } t_{\mathcal{W}}; x)\end{aligned}$$

Then we have $\mathit{deref}_{\Gamma_{\mathcal{W}}, x \mapsto \text{lin } t_{\mathcal{W}}}(x) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}})$. So the property holds.
- Given $(\Gamma, x :_\omega A = t; x)$, we have

$$(\Gamma_{\mathcal{W}}, x \mapsto \text{un } t_{\mathcal{W}}; x_{\mathcal{W}}) = (\Gamma_{\mathcal{W}}, x \mapsto \text{un } t_{\mathcal{W}}; x)$$

and

$$\begin{aligned}(\Gamma_{\mathcal{W}}, x \mapsto \text{un } t_{\mathcal{W}}; (x^*)_{\mathcal{W}}) &= (\Gamma_{\mathcal{W}}, x \mapsto \text{un } t_{\mathcal{W}}; x_{\mathcal{W}}) \\ &= (\Gamma_{\mathcal{W}}, x \mapsto \text{ un } t_{\mathcal{W}}; x)\end{aligned}$$

Then we have $\mathit{deref}_{\Gamma_{\mathcal{W}}, x \mapsto \text{un } t_{\mathcal{W}}}(x) = \mathit{deref}_{\Gamma_{\mathcal{W}}, x \mapsto \text{un } t_{\mathcal{W}}}(t_{\mathcal{W}})$. So the property holds.
- Given $(\Gamma; \lambda_\pi y : P.t)$, we have $(\Gamma_{\mathcal{W}}; (\lambda_\pi y : P.t)_{\mathcal{W}}) = (\Gamma_{\mathcal{W}}; \pi_{\mathcal{W}} \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.t_{\mathcal{W}})$ and

$$\begin{aligned}(\Gamma_{\mathcal{W}}; ((\lambda_\pi y : P.t)^*)_{\mathcal{W}}) &= (\Gamma_{\mathcal{W}}; (\lambda_\pi y : P.t^*)_{\mathcal{W}}) \\ &= (\Gamma_{\mathcal{W}}; \pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.(t^*)_{\mathcal{W}})\end{aligned}$$

We have

$$\begin{aligned}&\mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.t_{\mathcal{W}}) \\ =\ &\text{let } (t', S) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}}) \\ &\text{in } (\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.t', S)\end{aligned}$$

and

$$\begin{aligned}&\mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.(t^*)_{\mathcal{W}}) \\ =\ &\text{let } (t'', S') = \mathit{deref}_{\Gamma_{\mathcal{W}}}((t^*)_{\mathcal{W}}) \\ &\text{in } (\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.t'', S')\end{aligned}$$

By induction hypothesis, $\mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}}) = \mathit{deref}_{\Gamma_{\mathcal{W}}}((t^*)_{\mathcal{W}})$. Therefore, we have

$$\mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.t_{\mathcal{W}}) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.(t^*)_{\mathcal{W}})$$

so the property holds.

– Given $(\Gamma; < t_1, t_2 >_\pi)$, we have $(\Gamma_\mathcal{W}; \pi_\mathcal{W} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >)$ and

$$\begin{aligned}
&(\Gamma_\mathcal{W}; ((< t_1, t_2 >_\pi)^*)_\mathcal{W}) \\
&= (\Gamma_\mathcal{W}; (\text{let}_\pi\ a_1 : A_1 = {t_1}^*, a_2 : A_2 = {t_2}^* \text{ in } < a_1, a_2 >_\pi)_\mathcal{W}) \\
&= (\Gamma_\mathcal{W}; (< a_1, a_2 >_\pi)_\mathcal{W}[a_1 \mapsto ({t_1}^*)_\mathcal{W}][a_2 \mapsto ({t_2}^*)_\mathcal{W}]) \\
&= (\Gamma_\mathcal{W}; (\pi_\mathcal{W} < a_1, a_2 >)[a_1 \mapsto ({t_1}^*)_\mathcal{W}][a_2 \mapsto ({t_2}^*)_\mathcal{W}]) \\
&\quad \text{since we know that } a_1 \notin \mathit{fv}(({t_2}^*)_\mathcal{W}) \text{ and } a_2 \notin \mathit{fv}(({t_1}^*)_\mathcal{W}) \\
&= (\Gamma_\mathcal{W}; \pi_\mathcal{W} < a_1[a_1 \mapsto ({t_1}^*)_\mathcal{W}], a_2[a_2 \mapsto ({t_2}^*)_\mathcal{W}] >) \\
&= (\Gamma_\mathcal{W}; \pi_\mathcal{W} < ({t_1}^*)_\mathcal{W}, ({t_2}^*)_\mathcal{W} >)
\end{aligned}$$

We now know that

$$\begin{aligned}
&\mathit{deref}_{\Gamma_\mathcal{W}}(\pi_\mathcal{W} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >) \\
&= \text{let } ({t_1}', S_1) = \mathit{deref}_{\Gamma_\mathcal{W}}(t_{1\mathcal{W}}) \\
&\qquad ({t_2}', S_2) = \mathit{deref}_{S_1}(t_{2\mathcal{W}}) \\
&\quad \text{in } (\pi_\mathcal{W} < {t_1}', {t_2}' >, S_2)
\end{aligned}$$

and

$$\begin{aligned}
&\mathit{deref}_{\Gamma_\mathcal{W}}(\pi_\mathcal{W} < ({t_1}^*)_\mathcal{W}, ({t_2}^*)_\mathcal{W} >) \\
&= \text{let } ({t_1}'', {S_1}') = \mathit{deref}_{\Gamma_\mathcal{W}}(({t_1}^*)_\mathcal{W}) \\
&\qquad ({t_2}'', {S_2}') = \mathit{deref}_{{S_1}'}(({t_2}^*)_\mathcal{W}) \\
&\quad \text{in } (\pi_\mathcal{W} < {t_1}'', {t_2}'' >, {S_2}')
\end{aligned}$$

By induction hypothesis,

$$\mathit{deref}_{\Gamma_\mathcal{W}}(t_1)_\mathcal{W} = \mathit{deref}_{\Gamma_\mathcal{W}}(({t_1}^*)_\mathcal{W})$$

Then we know that $S_1 = {S_1}'$. So, similarly, by induction hypothesis

$$\mathit{deref}_{S_1}(t_{2\mathcal{W}}) = \mathit{deref}_{{S_1}'}(({t_2}^*)_\mathcal{W})$$

Therefore, we can conclude that

$$\mathit{deref}_{\Gamma_\mathcal{W}}(\pi_\mathcal{W} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >) = \mathit{deref}_{\Gamma_\mathcal{W}}(\pi_\mathcal{W} < ({t_1}^*)_\mathcal{W}, ({t_2}^*)_\mathcal{W} >)$$

The property holds.

– Given $(\Gamma; t\ x)$, we have $(\Gamma_\mathcal{W}; (t\ x)_\mathcal{W}) = (\Gamma_\mathcal{W}; t_\mathcal{W}\ x_\mathcal{W}) = (\Gamma_\mathcal{W}; t_\mathcal{W}\ x)$ and $(\Gamma_\mathcal{W}; ((t\ x)^*)_\mathcal{W}) = (\Gamma_\mathcal{W}; (t^*\ x^*)_\mathcal{W}) = (\Gamma_\mathcal{W}; (t^*)_\mathcal{W}\ x_\mathcal{W}) = (\Gamma_\mathcal{W}; (t^*)_\mathcal{W}\ x)$. We know that

$$\begin{aligned}
\mathit{deref}_{\Gamma_\mathcal{W}}(t_\mathcal{W}\ x) = \text{let } &({t_1}', S_1) = \mathit{deref}_{\Gamma_\mathcal{W}}(t_\mathcal{W}) \\
&({t_2}', S_2) = \mathit{deref}_{S_1}(x) \\
\text{in } &({t_1}'{t_2}', S_2)
\end{aligned}$$

and

$$\begin{aligned}deref_{\Gamma_\mathcal{W}}((t^*)_\mathcal{W}\ x) = \text{let } &(t_1{}'', S_1{}') = deref_{\Gamma_\mathcal{W}}((t^*)_\mathcal{W})\\ &(t_2{}'', S_2{}') = deref_{S_1{}'}(x)\\ \text{in } &(t_1{}''t_2{}'', S_2{}')\end{aligned}$$

By induction hypothesis, $deref_{\Gamma_\mathcal{W}}(t_\mathcal{W}) = deref_{\Gamma_\mathcal{W}}((t^*)_\mathcal{W})$ and $S_1 = S_1{}'$, therefore, we can conclude that $deref_{\Gamma_\mathcal{W}}(t_\mathcal{W}\ x) = deref_{\Gamma_\mathcal{W}}((t^*)_\mathcal{W}\ x)$, so the property holds.

- Given $(\Gamma; t_1\ t_2)$, we have $(\Gamma_\mathcal{W}; (t_1\ t_2)_\mathcal{W}) = (\Gamma_\mathcal{W}; t_{1\mathcal{W}}\ t_{2\mathcal{W}})$ and

$$\begin{aligned}(\Gamma_\mathcal{W}; ((t_1\ t_2)^*)_\mathcal{W}) &= (\Gamma_\mathcal{W}; (\text{let}_\pi\ a : A = t_2{}^* \text{ in } t_1{}^*\ a)_\mathcal{W})\\ &= (\Gamma_\mathcal{W}; (t_1{}^*\ a)_\mathcal{W}[a \mapsto (t_2{}^*)_\mathcal{W}])\\ &= (\Gamma_\mathcal{W}; ((t_1{}^*)_\mathcal{W}\ a_\mathcal{W})[a \mapsto (t_2{}^*)_\mathcal{W}]\\ &\text{we know that } a \notin fv((t_1{}^*)_\mathcal{W})\\ &= (\Gamma_\mathcal{W}; (t_1{}^*)_\mathcal{W}\ a[a \mapsto (t_2{}^*)_\mathcal{W}])\\ &= (\Gamma_\mathcal{W}; (t_1{}^*)_\mathcal{W}\ (t_2{}^*)_\mathcal{W})\end{aligned}$$

We know that

$$\begin{aligned}deref_{\Gamma_\mathcal{W}}(t_{1\mathcal{W}}\ t_{2\mathcal{W}}) = \text{let } &(t_1{}', S_1) = deref_{\Gamma_\mathcal{W}}(t_{1\mathcal{W}})\\ &(t_2{}', S_2) = deref_{S_1}(t_{2\mathcal{W}})\\ \text{in } &(t_1{}'\ t_2{}', S_2)\end{aligned}$$

and

$$\begin{aligned}deref_{\Gamma_\mathcal{W}}((t_1{}^*)_\mathcal{W}\ (t_2{}^*)_\mathcal{W}) = \text{let } &(t_1{}'', S_1{}') = deref_{\Gamma_\mathcal{W}}((t_1{}^*)_\mathcal{W})\\ &(t_2{}'', S_2{}') = deref_{S_1{}'}((t_2{}^*)_\mathcal{W})\\ \text{in } &(t_1{}''t_2{}'', S_2{}')\end{aligned}$$

By induction hypothesis, $deref_{\Gamma_\mathcal{W}}(t_{1\mathcal{W}}) = deref_{\Gamma_\mathcal{W}}((t_1{}^*)_\mathcal{W})$ and $S_1 = S_1{}'$, then we also know that $deref_{S_1}(t_{2\mathcal{W}}) = deref_{S_1{}'}((t_2{}^*)_\mathcal{W})$.
We can then conclude $deref_{\Gamma_\mathcal{W}}(t_{1\mathcal{W}}\ t_{2\mathcal{W}}) = deref_{\Gamma_\mathcal{W}}((t_1{}^*)_\mathcal{W}\ (t_2{}^*)_\mathcal{W})$.

- Given $(\Gamma; \text{split } t_1 \text{ as } y, z \text{ in } t_2)$, we have

$$(\Gamma_\mathcal{W}; (\text{split } t_1 \text{ as } y, z \text{ in } t_2)_\mathcal{W}) = (\Gamma_\mathcal{W}; \text{split } t_{1\mathcal{W}} \text{ as } y, z \text{ in } t_{2\mathcal{W}})$$

and

$$\begin{aligned}(\Gamma_\mathcal{W}; ((\text{split } t_1 \text{ as } y, z \text{ in } t_2)^*)_\mathcal{W}) &= (\Gamma_\mathcal{W}; (\text{split } t_1{}^* \text{ as } y, z \text{ in } t_2{}^*)_\mathcal{W})\\ &= (\Gamma_\mathcal{W}; \text{split } (t_1{}^*)_\mathcal{W} \text{ as } y, z \text{ in } (t_2{}^*)_\mathcal{W})\end{aligned}$$

We know that

$$\begin{aligned}&deref_{\Gamma_\mathcal{W}}(\text{split } t_{1\mathcal{W}} \text{ as } y, z \text{ in } t_{2\mathcal{W}})\\ &= \text{let } (t_1{}', S_1) = deref_{\Gamma_\mathcal{W}}(t_{1\mathcal{W}})\\ &\qquad (t_2{}', S_2) = deref_{S_1}(t_{2\mathcal{W}})\\ &\quad \text{in } (\text{split } t_1{}' \text{ as } y, z \text{ in } t_2{}', S_2)\end{aligned}$$

and

$$\begin{aligned} & \mathit{deref}_{\Gamma_{\mathcal{W}}}(\text{split } ({t_1}^*)_{\mathcal{W}} \text{ as } y, z \text{ in } ({t_2}^*)_{\mathcal{W}}) \\ = & \text{ let } ({t_1}'', {S_1}') = \mathit{deref}_{\Gamma_{\mathcal{W}}}(({t_1}^*)_{\mathcal{W}}) \\ & \quad ({t_2}'', {S_2}') = \mathit{deref}_{{S_1}'}(({t_2}^*)_{\mathcal{W}}) \\ & \text{ in } (\text{split } {t_1}'' \text{ as } y, z \text{ in } {t_2}'', {S_2}') \end{aligned}$$

By induction hypothesis

$$\mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{1\mathcal{W}}) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(({t_1}^*)_{\mathcal{W}})$$

and $S_1 = {S_1}'$, then we also know that, by induction hypothesis,

$$\mathit{deref}_{S_1}(t_{2\mathcal{W}}) = \mathit{deref}_{{S_1}'}(({t_2}^*)_{\mathcal{W}})$$

We can conclude that

$$\mathit{deref}_{\Gamma_{\mathcal{W}}}(\text{split } t_{1\mathcal{W}} \text{ as } y, z \text{ in } t_{2\mathcal{W}}) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(\text{split } ({t_1}^*)_{\mathcal{W}} \text{ as } y, z \text{ in } ({t_2}^*)_{\mathcal{W}})$$

so the property holds.

– Given $(\Gamma; \text{let}_\pi\ a_1 : A_1 = t_1, \ldots, a_n : A_n = t_n \text{ in } t)$, we have

$$\begin{aligned} & (\Gamma_{\mathcal{W}}; (\text{let}_\pi\ a_1 : A_1 = t_1, \ldots, a_n : A_n = t_n \text{ in } t)_{\mathcal{W}}) \\ = & (\Gamma_{\mathcal{W}}; t_{\mathcal{W}}[a_1 \mapsto t_{1\mathcal{W}}] \ldots [a_n \mapsto t_{n\mathcal{W}}]) \end{aligned}$$

and

$$\begin{aligned} & (\Gamma_{\mathcal{W}}; ((\text{let}_\pi\ a_1 : A_1 = t_1, \ldots, a_n : A_n = t_n \text{ in } t)^*)_{\mathcal{W}}) \\ = & (\Gamma_{\mathcal{W}}; (t^*)_{\mathcal{W}}[a_1 \mapsto ({t_1}^*)_{\mathcal{W}}] \ldots [a_n \mapsto ({t_n}^*)_{\mathcal{W}}]) \end{aligned}$$

Now we have

$$\mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}}[a_1 \mapsto t_{1\mathcal{W}}] \ldots [a_n \mapsto t_{n\mathcal{W}}])$$

and

$$\mathit{deref}_{\Gamma_{\mathcal{W}}}((t^*)_{\mathcal{W}}[a_1 \mapsto ({t_1}^*)_{\mathcal{W}}] \ldots [a_n \mapsto ({t_n}^*)_{\mathcal{W}}])$$

Let us consider the configuration $(\Gamma; t[a_1 \mapsto t_1] \ldots [a_n \mapsto t_n])$, which is obtained by decomposing *let*, then by induction hypothesis,

$$\begin{aligned} & \mathit{deref}_{\Gamma_{\mathcal{W}}}((t^*)_{\mathcal{W}}[a_1 \mapsto ({t_1}^*)_{\mathcal{W}}] \ldots [a_n \mapsto ({t_n}^*)_{\mathcal{W}}]) \\ = & \ \mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}}[a_1 \mapsto t_{1\mathcal{W}}] \ldots [a_n \mapsto t_{n\mathcal{W}}]) \end{aligned}$$

So the property holds.

□

Lemma 9.
If $(S, a \mapsto q\ (t^*)_\mathcal{W}; M) \twoheadrightarrow (S_1; N_1)$ *then*

$$(S, a \mapsto q\ t_\mathcal{W}; M) \twoheadrightarrow (S_2; N_2) \text{ and } \mathit{deref}_{S_2}(N_2) = \mathit{deref}_{S_1}(N_1)$$

Proof. By induction on $(S; t) \twoheadrightarrow (S'; v')$:

– Given $(S, a \mapsto q\ ({t_1}^*)_\mathcal{W}; q_1\ \lambda y : q_2\ P.\ t)$, we have, by rule $s\text{-}abs_\mathcal{W}$

$$\begin{aligned} & (S, a \mapsto q\ ({t_1}^*)_\mathcal{W}; q_1\ \lambda y : q_2\ P.\ t) \\ \rightarrow\ & (S, a \mapsto q\ ({t_1}^*)_\mathcal{W}, a_1 \mapsto q_1\ \lambda y : q_2\ P.\ t; a_1) \end{aligned}$$

We also know that

$$\frac{}{(S, a \mapsto q\ t_{1\mathcal{W}}; q_1\ \lambda y : q_2\ P.\ t) \rightarrow (S, a \mapsto q\ t_{1\mathcal{W}}, a_2 \mapsto q_1\ \lambda y : q_2\ P.\ t; a_2)}\ s\text{-}abs_\mathcal{W}$$

Then

$$\begin{aligned} & \mathit{deref}_{S, a \mapsto q\ ({t_1}^*)_\mathcal{W}, a_1 \mapsto q_1\ \lambda y : q_2\ P.\ t}(a_1) \\ = & \mathit{deref}_{S, a \mapsto q\ ({t_1}^*)_\mathcal{W}}(q_1\ \lambda y : q_2\ P.\ t) \\ & \text{since } a_1 \text{ is a fresh variable, it can be removed from the heap.} \end{aligned}$$

We know that $q_1\ \lambda y : q_2\ P.$ t is not evaluated, therefore $a \notin \mathit{fv}(t)$. Then, by Lemma 5.3,

$$\mathit{deref}_{S, a \mapsto q\ ({t_1}^*)_\mathcal{W}}(q_1\ \lambda y : q_2\ P.\ t) = \mathit{deref}_S(q_1\ \lambda y : q_2\ P.\ t)$$

and

$$\begin{aligned} & \mathit{deref}_{S, a \mapsto q\ t_{1\mathcal{W}}, a_2 \mapsto q_1\ \lambda y : q_2\ P.\ t}(a_2) \\ = & \mathit{deref}_{S, a \mapsto q\ t_{1\mathcal{W}}}(q_1\ \lambda y : q_2\ P.\ t) \text{ because } a_2 \text{ is a fresh variable} \\ & \text{since } a \notin \mathit{fv}(t), \text{ by Lemma 5.3} \\ = & \mathit{deref}_S(q_1\ \lambda y : q_2\ P.\ t) \end{aligned}$$

The property holds.

– Given $(S, a \mapsto q\ ({t_1}^*)_\mathcal{W}; q_1 < e_1, e_2 >)$, we have, by rule $s\text{-}pair_\mathcal{W}$

$$\begin{aligned} & (S, a \mapsto q\ ({t_1}^*)_\mathcal{W}; q_1 < e_1, e_2 >) \\ \rightarrow\ & (S, a \mapsto q\ ({t_1}^*)_\mathcal{W}, a_1 \mapsto q_1 < e_1, e_2 >; a_1) \end{aligned}$$

We also know that, by rule $s\text{-}pair_\mathcal{W}$

$$\begin{aligned} & (S, a \mapsto q\ t_{1\mathcal{W}}; q_1 < e_1, e_2 >) \\ \rightarrow\ & (S, a \mapsto q\ t_{1\mathcal{W}}, a_2 \mapsto q_1 < e_1, e_2 >; a_2) \end{aligned}$$

Then

$$deref_{S,a\mapsto q\ (t_1{}^*)_{\mathcal{W}},a_1\mapsto q_1<e_1,e_2>}(a_1) = deref_{S,a\mapsto q\ (t_1{}^*)_{\mathcal{W}}}(q_1 < e_1, e_2 >)$$

Since a_1 is a fresh variable, it can be removed from the heap. We know that $q_1 < e_1, e_2 >$ is not evaluated, therefore $a \notin fv(q_1 < e_1, e_2 >)$. Then, by Lemma 5.3,

$$deref_{S,a\mapsto q\ (t_1{}^*)_{\mathcal{W}}}(q_1 < e_1, e_2 >) = deref_S(q_1 < e_1, e_2 >)$$

and

$$\begin{aligned} & deref_{S,a\mapsto q\ t_{1\mathcal{W}},a_1\mapsto q_1<e_1,e_2>}(a_1) \\ = {} & deref_{S,a\mapsto q\ t_{1\mathcal{W}}}(q_1 < e_1, e_2 >) \\ & \text{ since } a_2 \text{ is a fresh variable, it can be removed from the heap} \\ = {} & deref_S(q_1 < e_1, e_2 >) \text{ by Lemma 5.3, because } a \notin fv(q_1 < e_1, e_2 >) \end{aligned}$$

So the property holds.

– Given $(S, a \mapsto q\ (t_1{}^*)_{\mathcal{W}}; x\ t)$, we have

$$\frac{S(x) = q_1\ \lambda y : q_2\ P.t'}{(S, a \mapsto q\ (t_1{}^*)_{\mathcal{W}}; x\ t) \to (S \overset{q_1}{\sim} x, a \mapsto q\ (t_1{}^*)_{\mathcal{W}}; t'[y \mapsto t])}\ \textit{s-app}_{\mathcal{W}}$$

We also know that

$$\frac{S(x) = q_1\ \lambda y : q_2\ P.t'}{(S, a \mapsto q\ t_{1\mathcal{W}}; x\ t) \to (S \overset{q_1}{\sim} x, a \mapsto q\ t_{1\mathcal{W}}; t'[y \mapsto t])}\ \textit{s-app}_{\mathcal{W}}$$

We know that if x occurs unrestricted then it will stay in S but, if x occurs linearly then it will not be used again, thus we can use $S \overset{q_1}{\sim} x = S$ instead of distinguishing both cases.
Then

$$\begin{aligned} & deref_{S,a\mapsto q\ (t_1{}^*)_{\mathcal{W}}}(t'[y \mapsto t]) \\ = {} & deref_S(t'[y \mapsto t][a \mapsto (t_1{}^*)_{\mathcal{W}}]) \text{ by Lemma 5.3} \\ & \text{ by Lemma 5.8, we know that} \\ = {} & deref_S(t'[y \mapsto t][a \mapsto t_{1\mathcal{W}}]) \end{aligned}$$

and

$$deref_{S,a\mapsto q\ t_{1\mathcal{W}}}(t'[y \mapsto t]) = deref_S(t'[y \mapsto t][a \mapsto t_{1\mathcal{W}}]) \text{ by Lemma 5.3}$$

Since we know that $(S, a \mapsto q\ (t_1{}^*)_{\mathcal{W}}; t'[y \mapsto t]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis,

$$(S, a \mapsto \text{ q } t_{1\mathcal{W}}; t'[y \mapsto t]) \twoheadrightarrow (S_2; v_2)$$

and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$. And the property holds.

– Given $(S, a \mapsto q\ (e^*)_{\mathcal{W}}; t_1 t_2)$, we have

$$(S, a \mapsto q\ (e^*)_{\mathcal{W}}; t_1 t_2) \twoheadrightarrow (S', a \mapsto q\ (e^*)_{\mathcal{W}}; {t_1}' t_2)$$

because

$$(S, a \mapsto q\ (e^*)_{\mathcal{W}}; t_1) \twoheadrightarrow (S', a \mapsto q\ (e^*)_{\mathcal{W}}; {t_1}')$$

where ${t_1}' \equiv q_1\ \lambda y : q_2\ P.t$ necessarily.
Now we have

$$(S', a \mapsto q\ (e^*)_{\mathcal{W}}; {t_1}' t_2) \to (S', a \mapsto q\ (e^*)_{\mathcal{W}}, a_1 \mapsto {t_1}'; a_1 t_2)$$

because, by rule *s-abs*$_{\mathcal{W}}$,

$$(S', a \mapsto q\ (e^*)_{\mathcal{W}}; {t_1}') \to (S', a \mapsto q\ (e^*)_{\mathcal{W}}, a_1 \mapsto {t_1}'; a_1)$$

and, by rule *s-app*$_{\mathcal{W}}$,

$$(S', a \mapsto q\ (e^*)_{\mathcal{W}}, a_1 \mapsto {t_1}'; a_1 t_2) \to (S' \overset{q_1}{\sim} a_1, a \mapsto q\ (e^*)_{\mathcal{W}}; t[y \mapsto t_2])$$

Since we know that $(S, a \mapsto q\ (e^*)_{\mathcal{W}}; t_1) \twoheadrightarrow (S', a \mapsto q\ (e^*)_{\mathcal{W}}; {t_1}')$ then, by induction hypothesis,

$$(S, a \mapsto q\ e_{\mathcal{W}}; t_1) \twoheadrightarrow (S'', a \mapsto q\ e_{\mathcal{W}}; {t_1}'')$$

and $\mathit{deref}_{S', a \mapsto q\ (e^*)_{\mathcal{W}}}({t_1}') = \mathit{deref}_{S'', a \mapsto q\ e_{\mathcal{W}}}({t_1}'')$. Therefore

$$(S, a \mapsto q\ e_{\mathcal{W}}; t_1 t_2) \twoheadrightarrow (S'', a \mapsto q\ e_{\mathcal{W}}; {t_1}'' t_2)$$

and, necessarily, ${t_1}'' \equiv {q_1}'\ \lambda z : {q_2}'\ P'.t'$. Now we have

$$(S'', a \mapsto q\ e_{\mathcal{W}}; {t_1}'' t_2) \to (S'', a \mapsto q\ e_{\mathcal{W}}, a_2 \mapsto {t_1}''; a_2 t_2)$$

because, by rule *s-abs*$_{\mathcal{W}}$,

$$(S'', a \mapsto q\ e_{\mathcal{W}}; {t_1}'') \to (S'', a \mapsto q\ e_{\mathcal{W}}, a_2 \mapsto {t_1}''; a_2)$$

and, by rule *s-app*$_{\mathcal{W}}$,

$$(S'', a \mapsto q\ e_{\mathcal{W}}, a_2 \mapsto {t_1}''; a_2 t_2) \to (S'' \overset{{q_1}'}{\sim} a_2, a \mapsto q\ e_{\mathcal{W}}; t'[z \mapsto t_2])$$

Since a_1 and a_2 are fresh variables, they will not be used again regardless of q_1 and ${q_1}'$, so we can use $S' \overset{q_1}{\sim} a_1 = S'$ and $S'' \overset{{q_1}'}{\sim} a_2 = S''$.
Then

$$\begin{aligned} & \mathit{deref}_{S', a \mapsto q\ (e^*)_{\mathcal{W}}}(t[y \mapsto t_2]) \\ = {} & \mathit{deref}_{S'}(t[y \mapsto t_2][a \mapsto (e^*)_{\mathcal{W}}]) \text{ by Lemma 5.3} \\ = {} & \mathit{deref}_{S'}(t[y \mapsto t_2][a \mapsto e_{\mathcal{W}}]) \text{ by Lemma 5.8} \end{aligned}$$

Since we know that $deref_{S',a\mapsto q\ (e^*)_{\mathcal{W}}}(q_1\ \lambda y : q_2\ P.t) = deref_{S'',a\mapsto q\ e_{\mathcal{W}}}({q_1}'\ \lambda z : {q_2}'\ P'.t')$, then we can conclude that

$$deref_{S'}(t[y \mapsto t_2][a \mapsto e_{\mathcal{W}}]) = deref_{S''}(t'[z \mapsto t_2][a \mapsto e_{\mathcal{W}}])$$

Since we know that if $(S', a \mapsto q\ (e^*)_{\mathcal{W}}; t[y \mapsto t_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis, $(S'', a \mapsto \text{ q } e_{\mathcal{W}}; t'[z \mapsto t_2]) \twoheadrightarrow (S_2; v_2)$ and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$. And the property holds.

– Given $(S, a \mapsto q\ ({t_1}^*)_{\mathcal{W}}; \text{split } x \text{ as } y, z \text{ in } t_2)$, we have, by rule *s-split*$_{\mathcal{W}}$

$$(S, a \mapsto q\ ({t_1}^*)_{\mathcal{W}}; \text{split } x \text{ as } y, z \text{ in } t_2)$$

$$\rightarrow (S \overset{q_1}{\sim} x, a \mapsto q\ ({t_1}^*)_{\mathcal{W}}; t_2[y \mapsto e_1][z \mapsto e_2])$$

where $S(x) = q_1 < e_1, e_2 >$.
We also know that, by rule *s-split*$_{\mathcal{W}}$

$$(S, a \mapsto q\ t_{1\mathcal{W}}; \text{split } x \text{ as } y, z \text{ in } t_2)$$

$$\rightarrow (S \overset{q_1}{\sim} x, a \mapsto q\ t_{1\mathcal{W}}; t_2[y \mapsto e_1][z \mapsto e_2])$$

where $S(x) = q_1 < e_1, e_2 >$.
We know that if x occurs unrestricted then it will stay in S but, if x occurs linearly then it will not be used again, thus we can use $S \overset{q_1}{\sim} x = S$ instead of distinguishing both cases.
Then,

$$\begin{aligned} &deref_{S,a\mapsto q\ ({t_1}^*)_{\mathcal{W}}}(t_2[y \mapsto e_1][z \mapsto e_2]) \\ =\ &deref_S(t_2[y \mapsto e_1][z \mapsto e_2][a \mapsto ({t_1}^*)_{\mathcal{W}}]) \text{ by Lemma 5.3} \\ =\ &deref_S(t_2[y \mapsto e_1][z \mapsto e_2][a \mapsto t_{1\mathcal{W}}]) \text{ by Lemma 5.8} \end{aligned}$$

and

$$\begin{aligned} &deref_{S,a\mapsto q\ t_{1\mathcal{W}}}(t_2[y \mapsto e_1][z \mapsto e_2]) \\ =\ &deref_S(t_2[y \mapsto e_1][z \mapsto e_2][a \mapsto t_{1\mathcal{W}}]) \text{ by Lemma 5.3} \end{aligned}$$

Since we know that $(S, a \mapsto q\ ({t_1}^*)_{\mathcal{W}}; t_2[y \mapsto e_1][z \mapsto e_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis,

$$(S, a \mapsto \text{ q } t_{1\mathcal{W}}; t_2[y \mapsto e_1][z \mapsto e_2]) \twoheadrightarrow (S_2; v_2)$$

and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$. And the property holds.

– Given $(S, a \mapsto q\ (e^*)_{\mathcal{W}}; \text{split } t_1 \text{ as } y, z \text{ in } t_2)$, we have

$$(S, a \mapsto q\ (e^*)_{\mathcal{W}}; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \twoheadrightarrow (S', a \mapsto q\ (e^*)_{\mathcal{W}}; \text{split } {t_1}' \text{ as } y, z \text{ in } t_2)$$

because

$$(S, a \mapsto q\ (e^*)_{\mathcal{W}}; t_1) \twoheadrightarrow (S', a \mapsto q\ (e^*)_{\mathcal{W}}; {t_1}')$$

where, necessarily, $t_1{}' \equiv q_1 < e_1, e_2 >$.
Now we have

$$(S', a \mapsto q\ (e^*)_{\mathcal{W}}; \text{split } t_1{}' \text{ as } y, z \text{ in } t_2) \to (S', a \mapsto q\ (e^*)_{\mathcal{W}}, a_1 \mapsto t_1{}'; \text{split } a_1 \text{ as } y, z \text{ in } t_2)$$

because, by rule *s-pair*$_{\mathcal{W}}$,

$$(S', a \mapsto q\ (e^*)_{\mathcal{W}}; t_1{}') \to (S', a \mapsto q\ (e^*)_{\mathcal{W}}, a_1 \mapsto t_1{}'; a_1)$$

and, by rule *s-split*$_{\mathcal{W}}$,

$$(S', a \mapsto q\ (e^*)_{\mathcal{W}}, a_1 \mapsto t_1{}'; \text{split } a_1 \text{ as } y, z \text{ in } t_2) \to (S' \overset{q_1}{\sim} a_1, a \mapsto q\ (e^*)_{\mathcal{W}}; t_2[y \mapsto e_1][z \mapsto e_2])$$

Since we know that $(S, a \mapsto q\ (e^*)_{\mathcal{W}}; t_1) \twoheadrightarrow (S', a \mapsto q\ (e^*)_{\mathcal{W}}; t_1{}')$ then, by induction hypothesis,

$$(S, a \mapsto q\ e_{\mathcal{W}}; t_1) \twoheadrightarrow (S'', a \mapsto q\ e_{\mathcal{W}}; t_1{}'')$$

and $\mathit{deref}_{S', a \mapsto q\ (e^*)_{\mathcal{W}}}(t_1{}') = \mathit{deref}_{S'', a \mapsto q\ e_{\mathcal{W}}}(t_1{}'')$. Therefore,

$$(S, a \mapsto q\ e_{\mathcal{W}}; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \twoheadrightarrow (S'', a \mapsto q\ e_{\mathcal{W}}; \text{split } t_1{}'' \text{ as } y, z \text{ in } t_2)$$

and, necessarily, $t_1{}'' \equiv q_1{}' < e_1{}', e_2{}' >$. Now we have

$$(S'', a \mapsto q\ e_{\mathcal{W}}; \text{split } t_1{}'' \text{ as } y, z \text{ in } t_2) \to (S'', a \mapsto q\ e_{\mathcal{W}}, a_2 \mapsto t_1{}''; \text{split } a_2 \text{ as } y, z \text{ in } t_2)$$

because, by rule *s-pair*$_{\mathcal{W}}$,

$$(S'', a \mapsto q\ e_{\mathcal{W}}; t_1{}'') \to (S'', a \mapsto q\ e_{\mathcal{W}}, a_2 \mapsto t_1{}''; a_2)$$

and, by rule *s-split*$_{\mathcal{W}}$,

$$(S'', a \mapsto q\ e_{\mathcal{W}}, a_2 \mapsto t_1{}''; \text{split } a_2 \text{ as } y, z \text{ in } t_2) \to (S'' \overset{q_1{}'}{\sim} a_2, a \mapsto q\ e_{\mathcal{W}}; t_2[y \mapsto e_1{}'][z \mapsto e_2{}'])$$

Since both a_1 and a_2 are fresh variables, they will not be used again regardless of q_1 and $q_1{}'$, so we can conclude that $S' \overset{q_1}{\sim} a_1 = S'$ and $S'' \overset{q_1{}'}{\sim} a_2 = S''$.
Then,

$$\begin{aligned}
& \mathit{deref}_{S', a \mapsto q\ (e^*)_{\mathcal{W}}}(t_2[y \mapsto e_1][z \mapsto e_2]) \\
= & \ \mathit{deref}_{S'}(t_2[y \mapsto e_1][z \mapsto e_2][a \mapsto (e^*)_{\mathcal{W}}]) \text{ by Lemma 5.3} \\
= & \ \mathit{deref}_{S'}(t_2[y \mapsto e_1][z \mapsto e_2][a \mapsto e_{\mathcal{W}}]) \text{ by Lemma 5.8}
\end{aligned}$$

We know that $\mathit{deref}_{S', a \mapsto q\ (e^*)_{\mathcal{W}}}(q_1 < e_1, e_2 >) = \mathit{deref}_{S'', a \mapsto q\ e_{\mathcal{W}}}(q_1{}' < e_1{}', e_2{}' >)$, thus we can conclude

$$\mathit{deref}_{S'}(t_2[y \mapsto e_1][z \mapsto e_2][a \mapsto e_{\mathcal{W}}]) = \mathit{deref}_{S''}(t_2[y \mapsto e_1{}'][z \mapsto e_2{}'][a \mapsto e_{\mathcal{W}}])$$

Since we know that if $(S', a \mapsto q\ (e^*)_{\mathcal{W}}; t_2[y \mapsto e_1][z \mapsto e_2]) \twoheadrightarrow (S_1; v_1)$ then, by induction hypothesis,

$$(S'', a \mapsto \text{ q } e_{\mathcal{W}}; t_2[y \mapsto e_1{}'][z \mapsto e_2{}']) \twoheadrightarrow (S_2; v_2)$$

and $\mathit{deref}_{S_2}(v_2) = \mathit{deref}_{S_1}(v_1)$. And the property holds.

$\square$

Theorem 1.
If $(S; M_{\mathcal{W}}) \to_k (S'; N_{\mathcal{W}})$ *then*

$$(S^*_{\mathcal{L}}; (M_{\mathcal{W}})^*_{\mathcal{L}}) \Downarrow (\Gamma; N_{\mathcal{L}}) \text{ and } \mathit{deref}_{\Gamma}(N_{\mathcal{L}}) = \mathit{deref}_{S'^*_{\mathcal{L}}}((N_{\mathcal{W}})^*_{\mathcal{L}})$$

Proof. By induction on $(S; t) \to_k (S'; t')$:

– Base case: $(k = 1)$
  - Given $(S; q\ \lambda y : q'\ P.\ t)$, by applying rule $s\text{-}abs_{\mathcal{W}}$, we have

$$\frac{}{(S; q\ \lambda y : q'\ P.\ t) \to (S, a \mapsto q\ \lambda y : q'\ P.\ t; a)}$$

    Since $(q\ \lambda y : q'\ \mathrm{P}.t)^*_{\mathcal{L}} = \lambda_\pi y :\ P_{\mathcal{L}}.t^*_{\mathcal{L}}$, with $\pi = q'_{\mathcal{L}}$, then by applying rule $s\text{-}abs_{\mathcal{L}}$,

$$\frac{}{(S^*_{\mathcal{L}}; \lambda_\pi y :\ P_{\mathcal{L}}.\ t^*_{\mathcal{L}}) \Downarrow (S^*_{\mathcal{L}}; \lambda_\pi y :\ P_{\mathcal{L}}.\ t^*_{\mathcal{L}})}$$

    Now, we know that

$$\mathit{deref}_{S^*_{\mathcal{L}}, a :_\pi A = \lambda_\pi y :\ P_{\mathcal{L}}.\ t^*_{\mathcal{L}}}(a) = \mathit{deref}_{S^*_{\mathcal{L}}}(\lambda_\pi y :\ P_{\mathcal{L}}.\ t^*_{\mathcal{L}})$$

    Since a is a fresh variable, it occurs linearly. So, the property holds.
  - Given $(S; q < t_1, t_2 >)$, by applying rule $s\text{-}pair_{\mathcal{W}}$, we know

$$\frac{}{(S; q < t_1, t_2 >) \to (S, a \mapsto q < t_1, t_2 >; a)}$$

    Since $(q < t_1, t_2 >)^*_{\mathcal{L}} = \mathrm{let}_\pi\ a_1 : A_1 = t^*_{1\mathcal{L}}, a_2 : A_2 = t^*_{2\mathcal{L}}\ \mathrm{in}\ < a_1, a_2 >_\pi$, with $\pi = q_{\mathcal{L}}$.
    Let $\Gamma = \{S^*_{\mathcal{L}}, a_1 :_\pi A_1 = {t_1}^*_{\mathcal{L}}, a_2 :_\pi A_2 = {t_2}^*_{\mathcal{L}}\}$, then by applying rule $s\text{-}pair_{\mathcal{L}}$

$$\dfrac{\dfrac{}{(\Gamma; < a_1, a_2 >_\pi) \Downarrow (\Gamma; < a_1, a_2 >_\pi)}\ s\text{-}pair_{\mathcal{L}}}{(S^*_{\mathcal{L}}; \mathrm{let}_\pi\ a_1 : A_1 = {t_1}^*_{\mathcal{L}}, a_2 : A_2 = {t_2}^*_{\mathcal{L}}\ \mathrm{in}\ < a_1, a_2 >_\pi) \Downarrow (\Gamma; < a_1, a_2 >_\pi)}\ s\text{-}let_{\mathcal{L}}$$

    We now have,

$$\begin{aligned}
&\mathit{deref}_{S^*_{\mathcal{L}}, a_1 :_\pi A_1 = {t_1}^*_{\mathcal{L}}, a_2 :_\pi A_2 = {t_2}^*_{\mathcal{L}}}(< a_1, a_2 >_\pi) \\
&= \mathrm{let}\ ({t_1}', \Gamma_1) = \mathit{deref}_{S^*_{\mathcal{L}}, a_1 :_\pi A_1 = {t_1}^*_{\mathcal{L}}, a_2 :_\pi A_2 = {t_2}^*_{\mathcal{L}}}(a_1) \\
&\qquad\qquad = \mathit{deref}_{S^*_{\mathcal{L}}, a_2 :_\pi A_2 = {t_2}^*_{\mathcal{L}}}({t_1}^*_{\mathcal{L}}) \text{ by Lemma 2} \\
&\qquad\qquad \text{since } a_2 \notin \mathit{fv}({t_1}^*_{\mathcal{L}}) \\
&\qquad\qquad = \mathit{deref}_{S^*_{\mathcal{L}}}({t_1}^*_{\mathcal{L}}) \\
&\qquad ({t_2}', \Gamma_2) = \mathit{deref}_{\Gamma_1, a_1 :_\pi A_1 = {t_1}^*_{\mathcal{L}}, a_2 :_\pi A_2 = {t_2}^*_{\mathcal{L}}}(a_2) \\
&\qquad\qquad = \mathit{deref}_{S^*_{\mathcal{L}}, a_1 :_\pi A_1 = {t_1}^*_{\mathcal{L}}}({t_2}^*_{\mathcal{L}}) \text{ by Lemma 2} \\
&\qquad\qquad \text{since } a_1 \notin \mathit{fv}({t_2}^*_{\mathcal{L}}) \\
&\qquad\qquad = \mathit{deref}_{S^*_{\mathcal{L}}}({t_2}^*_{\mathcal{L}}) \\
&\quad \mathrm{in}\ (< {t_1}', {t_2}' >_\pi, \Gamma_2)
\end{aligned}$$

and

$$deref_{S^*_{\mathcal{L}}, a :_\pi A_1 *_\pi A_2 = < {t_1}^*_{\mathcal{L}}, {t_2}^*_{\mathcal{L}} >_\pi}(a) = deref_{S^*_{\mathcal{L}}}(< {t_1}^*_{\mathcal{L}}, {t_2}^*_{\mathcal{L}} >_\pi)$$

a is a fresh variable, so it occurs linearly. Since we already know that

$$({t_1}', \Gamma_1) = deref_{S^*_{\mathcal{L}}}({t_1}^*_{\mathcal{L}}) \text{ and } ({t_2}', \Gamma_2) = deref_{S^*_{\mathcal{L}}}({t_2}^*_{\mathcal{L}})$$

then the property holds.

– Inductive case:
  - Given $(S; x\ t)$, by applying rule *s-app*$_{\mathcal{W}}$, we have
$$\frac{S(x) = q\ \lambda y : q'\ P.\ t_1}{(S; x\ t) \to (S \overset{q}{\sim} x; t_1[y \mapsto t])}$$
where $var(x)$. And we know that $\exists k \in \mathbb{N}$ such that
$$(S \overset{q}{\sim} x; t_1[y \mapsto t]) \to_k (S''; v)$$
Since $(x\ t)^*_{\mathcal{L}} = \text{let}_\pi\ a_1 : A_1 = t^*_{\mathcal{L}} \text{ in } x\ a_1$, where $\pi = q'_{\mathcal{L}}$, and
$$\{x :_\pi A_2 = \lambda_\pi y : P_{\mathcal{L}}.\ {t_1}^*_{\mathcal{L}}\} \in S^*_{\mathcal{L}}$$
    * $q = \text{lin}$
    Let $\Gamma_1 = \{S^*_{\mathcal{L}} \setminus \{x\}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}\}$ and let us call the following evaluation Ω.
$$\dfrac{\dfrac{}{(\Gamma_1; \lambda_\pi y : P_{\mathcal{L}}.\ {t_1}^*_{\mathcal{L}}) \Downarrow (\Gamma_1; \lambda_\pi y : P_{\mathcal{L}}.\ {t_1}^*_{\mathcal{L}})}\ \textit{s-abs}_{\mathcal{L}}}{(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; x) \Downarrow (\Gamma_1; \lambda_\pi y : P_{\mathcal{L}}.\ {t_1}^*_{\mathcal{L}})}\ \textit{s-linvar}_{\mathcal{L}}$$
    and
$$\dfrac{\dfrac{\Omega \qquad (\Gamma_1; {t_1}^*_{\mathcal{L}}[y \mapsto a_1]) \Downarrow (\Gamma; \sigma_1)}{(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; x\ a_1) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-app}_{\mathcal{L}}}{(S^*_{\mathcal{L}}; \text{let}_\pi\ a_1 : A_1 = t^*_{\mathcal{L}} \text{ in } x\ a_1) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-let}_{\mathcal{L}}$$
    We know that $(S \setminus \{x\}; t_1[y \mapsto t]) \to_k (S''; v)$ then, by induction hypothesis, we have
$$(S^*_{\mathcal{L}} \setminus \{x\}; (t_1[y \mapsto t])^*_{\mathcal{L}}) \Downarrow (\Delta; \phi)$$
    and $deref_\Delta(\phi) = deref_{{S''}^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$, for some Δ and ϕ.
    Since we know that $(t_1[y \mapsto t])^*_{\mathcal{L}} = {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$ by Lemma 7 and, since a_1 is a fresh variable, $a_1 \notin fv({t_1}^*_{\mathcal{L}})$, so by the Γ-Substitution Lemma, if
$$(\Gamma_1; {t_1}^*_{\mathcal{L}}[y \mapsto a_1]) \Downarrow (\Gamma; \sigma_1)$$
    then $(S^*_{\mathcal{L}} \setminus \{x\}; {t_1}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]) \Downarrow (\Gamma; \sigma_1)$ and $deref_\Gamma(\sigma_1) = deref_{{S''}^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$. And the property holds.
    * $q = \text{un}$
    Let $\Gamma_2 = \{S^*_{\mathcal{L}} \setminus \{x\}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}\}$ and let us call the following the following evaluation Ω.

$$\dfrac{\dfrac{}{(\Gamma_2; \lambda_\pi y : P_{\mathcal{L}}.\ t_1{}^*_{\mathcal{L}}) \Downarrow (\Gamma_2; \lambda_\pi y : P_{\mathcal{L}}.\ t_1{}^*_{\mathcal{L}})}\ \textit{s-abs}_{\mathcal{L}}}{(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; x) \Downarrow (S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; \lambda_\pi y : P_{\mathcal{L}}.\ t_1{}^*_{\mathcal{L}})}\ \textit{s-unvar}_{\mathcal{L}}$$

and

$$\dfrac{\dfrac{\Omega \qquad (S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; t_1{}^*_{\mathcal{L}}[y \mapsto a_1]) \Downarrow (\Gamma'; \sigma_2)}{(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; x\ a_1) \Downarrow (\Gamma'; \sigma_2)}\ \textit{s-app}_{\mathcal{L}}}{(S^*_{\mathcal{L}}; \mathrm{let}_\pi\ a_1 : A_1 = t^*_{\mathcal{L}}\ \mathrm{in}\ x\ a_1) \Downarrow (\Gamma'; \sigma_2)}\ \textit{s-let}_{\mathcal{L}}$$

We know that $(S; t_1[y \mapsto t]) \to_k (S''; v)$ then, by induction hypothesis, we have

$$(S^*_{\mathcal{L}}; (t_1[y \mapsto t])^*_{\mathcal{L}}) \Downarrow (\Delta; \phi)$$

and $\mathit{deref}_\Delta(\phi) = \mathit{deref}_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$, for some Δ and ϕ.
Since we know that $(t_1[y \mapsto t])^*_{\mathcal{L}} = t_1{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]$ by Lemma 7 and, since a_1 is a fresh variable, $a_1 \notin \mathit{fv}(t_1{}^*_{\mathcal{L}})$, so by the Γ-Substitution Lemma, if

$$(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t^*_{\mathcal{L}}; t_1{}^*_{\mathcal{L}}[y \mapsto a_1]) \Downarrow (\Gamma'; \sigma_2)$$

then $(S^*_{\mathcal{L}}; t_1{}^*_{\mathcal{L}}[y \mapsto t^*_{\mathcal{L}}]) \Downarrow (\Gamma'; \sigma_2)$ and $\mathit{deref}_{\Gamma'}(\sigma_2) = \mathit{deref}_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$. So the property holds.

- Given $(S; t_1\ t_2)$, where $\neg var(t_1)$ and $\neg var(t_2)$, then, by Lemma 1.1, $\exists k_1, k_2 \in \mathbb{N}$ such that $k = k_1 + k_2$ and $(S; t_1\ t_2) \to_{k_1} (S'; x\ t_2) \to_{k_2} (S''; v)$. Since, by applying rule $\textit{s-app}_{\mathcal{W}}$, we have

$$\dfrac{S'(x) = q\ \lambda y : q'\ P.\ \mathrm{t}}{(S'; x\ t_2) \to (S' \overset{q}{\sim} x; t[y \mapsto t_2])}$$

we can conclude that

$$\dfrac{\begin{array}{c}(S; t_1\ t_2) \to_{k_1} (S'; x\ t_2) \\ (S'; x\ t_2) \to (S' \overset{q}{\sim} x; t[y \mapsto t_2]) \\ (S' \overset{q}{\sim} x; t[y \mapsto t_2]) \to_{k_2 - 1} (S''; v)\end{array}}{(S; t_1\ t_2) \to_k (S''; v)}$$

Now we have $(t_1\ t_2)^*_{\mathcal{L}} = \mathrm{let}_\pi\ a_1 : A_1 = t_2{}^*_{\mathcal{L}}\ \mathrm{in}\ t_1{}^*_{\mathcal{L}}\ a_1$ and, since we know that $(S; t_1) \to_{k_1} (S'; x)$ then, by induction hypothesis,

$$\mathit{deref}_{S'^*_{\mathcal{L}}}(x) = \mathit{deref}_{S'^*_{\mathcal{L}}}(\lambda_\pi y : P_{\mathcal{L}}.\ t^*_{\mathcal{L}})$$

where $\pi = q'_{\mathcal{L}}$, and $(S^*_{\mathcal{L}}; t_1{}^*_{\mathcal{L}}) \Downarrow (S'^*_{\mathcal{L}}; \lambda_\pi y : P_{\mathcal{L}}.\ t^*_{\mathcal{L}})$. Now we have,

$$\dfrac{\dfrac{\begin{array}{c}(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t_2{}^*_{\mathcal{L}}; t_1{}^*_{\mathcal{L}}) \Downarrow (S'^*_{\mathcal{L}}, a_1 :_\pi A_1 = t_2{}^*_{\mathcal{L}}; \lambda_\pi y : P_{\mathcal{L}}.\ t^*_{\mathcal{L}}) \\ (S'^*_{\mathcal{L}}, a_1 :_\pi A_1 = t_2{}^*_{\mathcal{L}}; t^*_{\mathcal{L}}[y \mapsto a_1]) \Downarrow (\Gamma; \sigma_1)\end{array}}{(S^*_{\mathcal{L}}, a_1 :_\pi A_1 = t_2{}^*_{\mathcal{L}}; t_1{}^*_{\mathcal{L}}\ a_1) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-app}_{\mathcal{L}}}{(S^*_{\mathcal{L}}; \mathrm{let}_\pi\ a_1 : A_1 = t_2{}^*_{\mathcal{L}}\ \mathrm{in}\ t_1{}^*_{\mathcal{L}}\ a_1) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-let}_{\mathcal{L}}$$

We know that $(S' \overset{q}{\sim} x; t[y \mapsto t_2]) \to_{k_2-1} (S''; v)$ then, by induction hypothesis, $((S' \overset{q}{\sim} x)^*_{\mathcal{L}}; (t[y \mapsto t_2])^*_{\mathcal{L}}) \Downarrow (\Gamma'; \sigma_2)$, for some Γ' and σ_2, and $\mathit{deref}_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}}) = \mathit{deref}_{\Gamma'}(\sigma_2)$. If x occurs unrestricted then it will stay in S but, if x occurs linearly then it will not be used again, thus we can use $S' \overset{q}{\sim} x = S'$.
We know, by Lemma 7, that $(t[y \mapsto t_2])^*_{\mathcal{L}} = t^*_{\mathcal{L}}[y \mapsto t_2{}^*_{\mathcal{L}}]$ and, by the Γ-Substitution Lemma, if

$$({S'}^*_{\mathcal{L}}, a_1 :_\pi A_1 = t_2{}^*_{\mathcal{L}}; t^*_{\mathcal{L}}[y \mapsto a_1]) \Downarrow (\Gamma; \sigma_1)$$

then $({S'}^*_{\mathcal{L}}; t^*_{\mathcal{L}}[y \mapsto t_2{}^*_{\mathcal{L}}]) \Downarrow (\Gamma; \sigma_1)$ and $\mathit{deref}_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}}) = \mathit{deref}_{\Gamma}(\sigma_1)$. So, the property holds.

- Given (S; split x as y, z in t), by applying rule *s-split*$_{\mathcal{W}}$, we have

$$\frac{\mathrm{S}(x) = q < t_1, t_2 >}{(\mathrm{S};\ \text{split } x \text{ as } y, z \text{ in t}) \to (\mathrm{S} \overset{q}{\sim} x;\ \mathrm{t}[y \mapsto t_1][z \mapsto t_2])}$$

where $\mathit{var}(x)$. And we know that $\exists k \in \mathbb{N}$ such that

$$(S \overset{q}{\sim} x; t[y \mapsto t_1][z \mapsto t_2]) \to_k (S'; v)$$

Since (split x as y, z in t)$^*_{\mathcal{L}}$ = split x as y, z in $t^*_{\mathcal{L}}$ and

$$\{x :_\pi A_1 = \mathrm{let}_\pi\ y_1 : A_2 = t_1{}^*_{\mathcal{L}}, z_1 : A_3 = t_2{}^*_{\mathcal{L}} \text{ in } < y_1, z_1 >_\pi\} \in S^*_{\mathcal{L}}$$

where $\pi = q_{\mathcal{L}}$, then we have two cases:

  - $*$ $q = \mathrm{lin}$
  Let $\Gamma_1 = \{S^*_{\mathcal{L}} \setminus \{x\}, y_1 :_\pi A_2 = t_1{}^*_{\mathcal{L}}, z_1 :_\pi A_3 = t_2{}^*_{\mathcal{L}}\}$ and let us consider the derivation Ω.

$$\dfrac{\dfrac{\dfrac{}{(\Gamma_1; < y_1, z_1 >_\pi) \Downarrow (\Gamma_1; < y_1, z_1 >_\pi)}\ \textit{s-pair}_{\mathcal{L}}}{(S^*_{\mathcal{L}} \setminus \{x\}; \mathrm{let}_\pi\ y_1 : A_2 = t_1{}^*_{\mathcal{L}}, z_1 : A_3 = t_2{}^*_{\mathcal{L}} \text{ in } < y_1, z_1 >_\pi) \Downarrow (\Gamma_1; < y_1, z_1 >_\pi)}\ \textit{s-let}_{\mathcal{L}}}{(S^*_{\mathcal{L}}; x) \Downarrow (\Gamma_1; < y_1, z_1 >_\pi)}\ \textit{s-linvar}_{\mathcal{L}}$$

  and

$$\frac{\Omega \qquad (\Gamma_1; t^*_{\mathcal{L}}[y \mapsto y_1][z \mapsto z_1]) \Downarrow (\Gamma; \sigma_1)}{(S^*_{\mathcal{L}}; \text{split } x \text{ as } y, z \text{ in } t^*_{\mathcal{L}}) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-split}_{\mathcal{L}}$$

  We know that $(S \setminus \{x\}; t[y \mapsto t_1][z \mapsto t_2]) \to_k (S'; v)$ then, by induction hypothesis, we have

$$(S^*_{\mathcal{L}} \setminus \{x\}; (t[y \mapsto t_1][z \mapsto t_2])^*_{\mathcal{L}}) \Downarrow (\Delta; \phi)$$

  and $\mathit{deref}_{\Delta}(\phi) = \mathit{deref}_{S'^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$, for some Δ and ϕ.
  Since Linear Haskell evaluates applications where the argument is a variable and since we know that $y \notin \mathit{fv}(t_2)$ and $z \notin \mathit{fv}(t_1)$, because they are bounded by the split, then, by the Substitution Lemma,

$$t[y \mapsto t_1][z \mapsto t_2] \equiv t[z \mapsto t_2][y \mapsto t_1]$$

We know that $(t[z \mapsto t_2][y \mapsto t_1])^*_{\mathcal{L}} = (t[z \mapsto t_2][y \mapsto t_1])^*_{\mathcal{L}}$ and, by Lemma 7, $(t[z \mapsto t_2][y \mapsto t_1])^*_{\mathcal{L}} = (t[z \mapsto t_2])^*_{\mathcal{L}}[y \mapsto {t_1}^*_{\mathcal{L}}]$. We also know that, by Lemma 7, $(t[z \mapsto t_2])^*_{\mathcal{L}} = t^*_{\mathcal{L}}[z \mapsto {t_2}^*_{\mathcal{L}}]$. Since y_1 and z_1 are fresh variables, $y_1 \notin fv(t^*_{\mathcal{L}})$ and $z_1 \notin fv(t^*_{\mathcal{L}})$, so by the Γ-Substitution Lemma, if

$$(\Gamma_1; t^*_{\mathcal{L}}[y \mapsto y_1][z \mapsto z_1]) \Downarrow (\Gamma; \sigma_1)$$

then

$$(S^*_{\mathcal{L}} \setminus \{x\}; (t[y \mapsto t_1][z \mapsto t_2])^*_{\mathcal{L}}) \Downarrow (\Delta; \phi)$$

and $deref_{\Gamma}(\sigma_1) = deref_{\Delta}(\phi)$. Since we know that $deref_{\Delta}(\phi) = deref_{S'^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$ then $deref_{\Gamma}(\sigma_1) = deref_{S'^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$. And the property holds.

* $q = \text{un}$
Let $\Gamma_2 = \{S^*_{\mathcal{L}} \setminus \{x\}, y_1 :_\pi A_2 = {t_1}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_2}^*_{\mathcal{L}}\}$ and let us consider the following evaluation Ω.

$$\dfrac{\dfrac{\overline{(\Gamma_2; < y_1, z_1 >_\pi) \Downarrow (\Gamma_2; < y_1, z_1 >_\pi)}\ \textit{s-pair}_{\mathcal{L}}}{(S^*_{\mathcal{L}} \setminus \{x\}; \text{let}_\pi\ y_1 : A_2 = {t_1}^*_{\mathcal{L}}, z_1 : A_3 = {t_2}^*_{\mathcal{L}}\ \text{in}\ < y_1, z_1 >_\pi) \Downarrow (\Gamma_2; < y_1, z_1 >_\pi)}\ \textit{s-let}_{\mathcal{L}}}{(S^*_{\mathcal{L}}; x) \Downarrow (S^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_1}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_2}^*_{\mathcal{L}}; < y_1, z_1 >_\pi)}\ \textit{s-unvar}_{\mathcal{L}}$$

and

$$\dfrac{\Omega \qquad (S^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_1}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_2}^*_{\mathcal{L}}; t^*_{\mathcal{L}}[y \mapsto y_1][z \mapsto z_1]) \Downarrow (\Gamma; \sigma_1)}{(S^*_{\mathcal{L}}; \text{split}\ x\ \text{as}\ y, z\ \text{in}\ t^*_{\mathcal{L}}) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-split}_{\mathcal{L}}$$

We know that (S; t$[y \mapsto t_1][z \mapsto t_2]$) $\to_k$ (S′; v) then, by induction hypothesis, we have

$$(S^*_{\mathcal{L}}; (t[y \mapsto t_1][z \mapsto t_2])^*_{\mathcal{L}}) \Downarrow (\Delta; \phi)$$

and $deref_{\Delta}(\phi) = deref_{S'^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$, for some Δ and ϕ.
Since Linear Haskell evaluates applications where the argument is a variable and since we know that $y \notin fv(t_2)$ and $z \notin fv(t_1)$, because they are bounded by the split, then, by the Substitution Lemma,

$$t[y \mapsto t_1][z \mapsto t_2] \equiv t[z \mapsto t_2][y \mapsto t_1]$$

We know that $(t[z \mapsto t_2][y \mapsto t_1])^*_{\mathcal{L}} = (t[z \mapsto t_2][y \mapsto t_1])^*_{\mathcal{L}}$ and, by Lemma 7, $(t[z \mapsto t_2][y \mapsto t_1])^*_{\mathcal{L}} = (t[z \mapsto t_2])^*_{\mathcal{L}}[y \mapsto {t_1}^*_{\mathcal{L}}]$. We also know that, by Lemma 7, $(t[z \mapsto t_2])^*_{\mathcal{L}} = t^*_{\mathcal{L}}[z \mapsto {t_2}^*_{\mathcal{L}}]$.
Since y_1 and z_1 are fresh variables, $y_1 \notin fv(t^*_{\mathcal{L}})$ and $z_1 \notin fv(t^*_{\mathcal{L}})$, so by the Γ-Substitution Lemma, if

$$(S^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_1}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_2}^*_{\mathcal{L}}; t^*_{\mathcal{L}}[y \mapsto y_1][z \mapsto z_1]) \Downarrow (\Gamma'; \sigma_2)$$

then

$$(S^*_{\mathcal{L}}; (t[y \mapsto t_1][z \mapsto t_2])^*_{\mathcal{L}}) \Downarrow (\Delta; \phi)$$

and $deref_{\Gamma'}(\sigma_2) = deref_{\Delta}(\phi)$. Since we know that $deref_{\Delta}(\phi) = deref_{S'^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$ then $deref_{\Gamma'}(\sigma_2) = deref_{S'^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$. And the property holds.

- Given $(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2)$, where $\neg var(t_1)$, then, by Lemma 1.2, $\exists k_1, k_2 \in \mathbb{N}$ such that $k = k_1 + k_2$ and $(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \to_{k_1} (S'; \text{split } x \text{ as } y, z \text{ in } t_2) \to_{k_2} (S''; v)$. Since, by applying rule *s-split*$_{\mathcal{W}}$, we have

$$\frac{S'(x) = q < t_3, t_4 >}{(S'; \text{split } x \text{ as } y, z \text{ in } t_2) \to (S' \overset{q}{\sim} x; t[y \mapsto t_3][z \mapsto t_4])}$$

we can conclude that

$$\frac{\begin{array}{c}(S; \text{split } t_1 \text{ as } y, z \text{ in } t_2) \to_{k_1} (S'; \text{split } x \text{ as } y, z \text{ in } t_2)\\ (S'; \text{ split } x \text{ as } y, z \text{ in } t_2) \to (S' \overset{q}{\sim} x;\ t[y \mapsto t_3][z \mapsto t_4])\\ (S' \overset{q}{\sim} x;\ t[y \mapsto t_3][z \mapsto t_4]) \to_{k_2 - 1} (S'';\ v)\end{array}}{(S;\ \text{split } t_1 \text{ as } y, z \text{ in } t_2) \to_k (S'';\ v)}$$

Now we have $(\text{split } t_1 \text{ as } y, z \text{ in } t_2)^*_{\mathcal{L}} = \text{split } {t_1}^*_{\mathcal{L}} \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}$ and, since we know that $(S; t_1) \to_{k_1} (S'; x)$ then, by induction hypothesis,

$$\begin{aligned} deref_{S'^*_{\mathcal{L}}}(x) &= deref_{S'^*_{\mathcal{L}}}(\text{let}_\pi\ y_1 : A_2 = {t_3}^*_{\mathcal{L}}, z_1 : A_3 = {t_4}^*_{\mathcal{L}} \text{ in } < y_1, z_1 >_\pi) \\ &= deref_{S'^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_3}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_4}^*_{\mathcal{L}}}(< y_1, z_1 >_\pi) \end{aligned}$$

where $\pi = q_{\mathcal{L}}$, and $(S^*_{\mathcal{L}};\ {t_1}^*_{\mathcal{L}}) \Downarrow (S'^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_3}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_4}^*_{\mathcal{L}}; < y_1, z_1 >_\pi)$. Now we have,

$$\frac{\begin{array}{c}(S^*_{\mathcal{L}}; {t_1}^*_{\mathcal{L}}) \Downarrow (S'^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_3}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_4}^*_{\mathcal{L}}; < y_1, z_1 >_\pi)\\ (S'^*_{\mathcal{L}}, y_1 :_\pi A_2 = {t_3}^*_{\mathcal{L}}, z_1 :_\pi A_3 = {t_4}^*_{\mathcal{L}}; {t_2}^*_{\mathcal{L}}[y \mapsto y_1][z \mapsto z_1]) \Downarrow (\Gamma; \sigma_1)\end{array}}{(S^*_{\mathcal{L}}; \text{split } {t_1}^*_{\mathcal{L}} \text{ as } y, z \text{ in } {t_2}^*_{\mathcal{L}}) \Downarrow (\Gamma; \sigma_1)}\ \textit{s-split}_{\mathcal{L}}$$

We know that $(S' \overset{q}{\sim} x; t_2[y \mapsto t_3][z \mapsto t_4]) \to_{k_2 - 1} (S''; v)$ then, by induction hypothesis,

$$((S' \sim^q x)^*_{\mathcal{L}}; (t_2[y \mapsto t_3][z \mapsto t_4])^*_{\mathcal{L}}) \Downarrow (\Gamma'; \sigma_2)$$

for some Γ' and σ_2, and $deref_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}}) = deref_{\Gamma'}(\sigma_2)$. If x occurs unrestricted then it will stay in S but, if x occurs linearly then it will not be used again, thus we can use $S' \overset{q}{\sim} x = S'$.
Since Linear Haskell evaluates applications where the argument is a variable and since we know that $y \notin fv(t_4)$ and $z \notin fv(t_3)$, because they are bounded by the split, then, by the Substitution Lemma,

$$t_2[y \mapsto t_3][z \mapsto t_4] \equiv t[z \mapsto t_2][y \mapsto t_1]$$

We know that $(t_2[y \mapsto t_3][z \mapsto t_4])^*_{\mathcal{L}} = (t_2[z \mapsto t_4][y \mapsto t_3])^*_{\mathcal{L}}$ and, by Lemma 7, $(t_2[z \mapsto t_4][y \mapsto t_3])^*_{\mathcal{L}} = (t_2[z \mapsto t_4])^*_{\mathcal{L}}[y \mapsto {t_3}^*_{\mathcal{L}}]$.

Now, by the $\mathit{\Gamma}$-Substitution Lemma, if

$$(S'^*_{\mathcal{L}}, y_1 :_\pi A_2 = t_3{}^*_{\mathcal{L}}, z_1 :_\pi A_3 = t_4{}^*_{\mathcal{L}}; t_2{}^*_{\mathcal{L}}[y \mapsto y_1][z \mapsto z_1]) \Downarrow (\Gamma; \sigma_1)$$

then $(S'^*_{\mathcal{L}}; (t_2[y \mapsto t_3][z \mapsto t_4])^*_{\mathcal{L}}) \Downarrow (\Gamma'; \sigma_2)$ and $\mathit{deref}_{\Gamma}(\sigma_1) = \mathit{deref}_{\Gamma'}(\sigma_2)$. Since we know that $\mathit{deref}_{\Gamma'}(\sigma_2) = \mathit{deref}_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}})$, then $\mathit{deref}_{S''^*_{\mathcal{L}}}(v^*_{\mathcal{L}}) = \mathit{deref}_{\Gamma}(\sigma_1)$. So, the property holds.

□

Theorem 2.
If $(\Gamma; M^*_{\mathcal{L}}) \Downarrow (\Gamma'; N_{\mathcal{L}})$ *then*

$$(\Gamma_{\mathcal{W}}; (M_{\mathcal{L}})_{\mathcal{W}}) \twoheadrightarrow (S; N_{\mathcal{W}}) \text{ and } \mathit{deref}_{\mathcal{S}}(N_{\mathcal{W}}) = \mathit{deref}_{\Gamma'_{\mathcal{W}}}((N_{\mathcal{L}})_{\mathcal{W}})$$

Proof. By induction on $(\Gamma; t) \Downarrow (\Gamma'; t')$:

- Base case:
  - Given $(\Gamma; (\lambda_\pi y : P.\ t)^*) = (\Gamma; \lambda_\pi y : P.\ t^*)$, we have

$$\frac{}{(\Gamma; \lambda_\pi y : P.\ t^*) \Downarrow (\Gamma; \lambda_\pi y : P.\ t^*)}\ \mathit{s\text{-}abs}_{\mathcal{L}}$$

    Now, we know that $(\Gamma_{\mathcal{W}}; (\lambda_\pi y : P.\ t)_{\mathcal{W}}) = (\Gamma_{\mathcal{W}}; \pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ t_{\mathcal{W}})$. By rule *s-abs*$_{\mathcal{W}}$,

$$\frac{}{(\Gamma_{\mathcal{W}}; \pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ t_{\mathcal{W}}) \to (\Gamma_{\mathcal{W}}, a \mapsto \pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ t_{\mathcal{W}}; a)}$$

    and

$$\begin{aligned}\mathit{deref}_{\Gamma_{\mathcal{W}}, a \mapsto \pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ t_{\mathcal{W}}}(a) &= \mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ t_{\mathcal{W}}) \\ &= \text{let } (v, S) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}}) \\ &\quad \text{in } (\pi_{\mathcal{W}}\ \lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ v, S)\end{aligned}$$

    Since a is a fresh variable, it occurs linearly.
    We know that, by Lemma 9, $\mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{\mathcal{W}}) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(t^*{}_{\mathcal{W}})$, so the property holds.
  - Given

$$(\Gamma; (< t_1, t_2 >_\pi)^*) = (\Gamma; \text{let}_\pi\ a_1 : A_1 = t_1{}^*, a_2 : A_2 = t_2{}^* \text{ in } < a_1, a_2 >_\pi)$$

    Let $\Gamma_1 = \{\Gamma, a_1 :_\pi A_1 = t_1{}^*, a_2 :_\pi A_2 = t_2{}^*\}$, we have

$$\frac{\dfrac{}{(\Gamma_1; < a_1, a_2 >_\pi) \Downarrow (\Gamma_1; < a_1, a_2 >_\pi)}\ \mathit{s\text{-}pair}_{\mathcal{L}}}{(\Gamma; \text{let}_\pi\ a_1 : A_1 = t_1{}^*, a_2 : A_2 = t_2{}^* \text{ in } < a_1, a_2 >_\pi) \Downarrow (\Gamma_1; < a_1, a_2 >_\pi)}\ \mathit{s\text{-}let}_{\mathcal{L}}$$

    Now, we know that $(\Gamma_{\mathcal{W}}; (< t_1, t_2 >_\pi)_{\mathcal{W}}) = (\Gamma_{\mathcal{W}}; \pi_{\mathcal{W}} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >)$. By rule *s-pair*$_{\mathcal{W}}$,

$$\frac{}{(\Gamma_{\mathcal{W}}; \pi_{\mathcal{W}} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >) \to (\Gamma_{\mathcal{W}}, a \mapsto \pi_{\mathcal{W}} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >; a)}$$

and

$$\begin{aligned}
\mathit{deref}_{\Gamma_{\mathcal{W}}, a \mapsto \pi_{\mathcal{W}} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >}(a) &= \mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}} < t_{1\mathcal{W}}, t_{2\mathcal{W}} >) \\
&= \text{let } (v_1, S_1) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{1\mathcal{W}}) \\
&\qquad\quad (v_2, S_2) = \mathit{deref}_{S_1}(t_{2\mathcal{W}}) \\
&\quad \text{in } (\pi_{\mathcal{W}} < v_1, v_2 >, S_2)
\end{aligned}$$

Since a is a fresh variable, it occurs linearly.
And we have

$$\begin{aligned}
&\mathit{deref}_{\Gamma_{\mathcal{W}}, a_1 \mapsto \pi_{\mathcal{W}}(t_1{}^*)_{\mathcal{W}}, a_2 \mapsto \pi_{\mathcal{W}}(t_2{}^*)_{\mathcal{W}}}((< a_1, a_2 >_\pi)_{\mathcal{W}}) \\
= &\mathit{deref}_{\Gamma_{\mathcal{W}}, a_1 \mapsto \pi_{\mathcal{W}}(t_1{}^*)_{\mathcal{W}}, a_2 \mapsto \pi_{\mathcal{W}}(t_2{}^*)_{\mathcal{W}}}(\pi_{\mathcal{W}} \ < a_1, a_2 >)
\end{aligned}$$

since a_1 and a_2 are fresh variables, they can be removed from the heap

$$\begin{aligned}
= &\mathit{deref}_{\Gamma_{\mathcal{W}}}(\pi_{\mathcal{W}} < (t_1{}^*)_{\mathcal{W}}, (t_2{}^*)_{\mathcal{W}} >) \\
= &\text{let } (v_1{}', S_1{}') = \mathit{deref}_{\Gamma_{\mathcal{W}}}((t_1{}^*)_{\mathcal{W}}) \\
&\quad (v_2{}', S_2{}') = \mathit{deref}_{S_2}((t_2{}^*)_{\mathcal{W}}) \\
&\text{in } (\pi_{\mathcal{W}} < v_1{}', v_2{}' >, S_2{}')
\end{aligned}$$

By Lemma 9, we know that $\mathit{deref}_{\Gamma_{\mathcal{W}}}((t_1{}^*)_{\mathcal{W}}) = \mathit{deref}_{\Gamma_{\mathcal{W}}}(t_{1\mathcal{W}})$, which means that $S_1 = S_1{}'$, then we also know, by Lemma 9, that

$$\mathit{deref}_{S_1{}'}((t_2{}^*)_{\mathcal{W}}) = \mathit{deref}_{S_1}(t_{2\mathcal{W}})$$

And the property holds.

– Inductive case:
  - Given $(\Gamma, x :_\pi A = t; x^*) = (\Gamma, x :_\pi A = t; x)$, there are two cases:
    * $\pi = 1$
      We have
      $$\frac{(\Gamma; t) \Downarrow (\Delta; \sigma)}{(\Gamma, x :_1 A = t; x) \Downarrow (\Delta; \sigma)}\ \textit{s-linvar}_{\mathcal{L}}$$
      Since we know that $(\Gamma; t) \Downarrow (\Delta; \sigma)$ then, by induction hypothesis, $(\Gamma_{\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S; v)$ and $\mathit{deref}_S(v) = \mathit{deref}_{\Delta_{\mathcal{W}}}(\sigma_{\mathcal{W}})$. So the property holds.
    * $\pi = \omega$
      We have
      $$\frac{(\Gamma; t) \Downarrow (\Delta; \sigma)}{(\Gamma, x :_\omega A = t; x) \Downarrow (\Delta, x :_\omega A = \sigma; \sigma)}\ \textit{s-unvar}_{\mathcal{L}}$$
      Since we know that $(\Gamma; t) \Downarrow (\Delta; \sigma)$ then, by induction hypothesis, $(\Gamma_{\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S; v)$ and $\mathit{deref}_S(v) = \mathit{deref}_{\Delta_{\mathcal{W}}}(\sigma_{\mathcal{W}})$. We know that $x \notin \mathit{fv}(\sigma)$, so we can conclude that
      $$\begin{aligned}
      \mathit{deref}_{\Delta_{\mathcal{W}}, x \mapsto \text{ un } \sigma_{\mathcal{W}}}(\sigma_{\mathcal{W}}) &= \mathit{deref}_{\Delta_{\mathcal{W}}}(\sigma_{\mathcal{W}}) \\
      &= \mathit{deref}_S(v)
      \end{aligned}$$
      So the property holds.

- Given $(\Gamma; (t\ x)^*) = (\Gamma; t^*\ x)$, we have

$$\dfrac{(\Gamma; t^*) \Downarrow (\Delta; \lambda_\pi y_1 : P.t'^*) \qquad (\Delta; t'^*[y_1 \mapsto x]) \Downarrow (\Theta; \sigma)}{(\Gamma; t^*\ x) \Downarrow (\Theta; \sigma)}\ \textit{s-app}_{\mathcal{W}}$$

Now, we have

$$\begin{aligned}(\Gamma_{\mathcal{W}}; (t\ x)_{\mathcal{W}}) &= (\Gamma_{\mathcal{W}}; t_{\mathcal{W}}\ x_{\mathcal{W}})\\ &= (\Gamma_{\mathcal{W}}; t_{\mathcal{W}}\ x)\end{aligned}$$

Since we know that $(\Gamma; t^*) \Downarrow (\Delta; \lambda_\pi y_1 : P.t'^*)$ then, by induction hypothesis, $(\Gamma_{\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S; \pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e)$ and, since we know that

$$(\Delta_{\mathcal{W}}; (\lambda_\pi y_2 : P.t'^*)_{\mathcal{W}}) = (\Delta_{\mathcal{W}}; \pi_{\mathcal{W}}\lambda y : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.(t'^*)_{\mathcal{W}})$$

then

$$\textit{deref}_S(\pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e) = \textit{deref}_{\Delta_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y_1 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.(t'^*)_{\mathcal{W}})$$

We know that $(\Gamma_{\mathcal{W}}; t_{\mathcal{W}}\ x) \twoheadrightarrow (S; (\pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e)\ x)$.
We also know that, by rule *s-abs*$_{\mathcal{W}}$

$$\dfrac{}{(S; \pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e) \to (S, a \mapsto \pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e; a)}$$

which means that

$$(S; (\pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e)\ x) \to (S, a \mapsto \pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e; a\ x)$$

Since a is a fresh variable, we can remove it from the heap, then we have

$$\dfrac{}{(S, a \mapsto \pi_{\mathcal{W}}\ \lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e; a\ x) \to (S; e[y_2 \mapsto x])}$$

We know that $(\Delta; t'^*[y_1 \mapsto x]) \Downarrow (\Theta; \sigma)$ and we also know that

$$\textit{deref}_S(e) = \textit{deref}_{\Delta_{\mathcal{W}}}((t'^*)_{\mathcal{W}})$$

since, by Lemma 7, $t'^*[y_1 \mapsto x] = (t'[y_1 \mapsto x])^*$, then, by induction hypothesis, we have that, given $(\Delta; (t'[y_1 \mapsto x])^*) \Downarrow (\Theta; \sigma)$, we know, by α-conversion,

$$(S; e[y_2 \mapsto x]) \twoheadrightarrow (S'; v)$$

and $\textit{deref}_{S'}(v) = \textit{deref}_{\Theta_{\mathcal{W}}}(\sigma_{\mathcal{W}})$. And the property holds.

- Given $(\Gamma; (t_1\ t_2)^*) = (\Gamma; \mathrm{let}_\pi\ a : A = t_2{}^*\ \mathrm{in}\ t_1{}^*\ a)$, we have

$$\dfrac{\dfrac{(\Gamma, a :_\pi A = t_2{}^*; t_1{}^*) \Downarrow (\Delta; \lambda_\pi y_1 : P.t_1{}'^*) \qquad (\Delta; t_1{}'^*[y_1 \mapsto a]) \Downarrow (\Theta; \sigma)}{(\Gamma, a :_\pi A = t_2{}^*; t_1{}^*\ a) \Downarrow (\Theta; \sigma)}\ \textit{s-app}_{\mathcal{L}}}{(\Gamma; \mathrm{let}_\pi\ a : A = t_2{}^*\ \mathrm{in}\ t_1{}^*\ a) \Downarrow (\Theta; \sigma)}\ \textit{s-let}_{\mathcal{L}}$$

Now we have $(\Gamma_{\mathcal{W}}; (t_1\ t_2)_{\mathcal{W}}) = (\Gamma_{\mathcal{W}}; t_{1\mathcal{W}}\ t_{2\mathcal{W}})$. By Proposition 1, we know that, because $a \notin \mathit{fv}(t_1{}^*)$, $(\Gamma; t_1{}^*) \Downarrow (\Delta; \lambda_\pi y_1 : P.t_1{}'^*)$ then, by induction hypothesis,

$$(\Gamma_{\mathcal{W}}; t_{1\mathcal{W}}) \twoheadrightarrow (S; \pi_{\mathcal{W}}\ \lambda y_2 : \pi_{\mathcal{W}}\ A.\ e)$$

and $\textit{deref}_S(\pi_{\mathcal{W}}\ \lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e) = \textit{deref}_{\Delta_{\mathcal{W}}}(\pi_{\mathcal{W}}\ \lambda y_1 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ (t_1{}'^*)_{\mathcal{W}})$.
This means that $(\Gamma_{\mathcal{W}}; t_{1\mathcal{W}}\ t_{2\mathcal{W}}) \twoheadrightarrow (S; (\pi_{\mathcal{W}}\lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.e)\ t_{2\mathcal{W}})$.
Now we have, by rule *s-abs*$_{\mathcal{W}}$,

$$\frac{}{(S;(\pi_{\mathcal{W}}\lambda y_2:\pi_{\mathcal{W}}\ P.e)\rightarrow(S,a_1\mapsto\pi_{\mathcal{W}}\ \lambda y_2:\pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e;a_1)}$$

So we have $(S;(\pi_{\mathcal{W}}\ \lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e)\ t_{2\mathcal{W}}) \rightarrow (S, a_1 \mapsto \pi_{\mathcal{W}}\ \lambda y_2 : \pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e; a_1\ t_{2\mathcal{W}})$. By rule *s-app*$_{\mathcal{W}}$,

$$\frac{}{(S,a_1\mapsto\pi_{\mathcal{W}}\ \lambda y_2:\pi_{\mathcal{W}}\ P_{\mathcal{W}}.\ e;z\ t_{2\mathcal{W}})\rightarrow(S\overset{\pi_{\mathcal{W}}}{\sim}a_1;e[y_2\mapsto t_{2\mathcal{W}}])}$$

Since a_1 is a fresh variable, it occurs linearly regardless of $\pi_{\mathcal{W}}$, so we can use

$$S\overset{\pi_{\mathcal{W}}}{\sim}a_1 = S$$

We know that $(\Delta; {t_1}'^{*}[y_1 \mapsto a]) \Downarrow (\Theta;\sigma)$, we have $\mathit{deref}_S(e) = \mathit{deref}_{\Delta_{\mathcal{W}}}(({t_1}'^{*})_{\mathcal{W}})$ and, by Lemma 7, ${t_1}'^{*}[y_1 \mapsto a] = ({t_1}'[y_1 \mapsto a])^{*}$. We also know that, since a is a fresh variable then $a \notin \mathit{fv}(({t_1}'[y_1 \mapsto a])^{*})$, so by the Γ-Substitution Lemma, if $(\Delta;({t_1}'[y_1 \mapsto a])^{*}) \Downarrow (\Theta;\sigma)$ then

$$(\Delta\setminus\{a\};({t_1}'[y_1\mapsto t_2])^{*})\Downarrow(\Theta';\sigma')$$

and $\mathit{deref}_{\Theta}(\sigma) = \mathit{deref}_{\Theta'}(\sigma')$.
We know that $a \notin \mathit{fv}(\mathrm{e})$, so if $(\Delta\setminus\{a\};({t_1}'[y_1\mapsto t_2])^{*})\Downarrow(\Theta';\sigma')$ then, by induction hypothesis, we have that,

$$(S;e[y_2\mapsto t_{2\mathcal{W}}])\twoheadrightarrow(S';v)$$

which is true by α-conversion, and $\mathit{deref}_{S'}(v) = \mathit{deref}_{\Theta_{\mathcal{W}}}(\sigma_{\mathcal{W}})$. And the property holds.

- Given $(\Gamma;\ (\text{split}\ t_1\ \text{as}\ y,z\ \text{in}\ t_2)^{*}) = (\Gamma;\ \text{split}\ t_1^{*}\ \text{as}\ y,z\ \text{in}\ t_2^{*})$, we have

$$\frac{(\Gamma;\ t_1^{*})\Downarrow(\Delta;\ <y_1,z_1>_{\pi})\qquad(\Delta;\ t_2^{*}[y\mapsto y_1][z\mapsto z_1])\Downarrow(\Theta;\ z)}{(\Gamma;\ \text{split}\ t_1^{*}\ \text{as}\ y,z\ \text{in}\ t_2^{*})\Downarrow(\Theta;\ z)}\ \textit{s-split}_{\mathcal{L}}$$

Now we have

$$(\Gamma_{\mathcal{W}};(\text{split}\ t_1\ \text{as}\ y,z\ \text{in}\ t_2)_{\mathcal{W}})=(\Gamma_{\mathcal{W}};\text{split}\ t_{1\mathcal{W}}\ \text{as}\ y,z\ \text{in}\ t_{2\mathcal{W}})$$

We know that $(\Gamma;{t_1}^{*})\Downarrow(\Delta;<y_1,z_1>_{\pi})$ then, by induction hypothesis,

$$(\Gamma_{\mathcal{W}};t_{1\mathcal{W}})\twoheadrightarrow(S;\pi_{\mathcal{W}}<e_1,e_2>)$$

and $\mathit{deref}_S(\pi_{\mathcal{W}}<e_1,e_2>) = \mathit{deref}_{\Delta_{\mathcal{W}}}(\pi_{\mathcal{W}}<y_1,z_1>)$.
This means that

$$(\Gamma_{\mathcal{W}};\text{split}\ t_{1\mathcal{W}}\ \text{as}\ y,z\ \text{in}\ t_{2\mathcal{W}})\twoheadrightarrow(S;\text{split}\ \pi_{\mathcal{W}}<e_1,e_2>\ \text{as}\ y,z\ \text{in}\ t_{2\mathcal{W}})$$

Now we have, by rule *s-pair*$_{\mathcal{W}}$,

$$\frac{}{(S;\pi_{\mathcal{W}}<e_1,e_2>)\rightarrow(S,a\mapsto\pi_{\mathcal{W}}<e_1,e_2>;a)}$$

So we have

$$\begin{aligned}&(S; \text{split } \pi_{\mathcal{W}} < e_1, e_2 > \text{ as } y, z \text{ in } t_{2\mathcal{W}})\\ \to\ &(S, a \mapsto \pi_{\mathcal{W}} < e_1, e_2 >; \text{split } a \text{ as } y, z \text{ in } t_{2\mathcal{W}})\end{aligned}$$

By rule *s-split*$_{\mathcal{W}}$,

$$\begin{aligned}&(S, a \mapsto \pi_{\mathcal{W}} < e_1, e_2 >; \text{split } a \text{ as } y, z \text{ in } t_{2\mathcal{W}}))\\ \to\ &(S \overset{\pi_{\mathcal{W}}}{\sim} a; t_{2\mathcal{W}}[y \mapsto e_1][z \mapsto e_2])\end{aligned}$$

We know that a is a fresh variable, so it will not be used again regardless of $\pi_{\mathcal{W}}$, therefore we can use $S \overset{\pi_{\mathcal{W}}}{\sim} a = S$.
We know that, by Lemma 7,

$$t_2{}^*[y \mapsto y_1][z \mapsto z_1] = (t_2[y \mapsto y_1][z \mapsto z_1])^*$$

Then, since $(\Delta; (t_2[y \mapsto y_1][z \mapsto z_1])^*) \Downarrow (\Theta; \sigma)$ then, by induction hypothesis,

$$(\Delta_{\mathcal{W}}; (t_2[y \mapsto y_1][z \mapsto z_1])_{\mathcal{W}}) \twoheadrightarrow (S'; v')$$

and $deref_{\Theta_{\mathcal{W}}}(\sigma_{\mathcal{W}}) = deref_{S'}(v')$.
Since we know that $deref_S(\pi_{\mathcal{W}} < e_1, e_2 >) = deref_{\Delta_{\mathcal{W}}}(\pi_{\mathcal{W}} < y_1, z_1 >)$, the property holds.

- Given

$$\begin{aligned}&(\Gamma; (\text{let}_\pi\ a_1 : A_1 = t_1, \ldots, a_n : A_n = t_n \text{ in } t)^*)\\ =\ &(\Gamma; \text{let}_\pi\ a_1 : A_1 = t_1{}^*, \ldots, a_n : A_n = t_n{}^* \text{ in } t^*)\end{aligned}$$

we have

$$\frac{(\Gamma, a_1 :_\pi A_1 = t_1{}^*, \ldots, a_n :_\pi A_n = t_n{}^*; t^*) \Downarrow (\Delta; \sigma)}{(\Gamma; \text{let}_\pi\ a_1 : A_1 = t_1{}^*, \ldots, a_n : A_n = t_n{}^* \text{ in } t^*) \Downarrow (\Delta; \sigma)}\ \textit{s-let}_{\mathcal{L}}$$

Now we have

$$\begin{aligned}&(\Gamma_{\mathcal{W}}; (\text{let}_\pi\ a_1 : A_1 = t_1, \ldots, a_n : A_n = t_n \text{ in } t)_{\mathcal{W}})\\ =\ &(\Gamma_{\mathcal{W}}; t_{\mathcal{W}}[a_1 \mapsto t_{1\mathcal{W}}] \ldots [a_n \mapsto t_{n\mathcal{W}}])\end{aligned}$$

We know that if $(\Gamma, a_1 :_\pi A_1 = t_1{}^*, \ldots, a_n :_\pi A_n = t_n{}^*; t^*) \Downarrow (\Delta; \sigma)$ then, by induction hypothesis,

$$(\Gamma_{\mathcal{W}}, a_1 \mapsto \pi_{\mathcal{W}}(t_1{}^*)_{\mathcal{W}}, \ldots, a_n \mapsto \pi_{\mathcal{W}}(t_n{}^*)_{\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S; v)$$

and $deref_S(v) = deref_{\Delta_{\mathcal{W}}}(\sigma_{\mathcal{W}})$.
By Lemma 9, if $(\Gamma_{\mathcal{W}}, a_1 \mapsto \pi_{\mathcal{W}}(t_1{}^*)_{\mathcal{W}}, \ldots, a_n \mapsto \pi_{\mathcal{W}}(t_n{}^*)_{\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S; v)$ then

$$(\Gamma_{\mathcal{W}}, a_1 \mapsto \pi_{\mathcal{W}}\ t_{1\mathcal{W}}, \ldots, a_n \mapsto \pi_{\mathcal{W}}\ t_{n\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S_1; v_1)$$

and $deref_{S_1}(v_1) = deref_S(v)$. And by S-Substitution Lemma, we know that if $(\Gamma_{\mathcal{W}}, a_1 \mapsto \pi_{\mathcal{W}}\ t_{1\mathcal{W}}, \ldots, a_n \mapsto \pi_{\mathcal{W}}\ t_{n\mathcal{W}}; t_{\mathcal{W}}) \twoheadrightarrow (S_1; v_1)$ then

$$(\Gamma_{\mathcal{W}}; t_{\mathcal{W}}[a_1 \mapsto t_{1\mathcal{W}}] \ldots [a_n \mapsto t_{n\mathcal{W}}]) \twoheadrightarrow (S_2; v_2)$$

and $deref_{S_2}(v_2) = deref_{S_1}(v_1)$.
We can then conclude that $deref_{\Delta_{\mathcal{W}}}(\sigma_{\mathcal{W}}) = deref_{S_2}(v_2)$, so the property holds.

$\square$